Available online at www.sciencedirect.com

ScienceDirect

IFAC PapersOnLine 53-2 (2020) 4582–4587

Towards a Simple Sampled-Data Control Law for Stably Invertible Linear Systems ⋆

Claudia Sánchez * **Graham C. Goodwin** ** **Juan I. Yuz** *
María Serón ** **Diego Carrasco** **

* *Department of Electronic Engineering, Universidad Técnica Federico Santa María, Avenida España 1680, Casilla 110-V, Valparaíso, Chile (e-mails: claudia.sanchez@sansano.usm.cl, juan.yuz@usm.cl)*
** *Priority Research Centre CDSC, School of Electrical Engineering and Computing, University of Newcastle, Callaghan, NSW 2308, Australia (e-mails: graham.goodwin@newcastle.edu.au, maria.seron@newcastle.edu.au, diego.carrascoyanez@newcastle.edu.au)*

Abstract: A new high gain control law is proposed for stably invertible linear systems. The continuous-time case is first studied to set ideas. The extension to the sampled-data case is made difficult by the presence of sampling zeros. For continuous-time systems having relative degree greater than or equal to two, these zeros converge, as the sampling rate approaches zero, to either marginally stable or unstable locations. A methodology which specifically addresses the sampling zero issue is developed. The methodology uses an approximate model which includes, when appropriate, the asymptotic sampling zeros. The core idea is supported by simulation studies. Also, a preliminary theoretical analysis is provided for degree two, showing that the design based on the approximate model stabilizes the true system for the continuous and sampled-data cases.

Keywords: Linear systems, sampled-data systems, sampling zeros, sampled-data control, robustness.

1. INTRODUCTION

The heuristic idea underlying the current paper is that, at high frequencies, a continuous-time system of relative degree r behaves similarly to an r-th order pure integrator. The latter approximation has been applied, for example, in Gayaka et al. (2012); Zhou and Duan (2008); Sánchez and Yuz (2019). The idea is also central to the development of the theory of sampling zeros (see Åström et al. (1984)). This heuristic principle implies that one should be able to design a wide-bandwidth control law for stably invertible linear system by knowing only the relative degree and high frequency gain.

The extension of the above idea to the discrete-time domain faces an extra difficulty, namely the existence of sampling zeros. In Åström et al. (1984) and Weller et al. (2001) it is shown that, for fast sampling rate, these zeros can be asymptotically characterized in terms of the roots of the Euler-Frobenius polynomials. Moreover, substantial literature exists regarding the asymptotic location of those sampling zeros, which depends, inter-alia, on the sampling period and the nature of the hold device that generates the input signal.

For the purpose of the current paper, the input is considered to be piecewise constant, i.e., generated by a zero-order hold (ZOH). Related results on the asymptotic location of the sampling zeros are available for cases when the input is generated by a first-order hold (Hagiwara et al., 1993), by a fractional-order hold (Bárcena et al., 2000; Ishitobi and Kunimatsu, 2016) or by B-spline functions (Sánchez and Yuz, 2019). Sampling zeros are known to be relevant to cases when a more accurate model is needed for higher frequencies, in particular near the Nyquist rate (Yucra and Yuz, 2011; Goodwin et al., 2013).

For continuous-time systems having relative degree two, the asymptotic sampling zero is on the stability boundary and for relative degree larger than two, the asymptotic zeros lie outside the stability region. The existence of these non-minimum phase zeros in the model represents a major stumbling block, in particular for wide-bandwidth control when the closed loop bandwidth approaches the Nyquist rate for the given sampling period (Middleton, 1991).

The current paper explores the above ideas for SISO systems. We first analyse the continuous-time case. Some motivating examples are presented supporting the claim that wide-bandwidth control laws can be designed based only on the relative degree and high frequency gain. Next the sampled-data version of the problem is examined when the sampling period is small. Two approximate models are studied: the first model covers the case where the closed loop bandwidth is chosen to be significantly less than the

⋆ This work was supported by ANID through scholarship ANID-PFCHA/2018-21180825 and ANID-Basal Project FB0008, FONDECYT grant 1181090, UTFSM research scholarship and visiting student scholarship from the Priority Research Centre for Complex Dynamic Systems and Control, UoN, Australia.

10.1016/j.ifacol.2020.12.483

Nyquist frequency. The second model covers the case when the closed loop bandwidth is near the Nyquist frequency. In the latter case, the sampling zeros are included in the approximate model. The robustness properties of these two models differ due to the presence of sampling zeros.

2. CONTINUOUS-TIME SYSTEMS

In this section we explore the design of a control law for continuous-time systems having relative degree r.

2.1 System Having No Zeros

Consider an r-th order continuous-time system having transfer function

$$G_c(s) = \frac{B(s)}{A(s)} = \frac{b}{s^r + a_1 s^{r-1} + \dots a_r}. \tag{1}$$

We note that the high frequency gain is b, with associated roll-off of $20r$ dB per decade. Our goal is to carry out a continuous-time design based on the following approximate model

$$G_0(s) = \frac{B_0(s)}{A_0(s)} = \frac{b}{s^r}. \tag{2}$$

A multitude of continuous-time designs are possible. By way of illustration, we will use pole-assignment (see, for example, Goodwin et al. (2001)). A biproper controller of order $r-1$ can be used

$$C(s) = \frac{P(s)}{L(s)} = \frac{p_0 s^{r-1} + \dots + p_{r-1}}{s^{r-1} + \dots + l_{r-1}}, \tag{3}$$

with a target closed loop polynomial of the form

$$A^*_{cl}(s) \doteq A_0(s)L(s) + B_0(s)P(s) = (s+\alpha^*)^{2r-1}. \tag{4}$$

By equating coefficients, the controller parameters can be obtained as follows:

$$l_1 = (2r-1)\alpha^* \tag{5a}$$

$$\vdots$$

$$p_{r-1} = \frac{(\alpha^*)^{2r-1}}{b}. \tag{5b}$$

Our core hypothesis is that for $\alpha* > 0$ and sufficiently large, a control law based on the model (2) will stabilize all systems of the form (1) provided the open loop poles, i.e., the roots of the polynomial $A(s)$, lie in a restricted region. Moreover, for α^* large enough, the closed loop performance of the 'true' and approximate models will be essentially indistinguishable. We introduce the following assumption on the location of the open loop poles of the 'true' system.

Assumption 1. We assume that the poles, α_i, $i = 1, \dots, r$ of $G(s)$, i.e., the roots of the polynomial $A(s)$ in (1), belong to a bounded region in the complex plane, such that $|\alpha_i| < M$.

We study the second order case in detail. Thus, we consider the system

$$G_c(s) = \frac{b}{(s+\alpha_1)(s+\alpha_2)}, \tag{6}$$

which leads to the model (2), with $r = 2$, i.e.,

$$G_0(s) = \frac{b}{s^2}. \tag{7}$$

Considering (3), a suitable proper controller is given by

$$C(s) = \frac{p_0 s + p_1}{s + l_1}. \tag{8}$$

The pole assignment equations (5), particularized to this case, yield the controller parameters

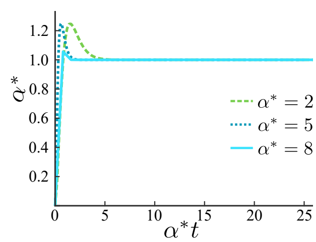
$$p_1 = \frac{(\alpha^*)^3}{b}, \qquad p_0 = \frac{3(\alpha^*)^2}{b}, \qquad l_1 = 3\alpha^*. \tag{9}$$

2.2 Numerical Example (Continuous-time)

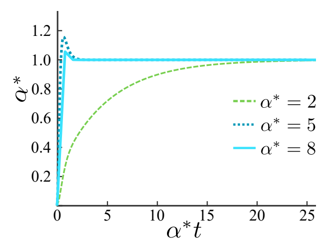
As a specific numerical example, we consider $b = 1$, $\alpha_1 = 1$, $\alpha_2 = -1$. Based on the above discussion and Assumption 1, we consider several values of α^* satisfying $\alpha^* > M > |\alpha_{1,2}|$. Figure 1(a) shows the resulting normalized step responses of the closed loop transfer function

$$T_0(s) = \frac{G_0(s)C(s)}{1 + G_0(s)C(s)}, \tag{10}$$

achieved when using the 'nominal' model (7). On the other hand, Figure 1(b) shows the normalized step responses of the closed loop system $T(s)/T(0) = G_c(s)C(s)/(T(0)(1 + G_c(s)C(s)))$ when the same controller is applied to the 'true' plant (6).



(a) Nominal Closed Loop



(b) True Closed Loop

Fig. 1. Step response for nominal and true closed loop.

It can be seen in Figure 1(b) that the 'true' closed loop responses are very similar to the nominal responses in Figure 1(a) except for the case $\alpha^* = 2$. An important question to be studied in the remainder of the paper is to derive a relationship between the bound M in Assumption 1 and the target closed loop pole location α^*, which is sufficient for the controller (8)–(9), designed for the approximate system (7), to also stabilize the true system (6).

3. A FIRST APPROACH USING THE ROBUST STABILITY THEOREM

An initial attempt to develop a suitable theory was to use the well-known sufficient condition for robust stability (see, for example, Goodwin et al. (2001)), namely

$$|T_0(s)G_\Delta(s)| < 1; \qquad \forall\, s = jw, \tag{11}$$

where $T_0(s)$ is the nominal closed loop complementary sensitivity function, defined in (10), and $G_\Delta(s)$ is the relative model error, i.e.,

$$G_\Delta(s) = \left| \frac{G(s) - G_0(s)}{G_0(s)} \right|. \tag{12}$$

Particularizing to the above problem, we have that

$$T_0(s) = \frac{B_0(s)P(s)}{A^*_{cl}(s)}, \tag{13}$$

$$G_\Delta(s) = -\frac{(\alpha_1+\alpha_2)s + \alpha_1\alpha_2}{(s+\alpha_1)(s+\alpha_2)} \tag{14}$$

Taking $s = jw$ and considering $w = 0$, yields to

$$T_0(j0) = 1, \qquad G_\Delta(j0) = -1. \tag{15}$$

Hence, condition (11) is not satisfied and robust stability cannot be guaranteed using this approach.

4. OSTROWSKI'S THEOREM

A second approach is to study whether, under some conditions, the roots of the 'nominal' and 'true' closed-loop polynomials are close in some sense. This is one of the core problems in perturbation theory. A useful result in this context is the Ostrowski's Theorem (Ostrowski, 1973, page 276), stated below.

Theorem 2. Consider two polynomials

$$f(s) = a_0 s^n + a_1 s^{n-1} + \cdots + a_n = \prod_{i=1}^{n}(s - \theta_i^f) \tag{16}$$

$$g(s) = b_0 s^n + b_1 s^{n-1} + \cdots + b_n = \prod_{i=1}^{n}(s - \theta_i^g), \tag{17}$$

where $a_0 = b_0 = 1$. Let

$$T = 2 \max_{1\le k\le n} \left(|a_k|^{1/k}, |b_k|^{1/k}\right). \tag{18}$$

Then the roots θ_i^f and θ_i^g of $f(s)$ and $g(s)$ can be enumerated in such a way that

$$\max_i |\theta_i^f - \theta_i^g| \le (2n-1)\left\{\sum_{k=1}^{n} |a_k - b_k| T^{n-k}\right\}^{1/n}. \tag{19}$$

Proof. See Ostrowski (1973). □

Remark 3. The above result provides a means of estimating the differences between the roots θ_i^f and θ_i^g in terms of the coefficients a_i and b_i. The result was embellished by Elsner (1982), who showed that the factor $(2n-1)$ can be replaced by $(n-1)$ if n is even and by n when is odd.

We will use Theorem 2 to obtain a sufficient condition for asymptotic stability of the closed-loop system formed by the true system (6) when the controller (8)–(9), designed for the approximate system (7), is deployed. Again, we study the second order case for the model (6). However, a similar strategy is anticipated to apply for the general case (1).

Theorem 4. Subject to Assumption 1, a sufficient condition for closed loop stability of the true plant $G_c(s)$, given in (6), under controller (8)-(9), is $\alpha^* \ge \kappa M$, for some sufficiently large positive real number κ.

Proof. The gain cancels in the controller, so without loss of generality, we take $b = 1$. Thus, the system (6) can by rewritten as follows

$$G_c(s) = \frac{1}{s^2 + t_1 s + t_2} = \frac{B(s)}{A(s)}, \tag{20}$$

where $t_1 = \alpha_1 + \alpha_2$ and $t_2 = \alpha_1\alpha_2$. Also, we notice that the polynomials (16)-(17) for this particular problem are given by

$$f(s) = s^3 + 3(\alpha^*)s^2 + 3(\alpha^*)^2 s + (\alpha^*)^3 \tag{21}$$

$$\begin{aligned} g(s) &= A(s)L(s) + B(s)P(s) \\ &= s^3 + (l_1+t_1)s^2 + (t_1 l_1 + t_2 + p_0)s + (t_2 l_1 + p_1). \end{aligned} \tag{22}$$

Replacing the parameters in (9), considering Assumption 1 and defining $S = \alpha^*/M$, then bounds for the coefficients in (21) and (22) are given by

$$|a_1| \le 3SM, \qquad |b_1| \le 2M + 3SM, \tag{23a}$$

$$|a_2| \le 3S^2M^2, \qquad |b_2| \le 6SM^2 + M^2 + 3S^2M^2, \tag{23b}$$

$$|a_3| \le S^3M^3, \qquad |b_3| \le 3SM^3 + S^3M^3. \tag{23c}$$

Then, based on Remark 3 and using (19), we have

$$\max_i |\theta_i^f - \theta_i^g| \le 3\left\{\sum_{k=1}^{3} |a_k - b_k| T^{n-k}\right\}^{1/3}, \tag{24}$$

where

$$\begin{aligned} T \le{}& 2\max\Big(3SM, \sqrt{3}SM, SM, 3SM + 2M, \\ & (6SM^2 + M^2 + 3S^2M^2)^{1/2}, (3SM^3 + S^3M^3)^{1/3}\Big). \end{aligned}$$

The first 3 elements do not contribute to the maximization since they are smaller than the fourth element, yielding:

$$\begin{aligned} T &\le 2\max\Big((3S+2)M, ((3S^2+6S+1)M^2)^{1/2}, \\ &\qquad ((S^3+3S)M^3)^{1/3}\Big) \\ &\le 2(3S+2)M. \end{aligned} \tag{25}$$

where the last inequality holds true for any $S \ge 0$. Replacing T in (24) and bounding each term $|a_k - b_k|$, we obtain

$$\begin{aligned} &\max_i |\theta_i^f - \theta_i^g| \\ &\le 3\big\{(2M)T^2 + (6SM^2 + M^2)T + (3SM^3)\big\}^{1/3} \\ &\le 3M\left\{8(3S+2)^2 + 2(6S+1)(3S+2) + 3S\right\}^{1/3} \\ &\le 3M\left\{108S^2 + 129S + 36\right\}^{1/3} \end{aligned} \tag{26}$$

Thus, closed loop stability is ensured if the right hand side of the above inequality is less than $|\alpha^*| = SM$. This yields

$$\begin{aligned} &\max_i |\theta_i^f - \theta_i^g| < |\alpha^*| \\ &\iff 3M\left(108S^2 + 129S + 36\right)^{1/3} < SM \\ &\iff S^3 - 27(36 + 129S + 108S^2) > 0 \end{aligned} \tag{27}$$

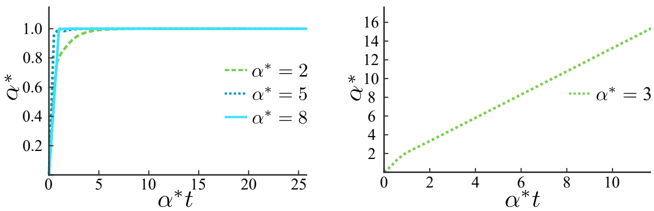
which holds true for $S = \kappa \ge 2918$. □

5. STABLY INVERTIBLE CONTINUOUS-TIME SYSTEMS

We hypothesize that a similar idea holds for general systems provided that the polynomial $B(s)$ is Hurwitz. We illustrate via a numerical study based on the following third order system

$$G_c(s) = \frac{b(s+\beta_1)}{(s+\alpha_1)(s+\alpha_2)(s+\alpha_3)} \tag{28}$$

where $\beta_1 > 0$. Note that the system has relative degree $r = 2$. We consider again the approximate model (7) and design a control law to place all the closed loop poles at $s = -\alpha^*$. This leads to the same control law (8) with parameters as in (9). As a specific case of the system (28) we choose $b = 2$ $\beta_1 = 4$, $\alpha_1 = -2$, $\alpha_2 = 2$, $\alpha_3 = 3$. Simulations of the normalized true closed loop response for different values of α^* are shown in Figure 2. We notice that, similarly to Example 2.2, as long as α^* is sufficiently large the true closed loop system is stable (see Figure 2(a)). When α^* is not sufficiently large, then the system can be unstable Figure 2(b).



(a) $\alpha^* \geq 4$ (b) $\alpha^* = 3$

Fig. 2. Step response for normalized true closed loop.

6. DISCRETE-TIME SYSTEMS

We consider the same set of underlying continuous-time systems as studied in Section 2. We express the time discretization of these systems in the δ-domain, where $\delta = \frac{q-1}{h}$ and q is the usual forward shift operator. In the associated complex variable domain we use the variable $\gamma = \frac{z-1}{h}$, where z is the complex variable in the Z-transform domain Middleton and Goodwin (1990).

With sampling period h and assuming that the input is generated by a ZOH, we obtain an exact discrete-time model of the form

$$G(\gamma) = \frac{b'\, P_r(h\gamma)(\gamma^m + c'_{m-1}\gamma^{m-1} + \cdots + c'_0)}{\gamma^n + d'_{n-1}\gamma^{n-1} + \cdots + d'_0} \tag{29}$$

where $P_r(h\gamma)$ is the sampling zeros polynomial. It is well-known that for small h, $P_r(h\gamma)$ converges to the asymptotic sampling zeros polynomial, $\bar{P}_r(h\gamma)$. The first few of these are (Yuz and Goodwin, 2014, 2005):

$$\bar{P}_1(h\gamma) = 1 \tag{30a}$$

$$\bar{P}_2(h\gamma) = 1 + \frac{h}{2}\gamma \tag{30b}$$

$$\bar{P}_3(h\gamma) = 1 + h\gamma + \frac{h^2}{6}\gamma^2 \tag{30c}$$

In order to design a discrete-time control law, we again use pole assignment. As before, the proposed biproper controller takes the form:

$$C(\gamma) = \frac{P'(\gamma)}{L'(\gamma)} = \frac{p'_0\gamma^{r-1} + \ldots + p'_{r-1}}{\gamma^{r-1} + \ldots + l'_{r-1}}. \tag{31}$$

Let the poles and the zeros of the discrete-time system be $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_m$, respectively. For the discrete-time design, we place the closed loop poles at the roots of $(\gamma + \alpha^*)^{2r-1}$ where $r = n - m$. We make the following assumption which defines a region where the open loop poles and zeros lie.

Assumption 5. The poles and zeros of the discretized system belong to a bounded region in the complex plane, such that $|\alpha_i, \beta_i| < R < \frac{1}{h}$.

For the control law design we consider two approximate discrete-time models:

$$G_d^1(\gamma) = \frac{b'}{\gamma^r} \tag{32}$$

$$G_d^2(\gamma) = \frac{b'\,\bar{P}_r(h\gamma)}{\gamma^r} \tag{33}$$

Note that the model (32) does not have any zeros whilst the model (33) includes the asymptotic sampling zeros. When carrying out pole assignment, for the model (32) we choose $\alpha^* \ll \frac{1}{h}$ and for the model (33) $\alpha^* \leq \frac{1}{h}$.

6.1 The Case of Continuous-time Second Order Systems with no Finite Zeros

Analogously to Section 2.1, we illustrate ideas by the second order case. Thus, consider a true discrete-time system given by

$$G_d(\gamma) = \frac{b'\,P_r(h\gamma)}{\gamma^2 + t'_1\gamma + t'_2} = \frac{b'\,(1 + \nu\frac{h}{2}\gamma)}{\gamma^2 + t'_1\gamma + t'_2} \tag{34}$$

Note that ν tends to 1 as h approaches zero. For the system (34), the proposed controller is given by

$$C(\gamma) = \frac{p'_0\gamma + p'_1}{\gamma + l'_1}. \tag{35}$$

Notice that, based on the approximate model $G_d^1(\gamma)$ in (32), the parameters of the controller are as in (9) with s replaced by γ. On the other hand, for the approximate model (33), with $\bar{P}_2(h\gamma) = (1 + \gamma\frac{h}{2})$, the parameters are

$$p'_1 = \frac{(\alpha^*)^3}{b'}, \qquad p'_0 = \frac{3(\alpha^*)^2}{b'} - \frac{h}{2}\frac{(\alpha^*)^3}{b'}, \tag{36}$$

$$l'_1 = 3\alpha^* - 3\frac{h}{2}(\alpha^*)^2 + \frac{h^2}{4}(\alpha^*)^3. \tag{37}$$

In what follows we often use the approximation $\nu \approx 1$, which is valid as the sampling period approaches zero. We have the following result:

Theorem 6. Consider the discrete-time model of the form (34). Then, for the control law design based on $G_d^1(\gamma)$ in (32) and subject to Assumption 5, a sufficient condition for closed loop stability of the true closed loop is that the nominal closed loop poles at $\gamma = -\alpha^*$ satisfy $\frac{1}{h} \gg \alpha^* \gg |\alpha_{1,2}|$, i.e., $\frac{1}{h} \gg \alpha^* > \kappa' R$ for some sufficiently large positive real number κ'.

Proof. As before, without loss of generality, we take $b' = 1$ and we consider a controller designed for the approximate model given by

$$G_d^1(\gamma) = \frac{1}{\gamma^2}, \tag{38}$$

and apply the controller to the true system (34). We use again the Ostrowski's Theorem, where the polynomials are

$$f(\gamma) = \gamma^3 + 3(\alpha^*)\gamma^2 + 3(\alpha^*)^2\gamma + (\alpha^*)^3 \tag{39}$$

$$g(\gamma) = \gamma^3 + (l'_1 + t'_1 + p'_0\frac{\nu h}{2})\gamma^2 + (t'_1 l'_1 + t'_2 + p'_0 + p'_1\frac{\nu h}{2})\gamma + (t'_2 l'_1 + p'_1). \tag{40}$$

Substituting parameters from (9), we consider Assumption 5 and define $K = \alpha^*/R$. Then,

$$|a_1| \le 3KR, \quad |b_1| \le 2R + 3KR + 3\tfrac{\nu h}{2}K^2R^2, \quad (41a)$$
$$|a_2| \le 3K^2R^2, \quad |b_2| \le 6KR^2 + R^2 + 3K^2R^2 + \tfrac{\nu h}{2}K^3R^3 \quad (41b)$$
$$|a_3| \le K^3R^3, \quad |b_3| \le 3KR^3 + K^3R^3. \quad (41c)$$

Then, proceeding as in the proof of Theorem 4, we have that

$$T \le 2\max\Big(3KR + 2R + 3\tfrac{\nu h}{2}(KR)^2, (3KR^3 + K^3R^3)^{1/3}, (6KR^2 + R^2 + 3K^2R^2 + \tfrac{\nu h}{2}(KR)^3)^{1/2}\Big)$$

where, as mentioned before, the first 3 terms in the maximization have been discarded. Moreover, we have that $\nu \approx 1$, for small h, and hence $\nu hKR \approx hKR = h\alpha^* \ll 1$. Thus,

$$\begin{aligned} T &\le 2\max\Big((4.5K+2)R, (3K+K^3)^{1/3}R \\ &\quad (6K + 1 + 3K^2 + 0.5K^2)^{1/2}R\Big) \\ &\le 2(4.5K+2)R, \end{aligned} \quad (42)$$

where the last inequality holds true for any $K \ge 0$. Thus, the distance between the roots of the polynomials is bounded as

$$\begin{aligned} \max_i |\theta_i^f - \theta_i^g| &\le 3\left\{(2R + 3\tfrac{\nu h}{2}K^2R^2)T^2 + \right. \\ &\quad \left.(6KR^2 + R^2 + \tfrac{\nu h}{2}K^3R^3)T + 3KR^3\right\}^{1/3} \\ &\le 3\left\{(2R + \tfrac{3}{2}R)(9K+4)^2R^2 + \right. \\ &\quad \left.(6KR^2 + R^2 + \tfrac{1}{2}KR^2)(9K+4)R + 3KR^3\right\}^{1/3} \\ &\le 3R\left\{60 + 290K + 342K^2\right\}^{1/3}. \end{aligned} \quad (43)$$

where we have used that, since $\frac{1}{h} \gg KR$, then we have that

$$hKR \ll 1 \implies hK^2R < 1 \implies \nu hK^2R < 1 \quad (44)$$

since $\nu \approx 1$ as $h \to 0$. Then, closed loop stability is guaranteed if the right hand side is less than $|\alpha^*| = KR$. This yields

$$\begin{aligned} &\max_i |\theta_i^f - \theta_i^g| < |\alpha^*| \\ \iff &3R\left(60 + 290K + 342K^2\right)^{1/3} < KR \\ \iff &K^3 - 27(342K^2 + 290K + 60) > 0 \end{aligned} \quad (45)$$

Thus, taking $K = \kappa' \ge 9235$ the theorem is proved. □

Theorem 7. Consider the discrete-time model of the form (34). Then, for a control law design based on (33) and subject to Assumption 5, a sufficient condition for closed loop stability of the corresponding sampled-data model is that the closed loop poles satisfy $\frac{1}{h} \ge \alpha^* > \kappa' R$ for some sufficiently large positive real number κ'.

Proof. Again, without loss of generality, we take $b' = 1$. Our focus is to design a controller based on the approximate model that includes the asymptotic sampling zeros. For the second order case we have

$$G_d^2(\gamma) = \frac{\bar{P}_2(h\gamma)}{\gamma^2} = \frac{1 + \gamma\frac{h}{2}}{\gamma^2}. \quad (46)$$

We use Ostrowski's Theorem, where the polynomials are the same as shown in (39)-(40), but with the parameters defined in (36)-(37). Then, considering Assumption 5, defining $K = \alpha^*/R$ and using the triangle inequality, we have that the bounds for a_i; $i = 1, 2, 3$ are given by (41) and

$$\begin{aligned} |b_1| &\le 2R + 3KR + \tfrac{3}{2}h(KR)^2|\nu - 1| \\ &\quad + \tfrac{h^2}{4}(KR)^3|1-\nu| \end{aligned} \quad (47a)$$
$$\begin{aligned} |b_2| &\le 6KR^2 + R^2 + \tfrac{h}{2}(KR)^3|1-\nu| + 3(KR)^2 \\ &\quad + \tfrac{h^2}{2}K^3R^4 + 3hK^2R^3 \end{aligned} \quad (47b)$$
$$|b_3| \le 3KR^3 + (KR)^3 + \tfrac{3}{2}hK^2R^4 + \tfrac{h^2}{4}K^3R^5. \quad (47c)$$

As before, we find a bound on T defined in (18). We then consider

$$T \le 2\max\left(3KR, \sqrt{3}KR, KR, |b_1|, |b_2|^{1/2}, |b_3|^{1/3}\right). \quad (48)$$

In order to find the maximum, we consider that

$$KR < 1/h \implies hKR < 1 \quad (49)$$
$$\nu \to 1 \implies |1 - v|K < 1 \quad (50)$$

Then, we have that

$$\begin{aligned} T &\le 2\max\big(R(2 + 3K + \tfrac{3}{2} + \tfrac{1}{4}), \\ &\quad R\left(6K + 1 + \tfrac{1}{2}K + 3K^2 + \tfrac{1}{2}K + 3K\right)^{1/2}, \\ &\quad R\left(3K + K^3 + \tfrac{3}{2}K + \tfrac{1}{4}K\right)^{1/3}\Big) \\ &\le 2(3K + \tfrac{15}{4})R, \end{aligned} \quad (51)$$

where the last inequality holds true for any $K \ge 0$. Replacing T in (19) and using Remark 3, the distance between the roots of the polynomials can be bounded as

$$\begin{aligned} &\max_i |\theta_i^f - \theta_i^g| \\ &\le 3\left\{\left(2R + \tfrac{3}{2}hK^2R^2|\nu - 1| + \tfrac{h^2}{4}K^3R^3|1-\nu|\right)T^2 + \right. \\ &\left(R^2 + 6KR^2 + 3hK^2R^3 + \tfrac{h^2}{2}K^3R^4 + \tfrac{h}{2}K^3R^3|1-\nu|\right)T \\ &\quad \left. + \left(3KR^3 + \tfrac{3}{2}hK^2R^4 + \tfrac{h^2}{4}K^3R^5\right)\right\}^{1/3} \\ &\le 3\left\{\left(2R + \tfrac{3}{2}R + \tfrac{1}{4}R\right)4(3K + \tfrac{15}{4})^2R^2 + \right. \\ &\left(R^2 + 6KR^2 + 3KR^2 + \tfrac{1}{2}KR^2 + \tfrac{1}{2}KR^2\right)2(3K + \tfrac{15}{4})R \\ &\quad \left. + \left(3KR^3 + \tfrac{3}{2}KR^3 + \tfrac{1}{4}KR^3\right)\right\}^{1/3} \\ &\le 3R\left\{\frac{3495}{16} + \frac{1693}{4}K + 195K^2\right\}^{1/3} \end{aligned} \quad (52)$$

Thus, closed loop stability is guaranteed if the right hand side is less than $|\alpha^*| = KR$. This yields

$$\begin{aligned} &\max_i |\theta_i^f - \theta_i^g| < |\alpha^*| \\ \iff &3R\left(\frac{3495}{16} + \frac{1693}{4}K + 195K^2\right)^{1/3} < KR \\ \iff &K^3 - 27\left(\frac{3495}{16} + \frac{1693}{4}K + 195K^2\right) > 0 \end{aligned} \quad (53)$$

Thus, taking $K = \kappa' \ge 5268$ the theorem is proved. □

6.2 Numerical Example (Sampled-Data)

We consider a second-order plant $G_c(s)$ of the form (6). For a specific numerical example we choose $b = -6$, $\alpha_1 = 3$, $\alpha_2 = -2$. Then, its discrete-time model depends on the sampling period chosen. When varying h and α^*, the nature of the closed loop response of the true system with the controller designed based on the approximation (32)

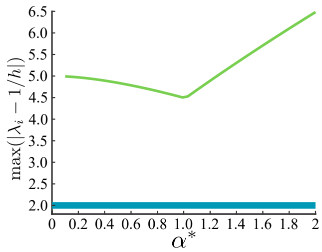


Fig. 3. Illustration of instability for $h = 0.5$.

is shown in Table 1. Note that the blank entries are not relevant since we consider only the case where $\alpha^* \leq 1/h$. The closed loop performance using the controller designed based on the model (33) is shown in Table 2.

α^* \ $1/h$	10	20	100	1,000	10,000
5	Stable	Stable	Stable	Stable	Stable
10	**Unstable**	Stable	Stable	Stable	Stable
16		**Unstable**	Stable	Stable	Stable
20		**Unstable**	Stable	Stable	Stable
50			Stable	Stable	Stable
70			**Unstable**	Stable	Stable
100			**Unstable**	Stable	Stable
500				Stable	Stable
800				**Unstable**	Stable
1000				**Unstable**	Stable
5000					Stable
10000					**Unstable**

Table 1. Robustness of the discrete-time model without considering sampling zeros

α^* \ $1/h$	10	20	100	1,000	10,000
5	Stable	Stable	Stable	Stable	Stable
10	Stable	Stable	Stable	Stable	Stable
16		Stable	Stable	Stable	Stable
20		Stable	Stable	Stable	Stable
50			Stable	Stable	Stable
70			Stable	Stable	Stable
100			Stable	Stable	Stable
500				Stable	Stable
800				Stable	Stable
1000				Stable	Stable
5000					Stable
10000					Stable

Table 2. Robustness of the discrete-time model considering sampling zeros

We conclude that, for h sufficiently small and for $\alpha^* > R$ and much less than $\frac{1}{h}$, then the design based on $G_d^1(\gamma)$ stabilizes the true system. However, when α^* approaches $\frac{1}{h}$, then it becomes necessary to use the controller designed using $G_d^2(\gamma)$ to achieve closed loop stability. Hence, the results suggest that for a robust controller design when α^* approaches $\frac{1}{h}$, the asymptotic sampling zeros must be included in the approximate model to ensure stability of the true closed loop system.

In addition, we notice that the results are only suitable for h small enough. For example, if we consider $h = 0.5$ and for different values of α^*, the closed loop response is unstable, even when the asymptotic sampling zeros are added in the control law. This is illustrated in Figure 3, where the maximum distance from the closed-loop eigenvalues from the point $-\frac{1}{h}$ is plotted as a function of $\alpha^* \in (0, \frac{1}{h})$. Note that this distance is larger than $\frac{1}{h} = 2$ in all cases, which implies that the closed-loop system is unstable.

7. CONCLUSION

This paper has presented a design procedure for a wide-bandwidth control law that depends only on the continuous-time relative degree and high frequency gain. Both continuous and sampled-data case have been analysed. We note that when the nominal closed loop poles approach the inverse of the sampling period, then it is necessary, for closed loop stability of the true system, that the control design is based on an approximate model which includes the asymptotic sampling zeros. Planned future work considers to extend the theory to higher order systems and systems having an unstable inverse.

REFERENCES

Åström, K.J., Hagander, P., and Sternby, J. (1984). Zeros of sampled systems. *Automatica*, 20(1), 31–38.

Bárcena, R., De la Sen, M., and Sagastabeitia, I. (2000). Improving the stability properties of the zeros of sampled systems with fractional order hold. volume 147, 456 – 464.

Elsner, L. (1982). On the variation of the spectra of matrices. *Linear Algebra and its Aplications*, 47, 127–138.

Gayaka, S., Lu, L., and Yao, B. (2012). Global stabilization of a chain of integrators with input saturation and disturbances: A new approach. *Automatica*, 48, 1389–1396.

Goodwin, G., Agüero, J., Cea, M., Salgado, M., and Yuz, J. (2013). Sampling and sampled-data models: The interface between the continuous world and digital algorithms. *IEEE Control Systems Magazine*, 33.

Goodwin, G., Graebe, S., and Salgado, M. (2001). *Control System Design*. Prentice Hall, New Jersey.

Hagiwara, T., Yuasa, T., and Araki, M. (1993). Stability of the limiting zeros of sampled-data systems with zero- and first- order holds. *International Journal of Control*, 58(6).

Ishitobi, M. and Kunimatsu, S. (2016). Zeros of sampled-data models for time delay MIMO systems. *IEEE Region 10 Conference (TENCON)*, 3410–3413.

Middleton, R. (1991). Trade-off in linear control systems. *Automatica*, 27, 281–292.

Middleton, R. and Goodwin, G. (1990). *Digital Control and Estimation. A Unified Approach*. Prentice Hall, Englewood Cliffs, New Jersey.

Ostrowski, A. (1973). *Solution of Equations in Euclidean and Banach Spaces*. Academic Press.

Sánchez, C. and Yuz, J. (2019). On the relationship between spline interpolation, sampling zeros and numerical integration in sampled-data models. *Control & System Letters*, 128, 1–8.

Weller, S., Moran, W., Ninness, B., and Pollington, A. (2001). Sampling zeros and the Euler-Frobenius polynomials. *IEEE Transactions on Automatic Control*, 46(2), 340–343.

Yucra, E. and Yuz, J. (2011). Frequency domain accuracy of approximate sampled-data models. *IFAC World Congress 2011*, 44, 8711–8717.

Yuz, J.I. and Goodwin, G.C. (2005). On sampled data models for nonlinear systems. *IEEE Transactions on Automatic Control*, 50(10), 1477–1489.

Yuz, J. and Goodwin, G. (2014). *Sampled-data Models for Linear and Nonlinear Systems*. Springer.

Zhou, B. and Duan, G.R. (2008). A novel nested non-linear feedback law for global stabilisation of linear systems with bounded controls. *International Journal of Control*, 81(9), 1352–1363.