The role of scalars and thermodynamic stability for hairy black holes

Memoria de Tesis Doctoral realizada por

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List of Publications

The following articles were published by the candidate during the realization of her doctoral studies and are included in this thesis:

1. D. Astefanesei, R.Ballesteros, P. Cabrera, G. Casanova and R. Rojas, **“Existence of thermodynamically stable asymptotically flat black holes”**, Phys. Rev. D 110 (2024) 2, 024045. arXiv:2404.15566 [hep-th].
2. R. Ballesteros and T. Ortín, **“Hairy black holes, scalar charges and extended thermodynamics”**, Class. Quant. Grav. 41 (2024) 5, 055007. arXiv:2308.04994 [gr-qc].
3. R. Ballesteros, C. Gómez-Fayrén, T. Ortín and M. Zatti, **“On scalar charges and black hole thermodynamics”**, JHEP 05 (2023) 158. arXiv:2302.11630 [hep-th].

The following work is not included in this manuscript:

1. R. Ballesteros and T. Ortín, **“Generalized Komar charges and Smarr formulas for black holes and boson stars”**, (In revision). arxiv:2409.08268[gr-qc].

Abstract

This work focuses on the thermodynamics of asymptotically flat black holes in different gravity theories, mainly those including scalar fields. It is divided into two parts: the first focuses on the thermodynamic stability of black holes, and the second on the identification of fundamental thermodynamic relations based on Wald's formalism. The objective is to understand the role of fields and scalar charges in the thermodynamics of black holes.

The main goal of the first part is to investigate the thermodynamic stability of black holes with a scalar field and a scalar potential in a 4-dimensional theory and compare it with the stability of a charged black hole that has Gauss-Bonnet corrections in 5 dimensions. The relevance of this study comes from the fact that these are examples of thermodynamically stable black holes in flat spacetime. The quasilocal (Brown-York) formalism supplemented with counterterms is used, and families of black hole solutions for both theories are analyzed, by evaluating the on-shell Euclidean action and the corresponding thermodynamic potential. The study focuses on identifying stability regions in phase space by analyzing the response functions in the canonical and grand canonical ensembles. It is observed that the family of black hole solutions with Gauss-Bonnet corrections exhibits thermodynamic similarities with black holes of the family with scalar field and scalar potential in regions near extremality.

The objective of the second part is to provide a description of the scalar charges for black holes, in a framework that allows an understanding of how scalar charges are included in thermodynamic relations. The scalar charge is defined as a closed 2-form on-shell in theories with and without global symmetries, and Wald's formalism is used to deduce the first law, where the term proportional to the scalar charge and the variations of the scalar field at infinity are obtained, as demonstrated by Gibbons, Kallosh, and Kol in 1996. Wald's formalism is also used to deduce the Smarr formula, where no term proportional to the scalar charge is found. In theories with global symmetries, the requirement of a bifurcate horizon in spacetime allows the definition of the scalar charge to explicitly reveal its dependence on other conserved charges, confirming its nature as secondary hair. On the other hand, in theories without global symmetries due to the presence of a scalar potential, the same requirement imposes specific conditions on the scalar potential to ensure that the theory admits black hole solutions. These conditions are consistent with the no-hair theorems present in the literature.

Resumen

Este trabajo abarca la termodinámica de agujeros negros asintóticamente planos, en diferentes teorías de gravedad, principalmente que incluyen campos escalares. Se divide en dos partes: la primera se centra en la estabilidad termodinámica y la segunda en la deducción de relaciones termodinámicas fundamentales a partir del formalismo de Wald. El objetivo es entender el rol que juegan los campos y cargas escalares en la termodinámica de agujeros negros.

La primera parte tiene como objetivo estudiar la estabilidad termodinámica de agujeros negros con campo escalar y potencial escalar en una teoría 4-dimensional, y comparar la estabilidad con otro agujero negro cargado que tiene correcciones de Gauss-Bonnet en 5-dimensiones. La relevancia de estudio viene del hecho de que estos son ejemplos de agujeros negros termodinámicamente estables en espaciotiempo plano. Se utiliza el formalismo cuasilocal de Brown-York suplementado con contratérminos y se analizan las familias de agujeros negros para ambas teorías, utilizando el valor de la acción euclidea on-shell y el potencial termodinámico correspondiente. El estudio se enfoca en identificar regiones de estabilidad en el espacio de fases analizando las funciones respuesta en los ensambles canónicos y gran canónico. Se observa que una familia de las soluciones del agujero negro con correcciones de Gauss-Bonnet exhibe similitudes termodinámicas con los agujeros negros de la familia con campo escalar y potencial escalar en regiones cercanas a la extremalidad.

El objetivo de la segunda parte es formular una descripción completa de las cargas escalares para agujeros negros asintóticamente planos, en un marco que permita entender cómo las cargas escalares se incluyen en las relaciones termodinámicas. Se define la carga escalar como una 2-forma cerrada on-shell en teorías con y sin simetrías globales y se utiliza el formalismo de Wald para deducir la primera ley, donde se obtiene el término proporcional a la carga escalar y a las variaciones del valor del campo escalar en el infinito, como demostraron Gibbons, Kallosh y Kol en 1996. El formalismo de Wald se utiliza también para deducir la fórmula de Smarr, donde no se observa un término proporcional a la carga escalar. En teorías con simetrías globales, la exigencia de un horizonte bifurcado en el espaciotiempo permite que la definición de carga escalar revele explícitamente su dependencia respecto de las demás cargas conservadas, confirmando así su naturaleza como pelo secundario. Por otro lado, en teorías sin simetrías globales debido a la presencia de un potencial escalar, la misma exigencia impone condiciones específicas sobre el potencial para que la teoría admita soluciones tipo agujero negro. Estas condiciones son coherentes con los teoremas de no-pelo existentes en la literatura.

Contents

1
Introduction

In 1687, Newton formulated the first mathematical theory [1] describing the motion of terrestrial and celestial objects, and the interaction between them, in a unified framework through universal laws. Among them, the universal law of gravitation

$$\vec{F} = -G\frac{m_1 m_2}{|r^3|}\vec{r},$$

described gravitational interactions between celestial masses as an attractive *force* and successfully predicted the orbits of planets. The work done by Newton, in his Principia Mathematica, constitutes the foundation of classical physics which governed the scientific thought up to the 20th century. It was John Michell in the 18th century, who just using Newtonian mechanics, speculated about the idea of "dark stars" [2]: if an object's escape velocity $v_e = \sqrt{\frac{2GM}{R}}$ exceeded the speed of light, not even light could escape its gravitational pull, rendering it invisible to an observer at infinity[1]. Despite its success, the universal law of gravitation was unable to explain the precession of Mercury's perihelion, and the notion of force acting at a distance intrigued scientists at the time, including Newton [3].

Today, we know that the theory which best describes gravitational interactions at large scales is Einstein's theory of General Relativity [4], which revolutionized our understanding of gravity and spacetime. Einstein's formulation no longer relies on forces but on the geometry of spacetime, making Newton's space and time dynamical quantities and explaining gravity as the curvature of spacetime (caused by the presence of mass or energy) through the field equations

$$G_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}.$$

The first test of General Relativity[2] was the calculation of Mercury's precession, which marked its triumph over Newtonian physics in describing gravitational phenomenona.

As soon as Einstein introduced General Relativity in 1915, exact solutions to its field equations began to be found. The earliest and most famous is the solution found by Karl Schwarzschild[3] in 1916, which describes the spherically symmetric spacetime around

[1] Dark stars would be the archaic version of black holes. Today we have a more precise definition.

[2] See [5] and references therein for experimental tests of general relativity

[3] We have to mention that four months after Schwarzschild published the first solution, Johannes Droste, a doctoral student of Lorentz, independently found the same solution more transparently and presented it even using the coordinates we usually use today [6]. To be fair, the solution should be called the Schwarzschild-Droste solution.

a non-rotating mass [7]. Initially, Schwarzschild's solution was met with some skepticism because of the singularity at $r = r_s$, where $r_s = 2GM/c^2$. In 1933 Georges Lemaître [8] presented a change of coordinates demonstrating that $r = r_s$ was nothing but a coordinate singularity. The hypersurface $r = r_s$ is null, dividing the spacetime into two causally disconnected regions, which is, in fact, the definition of the event horizon of a black hole. With the result of Lemaître, it was shown that the real singularity is at $r = 0$, where the curvature of the spacetime diverges. Later, in 1939, Oppenheimer and Snyder [9] found that a rotating spherical star could gravitationally collapse in such a way that it would lead to the Schwarzschild spacetime in the exterior, providing further evidence of the physical relevance of the Schwarzschild solution.

The development of General Relativity and its solutions also led to a deeper understanding of the causal structure of spacetime. Roger Penrose's singularity theorem [10], formulated in 1965, established that the existence of any trapped surface, such as those found inside a black hole, inevitably leads to a *singularity*[4] under certain conditions. These include the validity of the strong energy condition and reasonable causal properties of the spacetime.

All these works were fundamental for a better understanding of the Schwarzschild solution and to support black holes as a prediction[5] of the Einstein theory. Actually, it was not until 1967, that John Wheeler coined the term "black hole" and nowadays all observations[6] support its existence. Besides the Schwarzschild black hole, another black hole solutions including matter and rotation were found and are listed in table (1.1); these are called "classical" black holes. The Kerr solution [16] is the analogous to Schwarzschild but with a rotating mass. The Reissner-Nordström solution [17, 18] describes the charged, spherically symmetric spacetime around a non-rotating mass and its analogous charged model, but with a rotating mass is the Kerr-Newman solution [19]. The interested reader can check Refs. [20–24] and references therein for an exhaustive analysis of these classical black hole solutions.

Theory	**Black hole solution**	**Charges**
Einstein	Schwarzschild	M
Einstein-Maxwell	Reissner-Nordström	M, Q
Einstein	Kerr	M, J
Einstein-Maxwell	Kerr-Newman	M, Q, J

Table 1.1: Summary table of classical black holes. M is the mass, Q is the electric charge and J the angular momentum.

We restrict ourselves to the Schwarzschild spacetime which is the simplest solution of an asymptotically flat[7] uncharged spacetime around a non-rotating mass, as our toy model in order to study its causal structure in this paragraph. A *Penrose*, *Penrose-Carter* or *conformal diagram* is a useful representation of the causal structure of the

[4] The formal definition of a singularity in terms of geodesic incompleteness was given by Hakwing and Penrose in Ref. [11].

[5] General relativity also predicts gravitational waves [12,13] which have been detected in recent years [14].

[6] The first image of the supermassive black hole at the center of our galaxy, Sagittarius A*, was obtained by the Event Horizon Telescope (EHT) Collaboration and published in 2022 in Ref. [15].

[7] An asymptotically flat spacetime is a Lorentzian manifold indistinguishable from Minkowski spacetime at infinite distance, i.e., for $r \to \infty$ the metric is Minskowski. See Ref. [25] for a precise definition.

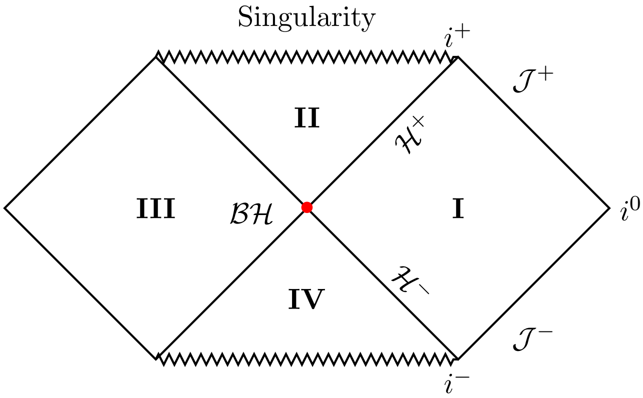

Figure 1.1: Penrose diagram for the Schwarzschild spacetime. The singularity at $r = 0$ is the zigzag line. Point i^+ (i^-) denotes the future (past) timelike infinity and point i^0 spacelike infinity. The line $\mathcal{J}^+$ ($\mathcal{J}^-$) is the future (past) null infinity. The line $\mathcal{H}^+$ ($\mathcal{H}^-$) is the future (past) oriented horizon at $r = r_s$. The red dot is the bifurcation surface, since each point in the diagram represents a 2−sphere. The causal future (past) is denoted J^+ (J^-).

spacetime. The construction[8] of these diagrams is based on the fact that the null geodesics of the metric $g_{\mu\nu}$ are also null geodesics of the metric $h_{\mu\nu} = \Omega^2(x)g_{\mu\nu}$, where $\Omega^2(x)$ is a global conformal factor; therefore it preserves the global causal structure of a given spacetime. We need to perform a change of coordinates to obtain the *maximally extended* Schwarzschild solution, removing the coordinate singularity at $r = r_s$ using the *Kruskal-Szekeres* [28] coordinates, where only the physical singularity at $r = 0$ remains; then we apply a conformal transformation to bring the infinities to a finite distance. Following these steps we obtain the Penrose diagram shown in Fig.(1). Region I represents the black hole exterior $r > r_s$. Region II is the interior of the black hole $r < r_s$, limited by the null hypersurface at $r = r_s$ where the event horizon is located (to be defined later). We can see that any observer that crosses $r = r_s$ will inevitably reach the spacelike singularity at $r = 0$ making it impossible to return to region I. Region III is described as a "another universe" and region IV is referred to as a "white hole" (the opposed to a black hole) since everything is expelled from it. These later two regions do not appear in a Penrose diagram of a black hole formed through gravitational collapse and will not be discussed further. With the information provided by the Penrose diagram, we can make some definitions which will be useful throughout this work[9]:

- The *future event horizon* $\mathcal{H}^+$ is defined as the boundary of the causal past of $\mathcal{J}^+$, i.e.,

$$\mathcal{H}^+ := \partial\left[J^-(\mathcal{J}^+)\right].$$

  A similar definition applies to the past event horizon $\mathcal{H}^-$ which is defined as the boundary of the causal future of $\mathcal{J}^+$. However, throughout this work, when referring to the event horizon, we will specifically mean the future event horizon.

- An asymptotically flat *black hole* region $\mathcal{B}$ is a region in the spacetime $\mathcal{M}$ (a

[8] Is assumed that the reader is familiarized with the concepts presented. Excellent references to study the causal structure of spacetimes and regarding the construction of Penrose diagrams are books in Refs. [25–27]

[9] This definitions are taken from Ref. [23].

Lorentzian manifold) defined as the complement of the causal past of $\mathscr{J}^+$, i.e.,

$$\mathcal{B} := \mathcal{M} - J^-(\mathscr{J}^+).$$

Then, by definition, a black hole is a region causally disconnected from the future null infinity, which means that there is no timelike or null trajectory connecting region I with region II.

- A *Killing horizon* $\mathcal{H}$ is a surface at which a Killing vector k becomes null

$$k^2 \stackrel{\mathcal{H}}{=} 0.$$[10]

Moreover, the Killing vector k is normal to the Killing horizon (k is orthogonal to $\mathcal{H}$ at every point). In what follows, we are going to denote the event horizon with $\mathcal{H}$ because the *rigidity theorem* [29] tell us that the event horizon of a stationary black hole[11] solution is also a Killing horizon. This relation is important since we only consider stationary black holes in this work.

- The *bifurcation surface* $\mathcal{BH}$ is the locus where the future and past horizons meet (red dot in Fig.(1)). It is a 2-sphere of radius $r = r_s$. At this surface, the Killing vector k vanishes

$$k \stackrel{\mathcal{H}}{=} 0.$$

When the bifurcation surface exists, the Killing horizon is called a *non-degenerate* or *bifurcate horizon.*

- The *surface gravity* κ of a Killing horizon with normal Killing vector k is defined as non-affinity parameter of the null geodesic[12]

$$k^\mu \nabla_\mu k^\nu := \kappa k^\nu.$$

When $\kappa = 0$ (for extremal black holes) the horizon is called a *degenerate horizon* and there is no bifurcation surface. We focus only in $\kappa \neq 0$ black holes.

Let us now discuss one of the most groundbreaking advances in black hole physics, which came from the early works on the "laws of black hole mechanics" formulated by Bekenstein, Christodoulou, Bardeen, Carter, and Hawking, among others [30–33]. These laws were very similar to those of standard thermodynamics, but it was not until Hawking's work, where he studied quantum fields in a black hole background and discovered that black holes emit radiation due to quantum effects (the Hawking radiation [34]) with temperature

$$T_H = \frac{\kappa}{2\pi},$$

that these laws were properly identified with *the laws of black hole thermodynamics.* Following Hawking's contribution, the entropy proposed by Bekenstein [35] was found to be

$$S = \frac{A}{4},$$

[10] The notation $\stackrel{\mathcal{H}}{=}$ means "evaluated at $\mathcal{H}$".

[11] A stationary black hole is whose admitting an asymptotically timelike Killing vector.

[12] Equivalently: $\kappa^2 \stackrel{\mathcal{H}}{=} -\frac{1}{2}\left(\nabla^\mu k^\nu\right)\left(\nabla_\mu k_\nu\right)$

where A is the area of the event horizon. All of this marks the beginning of black hole thermodynamics[13].

The thermodynamic description of black holes offers a theoretical framework for describing various equilibrium configurations involving charges and their conjugate chemical potentials through the first law. Understanding how these charges and their associated potentials are defined is crucial for revealing their roles and significance. Also, it has inspired new areas of study aimed at finding a microscopic interpretation of entropy and has been consolidated as an active area of research.

The four laws of black hole thermodynamics for the most general asymptotically flat, axisymmetric and stationary solution of the Einstein-Maxwell theory (Kerr-Newman black hole) are (these laws are taken from Ref. [21]):

- **0th law:** The temperature T_H (or the surface gravity κ) is constant[14] over the bifurcate horizon.

- **1st law:** When the black hole switches from one stationary state to another, its mass (or energy) changes by

$$\delta M = T_H \delta S + \Omega_H \delta J + \Phi_H \delta Q,$$

  where Ω_H is the angular velocity of the horizon, δJ is the variation of the angular momentum, Φ_H is the electrostatic potential (defined as the difference of the electric potential at infinity and at the event horizon) and δQ is the variation of the electric charge. The conjugate chemical potentials Ω_H and Φ_H are constant over the event horizon according to the generalized 0th law.

- **2nd law:** In any process, the area of a black hole (so its entropy), do not decrease

$$\delta S \geq 0.$$

- **3rd law:** It is impossible to reach a zero black hole temperature (or $\kappa = 0$) by a finite number of physical processes.[15].

  Note that if the temperature reaches zero, the entropy does not vanish since as it depends on the area A. Therefore the alternate version of this law does not hold for black holes.

The increasing interest in black holes has led to study in which way the physical charges involved characterize the solution. This gives rise to *no-hair theorems* (or *no-hair conjecture*) [37,38] which establish that any black hole is fully described by their conserved charges only[16]: the mass M, the electric charge Q (if the theory has a Maxwell field) and the angular momentum J (if it is rotating). The term "hair" is assigned to charges that are not protected by a conservation law. The uniqueness of the Schwarzschild [31], the

[13] For a historical and comprehensive review of black hole thermodynamics and future directions, the reader can check [36].

[14] There are another quantities constants over the event horizon which are enounced in the "generalized 0th law".

[15] This law is formulated as a conjecture, as no mathematical proof exists, and its validity remains under debate.

[16] With the remarkable counterexample of Einstein-Yang-Mills theory. See Refs. [39–42].

Reissner-Nordstrom [43], the Kerr [31, 44] and Kerr-Newman [45] solutions have been fully demonstrated and this fact is intrinsically related with the no-hair conjecture, as they demonstrate that no additional parameters are required to fully characterize these solutions.

Regarding theories with scalar fields, the existence of black hole solutions (*hairy black holes*[17]) are allowed for specific field configurations. Scalar charges are not related to any gauge symmetry and are not protected by any conservation law (are hair). However, there is a distinction between primary and secondary hair: primary hair would be an independent scalar charge, and secondary hair is a scalar charge which is not independent because it depends in a non-trivial way from the other parameters. The regular black hole solutions found for those theories are those with secondary hair since the primary hair is forbidden by the no hair-conjecture [47].

Today, black holes stand as natural laboratories where the most extreme conditions in the universe test our understanding of fundamental physics. Their study from a thermodynamical point of view, provides invaluable insights not only into the nature of gravity but also into the potential unification of quantum mechanics and gravity, two of the most successful yet seemingly incompatible theories of modern physics.

Since this introduction pretends to be a self-contained guide in order to understand the publications on which this thesis is based, it is organized as follows: In Section 1.1 we will review the conditions of thermodynamical stability for asymptotically flat black holes. Then, in Section 1.2 we will define conserved charges and discuss the special case for the scalar charges, which are not conserved and are the main concern of this thesis. We want to check what happens for charges that are related to global and local symmetries, so we will have to check Noether's theorems and also discuss the construction of generalized Komar charges. In 1.3 we will show how to obtain the Smarr formula starting with the generalized Komar charge. Finally, in Section 1.4 is presented a summary of the main results for each publication in Refs. [48–50] in which this thesis is based.

[17]See Ref. [46] and references therein for a nice review about no-hair theorems for asymptotically flat hairy black holes.

1.1 Thermodynamic stability for asymptotically flat black holes

Black holes are thermodynamic objects that obey the usual laws of thermodynamics. The study of its thermodynamic stability is fundamental due to its intrinsic connection with their quantum properties. Moreover, this analysis allows us to distinguish between configurations that can be studied within the framework of *equilibrium thermodynamics* and those that cannot be fully described using those tools.[18]

It is worth mentioning the distinction between *local* and *global* thermodynamic stability. Global stability is directly linked to phase transitions, as it determines the most favorable configuration among all possible states in a given thermodynamic ensemble. On the other hand, local stability refers to the response of the system to small perturbations in their parameters, such as the temperature or the electric potential.

Mathematically, local stability means that small fluctuations in the system's variables do not lead to an amplified response, allowing the system to remain in equilibrium. This requirement can be expressed in terms of the *convexity*[19] and *concavity*[20] of its thermodynamic potentials. Specifically, the thermodynamic potentials must be convex with respect to their extensive variables, such as the entropy S, the volume V, or the charge Q; and concave with respect to their intensive variables, such as the temperature T, the pressure P, or the electric potential Φ (see Ref. [51]). These properties ensure that the system does not deviate from equilibrium under small perturbations, providing us with a rigorous criterion for local thermodynamic stability.

The present section focuses on the local stability of asymptotically flat black holes whose conserved charges are the mass M and the electric charge Q. In Ref. [52], a general criterium to evaluate the local stability based on the positivity of the response functions is presented: the *heat capacity* and the *electric permittivity.* The heat capacity, defined as

$$C_X = T\left(\frac{\partial S}{\partial T}\right)_X , \tag{1.1}$$

measures the system's response to temperature variations, indicating how much energy is required to change its temperature by a small amount. Here, X represents the variable held constant, such as the electric charge Q (canonical ensemble) or the electric potential Φ (grand canonical ensemble). A positive specific heat ensures that entropy increases with temperature, indicating stability against thermal fluctuations and confirming consistency with the second law of thermodynamics, which requires that entropy does not decrease. Similarly, the electric permittivity

$$\epsilon_X = \left(\frac{\partial Q}{\partial \Phi}\right)_X \tag{1.2}$$

quantifies how the charge Q changes in response to variations in the electric potential Φ, keeping X constant. For thermodynamical stability ϵ_X must be positive; otherwise, adding charge would lead to a decrease in the electric potential, resulting in an unstable response.

[18] A thermodynamically unstable black hole will rapidly evolve toward a state of thermodynamic equilibrium, as is the case for any physical system in nature.

[19] A function $f(x)$ is said to be convex in an interval if its second derivative is positive $\frac{d^2 f}{dx^2} > 0$.

[20] A function $f(x)$ is concave if its second derivative is negative $\frac{d^2 f}{dx^2} < 0$.

This framework establishes the necessary conditions for ensuring local stability in different thermodynamic ensembles. This section aims to show these developments and is mainly based on Refs. [51–54]. For more details, the reader can check Appendix C.1.

1.1.1 Stability in the grand canonical ensemble: fixed Φ

In the grand canonical ensemble the electric potential Φ is held fixed, allowing the system to exchange energy and charge with a reservoir. The corresponding thermodynamic potential is the Gibbs free energy denoted by G, defined as

$$G = M - TS - Q\Phi. \tag{1.3}$$

Integrating by parts the first law, the derivative of the Gibbs free energy can be written as

$$dG = -SdT - Qd\Phi. \tag{1.4}$$

In the grand canonical ensemble, local thermodynamic stability is determined by analyzing small perturbations around equilibrium and ensuring that the system does not amplify such perturbations. Then, the Gibbs free energy must be convex respect to the entropy,

$$\left(\frac{\partial^2 G}{\partial S^2}\right) > 0, \tag{1.5}$$

but also must be concave respect to the electric potential,

$$\left(\frac{\partial^2 G}{\partial \Phi^2}\right) < 0. \tag{1.6}$$

These criteria are expressed in terms of the second derivatives of the mass M in Ref. [55]. After some algebra and using the thermodynamic relations, one arrives to the folloowing conditions:

1. The system's response to fluctuations in the electric potential Φ is measured by the electric permittivity ϵ_S. Local stability demands

$$\epsilon_S > 0. \tag{1.7}$$

2. The thermal stability is characterized by the heat capacity at constant Φ. For local stability, it must satisfy

$$C_\Phi > 0. \tag{1.8}$$

The fact that both stability conditions are satisfied simultaneously ensures that the system remains in equilibrium under small fluctuations of the electric potential or the entropy. Electrostatic stability, characterized by $\epsilon_S > 0$, ensures that the charge responds monotonically to variations in the potential Φ, avoiding electrostatic instability. Thermal stability, characterized by $C_\Phi > 0$, ensures that the entropy increases with temperature, reflecting stability against thermal perturbations. The determinant condition involving M guarantees stability under coupled perturbations in S and Q, providing a complete description of local stability in the grand canonical ensemble.

1.1.2 Stability in the canonical ensemble: fixed Q

The canonical ensemble is a thermodynamic framework where the electric charge Q is fixed. In this ensemble, the system can exchange energy with a thermal reservoir while maintaining a fixed charge. The appropriate thermodynamic potential for this ensemble is the Helmholtz free energy given by

$$F = M - TS. \tag{1.9}$$

Similar to the grand canonical ensemble, integrating by parts the first law yields

$$dF = \Phi dQ - SdT. \tag{1.10}$$

From the last equation, it can be seen that

$$\Phi = \left(\frac{\partial F}{\partial Q}\right)_T, \quad S = -\left(\frac{\partial F}{\partial T}\right)_Q. \tag{1.11}$$

Local stability is achieved by requiring that the potential F satisfies specific convexity or concavity properties with respect to its extensive and intensive variables, respectively. From Eq. (1.11), one obtains

$$\left(\frac{\partial \Phi}{\partial Q}\right)_T = \left(\frac{\partial^2 F}{\partial Q^2}\right)_T, \quad \left(\frac{\partial S}{\partial T}\right)_Q = -\left(\frac{\partial^2 F}{\partial T^2}\right)_Q \tag{1.12}$$

and, after some algebra, we find the two conditions for thermodynamic stability in the canonical ensemble for a black hole:

1. Electrostatic stability in the canonical ensemble is associated with the response of the system to fluctuations in the electric potential Φ at fixed Q, and it implies

   $$\epsilon_T > 0. \tag{1.13}$$

   This guarantees that the system responds stably to small fluctuations in the electric potential. In this way, the electric potential increases monotonically with the charge.

2. The positivity of C_Q guarantees that the entropy increases monotonically with temperature, reflecting thermal stability, that is,

   $$C_Q > 0. \tag{1.14}$$

The simultaneous satisfaction of both stability conditions ensures that the system remains in equilibrium under small perturbations of the temperature or the electric potential.

1.2 Charges, symmetries and conservation laws

The definition of charges and their conservation is probably one of the most fundamental topics in physics, as they enable the characterization of any system and the description of its thermodynamics. Noether theorems, which relate each continuous symmetry to a conserved charge, provide the mathematical basis for defining currents and charges in any field theory and are widely used in many fields of physics. The first and second theorems of Noether show us how to define charges depending on whether the symmetries are global or local, respectively.

Wald and his collaborators developed a formalism based on Noether theorems, that can be applied to general relativity and more complex gravity theories. They discovered that the first law can be obtained by using an identity involving the variation of the Noether-Wald charge, which is the Noether charge associated with the invariance under diffeomorphisms. This formalism gives a candidate to black hole entropy for generic diffeomorphism-invariant theories of gravity that reduces to the Bekenstein-Hawking entropy ($\propto 1/4$ of the area of the horizon) in general relativity. This entropy is known as Wald entropy.

Despite the success of this formalism—for example, in theories with higher-order derivatives—the original method fails to account for the work terms in the first law when applied to theories with matter fields, such as gauge or scalar fields. In the original formulation by Wald, it is assumed that all matter fields transform as tensors, but this is not correct for fields with some gauge freedom, as has been stressed in Refs. [56, 57]. Their transformations are not given by the standard Lie derivative. Instead, we applied a gauge-covariant derivative, consisting in the standard Lie derivative plus a compensating gauge transformation (see Ref. [58]).

In this thesis, we are going to focus on the role of scalar fields in black hole thermodynamics. We have found that using a covariant definition of scalar charge and correctly accounting for the symmetries, we can recover the terms involving the scalar charge in the first law as demonstrated by Gibbons, Kallosh and Kol in Ref. [59]. This covariant definition is also useful for proving no-hair theorems, as shown here.

Once Wald's approach is improved in such a way that it now deals with the symmetries of the fields in a gauge-covariant way, and assuming that the field configuration allows for the existence of a Killing vector field, it is possible to construct a closed $(d-2)$- form charge which we call the generalized Komar charge. As a consequence of the closedness of this charge, we can find a relationship between the thermodynamic quantities at infinity and at a black hole's horizon, which is the Smarr's formula, as we are going to show in this section. In particular, in this work we focus on how to define a generalized Komar charge in gravity theories with scalar fields and with a scalar potential.

1.2.1 Noether theorems: general overview

Consider an action S in a d-dimensional spacetime $\mathcal{M}$ (a Lorentzian manifold)

$$S = \int_{\mathcal{M}} \mathbf{L}(\varphi), \tag{1.15}$$

described by a d-form Lagrangian density $\mathbf{L}$ depending on some fields denoted collectively by φ. Under a general variation of the fields

$$\delta S = \int_{\mathcal{M}} \left[\mathbf{E}_\varphi \wedge \delta\varphi + d\mathbf{\Theta}(\varphi, \delta\varphi)\right], \quad \mathbf{E}_\varphi = \frac{\delta \mathbf{L}}{\delta\varphi} \tag{1.16}$$

where the Euler-Lagrange equations of motion are $\mathbf{E}_\varphi$ and where by definition $\mathbf{\Theta}$ is a $(d-1)$-form known as the *symplectic prepotential.* Now consider an arbitrary infinitesimal transformation $\delta_\sigma \varphi$ with parameter σ, that leaves invariant the action up to a total derivative[21]

$$\delta_\sigma S = -\int_{\mathcal{M}} d\mathbf{B}(\varphi, \sigma), \tag{1.17}$$

where $\mathbf{B}$ is a $(d-1)$-form depending on the fields and on σ. Applying the infinitesimal transformation to the general variation of the action

$$\delta_\sigma S = \int_{\mathcal{M}} \left[\mathbf{E}_\varphi \wedge \delta_\sigma \varphi + d\mathbf{\Theta}(\varphi, \delta_\sigma \varphi) \right]. \tag{1.18}$$

Comparing Eq. (1.17) with Eq. (1.18), we obtain the general off-shell *fundamental identity*

$$d \left[\mathbf{\Theta}(\varphi, \delta_\sigma \varphi) + \mathbf{B}(\varphi, \sigma) \right] = -\mathbf{E}_\varphi \wedge \delta_\sigma \varphi. \tag{1.19}$$

There are two cases depending on the form of the transformation in Eq. (1.18): if the parameters σ are constants, then the transformation is called global and we have to apply the *first Noether theorem*; if σ is an arbitrary function of the spacetime, then the transformation is called local and we have to apply the *second Noether theorem*. In what follows, we will briefly review each of these cases.

Global symmetries—First Noether theorem

Let us start considering the case in which the transformation $\delta_\sigma \varphi$ describes global symmetries of the field configuration, that is

$$\delta_\sigma \varphi = \sigma^A \delta_A \varphi, \tag{1.20}$$

where the σ^A are constant parameters and A is a label such that $A = 1, ..., n$. Since this is a linear transformation, we can always write

$$\mathbf{\Theta}_A(\varphi, \sigma^A) = \sigma^A \mathbf{\Theta}_A(\varphi), \quad \mathbf{B}_A(\varphi, \sigma^A) = \sigma^A \mathbf{B}_A(\varphi).$$

Replacing in the fundamental identity Eq. (1.19),

$$\sigma^A d \left\{ \mathbf{\Theta}_A(\varphi) + \mathbf{B}_A(\varphi) = -\mathbf{E}_\varphi \wedge \delta_A \varphi \right\}. \tag{1.21}$$

Since the identity Eq. (1.21) is satisfied for arbitrary values of σ^A, then it is independent of σ^A. Thus,

$$d \left[\mathbf{\Theta}_A(\varphi) + \mathbf{B}_A(\varphi) \right] = -\mathbf{E}_\varphi \wedge \delta_A \varphi, \tag{1.22}$$

and if the equations of motion are satisfied[22], the last equation reads

$$d \left[\mathbf{\Theta}_A(\varphi) + \mathbf{B}_A(\varphi) \right] \doteq 0. \tag{1.23}$$

Some observations are in order:

1. From Eq. (1.23) we define n $(d-1)$-forms $\mathbf{J}_A$ which obey

$$\mathbf{J}_A \equiv \mathbf{\Theta}(\varphi, \delta_A \varphi) + \mathbf{B}_A(\varphi), \tag{1.24}$$

and we call them the *Noether currents* associated to global symmetries.

[21] This is the definition of a symmetry of the action.
[22] In what follows, $\doteq$ means on-shell.

2. For every $(d-1)$-form current $\mathbf{J}_A$, we define a *Noether charge* q_A

$$q_A \equiv \int_{\Sigma^{(d-1)}} \mathbf{J}_A, \tag{1.25}$$

where $\Sigma^{(d-1)}$ is a spacelike hypersurface.

3. To determine if q_A are conserved charges, we take $\Sigma^{(d-1)}$ and sweep into a spacetime volume $\mathcal{M}$. Now we can apply the Stokes theorem to Eq. (1.24)

$$\int_{\mathcal{M}} d\mathbf{J}_A = \int_{\partial\mathcal{M}} \mathbf{J}_A, \tag{1.26}$$

where $\partial\mathcal{M}$ denotes the $(d-1)$-dimensional boundary of the spacetime. Decomposing this piece in (see Fig. 1.2.1):

- *i*) Two $(d-1)$-dimensional spacelike hypersurfaces at $t=0$ and another at $t=t'$ denoted, respectively by $\Sigma^{(d-1)}$ and $\Sigma'^{(d-1)}$.
- *ii*) A $(d-1)$-dimensional timelike hypersurface (which we call *the flux*) of $\partial\mathcal{M}$ denoted by $\Sigma^{(d-1)}_{flux}$,

then

$$\partial\mathcal{M} = \Sigma^{(d-1)} \cup \Sigma'^{(d-1)} \cup \Sigma^{(d-1)}_{flux}. \tag{1.27}$$

With the definition of the Noether charge Eq. (1.25) and the decomposition Eq. (1.27) in Eq. (1.26) we find

$$q_A - q'_A = \int_{\Sigma^{(d-1)}_{flux}} \mathbf{J}_A, \tag{1.28}$$

where $q'_A = \int_{\Sigma'^{(d-1)}} \mathbf{J}_A$ and $q_A = \int_{\Sigma^{(d-1)}} \mathbf{J}_A$. Then, the change in time of the Noether charge Eq. (1.25) is equal with the flux integral of the current.

Local symmetries—Second Noether theorem

If the transformation parameters σ^A are arbitrary functions of the spacetime i.e., $\sigma^A(x)$, from the fundamental identity Eq. (1.19) can be seen that, in general, the terms are not linear in $\sigma^A(x)$, since now the variation of the fields $(\delta_\sigma\varphi)$ may include derivatives of $\sigma^A(x)$. However, integrating by parts as many times as necessary, we can obtain a relation that is linear in σ^A plus a total derivative

$$\mathbf{E}_\varphi \wedge \delta_\sigma\varphi = \sigma^A F_A(\mathbf{E}_\varphi) + d\mathbf{S}(\varphi,\sigma). \tag{1.29}$$

Comparing this result with the fundamental identity Eq. (1.19), we find that those linear terms must vanish i.e.,

$$F_A(\mathbf{E}_\varphi) = 0. \tag{1.30}$$

These are the *Noether identities*, which relate the equations of motion (satisfied off-shell) and express that not all of them are independent. Finally, we find a relation between the equations of motion and a $(d-1)$- form $\mathbf{S}$

$$\mathbf{E}_\varphi \wedge \delta_\sigma\varphi = d\mathbf{S}(\varphi,\sigma), \tag{1.31}$$

which is the *Second Noether theorem.* Some comments are in order:

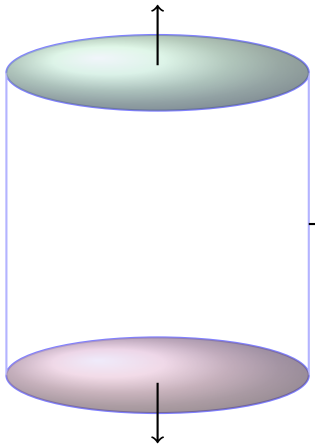

Figure 1.2: Schematic representation of the decomposition of a piece of spatial volume and its evolution through time. The spacelike hypersufaces $\Sigma^{(d-1)}$ and $\Sigma'^{(d-1)}$ have different orientations, and $\Sigma^{(d-1)}_{flux}$ is a timelike hypersurface.

1. Replacing Eq. (1.31) in Eq. (1.19), we find the off-shell identities that allow us to define closed $(d-1)$-form *Noether currents* $\mathbf{J}$

$$d\mathbf{J}[\sigma] = 0, \quad \mathbf{J}[\sigma] = \mathbf{S}(\varphi, \sigma) + \mathbf{\Theta}(\varphi, \delta_\sigma \varphi) + \mathbf{B}(\varphi, \sigma), \tag{1.32}$$

for each independent σ.

2. An off-shell closed $(d-1)$-form Noether current implies the local existence of $(d-2)$-form *Noether charges* $\mathbf{Q}[\sigma]$ such that

$$\mathbf{J}[\sigma] = d\mathbf{Q}[\sigma]. \tag{1.33}$$

It is worth noting that it is not possible to integrate the currents over a closed spacelike surface as we did in the first Noether theorem to compute the charge because we will obtain zero by Stokes theorem.

3. In general, the charges forms defined in Eq. (1.33) are not closed, i.e $\mathbf{J}[\sigma] = d\mathbf{Q}[\sigma]$, but under certain conditions it is possible to construct closed $(d-2)$-form charges associated to local symmetries [60]. Consider a solution for which exists *Killing* or *reducibility parameters* σ_k such that δ_{σ_k} annihilates all the fields (i.e., $\delta_{\sigma_k}\varphi = 0$). Then, from the current Eq. (1.32) and the charge Eq. (1.33) on-shell will satisfy

$$\mathbf{J}[\sigma_k] = d\mathbf{Q}[\sigma_k] \doteq \mathbf{B}(\varphi, \sigma_k), \tag{1.34}$$

because $\mathbf{S} \doteq 0$ for a field configuration that satisfies the equations of motion and since $\mathbf{\Theta}$ is linear in $\delta_{\sigma_k}\varphi$, then $\mathbf{\Theta} = 0$ for symmetric fields. Consider two cases:

(a) If $\mathbf{B}(\varphi, \sigma_k) = 0$ in Eq. (1.34), we will obtain closed $(d-2)$-form charges associated to each Killing parameter σ_k

$$d\mathbf{Q}[\sigma_k] \doteq 0. \tag{1.35}$$

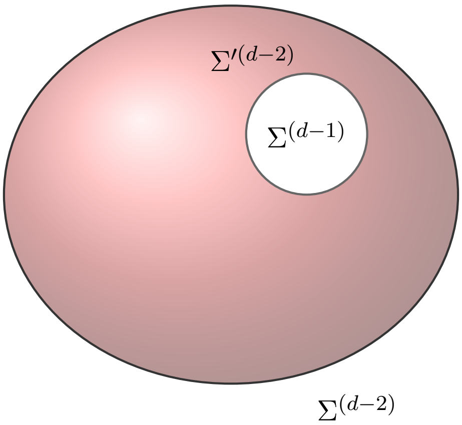

Figure 1.3: Schematic representation of two closed hypersurfaces. $\Sigma'^{(d-2)}$ is a smooth deformation of $\Sigma^{(d-2)}$. $\Sigma^{(d-1)}$ is a hypersurface of dimension $(d-1)$ that fills the space between the two $(d-2)$ closed surfaces.

We define the *charge* as the integral of the $(d-2)$-form charge

$$q_{\sigma_k} \equiv \int_{\Sigma^{(d-2)}} \mathbf{Q}[\sigma_k], \tag{1.36}$$

where $\Sigma^{(d-2)}$ is a closed $(d-2)$-dimensional hypersurface. Consider a closed hypersurface $\Sigma'^{(d-2)}$ obtained by a smooth deformation from $\Sigma^{(d-2)}$ (see Fig. 3a). The charges will satisfy

$$q'_{\sigma_k} - q_{\sigma_k} \doteq 0, \tag{1.37}$$

where $q'_{\sigma_k} = \int_{\Sigma'^{(d-2)}} \mathbf{Q}'[\sigma_k]$, $q_{\sigma_k} = \int_{\Sigma^{(d-2)}} \mathbf{Q}[\sigma_k]$ and the minus is because of the different orientations. By Stokes theorem in Eq. (1.35) and using Eq. (1.37), with $\partial\Sigma^{(d-1)} = \Sigma^{(d-2)} \cup \Sigma'^{(d-2)}$

$$\int_{\Sigma^{(d-1)}} d\mathbf{Q}[\sigma_k] = q'_{\sigma_k} - q_{\sigma_k} \doteq 0. \tag{1.38}$$

This is a generalization of the standard Gauss law. The value of the integral does not change under smooth deformations of the integration surfaces, as long as the deformations does not cross any point at which the equations of motion are not satisfied. We have to emphasize this result: *every charge defined as an integral of a closed* $(d-2)$*-form charge will satisfy a Gauss law.*

(b) If the $(d-2)$-form $\mathbf{B}(\varphi, \sigma_k)$ in Eq. (1.34) does not vanish, by definition of the transformation of the action under Killing parameters

$$\delta_{\sigma_k} S = 0. \tag{1.39}$$

This guarantees that $\mathbf{B}$ will be locally exact and we can write it in terms of a $(d-2)$-form $\mathbf{w}_k$ as

$$\mathbf{B} \doteq d\mathbf{w}_k. \tag{1.40}$$

Then, with Eq. (1.40) in Eq. (1.34) we can construct a $(d-2)$-form charge $\mathbf{K}$

$$\mathbf{K}[\sigma_k] = -\left(\mathbf{Q}[\sigma_k] - \mathbf{w}_{\sigma_k}\right), \tag{1.41}$$

which is closed on-shell by construction. Because of its closedness, it can be shown using the same procedure that in the previous item, that this charge will also satisfy a Gauss law.

4. If we are dealing with fields with gauge freedoms, it has been stressed in Refs. [56,57] that the variation of the fields under diffeomorphisms (denoted δ_ξ) includes the standard Lie derivative (denoted $\mathcal{L}_\xi$) plus (induced) gauge transformations (denoted δ_{σ_ξ})

$$\delta_\xi = -\mathcal{L}_\xi + \delta_{\sigma_\xi}, \tag{1.42}$$

where ξ is a vector field and represents the generator of infinitesimal *general coordinate transformations* (GCTs) or diffeomorphisms; δ_{σ_ξ} depends on ξ and on the fields on which the transformation acts. By hypothesis, is assumed that $\mathbf{L}$ is invariant under gauge transformations. Acting with δ_ξ on the action and using Cartan formula for the standard Lie derivative[23]

$$\delta_\xi S = -\int_{\mathcal{M}} (d\imath_\xi \mathbf{L} + \imath_\xi \mathbf{dL}) = -\int_{\mathcal{M}} d\imath_\xi \mathbf{L}, \tag{1.43}$$

we identify this total derivative with the one in Eq. (1.17). Replacing in Eq. (1.32), we obtain the $(d-1)$-form current Noether associated to diffeomorphisms

$$\mathbf{J}[\xi] = \mathbf{S}(\varphi, \xi) + \boldsymbol{\Theta}(\varphi, \delta_\xi \varphi) + \imath_\xi \mathbf{L}, \tag{1.44}$$

which is closed off-shell by construction. We define the $(d-2)$-form charge $\mathbf{Q}[\xi]$ related to the invariance of the theory under diffeomorphisms

$$\mathbf{J}[\xi] = d\mathbf{Q}[\xi], \tag{1.45}$$

which is the *Noether-Wald* charge and, in general, it is not necessarily closed.[24]

Note from the current Eq. (1.44) that for Killing parameters in a theory which obeys Eq. (1.43), the $(d-2)$-form $\mathbf{w}_k$ used to define the closed charge $\mathbf{K}$ in Eq. (1.41) reads $\imath_k \mathbf{L} = d\mathbf{w}_k$. With this identification we can define the $(d-2)$-form $\mathbf{K}[k]$

$$\mathbf{K}[k] = -\left(\mathbf{Q}[k] - \mathbf{w}_k\right), \qquad \imath_k \mathbf{L} = d\mathbf{w}_k, \tag{1.46}$$

which satisfies a Gauss law because of its closedness and we call it the *generalized Komar charge.*

This concludes the brief review about Noether theorems. Below, there is a table (1.2) summarizing the main results.

1.2.2 Scalar charges

In the previous section, we have shown how to construct charges using the Noether theorem, for both global and local symmetries. The construction of charges relies on the closedness of a current or a charge form. Then, we want to define scalar charges as an

[23] Cartan formula: $\mathcal{L}_\xi = \imath_\xi d + d\imath_\xi$.

[24] As an illustrative example, we show in Appendix (B.1) the explicit computation of the Noether-Wald charge for the d-dimensional Einstein-Maxwell theory for a $(p+2)$-form F.

	First Noether theorem	**Second Noether theorem**
Application	Applies to global symmetries of the Lagrangian.	Applies to local (gauge) symmetries of the Lagrangian.
$(d-1)$**-forms**	The Noether current, closed on-shell $$\mathbf{J}_A \equiv \mathbf{\Theta}(\varphi, \delta_A \varphi) + \mathbf{B}_A(\varphi).$$	The Noether current, closed off-shell $$\mathbf{J}[\sigma] = \mathbf{S}(\varphi, \sigma) + \mathbf{\Theta}(\varphi, \delta_\sigma \varphi) + \mathbf{B}(\varphi, \sigma).$$
$(d-2)$**-forms**	The scalar charge (to define in Sec. (1.2.2), closed on-shell $$\mathbf{Q}_s[k] \equiv \imath_k \mathbf{J}.$$	The Noether charge $\mathbf{Q}[\sigma]$ not necessarily closed $$\mathbf{J}[\sigma] = d\mathbf{Q}[\sigma].$$ The charge, closed on-shell: $$\mathbf{K}[\sigma_k] = -\left(\mathbf{Q}[\sigma_k] - \mathbf{w}_{\sigma_k}\right).$$
Charges for closed forms on-shell	The global Noether charges $$q_A = \int_{\Sigma^{(d-1)}} \mathbf{J}_A.$$	Every closed $(d-2)$-form charge $\mathbf{K}[\sigma_k]$ will obey a Gauss law $$q_{\sigma_k} = \int_{\Sigma^{(d-2)}} \mathbf{K}[\sigma_k].$$
The main ingredient	The transformation parameter is linear Eq. (1.21).	The existence of Noether identities Eq. (1.30).

Table 1.2: Comparison between the First and Second Noether Theorem.

integral of a closed form in order to have a covariant definition. The study of scalar charges is relevant since they are understood as "hair" in gravity theories evolving scalar fields, as we pointed out in the introduction.

There are many hairy black holes solutions for asymptotically flat spacetimes (see Ref. [46]) but in all of them the scalar charges are secondary hair (functions of another conserved physical parameters), since they are the only regular black hole solutions allowed by the no-hair conjecture. Also, as we are going to see, scalar charges are not protected by any conservation law, but they still appear in the first law. The role they play in black hole thermodynamics is still not clear and finding a good definition can give us some insights.

The standard definition of the scalar charge in the literature is in the asymptotic expansion of the scalar field $\phi(r)$ for asymptotically flat spacetimes:

$$\phi(r) = \phi_\infty + \frac{\Sigma}{r} + \mathcal{O}(r^{-2}) \tag{1.47}$$

where Σ is the scalar charge and r is the standard radial coordinate.

Our definition needs to recover this result. Let us start considering the simplest case (which will be our toy model in what follows) consisting in a 4-dimensional Einstein-Scalar theory given by the action

$$S[e^a, \phi] = \frac{1}{16\pi G_N^{(4)}} \int \left[-\star\left(e^a \wedge e^b\right) \wedge R_{ab} + \frac{1}{2} d\phi \wedge \star d\phi \right] \tag{1.48}$$

where ϕ is a real scalar field. A general variation of the action is

$$\delta S[e^a, \phi] = \int [\mathbf{E}_a \wedge \delta e^a + \mathbf{E}\delta\phi + d\mathbf{\Theta}(e^a, \phi, \delta e^a, \delta\phi)] \tag{1.49}$$

where $\mathbf{E}$ is the scalar equation of motion

$$\mathbf{E} = -d \star d\phi. \tag{1.50}$$

The action in Eq. (1.49) is exactly invariant under constants shifts of the scalar

$$\phi = \phi + c, \quad c \in \mathbb{R}; \tag{1.51}$$

that is, the action has a global symmetry given by the constant translations of the scalar field. If we take into account the global symmetry Eq. (1.51), we could define a $(d-1)$-form current $\mathbf{J}$ and write $d\mathbf{J}$ in terms of the equation of motion Eq. (1.50)

$$d\mathbf{J} = d\left(\star d\phi\right) = -\mathbf{E} \tag{1.52}$$

to obtain an on-shell closed current. We may define the scalar charge Σ integrating $\mathbf{J}$ over a 3-dimensional spacelike hypersurface $\Sigma^{(3)}$, up to a normalization factor, as

$$\Sigma \sim \int_{\Sigma^3} \star d\phi. \tag{1.53}$$

But we find that this scalar charge definition vanishes on stationary black hole spacetimes and does not allow us to recover the standard definition given in Eq. (1.47). Then, the definition given in Eq. (1.53) does not satisfy our requirements.

We are going to show how we can construct scalar charges as in Refs. [49, 50, 61]. Assuming that the field configuration is invariant under the diffeomorphisms generated by a Killing vector k,

$$\delta_k\phi = -\mathcal{L}_k\phi = -\imath_k d\phi = 0. \tag{1.54}$$

By hypothesis $\delta_k \star d\phi$ must be invariant also

$$-\mathcal{L}_k \star d\phi = -\imath_k d \star d\phi - d\imath_k \star d\phi = 0. \tag{1.55}$$

From the scalar equation of motion Eq. (1.50) and using Eq. (1.55), we find that

$$\imath_k\mathbf{E} = -\imath_k d \star d\phi = d\imath_k \star d\phi \doteq 0, \tag{1.56}$$

where $\imath_k \star d\phi$ is a 2-form which is closed on-shell. We define, up to a normalization factor, the 2-*form scalar charge* $\mathbf{Q}_s[k]$,

$$\mathbf{Q}_s[k] \equiv \imath_k \star d\phi \tag{1.57}$$

which will obey a Gauss law.[25] This is,

$$\Sigma \sim \int_{\Sigma^2} \imath_k \star d\phi \tag{1.58}$$

where Σ^2 is a 2-dimensional hypersurface. In the context of black holes, this hypersurface is commonly chosen as either a hypersphere at spatial infinity or the bifurcation surface.

[25] By the same arguments given in the second Noether theorem section, see Item 3a.

However, integration can also be extended to other appropriate hypersurfaces, depending on the specific application or geometric considerations. In this thesis we focus exclusively on the geometry of black holes.

Let us demonstrate that the scalar charge definition Eq. (1.57) leads to the result given in Eq. (1.47). Since the scalar charge must remain invariant regardless of the integration surface, we can evaluate it over a sphere at spatial infinity S^2_∞

$$\Sigma \sim \int_{S^2_\infty} \imath_k \star d\phi \sim -4\pi\Sigma, \tag{1.59}$$

where we recover the scalar charge Σ along with the normalization factor that must be taken into account. Now, if we integrate the scalar charge definition Eq. (1.57) over the bifurcation surface

$$\Sigma \sim \int_{\mathcal{BH}} \imath_k \star d\phi = 0, \tag{1.60}$$

this vanishes because by definition the bifurcation surface satisfies the property

$$k \stackrel{\mathcal{BH}}{=} 0. \tag{1.61}$$

By equating Eq. (1.59) with Eq. (1.60), we find

$$\Sigma = 0, \tag{1.62}$$

which physically implies that the only regular black hole solution in the theory Eq. (1.48) occurs when the scalar charge vanishes.

Unlike the naive procedure in Eq. (1.53), this definition proves more robust for studying no-hair theorems. The presence of a bifurcation surface is particularly crucial because it enforces $\Sigma = 0$. The vanishing of the scalar charge on the bifurcation surface arises directly from the fact that the Killing vector field k is null there, making the integrand to vanish. This property provides a key geometric constraint that plays a central role in validating no-hair theorems within this framework.

In this work, we have defined scalar charges associated to global symmetries in theories with Abelian gauge fields (see Ref. [49]) and in theories with a scalar potential and no global symmetries (see Ref. [50]). In any of them, the existence of regular hairy black hole solutions is restricted to the conditions imposed by the integral of the scalar charge over the bifurcation surface.

In theories with global symmetries, the integral over the bifurcation surface gives us the non-trivial relation of the scalar charge with the conserved quantities, while in theories with no global symmetries the integral over the bifurcation surface gives the restrictions that the scalar potential must satisfy in order to have hairy black holes.

1.3 Smarr formula from the generalized Komar charge

The Smarr formula was first derived as a consequence of the homogeneity of the entropy and Euler's theorem [62], resulting in a thermodynamic relation involving the mass and other conserved quantities. Moreover, Komar integrals [63] have also proven be useful for deriving the Smarr formula for stationary black holes [64]. In [65], a systematic framework is presented for constructing Komar integrals, starting from the Noether-Wald charge and employing the formalism outlined in Refs. [66–68]. This approach combines surface and volume integrals into a unified expression. Remarkably, as demonstrated in Ref. [69], the volume term can always be reformulated as a surface term which give rise to the Smarr formula, highlighting the consistency and versatility of Wald's formalism in gravitational theories.

In this section we will briefly review how to derive the Smarr formula using the improved Wald formalism. This involves deducing the generalized Komar charge from the Noether-Wald charge and applying the Stokes theorem to integrate it over well-defined black hole boundaries: the bifurcation surface and a sphere at spatial infinity.

The generalized $(d-2)$-form Komar charge in Eq. (1.46) is closed on-shell by construction

$$d\mathbf{K}[k] = 0. \tag{1.63}$$

Let us integrate the last equation, a $(d-1)$-form, over a hypersurface $\Sigma^{(d-1)}$ whose boundaries are the bifurcation surface and a $(d-2)$-sphere at spatial infinity. With the Stokes theorem in Eq. (1.63) we find

$$\int_{S^{(d-2)}_\infty} \mathbf{K}[k] - \int_{\mathcal{BH}} \mathbf{K}[k] \doteq 0, \tag{1.64}$$

To illustrate how this formula relates quantities at infinity and over the bifurcation surface, let us consider again the action in Eq. (1.48). The generalized Komar charge for that theory can be obtained following the observations made in the Noether theorem's section (item 4) and was defined in Eq. (1.46). Note that in this case the Noether-Wald charge is the same as in pure Einstein gravity (see Ref. [67])

$$\mathbf{Q}[\xi] = \frac{1}{16\pi G_N^{(4)}} \star \left(e^a \wedge e^b\right) P_{\xi ab}, \tag{1.65}$$

where $P_{\xi ab}$ is the *Lorentz momentum map*, defined by the equation

$$\mathcal{D}P_k{}^{ab} + \imath_k R^{ab} = 0 \tag{1.66}$$

for Killing vectors k. This equation is satisfied by the Killing bivector

$$P_k{}^{ab} = \nabla^a k^b, \tag{1.67}$$

and using the Noether-Wald charge in Eq. (1.46) it is easy to see that

$$\mathbf{K}[k] = \frac{-1}{16\pi G_N^{(4)}} \star \left(e^a \wedge e^b\right) P_{k\,ab} \tag{1.68}$$

is the generalized Komar charge for the Einstein-Scalar theory Eq. (1.48).

Now let us consider a static and asymptotically flat black hole spacetime admitting a Killing vector $k = \partial_t - \Omega_H \partial_\varphi$. Using the generalized zeroth law [33] for each term in Eq. (1.64), the integral evaluated at spatial infinity takes the form

$$\int_{S^2_\infty} \mathbf{K}[k] = \int_{S^2_\infty} \big(\mathbf{K}[\partial_t] - \Omega_H \mathbf{K}[\partial_\varphi]\big) = \frac{M}{2} - \Omega_H J. \tag{1.69}$$

The integral over the bifurcation surface is

$$\int_{\mathcal{BH}} \mathbf{K}[k] = -\frac{\kappa}{16\pi G_N} \int_{\mathcal{BH}} d^2\Sigma n_{ab} n^{ab} = \frac{\kappa A_H}{8\pi G_N} = T_H S. \tag{1.70}$$

Equating Eqs.(1.69) and (1.70) we arrive at the *Smarr formula*

$$2T_H S = M - 2\Omega_H J. \tag{1.71}$$

For the theory under consideration, the Smarr formula is the same as in pure Einstein gravity, since the generalized Komar charge remains unchanged. This is due to the absence of fields with gauge freedoms that could alter it, as happens in the case of gravity plus a Maxwell field (see Appendix C). It is important to emphasize that when the theory includes a scalar potential that breaks the global symmetry, the Noether-Wald charge remains unaffected, but the generalized Komar charge does change. This is due to the term $\mathbf{w}_k$, which no longer vanishes as it does in pure gravity. The scalar potential in Einstein-Scalar-Potential theories includes a dimensionful coupling constant α, which will explicitly appears in the Smarr formula as a new thermodynamics variable with a conjugate potential; in contrast to the scalar charge that does not contribute to the Smarr formula.

Although the theory under consideration is relatively simple, the Smarr formula offers a clear physical interpretation by establishing a thermodynamic relationship between quantities measured at spatial infinity and those defined at the horizon. Quantities measured at infinity, such as the energy M and angular momentum J, represent the global properties of the spacetime. In contrast, the thermodynamic quantities at the horizon are directly linked to the geometric characteristics of the black hole: the entropy S is proportional to the area of the event horizon and the temperature T depends on the surface gravity κ, which is constant across the horizon by virtue of the zeroth law.

1.4 Summary of the main results.

Part I: Thermodynamically stable asymptotically flat black holes

In the first part, which consists of Chapter 2 we study the thermodynamic stability for asymptotically flat black holes, comparing two black hole solutions:

1. A family of hairy black holes in a 4-dimensional Einstein-Maxwell-Scalar-Potential theory with action Eq.(2.1), and
2. a family of black holes in a 5-dimensional Einstein-Maxwell-Gauss-Bonnet theory whose action is Eq.(2.1).

The goal is to identify configurations that allow for thermodynamic stability.

For the hairy black holes, the thermodynamic stability is achieved in the positive branch of solutions, both in the canonical and grand canonical ensemble. For the Gauss-Bonnet black hole, the stability is found in the positive branch also (with positive coupling parameter α), but only in the canonical ensemble.

The striking similarity is that in both cases the stability regions are for small black holes, near extremality, where the corrections from scalar fields or higher-curvature terms play a crucial role in regulating the thermodynamic behavior.

Part II: Scalars in black hole thermodynamics

The second part consists of the Chapters 3 and 4, and it is devoted to the investigation of the effects produced by scalar fields in black hole thermodynamics.

- In Chapter 3, based on Ref. [49], we considered 4-dimensional ungauged supergravity-inspired theories containing n_S scalar fields ϕ^x that parametrize a symmetric space and n_V 1-form fields $A^A = A^A_\mu dx^\mu$ with 2-form field strengths $F^A = dA^A$ coupled to gravity, described by the generic action given in Eq.(3.2). Two particular families of black hole solutions contained in that action are the static dilaton and static axion-dilaton.

  We use the improved Wald's formalism to study black hole thermodynamics with scalar charges, introducing a new, covariant definition of scalar charge as an integral over closed 2-surfaces. These charges satisfy a Gauss law, are invariant under coordinate and gauge transformations, and are derived from global symmetries of the action (specifically in stationary spacetimes with timelike Killing vector).

  Also we recover consistently the scalar charge term proposed by Gibbons, Kallosh and Kol in Ref. [59].

  Finally, it is explicitly shown how the scalar charge definition works for static dilaton and static axion-dilaton black hole spacetimes. The key outcome of this work highlights the dependence of scalar charges on conserved charges, confirming their nature as secondary hair. Importantly, the existence of a bifurcation surface is shown to be essential for this conclusion, as it ensures the well-posedness of the scalar charge in these spacetimes.

- In Chapter 4, based on Ref. [50] we applied the procedure of the previous chapter to define scalar charges in a 4-dimensional theory for a real scalar field in the action known as Einstein-Scalar-Potential Eq.(4.1). We extend the covariant definition of scalar charge to include theories with scalar potentials that lack global symmetries.

  The Smarr formula is derived in such a way that incorporates a term proportional to the dimensionful parameter α present in the scalar potential, as expected, akin to the cosmological constant in black hole chemistry. The regularity of the conjugate potential Φ_α of α in the Smarr formula imposes the same conditions on the scalar potential found previously, confirming in this way the consistency of the constrains. Also, it is explicitly shown how the scalar charge appears in the first law associated with variations of the asymptotic scalar field using the Wald's formalism, confirming the results by Gibbons, Kallosh and Kol in Ref. [59].

  Finally, it is shown how these results are valid in the Anabalón-Oliva hairy black hole. The defined scalar charge must remain invariant regardless of the surface over which it is integrated. The main result of this work is that by performing the integration of the scalar charge over the bifurcation surface and equating it to the integral of the scalar charge at spatial infinity, one derives conditions that the scalar potential must satisfy to ensure the existence of a regular black hole, providing a powerful framework that complements classical no-hair theorems.

Part III: Conclusions and appendices

The last part of the thesis contains the conclusions in English and Spanish and several appendices with additional information.

Part I

Thermodynamic stability

2

Existence of thermodynamically stable asymptotically flat black holes

This chapter is based on:
Existence of thermodynamically stable asymptotically flat black holes
D. Astefanesei, R. Ballesteros, P. Cabrera, G. Casanova and R.Rojas
Phys. Rev. D 110 (2024) 2, 024045 (arXiv:2404.15566)

The black hole represents the equilibrium end state of gravitational collapse and the relationship between thermodynamic entropy and the area of an event horizon is one of the most robust results in gravitational physics. Once quantum effects were taken into account, it was understood by Hawking [34] that black holes can emit radiation. While the Hawking radiation produces a very small effect that is not relevant from an empirical point of view, the black hole thermodynamics makes sense only in a theoretical framework in which the black holes are thermodynamically stable. However, this is not the case for black holes in flat spacetime, e.g. Schwarzschild, Reissner-Nordström (RN), and Kerr black holes are thermodynamically unstable.

One way to circumvent this situation is to enclose the asymptotically flat black hole by a finite 'box' [70] such that the radiation can not escape to the asymptotic region. However, while this proposal is interesting from a theoretical point of view, it does not provide any concrete hint of physical situations where this kind of boundary conditions can appear naturally. A more general construction, for a theory with a non-trivial negative cosmological constant, was considered by Hawking and Page [71]. They found that, indeed, there exists a general family of 'large' black holes (with the horizon radius comparable with the radius of anti-de Sitter (AdS) spacetime) that are thermodynamically stable. In this case, the conformal boundary of AdS (where the boundary conditions should be imposed) plays the role of the 'box'. However, the boundary conditions are changed to correspond to a spacetime with negative cosmological constant and, while this is a very important result in the context of AdS-CFT duality [72], it can not be applied to asymptotically flat black holes.

Therefore, to obtain a well defined thermodynamic framework, the key point is the existence of asymptotically flat black holes, which are thermodynamically stable, without imposing artificially boundary conditions corresponding to a finite box. Elucidating this aspect could be relevant for understanding the existence of supermassive black holes that can not be formed by gravitational collapse. One can imagine a scenario where, in the early universe, small black holes exist in a suitable environment that makes them thermodynamically stable, e.g. surrounded by dark matter. Then, after a long time, they can grow up to become supermassive if there is enough matter around they can absorb.

In this work, we carefully investigate the existence of such an 'environment' and present a detailed analysis of the thermodynamic stability by using the quasilocal formalism supplemented with boundary counterterms. Both Gauss-Bonnet-Maxwell and Einstein-Maxwell-dilaton theories can be interpreted as particular completions of Einstein-Maxwell gravity, and so the question of how these theories modify some concrete features of black holes could be of significant interest. First, we consider a four-dimensional theory, with a scalar field and its self-interaction, in which there exist exact hairy black hole solutions. While this is a toy model for static black holes, it shows that, even in a simple theoretical set up, the hair can stabilize, dynamically and thermodynamically, the black holes (see, also, [52, 53, 55]). It can very well be that more complicated models for dark matter can have a similar effect. Second, we consider the five-dimensional gravity when the Einstein-Hilbert action is supplemented by GB corrections. These models can be understood, for example, as effective theories from string theory where the higher derivative corrections are tree level, depending of the string tension (there are also quantum corrections depending of the string coupling). It was found long time ago that there exist exact static neutral black hole solutions and they are thermodynamically stable [73] (see also [74, 75]).[1] We consider the generalization of RN black hole in GB theory, compute the conserved charges within the quasilocal formalism of Brown and York [77] that is self-consistent and so we do not need Wald formalism [67] to obtain the entropy.[2] We find again that there exist thermodynamically stable charged static black holes.

The quasilocal formalism supplemented with counterterms in flat spacetime was used in [78] to obtain the thermodynamics of the dipole black ring [79]. However, this formalism was put on a firm ground by Mann and Marolf in [80], where they have shown that the addition of an appropriate covariant boundary term to the gravitational action yields a well-defined variational principle for asymptotically flat spacetimes. This becomes the standard tool that leads to a natural definition of conserved quantities at spatial infinity, e.g. [81–87]. In this work, we use concrete counterterms to circumvent the problem with the background subtraction method when the 'background' can not be explicitly constructed. We construct the counterterm for GB gravity (and fix the ambiguity of an overall numerical factor), which is consistent with a correct variational principle and regularizes the Euclidean action. Interestingly, unlike the AdS case, there is no need of adding counterterms for the scalar field in flat spacetime when it vanishes at the boundary. We shall consider both the grand canonical ensemble, in which the electric potential difference between the event horizon and the infinity is fixed, and the canonical ensemble, in which the electric charge is fixed.

The remainder of the paper is organized as follows: in Section 2.1, we present the exact asymptotically flat hairy charged black hole solution and study in detail the local and global thermodynamic stability. We use the counterterm method to obtain the on-shell regularized Euclidean action and the quasilocal formalism to obtain the conserved charges. We identify the region of the phase space where there exist thermodynamically stable black holes. In Section 2.2, we present the charged black hole solution in GB gravity. We obtain the on-shell Euclidean action, conserved charges, Smarr formula, and verify the quantum-statistical relation. Again, we identify the region of the phase space where

[1] There is another physical mechanism that could be important for the stability of black holes in theories with higher derivative terms, namely the scalarization and existence of the scalar condensates [76].

[2] The Wald formalism and a concrete computation of scalar charge for various hairy black holes can be found, e.g., in [49, 56, 57].

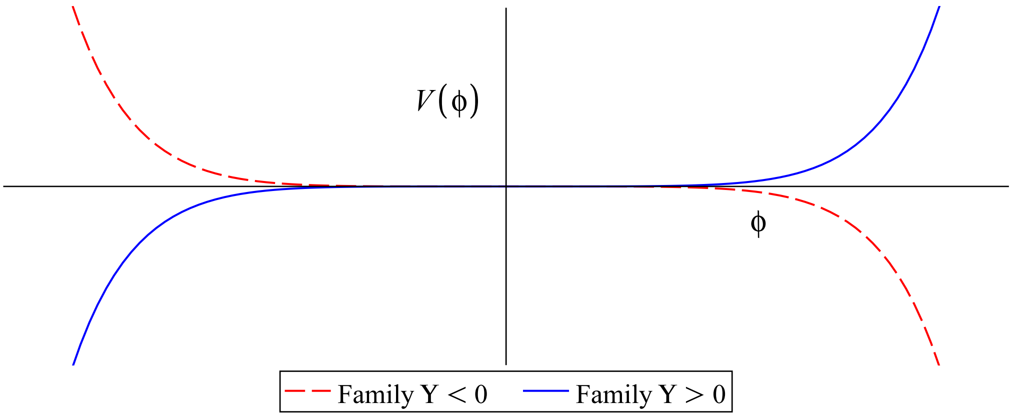

Figure 2.1: Schematic behaviour of the scalar field potential for the family $\Upsilon < 0$ and $\Upsilon > 0$. The region $\phi < 0$ corresponds to the negative branch and $\phi > 0$ corresponds to the positive branch.

there exist thermodynamically stable black holes. In the last section, we present a brief review of our main results, complement the analysis of thermodynamic behaviour with an analysis using the thermodynamic potential, and discuss to some extent the extremal limit that is important for the existence of a consistent canonical ensemble. In Appendix C.1, we summarize the general thermodynamic stability conditions of response functions for charged black holes. In Appendix C.2, we compare the counterterm method in AdS with the one in flat spacetime and explain why there is no need of scalar counterterms for regularizing the Euclidean action of asymptotically flat hairy black holes. In Appendix C.3, we review the basics of the Arnowitt-Deser-Misner (ADM) formalism in five dimensions, required for Sec. 2.2.

2.1 Black holes in Einstein-Maxwell-dilaton theory

Let us consider the theory given by the action

$$I = \frac{1}{2\kappa}\int_{\mathcal{M}} d^4x\sqrt{-g}\left[R - \frac{1}{2}(\partial\phi)^2 - e^{\phi}F^2 - V(\phi)\right] \tag{2.1}$$

where $\kappa = 8\pi$, in the unit system where $G = c = 1$; $F^2 = F_{\mu\nu}F^{\mu\nu}$, where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the gauge field and A_μ the gauge potential, and $(\partial\phi)^2 \equiv g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$.

The potential [88–90] is

$$V(\phi) \equiv 2\Upsilon\left(2\phi + \phi\cosh\phi - 3\sinh\phi\right) \tag{2.2}$$

where Υ is a dimensionful parameter that characterize the strength of the potential. Importantly, now it is well understood that in fact this is a model of a dilaton and its self-interaction, in the sense that the dilaton is endowed with a potential that originates from an electromagnetic Fayet-Iliopoulos (FI) term in $\mathcal{N} = 2$ extended supergravity in four spacetime dimensions [91, 92] (see, also, [93]). Therefore, the theory we consider is consistent and its ground state is stable (recently, exact hairy charged soliton solutions were constructed in [94–97]). The schematic behaviour of the potential (2.2) is depicted in Fig. 2.1. The potential is symmetric under $\phi \to -\phi$ and $\Upsilon \to -\Upsilon$ transformations.

The exact static hairy black hole solution was obtained in [90],

$$ds^2 = \Omega(x)\left[-f(x)dt^2 + \frac{\eta^2 dx^2}{x^2 f(x)} + d\Sigma_2^2\right], \qquad A_\mu = \left[\Phi - \frac{q(x-1)}{x}\right]\delta_\mu^t, \qquad \phi(x) = \ln(x) \tag{2.3}$$

where $d\Sigma_2^2 \equiv d\theta^2 + \sin^2\theta d\varphi^2$ and

$$f(x) = \Upsilon\left(\frac{x^2-1}{2x} - \ln x\right) + \frac{\eta^2(x-1)^2}{x}\left[1 - \frac{2q^2(x-1)}{x}\right], \qquad \Omega(x) = \frac{x}{\eta^2(x-1)^2} \tag{2.4}$$

In the expressions above, x is a dimensionless coordinate that is related to the standard canonical radial coordinate, r, by the transformation $\Omega(x) = r^2$. There are two constants of integration, η and q, which are related to the mass and electric charge of the solution, and Φ in the expression of gauge field is an arbitrary additive constant. The black hole event horizon is located at $x = x_+$ such that $f(x_+) = 0$ and x_+ is the biggest root. The asymptotic region is located at $x \to 1$, where the conformal factor Ω diverges, and the scalar field (along with its potential) vanishes.

We distinguish two families of solutions: the family for which $\Upsilon > 0$ and the family for which $\Upsilon < 0$. Within each family, there are two branches of solutions, characterized by the domain for the coordinate x, which in turn fixes the sign of ϕ. These two branches are characterized by distinct boundary condition for the scalar field. To obtain the asymptotic behaviour of the dilaton, we consider the transformation to the canonical radial coordinate, namely $\Omega(x) = r^2$ near the boundary. This gives rise to two possible changes of coordinates,

$$x = 1 \pm \frac{1}{\eta r} + \frac{1}{2\eta^2 r^2} \pm \frac{1}{8\eta^3 r^3} + \mathcal{O}(r^{-4}) \tag{2.5}$$

The negative branch corresponds to the case where $0 \le x < 1$ (and so $\phi < 0$). By using the corresponding change of coordinates, we obtain the following boundary expansion: $\phi = -\frac{1}{\eta r} + \mathcal{O}(r^{-3})$. The positive branch corresponds to the case where $1 < x \le \infty$ (and so $\phi > 0$), and the boundary expansion of the scalar field becomes $\phi = +\frac{1}{\eta r} + \mathcal{O}(r^{-3})$.[3]It is worth emphasizing that the "scalar charge" $\Sigma = \eta^{-1}$ is not conserved, and so there is no independent integration constant associated to the scalar field.

In the remainder of this section, we will be focusing on the family $\Upsilon > 0$, which contains thermodynamically stable black holes. The family $\Upsilon < 0$ only supports black hole solutions for the positive branch, but in this case the potential (2.2) becomes unbounded from below, as shown in Fig. 2.1.

2.1.1 Euclidean action and thermodynamics

Let us briefly present the computation of the on-shell Euclidean action. For concreteness, we perform the computations in the positive branch, that is, under the assumption that $x - 1 > 0$.

[3]Generally, for example in string theory where the scalar fields are moduli related to the coupling constants, the expansion of the scalar field in flat spacetime is $\phi = \phi_\infty + \Sigma/r + ...$, where Σ is the scalar charge. A discussion of why scalar charges (which are not conserved charges) [59], when ϕ_∞ can vary, do not appear in the first law of black hole thermodynamics was presented in [98, 99]. However, since the theory we are interested in contains the dilaton potential, the boundary conditions for the dilaton are such that $\phi_\infty = 0$ (see, also, [50]).

Let us consider first the bulk part of the action (2.1) and the Gibbons-Hawking boundary term [100], given by

$$I_{GH} = \frac{1}{\kappa} \int_{\partial\mathcal{M}} d^3x\sqrt{-h}K \tag{2.6}$$

where $K \equiv \nabla_\mu n^\mu$ is the trace of the extrinsic curvature and h is the determinant of the metric on the hypersurface $x = const$, with unit normal n_μ. Within this foliation of the spacetime, the boundary $\partial\mathcal{M}$ becomes the hypersurface $x = const$ in the limit $x \to 1$.

These two first contributions yield

$$I^E_{bulk} + I^E_{GH} = \beta(-TS - \Phi Q) + \frac{4\pi\beta}{\kappa}\left[\frac{2}{\eta(x-1)} + \frac{\Upsilon - 12\eta^2 q^2 + 3\eta^2}{3\eta^3} + \mathcal{O}(x-1)\right] \tag{2.7}$$

where the temperature and entropy are defined as usual: The temperature is obtained by removing the conical singularity in the Euclidean section, and the entropy is one quarter of the area of the event horizon,

$$T = -\frac{x_+}{4\pi\eta}\left.\frac{df(r)}{dr}\right|_{x=x_+}, \qquad S = \pi\Omega(x_+) \tag{2.8}$$

and Q and Φ are the electric charge and its conjugate potential. The electric charge, in terms of the constants of integration, is obtained by using the Gauss law

$$Q = \frac{1}{4\pi}\oint_{s^2_\infty} e^\phi * F = \frac{q}{\eta} \tag{2.9}$$

and the conjugate potential is defined as

$$\Phi = A_t(x \to 1) - A_t(x_+) = \frac{q(x_+ - 1)}{x_+} \tag{2.10}$$

Now, in order to remove the divergent contribution appearing in (2.7), it is necessary to add a gravitational counterterm for asymptotically flat spacetime [78, 80, 101–103][4]

$$I_{ct} = -\frac{1}{\kappa}\int_{\partial\mathcal{M}} d^3x\sqrt{-h}\sqrt{2\mathcal{R}^{(3)}} \tag{2.11}$$

where $\mathcal{R}^{(3)}$ is the Ricci scalar on $\partial\mathcal{M}$. In the Euclidean section, we have

$$I^E_{ct} = \frac{4\pi\beta}{\kappa}\left[-\frac{2}{\eta(x-1)} - \frac{\Upsilon - 12\eta^2 q^2 + 6\eta^2}{6\eta^3} + \mathcal{O}(x-1)\right]. \tag{2.12}$$

The total action is therefore

$$I^E = I^E_{bulk} + I^E_{GH} + I^E_{ct} = \beta(-TS - \Phi Q) + \beta\left(\frac{12\eta^2 q^2 - \Upsilon}{12\eta^3}\right). \tag{2.13}$$

[4] Unlike AdS spacetime where counterterms for the scalar field should be also included, in our case there is no need of counterterms associated to the dilaton. We discuss this point in great detail in Appendix C.2.

The last term in (2.13) is indeed the mass of the black hole, as follows from computing the conserved charges of the system. We compute this energy by using the Brown-York quasilocal formalism [77], with the quasilocal boundary stress tensor

$$\tau_{ab} \equiv -\frac{2}{\sqrt{-h}}\frac{\delta I}{\delta h^{ab}} \tag{2.14}$$

where $I = I_{bulk} + I_{GH} + I_{ct}$, and the index a stands for the coordinates on the hypersurface $x = const$. For this case, the concrete expression for τ_{ab} is [78]

$$\tau_{ab} = \frac{1}{\kappa}\left[K_{ab} - h_{ab}K - \Psi\left(\mathcal{R}^{(3)}_{ab} - \mathcal{R}^{(3)}h_{ab}\right) - h_{ab}\Box\Psi + \Psi_{;ab}\right] , \qquad \Psi \equiv \left(\frac{2}{\mathcal{R}^{(3)}}\right)^{\frac{1}{2}} \tag{2.15}$$

Now, according to the Brown-York formalism, the conserved charge associated to the isometry generated by the killing vector ξ^a is

$$E = \oint_{s^2_\infty} d^2\sigma\sqrt{\sigma}N^a\xi^b\tau_{ab} \tag{2.16}$$

where N^a is the timelike unit normal to the hypersurface $x = const$ and σ is the determinant of the metric with $x = const$ and $t = const$. The integration is performed in the limit $x \to 1$. For $\xi^a = \delta^a_t$ and the result is

$$E = \frac{12\eta^2q^2 - \Upsilon}{12\eta^3} \tag{2.17}$$

which also reproduces the mass read off from the expansion, in the canonical coordinates, of g_{tt} in the asymptotic region. Therefore, the Euclidean on-shell action satisfies the quantum-statistical relation

$$\frac{I^E}{\beta} = E - TS - \Phi Q \tag{2.18}$$

The computation made so far has assumed that the boundary condition for the gauge field is $\delta A_t|_{\partial\mathcal{M}} = 0$. This implies that the conjugate potential, Φ, is fixed. Thus, the thermodynamic potential obtained, namely, $\mathcal{G} \equiv E - TS - \Phi Q$, corresponds to the grand canonical ensemble. One can construct the canonical ensemble, in which Q is fixed instead, by performing a Legendre transform in (Φ, Q), which is equivalent to adding to the action the following boundary term

$$I_A = \frac{2}{\kappa}\int_{\partial\mathcal{M}} d^3x\sqrt{-h}\, n_\nu F^{\mu\nu}A_\nu \tag{2.19}$$

giving rise to the thermodynamic potential $\mathcal{F} \equiv E - TS$.

Now that we have all the thermodynamic quantities consistently computed, one can verify that the first law of black hole thermodynamics is satisfied,

$$dE = TdS + \Phi dQ \tag{2.20}$$

Notice that the scalar charge does not appear explicitly in the first law as an independent term. This comes as a consequence of the fact that this scalar field is a secondary hair with no independent integration constant associated to it. Explicitly, the scalar charge is $\Sigma = \eta^{-1}$ and, by rearranging (2.17), we get

$$E - \frac{Q^2}{\Sigma} + \frac{1}{12}\Upsilon\Sigma^3 = 0 \tag{2.21}$$

and so Σ depends on the conserved charges E and Q, and the parameter of the theory Υ.

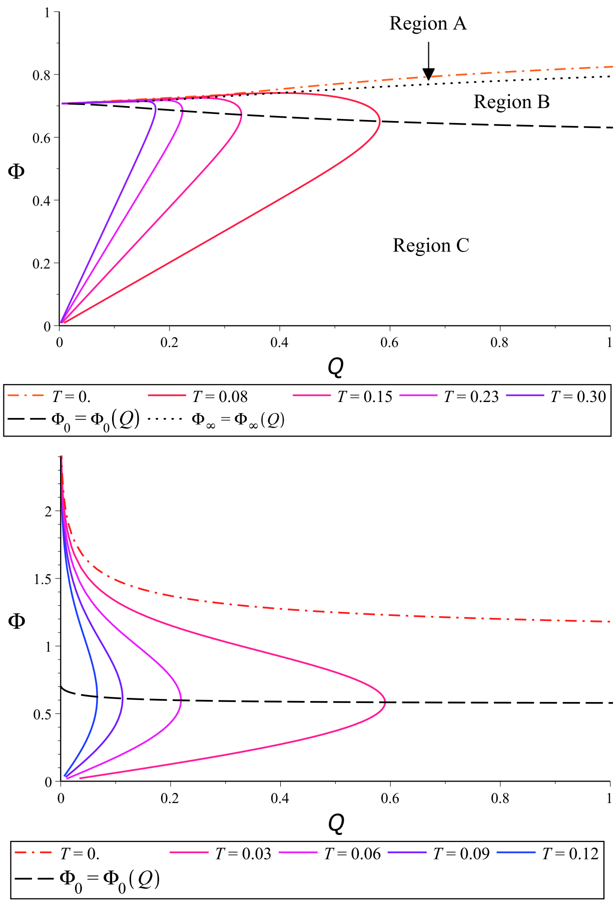

Figure 2.2: Equation of state $\Phi - Q$, for T fixed. Left hand side: Positive branch. Right hand side: Negative branch.

2.1.2 Equation of state $\Phi = \Phi(Q, T)$

To proceed further with the thermodynamic behaviour, let us construct the equation of state $\Phi = \Phi(Q, T)$. First, we replace $q = Q\eta$ in the expressions for the thermodynamic quantities, as follows from (2.9). We then have the parametric expressions

$$T = \frac{(x_+ - 1)\left[\left(\eta^4 Q^2 - \frac{1}{2}\eta^2 - \frac{1}{4}\Upsilon\right)x_+^2 + \left(\eta^4 Q^2 - \frac{1}{2}\eta^2 + \frac{1}{4}\Upsilon\right)x_+ - 2\eta^4 Q^2\right]}{2\pi\eta x_+^2}\,, \Phi = \frac{Q\eta(x_+ - 1)}{x_+} \tag{2.22}$$

where η must be isolated from the horizon equation $f(x_+) = 0$. In Fig. 2.2, we have depicted the equation of state for the positive and negative branch, respectively. In both cases we have considered the family $\Upsilon > 0$.

The equation of state contains relevant information on the local stability. Concretely, the response function

$$\epsilon_T \equiv \left(\frac{\partial Q}{\partial \Phi}\right)_T \tag{2.23}$$

known as isothermal permittivity is a measure of the stability of the configurations against small fluctuations of the electric charge (for more details, see [104, 105]). If $\epsilon_T > 0$, the system is locally stable against electric fluctuations. In both cases, the positive and negative branches contain regions where $\epsilon_T > 0$. For the negative branch (the plot at the right hand side of Fig. 2.2), the region for which $\epsilon_T > 0$ corresponds, as we are going to show in the next section, to the region where the second relevant response function, the heat capacity at constant electric charge, C_Q (and also C_Φ), is negatively defined, indicating a thermal instability. This is similar with the thermodynamics of RN black hole and so these configurations are not locally stable.

The interesting case is for the positive branch (the plot on the left hand side of Fig. 2.2), where a novel region with $\epsilon_T > 0$ develops. This is the region A, as shown in the plot. It looks a very small region in the phase space, but it is valid for any $Q > 0$ (and $T > 0$). Next, we show that in this particular region, the heat capacity is also positive, thus fulfilling the conditions for local stability. The regions A, B and C are separated by the curves characterized by $(\partial\Phi/\partial Q)_T = 0$ (the dotted curve, that separates A from B) and $(\partial Q/\partial\Phi)_T = 0$ (the dashed curve, that separates B from C).

2.1.3 Phase diagram and local stability

The second response function relevant for the local stability is the heat capacity, both at fixed electric charge, C_Q, and at fixed conjugate potential C_Φ,

$$C_Q \equiv T\left(\frac{\partial S}{\partial T}\right)_Q , \qquad C_\Phi \equiv T\left(\frac{\partial S}{\partial T}\right)_\Phi \tag{2.24}$$

Thermally stable configurations are those with $C_Q > 0$, for the canonical ensemble, and $C_\Phi > 0$, for the grand canonical (see Appendix C.1 for a brief summary on the criteria for local stability).

Canonical ensemble: Q fixed

In Fig. 2.3, we have depicted the phase diagram $T - S$ of the canonical ensemble for the positive (left hand side plot) and negative branch (right hand side plot), respectively. As commented above, the interesting behaviour occurs for the positive branch, where we observe that $C_Q > 0$ for both regions, A and B. However, only in region A we have, in addition, $\epsilon_T > 0$ and, hence, only region A contains fully locally stable configurations, for the positive branch. On the other hand, the negative branch is characterized by two regions, according to the sign of C_Q. These two regions are separated by exactly the same curve that divides the regions $\epsilon_T > 0$ and $\epsilon_T < 0$ in the $\Phi - Q$ phase space (the dashed curve, for which $(\partial Q/\partial\Phi)_T = 0$ and $C_Q^{-1} = 0$) and, thus, the response functions C_Q and ϵ_T can not be simultaneously positive for the negative branch.

Grand canonical ensemble: Φ fixed

A similar analysis can be done in the grand canonical ensemble. As shown in Fig. 2.4, since clearly $C_\Phi < 0$, the black holes of the negative branch are not stable. However, for the positive branch, we also observe that, in particular, the region A is characterized by $C_\Phi > 0$, which also corresponds to $\epsilon_T > 0$ and so there exist black hole solutions, which are thermodynamically stable (region B corresponds to $\epsilon_T < 0$ and so the response functions are not simultaneously positively defined). We would also like to point out that, for the positive branch, the stable black holes are also globally stable, as they minimize

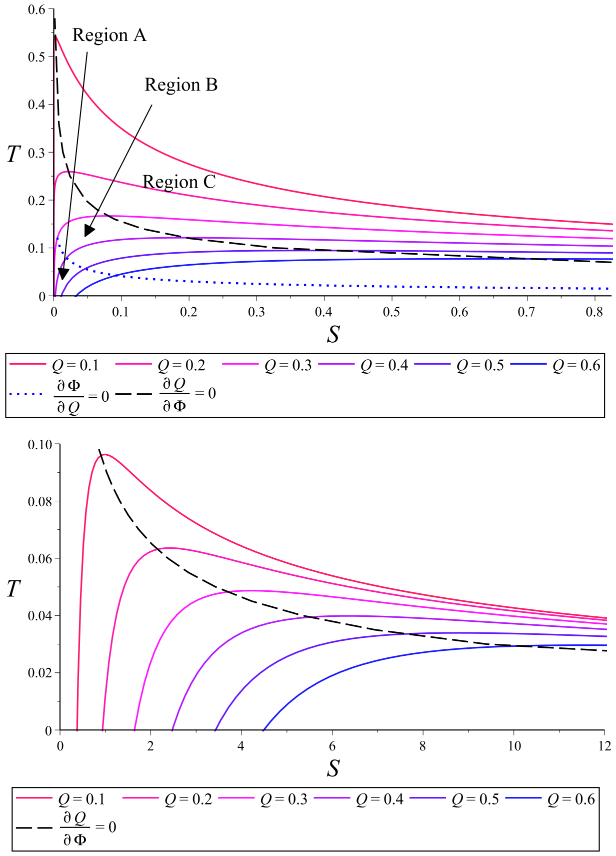

Figure 2.3: $T - S$ for fixed Q. Left hand side: Positive branch. Right hand side: Negative branch.

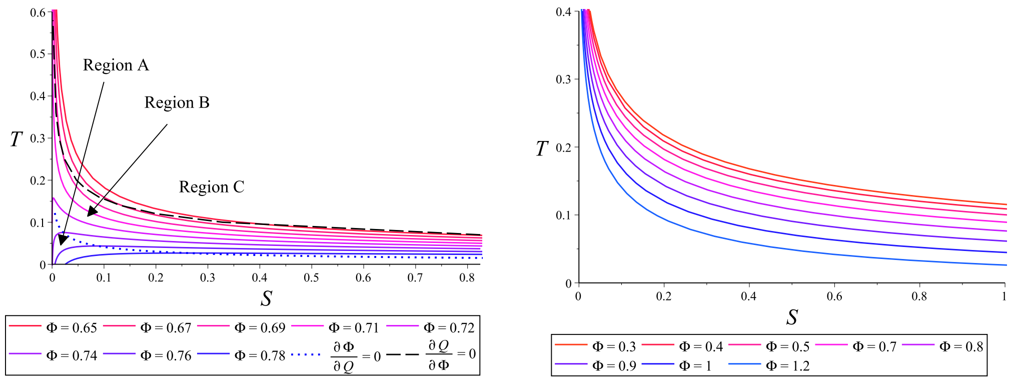

Figure 2.4: $T - S$ for fixed Φ. Left hand side: Positive branch. Right hand side: Negative branch.

the thermodynamic potential, $\mathcal{G}$. We are going to discuss this point in more detail in discussion section.

2.2 Black holes in Einstein-Maxwell-Gauss-Bonnet

Let us now consider the Einstein-Maxwell-Gauss-Bonnet action [73]

$$I = \frac{1}{2\kappa}\int_{\mathcal{M}} d^5x\sqrt{-g}\left(R + \frac{1}{4}\alpha\mathcal{R}_{GB} - F^2\right) \tag{2.25}$$

where $\kappa = 8\pi$, in the unit system where $G = c = 1$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, A_μ is the gauge potential, $\mathcal{R}_{GB} \equiv R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho}$ is the GB invariant and α is the coupling constant with GB sector. It is convenient to distinguish two branches, according to the sign of α, since they have different thermodynamic properties, as shown below.

The equations of motion for this system are

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \frac{1}{4}\alpha H_{\mu\nu} = \kappa T^{EM}_{\mu\nu}\,, \qquad \nabla_\mu F^{\mu\nu} = 0 \tag{2.26}$$

where $T^{EM}_{\mu\nu} \equiv \frac{1}{4\pi}\left(F_{\mu\alpha}F_\nu{}^\alpha - \frac{1}{4}g_{\mu\nu}F^2\right)$ is the energy-momentum tensor for the electromagnetic field, and

$$H_{\mu\nu} \equiv 2R_{\mu\alpha\beta\sigma}R_\nu{}^{\alpha\beta\sigma} - 4R_{\mu\sigma\nu\rho}R^{\sigma\rho} - 4R_{\mu\sigma}R^\sigma_\nu + 2RR_{\mu\nu} - \frac{1}{2}\mathcal{R}_{GB}g_{\mu\nu} \tag{2.27}$$

The spherically symmetric black hole solution was obtained in [106]

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Sigma_3^2\,, \qquad A_\mu = \left(\Phi - \frac{\sqrt{3}q}{2r^2}\right)\delta^t_\mu \tag{2.28}$$

where $d\Sigma_3^2 \equiv d\theta^2 + \sin^2\theta d\phi^2 + \sin^2\theta\sin^2\phi d\psi^2$ is the line element of the unit 3-sphere, Φ is an additive constant in the expression of gauge potential, and q is the charge parameter. Working with this ansatz, the differential equation for $f(r)$ becomes

$$\left(r^2 + \alpha - \alpha f\right)f' + 2r\left(f-1\right) + \frac{2q^2}{r^3} = 0 \tag{2.29}$$

and there exists two families characterized by $\epsilon = \pm 1$, which can be expressed in a compact form as

$$f(r) = 1 + \frac{r^2}{\alpha}\left[1 + \epsilon\left(1 + \frac{2\alpha\mu}{r^4} - \frac{2\alpha q^2}{r^6}\right)^{\frac{1}{2}}\right] \tag{2.30}$$

where μ is the constant of integration related to the mass. However, only the family $\epsilon = -1$ is consistent with asymptotic conditions of flat spacetime, namely $f(r \to \infty) = 1 + \mathcal{O}(r^{-2})$. The other family characterized by $\epsilon = 1$ is not asymptotically flat, namely $f(r \to \infty) = 2r^2/\alpha + 1 + \mathcal{O}(r^{-2})$. Since we are interested in asymptotically flat spacetime, in what follows we are going to consider the family $\epsilon = -1$.

As in general relativity, in GB theory the black hole configurations can also have at most two horizons for which $f(r) = 0$,

$$r_\pm = \frac{1}{2}\left[2\mu - \alpha \pm \sqrt{(2\mu-\alpha)^2 - 16q^2}\right]^{\frac{1}{2}} \tag{2.31}$$

with the outer one, r_+, corresponding to the event horizon. We would like to emphasize that the solution is regular when $2\mu - \alpha \geq 0$, otherwise it becomes a naked singularity.

Therefore, the extremal black holes exist when the constraint $(2\mu-\alpha)^2 = 16q^2$ is satisfied. In this case, the two horizons coincide ($r_+ = r_-$) and so the mass parameter can be written as $\mu = 2r_+^2 + (1/2)\alpha$.

In the next section, we shall explicitly compute the regularized quasilocal stress tensor and energy. As a consistency check, we prove that, once an ambiguity in the overall factor that multiplies the counterterm is fixed, the quasilocal mass matches, indeed, the Arnowitt-Deser-Misner (ADM) mass [107–110].

2.2.1 Euclidean action and thermodynamics

In this section, we use again the counterterm method and quasilocal formalism of Brown and York [77] to consistently compute the regularized Euclidean on-shell action, boundary stress tensor, and energy. We verify that the quantum-statistical relation and first law of black hole thermodynamics are consistently satisfied. We also obtain Smarr formula in this non-trivial case.

The regularized action of this theory is

$$I^E = I^E_{bulk} + I^E_{GH} + I^E_{ct} \tag{2.32}$$

where I^E_{bulk} is the bulk part of the action given by (2.25), I_{GH} and I_{ct} are the Gibbons-Hawking boundary term [111] and the gravitational counterterm, respectively, with their corresponding extension for the GB sector,

$$I^E_{GH} = -\frac{1}{\kappa}\int_{\partial\mathcal{M}} d^4x\sqrt{h^E}\left(K + \frac{1}{2}\alpha\mathcal{K}_{GB}\right) \tag{2.33}$$

$$I^E_{ct} = \frac{1}{\kappa}\int_{\partial\mathcal{M}} d^4x\sqrt{h^E}\left(\frac{3}{2}\mathcal{R}\right)^{\frac{1}{2}}\left(1 + \frac{j}{9}\alpha\mathcal{R}\right) \tag{2.34}$$

where h is the determinant of the induced metric h_{ab} on the boundary $\partial\mathcal{M}$, $K_{ab} = \nabla_a n_b$ is the extrinsic curvature, with n_a being the normal unit vector to $\partial\mathcal{M}$, $K \equiv h^{ab}K_{ab}$, and

$$\mathcal{K}_{GB} \equiv \left[\frac{2}{3}K_{ac}K_{db}\left(Kh^{cd} - K^{cd}\right) + \frac{1}{3}K_{ab}\left(K_{cd}K^{cd} - K^2\right)\right]h^{ab} - 2\left(\mathcal{R}_{ab} - \frac{1}{2}\mathcal{R}h_{ab}\right)K^{ab} \tag{2.35}$$

and $\mathcal{R} = h^{ab}\mathcal{R}_{ab}$ is the trace of the Ricci tensor on $\partial\mathcal{M}$.

For now, we use an arbitrary factor denoted by j in the GB counterterm because the variational principle is well defined for any j. This ambiguity can be eliminated on physical grounds, and we are going to fix this factor in a consistent manner later on so that the physical quantities are well defined. A similar couterterm was used in [112] (see, also, [113, 114] for GB counterterms in AdS).

The on-shell Euclidean action in the bulk can be written as[5]

$$I^E_{\text{bulk}} = \frac{\pi\beta}{8}\left[r^3 f' - 3\alpha r\left(f-1\right)f'\right]\Big|_{r_+}^{r_b} - \frac{3\pi\beta q^2}{4r_+^2} \tag{2.36}$$

where $f' \equiv df/dr$ and r_b is the location of the boundary. We shall take the limit $r_b \to \infty$ at the end of the computation. Specifically, for RNGB black hole, the result for the bulk

[5]Here, $\beta = \int_0^\beta d\tau$ is the periodicity of the imaginary time in the Euclidean section.

term can be written in the compact form as

$$I_{\rm bulk}^{E} = \frac{\pi\beta}{4}\left(\mu - \frac{r_+^4 + 3\alpha r_+^2 + 2q^2}{r_+^2 + \alpha}\right) \tag{2.37}$$

The Gibbons-Hawking boundary term is[6]

$$I_{GH}^{E} = \frac{\pi\beta}{2}(\mu - \alpha) - \frac{3\pi\beta}{4}r_b^2 + \mathcal{O}\left(r_b^{-2}\right) \tag{2.38}$$

Observe that I_{GH}^{E} contains a divergent part $\propto r_b^2$, which does not come from the GB sector. As expected, this divergent quantity is going to be removed with the gravitational counterterm that turns out to be

$$I_{ct}^{E} = -\frac{\pi\beta}{2}\left(\frac{3}{4}\mu - j\alpha\right) + \frac{3\pi\beta}{4}r_b^2 + \mathcal{O}\left(r_b^{-2}\right) \tag{2.39}$$

All the divergent contributions are now cancelled out and the final result for the Euclidean action is

$$I^{E} = \frac{3\pi\beta}{8}\mu + \frac{\pi\beta}{2}(j-1)\alpha - \frac{\pi\beta}{4}\left(\frac{r_+^4 + 3\alpha r_+^2 + 2q^2}{r_+^2 + \alpha}\right) \tag{2.40}$$

Notice that the second term in Eq. (2.40), proportional to α, is a finite contribution purely due to the GB sector. We shall see below that, for consistency with the asymptotically flat spacetime boundary conditions, we have to fix $j = 1$ such that we recover the usual ADM mass.

We now compute the quasilocal energy for this system at spatial infinity. According to the Brown-York formalism,

$$E_{quasi} = \int_{s_\infty^3} d^3x\sqrt{\sigma}n^a\tau_{ab}\xi^b \tag{2.41}$$

is the conserved charge associated with the time symmetry of the metric, given by the Killing vector $\xi^a = \delta_t^a$, where σ is the determinant of the metric on the three-sphere $ds_\sigma^2 = r^2 d\Sigma_3^2$, $n_a = \delta_a^t/\sqrt{-g^{tt}} = f^{\frac{1}{2}}\delta_a^t$ is the normal unit to the hypersurface $t =$ constant, and τ_{ab} is the boundary stress tensor

$$\tau_{ab} \equiv \frac{2}{\sqrt{-h}}\frac{\delta I}{\delta h^{ab}} = \frac{1}{\kappa}\left[K_{ab} - Kh_{ab} + \frac{1}{2}\alpha\left(\tilde{\mathcal{Q}}_{ab} - \frac{1}{2}\tilde{\mathcal{Q}}h_{ab}\right)\right] - \frac{2}{\kappa}\tilde{\Psi}_{ab} \tag{2.42}$$

where

$$\tilde{\mathcal{Q}}_{ab} \equiv 3\mathcal{J}_{ab} + 2K\mathcal{R}_{ab} - 4\mathcal{R}_{ac}K_b^c + \mathcal{R}K_{ab} - 2K^{cd}\mathcal{R}_{cadb}\,, \qquad \tilde{\mathcal{Q}} = \tilde{\mathcal{Q}}_{ab}h^{ab} \tag{2.43}$$

and

$$\tilde{\Psi}_{ab} \equiv \frac{d\Psi(\mathcal{R})}{d\mathcal{R}}\mathcal{R}_{ab} - \frac{1}{2}\Psi(\mathcal{R})h_{ab}\,, \qquad \Psi(\mathcal{R}) = \left(\frac{3}{2}\mathcal{R}\right)^{\frac{1}{2}}\left[1 + \frac{1}{9}j\alpha\mathcal{R}\right] \tag{2.44}$$

[6] Some intermediate results are

$$\mathcal{J}_{ab}h^{ab} = -\frac{f^{\frac{1}{2}}\left(3rf' + 2f\right)}{r^3}\,, \quad \mathcal{G}_{ab}K^{ab} = -\frac{3rf' + 6f}{2r^3 f^{\frac{1}{2}}}$$

where $\mathcal{G}_{ab} \equiv \mathcal{R}_{ab} - \frac{1}{2}\mathcal{R}h_{ab}$.

The components of the boundary stress tensor are

$$\tau_{tt} = \frac{f}{8\pi r_b^3}\left[3(r_b^2+\alpha)f^{\frac{1}{2}} - \alpha f^{\frac{3}{2}} - \left(2j\alpha + 3r_b^2\right)\right] = -\frac{1}{4\pi r_b^3}\left[\frac{3\mu}{4} + (j-1)\alpha\right] + \mathcal{O}(r_b^{-5}) \tag{2.45}$$

$$\tau_{\theta\theta} = -\frac{4r_b f^{\frac{1}{2}}\left(f^{\frac{1}{2}}-1\right) + \left(r_b^2+\alpha - f\alpha\right)f'}{16\pi f^{\frac{1}{2}}} = -\frac{\mu^2-4q^2}{32\pi r_b^3} + \mathcal{O}(r_b^{-5}) \tag{2.46}$$

where $f = f(r_b)$ and $\tau_{\psi\psi} = \sin^2\phi\tau_{\phi\phi} = \sin^2\theta\sin^2\phi\tau_{\theta\theta}$.

We have now all the necessary ingredients to compute the thermodynamic quantities. By using the formula (2.41), we get

$$E_{quasi} = \frac{3}{8}\pi\mu + \frac{1}{2}\pi(j-1)\alpha \tag{2.47}$$

This expression for the energy of RNGB black hole is consistent with the usual statistical formula obtained from the thermodynamic potential in the grand canonical ensemble:

$$E = \left(\frac{\partial I^E}{\partial\beta}\right)_\Phi - \frac{\Phi}{\beta}\left(\frac{\partial I^E}{\partial\Phi}\right)_\beta = E_{quasi} \tag{2.48}$$

A similar computation can be done in the canonical ensemble as $E = (\partial\tilde{I}^E/\partial\beta)_Q$, where [115]

$$\tilde{I} = I + \frac{2}{\kappa}\int_{\partial\mathcal{M}} d^4x\sqrt{-h}F^{\mu\nu}n_\mu A_\nu \quad\rightarrow\quad \tilde{I}^E = I^E + \beta Q\Phi \tag{2.49}$$

This action is compatible with the boundary condition for the gauge field that fixes the electric charge, which is given by the Gauss law,

$$Q = \epsilon_0\oint\star F = \frac{\sqrt{3}\pi}{2}q \tag{2.50}$$

and $\Phi \equiv A_t(\infty) - A_t(r_+) = \frac{\sqrt{3}}{2r_+^2}q$ is the conjugate potential. The Hawking temperature is computed as usual by removing the conical singularity in the Euclidean section,

$$T = \beta^{-1} = \frac{1}{4\pi}f'(r_+) = \frac{r_+^4 - q^2}{2\pi r_+^3(r_+^2+\alpha)} \tag{2.51}$$

Before comparing the ADM and quasilocal masses, let us first proceed further with computing the entropy directly from the regularized action

$$S = -I^E + \beta\left(\frac{\partial I^E}{\partial\beta}\right)_\Phi = \frac{1}{2}\pi^2 r_+\left(r_+^2+3\alpha\right) \tag{2.52}$$

which follows from statistical mechanics, after using the semi-classical approximation $\ln Z \approx e^{-I^E}$, where Z is the partition function. The entropy receives a contribution from the GB sector and can be also computed by Wald formalism [67, 116].

Before fixing j, let us first verify the quantum-statistical relation. There is some hope that we can eliminate the ambiguity related to j by using this relation, but since the

τ_{tt} component of the boundary stress tensor has a similar contribution, both contributions cancel each other in the quantum statistical relation. Concretely, we obtain[7]

$$\mathcal{G} \equiv E - TS - \Phi Q = \frac{3}{8}\pi\mu + \frac{1}{2}\pi(j-1)\alpha - \frac{1}{4}\pi\left(\frac{r_+^4 + 3\alpha r_+^2 + 2q^2}{r_+^2+\alpha}\right) = \beta^{-1} I^E \tag{2.53}$$

and so it is satisfied for any j.

Therefore, the correct way to fix j is to compare the quasilocal mass at spatial infinity with the ADM mass. Once j is fixed for one specific solution, the counterterm can be used to regularize the energy of all regular solutions with the same boundary conditions. We emphasize that the fall off of GB term is very fast at the boundary and that is why the asymptotic flat boundary conditions are permitted. For completeness, in Appendix C.3, we provide the basic details for computing the ADM mass that matches the holographic mass only for $j = 1$.

We would also like to emphasize that in the limit $r_b \to \infty$, the trace of the boundary stress tensor vanishes,

$$\tau_{ab}h^{ab} = \frac{3\mu}{16\pi r_b^3} + \mathcal{O}(r_b^{-5}) \tag{2.54}$$

and it is covariantly conserved, ${\tau^{ab}}_{;b} = 0$, which is compatible with our solution with no matter or conical defects at spatial infinity [117].

It is also straightforward to verify the first law of black hole thermodynamics,

$$dE = TdS + \Phi dQ. \tag{2.55}$$

The Smarr formula is

$$2E = 3TS + 2\Phi Q + 2\mathcal{B}\alpha, \tag{2.56}$$

where the last term is, in principle, consistent with an extension of the first law, $dE = TdS + \Phi dQ + \mathcal{B}d\alpha$,

$$\mathcal{B} \equiv \left(\frac{\partial E}{\partial \alpha}\right)_{S,Q} = -\frac{3}{8}\pi\left(\frac{1}{2} + \frac{3r_+^2 - 2\mu}{r_+^2+\alpha}\right) \tag{2.57}$$

when the parameter α can vary, which is not our study case.

2.2.2 Equation of state: $Q - \Phi$

In this section, we obtain the equation of state and study in detail the regions where the system achieves local stability.

The equation of state $\Phi = \Phi(Q, T)$ can be implicitly written as

$$T = \frac{1}{6}\left(\frac{Q\Phi}{\pi}\right)^{\frac{1}{2}} \frac{3 - 4\Phi^2}{\Phi\pi\alpha + Q} \tag{2.58}$$

In the limit $\alpha = 0$, the equation of state reduces to the one of charged black hole in general relativity. For this case, it is well known that there are no locally stable configurations, as C_Q and ϵ_T can not be simultaneously positive.

[7] We have assumed the boundary condition $\delta A_\mu|_{\partial\mathcal{M}} = 0$ for the gauge potential, that fixes Φ (the grand canonical ensemble). Similarly, $\mathcal{F} \equiv E - TS = \frac{3}{8}\pi\mu + \frac{1}{2}\pi(j-1)\alpha - \frac{1}{2}\pi^2 r_+\left(r_+^2 + 3\alpha\right) = \beta^{-1}\tilde{I}^E$ for the canonical ensemble.

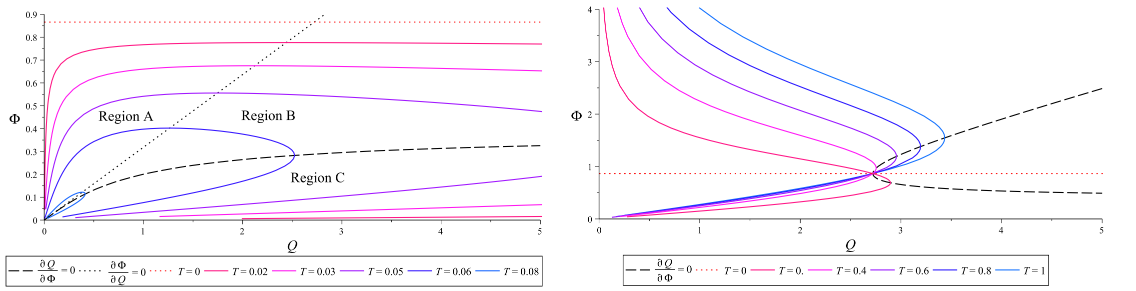

Figure 2.5: Equation of state $\Phi - Q$, for fixed T. Left hand side: $\alpha = 1$. Right hand side: $\alpha = -1$.

For the family $\alpha > 0$, the isotherms $T \neq 0$ in the $Q - \Phi$ plane are characterized by being 'closed', i.e, starting and ending at $Q = \Phi = 0$, as shown in the first plot of Fig. 2.5. This indicates that there is one curve, say $\Phi_0 = \Phi_0(Q)$, along which $\epsilon_T = 0$, and another curve, say $\Phi_\infty = \Phi_\infty(Q)$, along which ϵ_T diverges or, alternatively, $\epsilon_T^{-1} = 0$. From the expression of the isothermal permittivity,

$$\epsilon_T = \frac{Q\left(12\pi\alpha\Phi^3 + 20Q\Phi^2 + 3\pi\alpha\Phi - 3Q\right)}{(4\Phi^2 - 3)(Q - \pi\alpha\Phi)\Phi} \tag{2.59}$$

it follows that the $\Phi_0 = \Phi_0(Q)$ is given implicitly by the cubic equation

$$12\pi\alpha\Phi_0^3(Q) + 20Q\Phi_0^2(Q) + 3\pi\alpha\Phi_0(Q) - 3Q = 0 \tag{2.60}$$

whereas

$$\Phi_\infty(Q) = \frac{Q}{\pi\alpha} \tag{2.61}$$

Because ϵ_T changes twice its sign along a given isotherm $T \neq 0$, we can distinguish three regions. Regions A and C contain electrically stable configurations ($\epsilon_T > 0$), and region B contains electrically unstable ones. While region B and C mimic the behaviour for charged black holes in Einstein-Maxwell gravity, the presence of the GB invariant, moduled by the constant α, is responsible for new stable configurations within region A, which consists of small black holes characterized by $\Phi > \frac{Q}{\pi\alpha}$, which is equivalent to $r_+^2 < \alpha$.

Moreover, for $\alpha > 0$, there exists a maximum allowed temperature that depends only on α. By looking at the first plot in Fig. 2.5, we can observe that, as temperature approaches to its maximum, T_{max}, the corresponding isotherm shrinks to eventually disappears. It can be shown, but also follows from visual inspection, that in the limit $T \to T_{max}$, $\Phi \to Q/(\pi\alpha)$. In other words, the isotherms tend to align with the dotted curve in Fig. 2.5. Replacing $\Phi = Q/(\pi\alpha)$ in the equation of state and taking the limit $Q \to 0$ leads to

$$T_{max} = \frac{1}{4\pi\sqrt{\alpha}} \tag{2.62}$$

On the other hand, for $\alpha < 0$, there is one special curve in the $Q - \Phi$ phase space, namely, $\Phi_0 = \Phi_0(Q)$, where $\epsilon_T = 0$. This situation is depicted in the second plot of Fig. 2.5. As shown next, for the $\alpha < 0$ case, the region with $\epsilon_T > 0$ is thermally unstable, $C_Q < 0$.

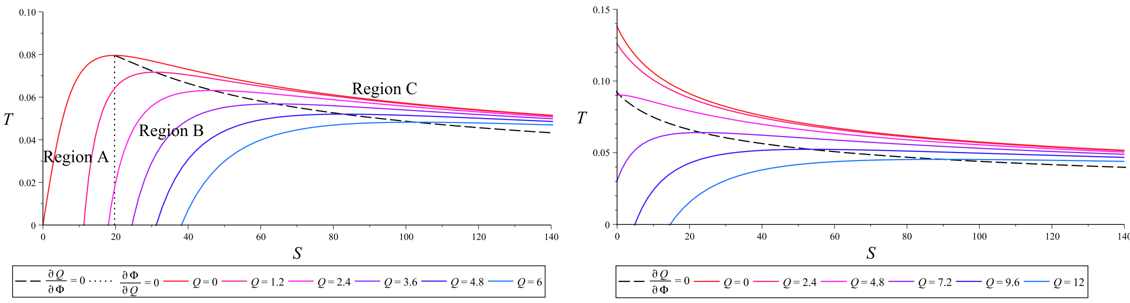

Figure 2.6: $T-S$ for fixed Q. Left hand side: $\alpha = 1$. Right hand side: $\alpha = -1$.

2.2.3 Phase diagram and local stability

Canonical ensemble: Q fixed

Let us focus on the case $\alpha > 0$. From the equation of state, we have that, for a given Q within the region A, the electrically stable configuration is the one that has the largest value of Φ. We now explicitly show that configurations within region A are also thermally stable. The heat capacity in this case is

$$C_Q = T\left(\frac{\partial S}{\partial T}\right)_Q = \frac{3\sqrt{\pi Q\Phi}(Q+\pi\alpha\Phi)^2\left(3-4\Phi^2\right)}{2\Phi^2(12\pi\alpha\Phi^3+20Q\Phi^2+3\pi\alpha\Phi-3Q)} \tag{2.63}$$

It is easy to translate the curves that divide region A, B and C, given by (2.60) and (2.61), into the diagrams $T-S$, as shown in the first plot in Fig. 2.6. As commented earlier, region A contains small black holes, $r_+^2 < \alpha$, that is $S < 2\pi^2\alpha^{\frac{3}{2}}$. These configurations are thermally stable as $C_Q > 0$, and thus, are locally thermodynamically stable.

Grand canonical ensemble: Φ fixed

The thermodynamic stability in the grand canonical ensemble can be investigated in the same way, though the analysis is simpler. The heat capacity for this ensemble is

$$C_\Phi = \frac{3\sqrt{\pi Q}(Q+\pi\alpha\Phi)^2}{2\Phi^{\frac{3}{2}}(\pi\alpha\Phi - Q)} \tag{2.64}$$

Consider the case of interest, $\alpha > 0$. Since region A is characterized by $\Phi > Q/(\pi\alpha)$, it immediately follows that $C_\Phi > 0$ within this region.

Thus, we conclude that region A contains fully thermodynamically stable configuration both in the canonical and grand canonical ensembles.

2.3 Discussion

In this paper, we provide examples of thermodynamically stable asymptotically flat black holes. We have shown that effective theories with the dilaton and its self-interaction, as well as gravity with GB corrections, can allow for thermodynamic stability regions in the phase diagram of charged static black holes.

The conserved charges of asymptotically flat black holes are usually computed by using Hamiltonian methods [107–110]. However, supplemented with counterterms, the quasilocal formalism of Brown and York [77] becomes a powerful framework for computing conserved quantities in general relativity. The basic idea in [77] is to define a 'quasilocal' energy inside a given finite region that can be directly derived from the gravitational action for that specific spatially bounded region. The quasilocal energy is the value of Hamiltonian which generates unit magnitude proper-time translations in a timelike direction orthogonal to spacelike hypersurfaces at some fixed spatial boundary and so it agrees with the ADM energy in the limit when the spatial boundary is pushed to infinity. We emphasize, though, that while within the ADM formalism the foliation is made by hypersurfaces that are Cauchy surfaces such that the data on a slice determine completely the future evolution of the system, that is not necessary the case for the quasilocal formalism. One important result of our work was to obtain a consistent counterterm for GB gravity in flat spacetime. While the variational principle is at the basis of this construction, a relevant subtlety we had to deal with was the existence of a finite contribution coming from the counterterm that is related to the ambiguity in defining its overall factor. To fix this ambiguity and construct the general counterterm that regularizes the action we had to rely on the Hamiltonian formalism. Once the quasilocal formalism is consistently supplemented with counterterms, it can be also used for theories with higher derivative corrections. We emphasize that, unlike in the case of Wald formalism, we do not need to use the first law a priori and so the quasilocal formalism is self-consistent providing all information about the thermodynamic behaviour of the gravitational system.

The existence of asymptotically flat hairy black holes in theories with a scalar field potential was proposed in [118]. We made this proposal concrete by analyzing exact solutions when the dilaton together with its potential are present in the theory. Since in flat spacetime, the effective potential (obtained from the self-interaction of dilaton together with the non-trvial coupling to the gauge field) plays the role of the local 'box', one can expect that there can exist thermodynamcally stable black holes in such theories. This is indeed true for 4-dimensional supergravity theory with FI terms for which, interestingly, there exist exact hairy solutions. Unlike the black holes in AdS spacetime, the small black holes (the parameter Υ coming from the FI sector provide a scale in the theory) are stable. This can be understood as follows: when the horizon radius is large, the dilaton potential gets weaker (it vanishes at the boundary) and so the large black holes are not stable, while for small ones, the self-interaction becomes relevant acting like a box allowing configurations in stable thermal equilibrium. There exists a family of solutions that contains two distinct branches, but only one branch contains thermodynamically stable hairy black holes (see the first plot in Fig. 2.2 and the first plot in Fig. 2.3, where the stable black holes reside within region A, for the positive branch). The range for which these black holes exist is close to the extremality.

Surprisingly, the charged black holes in a gravity theory with GB corrections have a similar thermodynamic behaviour. As discussed in Section 2.2, there is again a family of asymptotically flat black holes with two different branches, but only one branch contains thermodynamically stable black holes. In this case, it seems that the GB term in the action behaves as an 'effective potential'. This can be explicitly checked by comparing the behaviour in the asymptotic region of $V(\phi)$ vs GB term:

$$V(\phi) = \frac{\Upsilon}{30}\phi^5 + \frac{\Upsilon}{630}\phi^7 + \mathcal{O}\left(\phi^9\right) = \frac{\Upsilon}{30\eta^5 r^5} - \frac{3\Upsilon}{560\eta^7 r^7} + \mathcal{O}(r^{-9}) \tag{2.65}$$

and

$$\mathcal{R}_{GB} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho} = \frac{72\mu^2}{r^8} - \frac{360\mu q^2}{r^{10}} - \frac{336\left(\mu^3\alpha + q^4\right)}{r^{12}} + \mathcal{O}\left(r^{-14}\right) \quad (2.66)$$

We observe the rapid fall off that explains, in both cases, the existence of asymptotically flat solutions. However, in both cases, the backreaction deep in the bulk becomes relevant.

For GB theory, the small black holes in the range close to the extremality are again thermodynamically stable. However, there is now an extra constraint, namely the existence of a critical charge: only those with $Q < \frac{1}{2}\sqrt{3}\pi\alpha$ or, equivalently, $S < 2\pi^2\alpha^{3/2}$ are thermodynamically stable. This can be also understood from the first plot in Fig. 2.5 and Fig. 2.6, where we see that one zone that contains the near-extremal black holes (those close to the horizontal dotted curve) is within Region A (stable), but the other is in Region B (unstable).

To summarize this striking analogy, let us concretely emphasize which branches are relevant for the existence of thermodynamically stable black hole configurations. For the Einstein-Maxwell-scalar theory, stability criteria are met when the scalar field potential has positive concavity and is bounded from below. This occurs for one of two families, namely the $\Upsilon > 0$ one, where Υ is the (global) parameter that controls the strength of the self-interaction. Within this family, only the branch for which the scalar field is positively defined and satisfies some particular boundary condition contains thermodynamically stable black hole solutions. For charged black holes in the GB theory, stability criteria are met when the metric is asymptotically flat. This occurs for one of the two families, namely the one defined by $\epsilon = -1$, where $\epsilon = \pm 1$ defines the asymptotic structure of the metric, and, within this family, for one of the two branches, namely the one where $\alpha > 0$, where α controls the strength of the GB correction in the action.

To complement the thermodynamic analysis in the bulk of the paper that was based on the equation of state, we now briefly present an analysis of the thermodynamic potential. We start with the GB case and focus on the case of interest, namely $\alpha > 0$. The relevant information obtained from the equation of state can also be read off from the $\mathcal{F} - Q$ diagram when T is fixed, depicted in the first plot of Fig. 2.7. The locally thermodynamically stable black holes are the ones in region A, i.e., those with the biggest value of Φ, for a given $Q < \frac{1}{2}\sqrt{3}\pi\alpha$. Since $\Phi = (\partial\mathcal{F}/\partial Q)_T$, the stable configurations in $\mathcal{F} - Q$ diagram correspond to the lowest part of each isotherm, having the biggest slope. Also from Fig. 2.7, it follows that these configurations minimize the thermodynamic potential. From this observation, we conclude that locally stable black holes are also globally stable. We shall investigate the global stability using the $\mathcal{F} - T$ diagram, depicted in the second plot of Fig. 2.7. Since $C_Q = -T(\partial^2\mathcal{F}/\partial T^2)_Q$, it follows that locally stable configurations are those satisfying $(\partial^2\mathcal{F}/\partial T^2)_Q < 0$, which also corresponds to the lower part of the curve, containing again configurations minimizing $\mathcal{F}$.

The thermodynamic potential for the hairy black hole solution is depicted in Fig. 2.8 where we consider the positive branch. However, the thermodynamically stable configurations are present only in the positive branch, corresponding to the first plot in Fig. 2.8. Since $C_Q = -T(\partial^2\mathcal{F}/\partial T^2)_Q$, we observe that the locally stable configurations $C_Q > 0$ are those with negative concavity, which correspond to the configurations that minimizes $\mathcal{F}$. Therefore, in a similar manner as we did in the GB case, we obtain that these solutions are also globally stable.

For completeness, we present the plots of T vs r_+ for both cases corresponding to

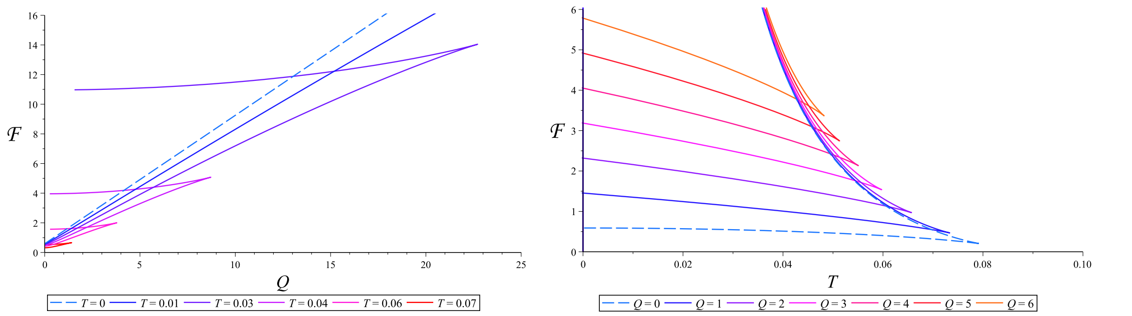

Figure 2.7: **Left hand side:** $\mathcal{F}-Q$ for fixed T for $\alpha=1$. **Right hand side:** $\mathcal{F}-T$ for fixed Q for $\alpha=1$.

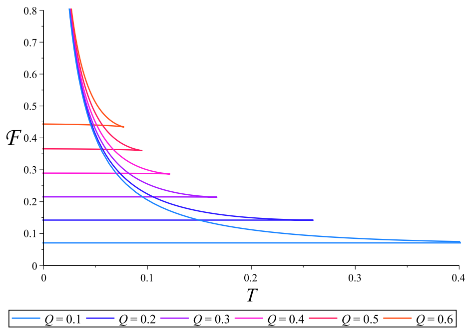

Figure 2.8: $\mathcal{F}-T$ for fixed Q, in the positive branch of the hairy solutions.

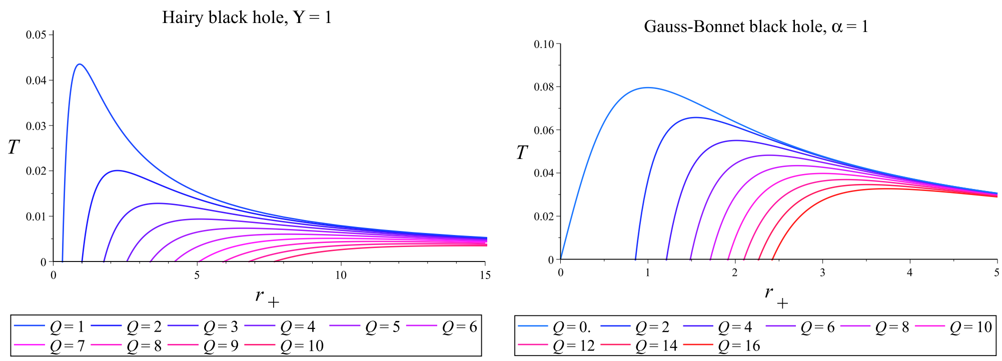

Figure 2.9: Horizon temperature vs r_+ for both scenarios, the hairy black hole (the plot at the left) and for the Gauss-Bonnet black hole (the plot at the right).

the hairy and GB black hole solutions. For the hairy case, we consider $r_+ \equiv \sqrt{\Omega(x_+)}$ that plays the role of the canonical radial coordinate of the horizon (for the positive branch, $1 < x_+ \leq \infty$). This is depicted in Fig. 2.9, where we observe again a clear similarity: In the presence of electric charge, there exists a maximum value of the temperature and two branches, of which the branch of small black holes contains the stable ones.

Since there is no solution of flat spacetime with constant charge, and unlike AdS spacetime where hairy charged solitons were explicitly constructed in some specific cases [94–96], making sense of the ground state in this case is not obvious. However, as in [104], one can consider the extremal black hole (otherwise the spacetime can collapse in a naked

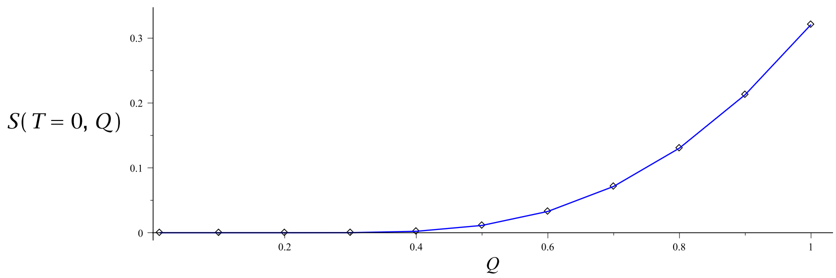

Figure 2.10: Entropy of extremal black hole for the hairy black holes in the positive branch. $\Upsilon = 1$.

singularity) as the state with respect to which we compute the energy. Therefore, to complete the analysis, let us consider the extremal limit that is relevant for the existence of the canonical ensemble. One can use the entropy function formalism [119–121] to obtain the near horizon geometry of black hole solutions in theories that are diffeomorphism and gauge invariant. This method is based on the existence of an enhanced AdS_2 in the near horizon geometry that also plays an important role for understanding the entropy of spinning [119] astrophysical black holes [122]. For asymptotically flat hairy black holes in a theory with a dilaton potential, this analysis was done in [55] (see, also, [123] for a related discussion in a different context) and for GB charged hairy black holes, the details can be found in [113] and so we do not repeat them here. We would only like to emphasize that, for the extremal black holes we have considered in our work, the entropy does not vanish and so these are regular solutions of the equations of motion. In the GB case, the results can be obtained analytically, as follows: consider $T = 0$ limit, namely $r_+^2 = q$, as follows from (2.51). By replacing this value in the horizon equation, $f(r_+) = 0$ or, equivalently, in (2.31), we get the following relation between the conserved charges for the extremal black hole:

$$E = \frac{1}{16}\left(3\pi\alpha + 8\sqrt{3}Q\right) \tag{2.67}$$

and the entropy becomes

$$S = \frac{(2\pi Q)^{\frac{1}{2}}}{2\cdot 3^{\frac{1}{4}}}\left(3\pi\alpha + \frac{2Q}{\sqrt{3}}\right) \tag{2.68}$$

In the hairy case, we can show, through numerical computations, that the entropy of extremal black holes is also positive, with its values depending of the electric charge. It turns out that, for increasing values of Q, the entropy of extremal hairy black hole rapidly approaches to zero, as depicted in Fig. 2.10. This can be also seen from the first plot of Fig. 2.3, though, from that plot and for curves with large values of Q, it is not as easy to discern that $S(T = 0) > 0$.

While the theories are completely different, we would like to point out that, generally, theories with higher derivatives $f(R)$ are equivalent with theories with a scalar field and its self-interaction [124, 125]. However, this consideration can not be applied for our work because one theory is four-dimensional, while the other is five-dimensional. The GB term is trivial in four-dimensions, but once it is coupled with the scalar field it contributes to the equations of motion and so we expect that the hairy black holes receive corrections. Since we were not able to generate exact solutions, we leave the detailed analysis of this specific case for a future work.

Part II

Role of scalar charges in black hole thermodynamics

3

On scalar charges and black hole thermodynamics

This chapter is based on:
On scalar charges and black hole thermodynamics
R. Ballesteros, C. Gómez-Fayrén, T. Ortín and M.Zatti
JHEP 05 (2023) 158 (arXiv:2302.11630)

It is widely believed that one of the defining characteristics of classical black holes is that they have no "hair". The concept of black-hole hair is a very broad one but, for the stationary black holes we will be concerned with in this paper it can be defined as any parameter that enters the metric and which cannot be eliminated through a coordinate transformation which is not a function of the charges of the theory which are conserved by virtue of a local symmetry (mass, angular momenta, electric charges) or a topological property (magnetic charges) or the asymptotic values of the scalars (*moduli*).

Scalar charges, typically defined through the asymptotic behavior at spatial infinity of the scalars in the black-hole spacetime, are not protected by any conservation law. In ungauged theories the only local symmetries scalar fields transform under are diffeomorphisms but the conserved charges associated to them are the gravitational ones: mass and linear and angular momenta. Scalar fields only transform under global symmetries of the action or of the equations of motion only to which we will refer to as dualities. However, the charges associated to those symmetries in stationary black-hole spacetimes vanish identically. They seem to have nothing to do with the conventionally-defined black-hole scalar charges. Gauging the global symmetries does not help because the gauge symmetry would be associated to some 1-form gauge fields and the conserved charges would have the interpretation of electric and magnetic charges.

Therefore, according to our definition of hair, scalar charges are understood as hair and, according to the *no-hair conjecture*, no black-hole solutions with regular horizons (henceforth to be referred to as "regular black holes") carrying scalar charges should be expected. Any scalar charges possessed by gravitationally collapsing matter should be radiated away in the black-hole formation. However, there are many regular black hole solutions carrying non-vanishing scalar charges such as dilaton black holes and their generalizations.[1]

The solution to this apparent counterexample of the no-hair conjecture lies in the distinction between primary and secondary hair [127]: in all the regular black-hole solutions with non-vanishing scalar charges, those charges are not independent parameters but very specific functions of the independent conserved charges which are allowed by the no-hair conjecture and they are (by definition) secondary hair. In the solutions in which

[1] For a review with many references, see Ref. [126].

the scalar charges are truly independent parameters, such as the Janis-Newman-Winicour solution [128] or the Agnese-La Camera solutions [129] and their generalizations [126], there are no regular horizons but naked singularities unless the scalar charge takes the value of the specific function of the conserved charges we mentioned above (simply zero in the JNW solution). This kind of scalar hair is, by definition, primary hair and it is the one which would actually be forbidden by the conjecture.

The scalar charges which are allowed by the no-hair conjecture remain, nevertheless, quite mysterious: What are the values of the scalar charges allowed in a given theory? Why are those values allowed and no others? And, even more basic: Is there a coordinate-independent definition of scalar charge?

This mystery only deepened when Gibbons, Kallosh and Kol (GKK) showed in Ref. [59] (see also Ref. [98]) that the allowed scalar charges occur in the first law of black hole mechanics [33] as thermodynamical potentials conjugate to the variations of the moduli. While it is not clear which kind of physical process may result in a change of the moduli,[2] it is a fact that varying the black-hole entropy formulae of known solutions with respect to the moduli one finds the scalar charges as coefficients of those variations.

Wald's formalism [66–68] opened a new venue for the study of black-hole thermodynamics that can be used to explore the role of scalar charges into it. The main observation, realized in the context of purely gravitational (matter-free) theories invariant under diffeomorphisms is that the properties of the Noether $(d-2)$-form charge associated to the invariance under diffeomorphisms (*Noether-Wald charge*) can be used to prove the first law of black-hole thermodynamics.

In theories with matter, this law includes work terms proportional to the variations of conserved charges and the GKK scalar term proportional to the variations of the moduli. In the last few years we have extended the formalism to handle theories in which there are matter fields with gauge symmetries coupled to gravity showing how the electric work terms appear [56, 130, 131],[3] showing how extended black-hole thermodynamics arises in this context [69, 138], how to include magnetic charges in the first law [57] and how to construct Komar integrals from which Smarr formulae can be derived [69, 139]. In all those cases each new work term in the first laws is associated to a gauge symmetry or an equivalent topological property. Since, as we have seen, scalar charges are not associated to neither, it is unclear how the GKK work term can be recovered in Wald's formalism.[4] The absence of a good coordinate-independent definition for the scalar charge complicates this problem.

In this paper we are going to show how this problem can be solved taking into account hitherto ignored contributions to the integrals at spatial infinity and using a definition of scalar charge as the integral of a $(d-2)$-form which is manifestly coordinate and gauge independent and which satisfies a Gauss law in stationary black-hole spacetimes. This definition relies in the existence of conserved charges associated to global symmetries and in the existence of a timelike Killing vector whose Killing horizon coincides with the black-

[2] The same could be said about magnetic charges.

[3] A slightly different approach to the one taken in those papers, which is the one used here as well, is the point of view of "invariance up to gauge transformations", taken in Refs. [132–137].

[4] In extended thermodynamics there are work terms associated to the variation of dimensionful constants which, apparently, unrelated to gauge symmetries.[5] However, those constants can be dualized into $(d-1)$-form potentials with a gauge freedom (for the cosmological constant, see Refs. [142, 143]) and this description leads to the work terms [69, 99, 138, 140, 144].

hole's event horizon and whose action leaves invariant all the physical fields. Therefore, there is a scalar charge associated to each global symmetry, and, therefore, the number of charges may or may not coincide with the number of scalar fields.

In this paper we have studied 4-dimensional theories[6] whose scalar kinetic terms are described by symmetric sigma models in which the scalar fields map spacetime into a target space which is a symmetric Riemannian homogeneous space G/H. These kinetic terms are very common in supergravity theories. Furthermore, our theories include Abelian 1-forms and we are going to assume that the couplings of the scalars to those 1-forms are such that the equations of motion, enhanced with the Bianchi identities satisfied by the 2-form field strengths are invariant under the duality group G.[7] Again, this is a fairly common situation in supergravity and include simple theories such as the Einstein-Maxwell-Dilaton ones. In these theories we can associate a conserved scalar charge to each of the generators of G, even if some of the transformations (the electric-magnetic duality rotations in particular) do not leave the action invariant. As a result, according to our definition, there are always more scalar charges than scalars. Nevertheless, we are going to show that the conventional scalar charges can be recovered as combinations of the ones we have defined and we are going to check these relations in particular black-hole solutions.

In this framework we are going to proof the first law of black-hole thermodynamics recovering the GKK results and, as a byproduct, we are going to find a general expression for the scalar charges in terms of the conserved charges and the position of the horizon, thus answering one of the long-standing questions posed above.[8] Observe that, since our definition of scalar charge satisfies a Gauss Law, the value obtained for those charges is the same whether we calculate the integrals over the horizon or at infinity.

This paper is organized as follows: in Section 3.1 we review the kind of theories that we are considering, their duality symmetries, the Gaillard-Zumino theorem [146] and the construction of the *Noether-Gaillard-Zumino* (NGZ) currents which will be used in Section 3.2 to define the scalar charges. In Section 3.3 we derive the first law recovering the GKK results and the general expression of scalar charges in terms of conserved charges and the position of the horizon. In Sections 3.4 and 3.5 we test our results on dilaton and axion-dilaton black holes respectively. Section 3.6 contains a discussion of our results.

3.1 The theory

In this section we are going to review the theories we are going to consider and their duality symmetries. Most of this material can be found elsewhere, but here we adapt it to our needs and conventions.

Throughout this paper we are going to consider 4-dimensional ungauged supergravity-inspired theories containing n_S scalar fields ϕ^x that parametrize a symmetric coset space G/H and n_V 1-form fields $A^\Lambda = A^\Lambda{}_\mu dx^\mu$ with 2-form field strengths

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[6] The extension to higher dimensions and higher-rank forms is straightforward using the results of Ref. [145] for the Noether-Gaillard-Zumino currents.

[7] The general form of the theories that we consider is identically to that of the theories considered by GKK in Ref. [59] but, in our approach it is crucial to know the global symmetries of the theory.

[8] After completion of this work, we found that a similar definition of scalar charge and similar result had been found in Ref. [61] in the context of the dilaton black holes of the EMD theories.

$$F^{\Lambda}=dA^{\Lambda}\,, \tag{3.1}$$

coupled to gravity which we will describe through the Vierbein $e^a = e^a{}_\mu dx^\mu$. Up to two derivatives, they can be described by the generic action

$$\begin{aligned} S &= \frac{1}{16\pi G_N^{(4)}}\int\left[-\star(e^a\wedge e^b)\wedge R_{ab}+\tfrac{1}{2}g_{xy}d\phi^x\wedge\star d\phi^y-\tfrac{1}{2}I_{\Lambda\Sigma}F^{\Lambda}\wedge\star F^{\Sigma}-\tfrac{1}{2}R_{\Lambda\Sigma}F^{\Lambda}\wedge F^{\Sigma}\right] \\ &\equiv\int\mathbf{L}\,, \end{aligned} \tag{3.2}$$

where the kinetic matrix $I = (I_{\Lambda\Sigma})$ is negative-definite and we are going to assume that the positive-definite σ-model metric $g_{xy}(\phi)$ is invariant under the action of G (the duality group) which also leaves invariant the set of all equations of motion plus the Bianchi identities of the theory. This assumption will be translated into conditions for the scalar-dependent matrices $I = (I_{\Lambda\Sigma})$ and $R = (R_{\Lambda\Sigma})$ shortly.

We will set $G_N^{(4)} = 1$ and we will ignore the normalization factor $(16\pi)^{-1}$ for the time being.

The equations of motion are defined by (here φ stands for all the fields of the theory)

$$\delta S=\int\left\{\mathbf{E}_a\wedge\delta e^a+\mathbf{E}_x\delta\phi^x+\mathbf{E}_{\Lambda}\delta A^{\Lambda}+d\mathbf{\Theta}(\varphi,\delta\varphi)\right\}\,, \tag{3.3}$$

and given by

$$\begin{aligned}\mathbf{E}_a &= \imath_a\star(e^b\wedge e^c)\wedge R_{bc}+\tfrac{1}{2}g_{xy}\left(\imath_a d\phi^x\star d\phi^y+d\phi^x\wedge\imath_a\star d\phi^y\right) \\ &\quad -\tfrac{1}{2}I_{\Lambda\Sigma}\left(\imath_a F^{\Lambda}\wedge\star F^{\Sigma}-F^{\Lambda}\wedge\imath_a\star F^{\Sigma}\right)\,,\end{aligned} \tag{3.4a}$$

$$\mathbf{E}_x=-g_{xy}\left\{d\star d\phi^y+\Gamma_{zw}{}^{y}d\phi^z\wedge\star d\phi^w\right\}-\tfrac{1}{2}\partial_x I_{\Lambda\Sigma}F^{\Lambda}\wedge\star F^{\Sigma}-\tfrac{1}{2}\partial_x R_{\Lambda\Sigma}F^{\Lambda}\wedge F^{\Sigma}\,, \tag{3.4b}$$

$$\mathbf{E}_{\Lambda}=dF_{\Lambda}\,, \tag{3.4c}$$

where we have defined the dual 2-form field strength

$$F_{\Lambda}\equiv I_{\Lambda\Sigma}\star F^{\Sigma}+R_{\Lambda\Sigma}F^{\Sigma}\,. \tag{3.5}$$

Furthermore,

$$\mathbf{\Theta}(\varphi,\delta\varphi)=-\star(e^a\wedge e^b)\wedge\delta\omega_{ab}+g_{xy}\star d\phi^x\delta\phi^y-F_{\Lambda}\wedge\delta A^{\Lambda}\,. \tag{3.6}$$

The original and dual 2-forms can be combined into a symplectic vector of 2-forms[9]

[9] The symplectic nature of this vector will be proven shortly.

$$(F^M) \equiv \begin{pmatrix} F^\Lambda \\ F_\Lambda \end{pmatrix}, \tag{3.7}$$

and the Bianchi identities of the original 2-form field strength F^Λ

$$dF^\Lambda = 0\,, \tag{3.8}$$

and the Maxwell equations $\mathbf{E}_\Lambda = 0$ can be written as

$$dF^M = 0\,. \tag{3.9}$$

These equations can be interpreted as Bianchi identities implying the local existence of 1-form potentials

$$(A^M) \equiv \begin{pmatrix} A^\Lambda \\ A_\Lambda \end{pmatrix}, \tag{3.10}$$

such that

$$F^M = dA^M\,. \tag{3.11}$$

The set of equations (3.9) is invariant under arbitrary $\mathrm{GL}(2n_V, \mathbb{R})$ transformations

$$F^{M\,\prime} = S^M{}_N F^N\,, \tag{3.12}$$

but we have to take into account the rest of the equations and an important constraint: the components of F^M are not independent and, therefore, F^M satisfies the following *twisted self-duality constraint*

$$\star F^M = -\Omega^{MN}\mathcal{M}_{NP}F^P\,, \tag{3.13}$$

where $\mathcal{M}_{MN}$ is the $2n_V \times 2n_V$ symmetric symplectic matrix

$$\begin{aligned} \mathcal{M} = (\mathcal{M}_{MN}) &= \begin{pmatrix} I + RI^{-1}R & -RI^{-1} \\ -I^{-1}R & I^{-1} \end{pmatrix}, \\ \mathcal{M}^{-1} = \left(\mathcal{M}^{MN}\right) = \Omega^{-1\,T}\mathcal{M}\Omega &= \begin{pmatrix} I^{-1} & I^{-1}R \\ RI^{-1} & I + RI^{-1}R \end{pmatrix}, \end{aligned} \tag{3.14}$$

and

$$\Omega = (\Omega_{MN}) = \begin{pmatrix} 0 & \mathbb{1}_{n_V\times n_V} \\ -\mathbb{1}_{n_V\times n_V} & 0 \end{pmatrix}, \qquad \Omega^{-1} = \left(\Omega^{PN}\right)\,. \tag{3.15}$$

As a consequence, the set of Maxwell equations and Bianchi identities will only be invariant under the subset of $\mathrm{GL}(2n_V,\mathbb{R})$ transformations that preserve this constraint, which is possible provided that $\mathcal{M}$ transforms as

$$\mathcal{M}' = \left(\Omega^{-1}S\Omega\right)\mathcal{M}S^{-1}\,, \qquad S = \left(S^{M}{}_{N}\right)\,. \tag{3.16}$$

It is convenient to analyze the invariance of the Einstein equations first.

Using the identity

$$\mathcal{M}_{MN}\imath_a F^M \wedge \star F^N = I_{\Lambda\Sigma}\left(\imath_a F^\Lambda \wedge \star F^\Sigma - F^\Lambda \wedge \imath_a \star F^\Sigma\right)\,, \tag{3.17}$$

and the twisted self-duality constraint Eq. (3.13), the energy-momentum tensor of the 1-forms can be written in the form

$$-\Omega_{MN}\imath_a \star F^M \wedge \star F^N\,, \tag{3.18}$$

which is left invariant by the transformations that leave invariant Ω

$$S^T\Omega S = \Omega\,, \tag{3.19}$$

that is, by transformations that belong to $\mathrm{Sp}(2n_V,\mathbb{R})$ [146]. Defining the $n_V\times n_V$ blocks of the symplectic matrix S

$$S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\,, \tag{3.20}$$

the symplectic nature of S implies the following conditions for them:

$$A^TC - C^TA = 0\,, \tag{3.21a}$$

$$B^TD - D^TB = 0\,, \tag{3.21b}$$

$$D^TA - B^TC = \mathbb{1}_{n_V\times n_V}\,. \tag{3.21c}$$

It is not difficult to see that, if S symplectic, so is S^T. The symplectic nature of S^T implies

$$BA^T - AB^T = 0\,, \tag{3.22a}$$

$$DC^T - CD^T = 0\,, \tag{3.22b}$$

$$DA^T - CB^T = \mathbb{1}_{n_V\times n_V}\,. \tag{3.22c}$$

On the other hand, Eq. (3.19) implies

$$\Omega^{-1} S \Omega = S^{-1\,T}\,, \tag{3.23}$$

and, going back to Eq. (3.16), we find that

$$\mathcal{M}^{-1\,\prime} = S \mathcal{M}^{-1} S^{T}\,. \tag{3.24}$$

Defining the $n_V \times n_V$, symmetric, *period matrix*

$$\mathcal{N} = R + iI\,, \tag{3.25}$$

it can be seen that the transformation of $\mathcal{M}$ Eq. (3.24) is equivalent to the following generalized fractional-linear transformations of $\mathcal{N}$:

$$\mathcal{N}' = (C + D\mathcal{N})\,(A + B\mathcal{N})^{-1}\,. \tag{3.26}$$

It is clear that these transformations of the period matrix are associated to transformations of the scalars which we are going to study in their infinitesimal form. The transformations of the scalars that leave the equations of motion invariant must necessarily be generated by the Killing vectors of the σ-model metric g_{xy}, which we are going to denote by $\{k_A{}^x(\phi)\}$.[10] In some cases it is convenient to include in this set some vectors which are identically zero so that the index A can be used to label also transformations of the 1-form fields that do not involve the scalars, if necessary. Of course, additional conditions involving the kinetic matrices (hence, the period matrix) need to be satisfied.

The infinitesimal transformations of the 1-form fields are

$$S \sim \mathbb{1}_{2n_V \times 2n_V} + \alpha^A T_A\,,$$

$$T_A = \left(T_A{}^M{}_N\right) = \begin{pmatrix} T_A{}^\Lambda{}_\Sigma & T_A{}^{\Lambda\Sigma} \\ T_{A\,\Lambda\Sigma} & T_{A\,\Lambda}{}^\Sigma \end{pmatrix}. \tag{3.28}$$

S is symplectic if

$$T_A{}^T \Omega + \Omega T_A = 0\,, \quad \Rightarrow (\Omega T_A)^T = \Omega T_A\,, \tag{3.29}$$

(so $\Omega_{MP} T_A{}^P{}_N$ is symmetric in MN) which implies, for the block matrices

$$\begin{aligned} T_{A\,\Lambda\Sigma} &= T_{A\,\Sigma\Lambda}\,, \\ T_A{}^\Lambda{}_\Sigma &= -T_{A\Sigma}{}^\Lambda\,, \\ T_A{}^{\Lambda\Sigma} &= T_A{}^{\Sigma\Lambda}\,. \end{aligned} \tag{3.30}$$

[10] These transformations leave exactly invariant the energy-momentum tensor of the scalars, which is the only piece of the Einstein equations that we had not studied and transform covariantly the first two terms of the scalar equations of motion:

$$\delta_A \left\{ d \star d\phi^x + \Gamma_{yz}{}^x d\phi^y \wedge \star d\phi^z \right\} = \partial_w k_A^x \left\{ d \star d\phi^w + \Gamma_{yz}{}^w d\phi^y \wedge \star d\phi^z \right\}\,. \tag{3.27}$$

Then, the infinitesimal form of Eq. (3.26) is

$$\delta_A \mathcal{N}_{\Lambda\Sigma} = T_{A\,\Lambda\Sigma} + T_{A\,\Lambda}{}^{\Omega}\mathcal{N}_{\Omega\Sigma} - \mathcal{N}_{\Lambda\Omega}T_A{}^{\Omega}{}_{\Sigma} - \mathcal{N}_{\Lambda\Gamma}T_A{}^{\Gamma\Omega}\mathcal{N}_{\Omega\Sigma}\,, \tag{3.31}$$

and, for the kinetic matrices,

$$\delta_A R_{\Lambda\Sigma} = T_{A\,\Lambda\Sigma} + T_{A\,\Lambda}{}^{\Omega}R_{\Omega\Sigma} - R_{\Lambda\Omega}T_A{}^{\Omega}{}_{\Sigma} - R_{\Lambda\Gamma}T_A{}^{\Gamma\Omega}R_{\Omega\Sigma} + I_{\Lambda\Gamma}T_A{}^{\Gamma\Omega}I_{\Omega\Sigma}\,, \tag{3.32a}$$

$$\delta_A I_{\Lambda\Sigma} = T_{A\,\Lambda}{}^{\Omega}I_{\Omega\Sigma} - I_{\Lambda\Omega}T_A{}^{\Omega}{}_{\Sigma} - 2R_{(\Lambda|\Gamma}T_A{}^{\Gamma\Omega}I_{\Omega|\Sigma)}\,. \tag{3.32b}$$

Then, it can be easily seen that the whole scalar equations of motion transform as

$$\delta_A \mathbf{E}_x = -\partial_x k_A{}^y \mathbf{E}_y\,, \tag{3.33}$$

under the transformations

$$\delta_A \phi^x = k_A{}^x\,, \qquad \delta_A F^M = T_A{}^M{}_N F^N\,, \tag{3.34}$$

provided that

$$k_A{}^x \partial_x \mathcal{N} = \delta_A \mathcal{N}\,, \tag{3.35}$$

where $\delta_A \mathcal{N}$ is the infinitesimal generalized fractional-linear transformation in Eq. (3.31) (or equivalently, in Eqs. (3.32a) and (3.32b) for the kinetic matrices). This equivariance condition of the kinetic matrices is the condition we announced when we defined the theory.

3.2 A definition of scalar charge

Not all the symmetries of the equations of motion that we have studied are symmetries of the action: those generated by $T_A{}^{\Lambda\Sigma}$ do not leave the action invariant. Those generated by $T_{A\,\Lambda\Sigma}$ leave it invariant up to a total derivative. However, as shown in Ref. [146], there is an on-shell conserved current for each of them, the so-called *Noether-Gaillard-Zumino (NGZ) current*. The simplest way to construct them is by contracting the scalar equations of motion with the Killing vectors that generate them. Using the Killing vector equation and the equivariance conditions Eqs. (3.32a) and (3.32b) we get [145]

$$\begin{aligned} k_A{}^x \mathbf{E}_x &= -d\star \hat{k}_A - \tfrac{1}{2}\Omega_{MP}T_A{}^P{}_N F^M \wedge F^N \\ &= -d\left[\star\hat{k}_A + \tfrac{1}{2}\Omega_{MP}T_A{}^P{}_N A^M \wedge F^N\right] + \tfrac{1}{2}\Omega_{MP}T_A{}^P{}_N A^M \wedge \mathbf{E}^N\,, \end{aligned} \tag{3.36}$$

where we have collected in a symplectic vector of 3-forms the Maxwell equations and Bianchi identities:

$$(\mathbf{E}^M) \equiv \begin{pmatrix} \mathbf{E}^\Lambda \\ \mathbf{E}_\Lambda \end{pmatrix}\,, \tag{3.37}$$

and where we have denoted by $\hat{k}_A$ the pullback of the 1-form dual to the target space Killing vector k_A

$$\hat{k}_A \equiv k_A{}^x g_{xy} d\phi^y \,. \tag{3.38}$$

Therefore, we find that the NGZ currents

$$\star j_A \equiv - \star \hat{k}_A - \tfrac{1}{2}\Omega_{MP} T_A{}^P{}_N A^M \wedge F^N \,, \tag{3.39}$$

are conserved on-shell

$$d \star j_A = k_A{}^x \mathbf{E}_x - \tfrac{1}{2}\Omega_{MP} T_A{}^P{}_N A^M \wedge \mathbf{E}^N \doteq 0 \,. \tag{3.40}$$

The conservation of these currents follows from a global symmetry and the associated charges are expressed as integrals over spacelike hypersurfaces (volumes)

$$q_A \sim \int_{\Sigma^3} \star j_A \,. \tag{3.41}$$

However, it is not difficult to see that in static black hole solutions with non-trivial scalar fields ϕ^x whose charges Σ^x are conventionally defined[11] through the asymptotic behavior of the field at spatial infinity

$$\phi^x \overset{r\to\infty}{\sim} \phi^x_\infty + \frac{\Sigma^x}{r} \,, \tag{3.42}$$

the NGZ charges not only do not reproduce the charges Σ^x (or combinations of them) but vanish identically.

In stationary black hole spacetimes, though, there is another definition of scalar charge that satisfies a Gauss law. Let us assume that all the fields are invariant under the isometry generated by the spacetime vector k, $\delta_k \varphi = 0$. This implies, in particular, that k is a Killing vector and, for us, it will be the Killing vector associated to the Black hole's Killing horizon. For the scalar fields it means that their Lie derivatives with respect to that vector vanishes

$$\delta_k \phi^x = -\pounds_k \phi^x = -\imath_k d\phi^x = 0 \,. \tag{3.43}$$

As shown in [56, 130, 131],[12] for the 1-form fields, it means that their Lie derivatives with respect to k plus a gauge transformation with parameter

$$\chi_k = \imath_k A - P_k \,, \tag{3.44}$$

where the *Maxwell momentum map* P_k satisfies the *Maxwell momentum map equation*[13]

[11] See, for instance, Ref. [59].

[12] The work terms for the electric charges associated to p-forms were found using the covariant phase space formalism in Ref. [147]. See also [148] for a different, equivalent, approach based on the mathematics of principal bundles. The importance of the gauge- and diffeomorphism invariance of the charges and potentials that occur in the laws of black-hole thermodynamics has been stressed in [137, 149] and in the 5th chapter of [23].

[13] The local existence of a P_k satisfying this equation follows from the assumption:

$$\delta_k F = -\pounds_k F = -d\imath_k F = 0 \,. \tag{3.45}$$

$$\imath_k F + dP_k = 0\,, \tag{3.46}$$

vanish identically:

$$\begin{aligned}\delta_k A^M &= -\pounds_k A^M + d\chi_k^M = -\left(\imath_k d + d\imath_k\right) A^M + d\left(\imath_k A^M - P_k{}^M\right)\\ &= -\left(\imath_k F^M + dP_k{}^M\right) = 0\,,\end{aligned} \tag{3.47}$$

by virtue of the Maxwell momentum map equation (3.46).

If all the fields are invariant under δ_k, so must the NGZ currents be. Furthermore, since the NGZ currents are not gauge invariant, we must use this definition for δ_k:

$$\begin{aligned}\delta_k \star j_A &= -\pounds_k \star j_A + \delta_{\chi_k} \star j_A\\ &= -\left(\imath_k d + d\imath_k\right) \star j_A - \tfrac{1}{2}\Omega_{MP} T_A{}^P{}_N \delta_{\chi_k} A^M \wedge F^N\\ &\doteq -d\imath_k \star j_A - \tfrac{1}{2}\Omega_{MP} T_A{}^P{}_N d\chi_k{}^M \wedge F^N\\ &\doteq d\left\{-\imath_k \star j_A - \tfrac{1}{2}\Omega_{MP} T_A{}^P{}_N \chi_k{}^M F^N\right\}\\ &= 0\,,\end{aligned} \tag{3.48}$$

by assumption.

The expression in brackets is a 2-form that satisfies a Gauss law. Massaging it a bit, we find the following manifestly gauge-invariant expression for it:

$$\mathbf{Q}_A[k] = \imath_k \star \hat{k}_A + \Omega_{MP} T_A{}^P{}_N P_k{}^M F^N\,. \tag{3.49}$$

Now, integrating over 2-dimensional, spacelike, closed surfaces (and restoring the normalization) we get the charges associated to the NGZ currents:

$$Q_{A,k} = \frac{1}{16\pi G_N^{(4)}} \int_{\Sigma^2} \left\{\imath_k \star \hat{k}_A + \Omega_{MP} T_A{}^P{}_N P_k{}^M F^N\right\}\,. \tag{3.50}$$

This is our proposal for scalar charges. Observe that under a duality transformation generated by k_A, T_A with Lie brackets and commutation relations

$$[k_A, k_B] = -f_{AB}{}^C k_C\,, \qquad [T_A, T_B] = +f_{AB}{}^C T_C\,, \tag{3.51}$$

these charges transform in the adjoint representation of the duality group:

$$\delta_A Q_{B,k} = -f_{AB}{}^C Q_{C,k}\,. \tag{3.52}$$

In what follows we are going to show in several examples corresponding to static dilaton and axidilaton black holes that their values are non-vanishing and reproduce the values of the conventionally-defined scalar charges Eq. (3.42) but, before we set to do that, let us observe that this definition depends on the value of the momentum map over the integration surface. The Maxwell momentum map is defined only up to an additive constant. This constant can be chosen so that $P_k{}^M\big|_\infty = 0$. That is the choice that allows us to recover the values of the conventionally-defined scalar charges Eq. (3.42). However, other choices are possible. The form of the first law that we are going to find includes an additional term that takes into account that possibility so that the first law is invariant under a change of asymptotic value of the Maxwell momentum maps.

It is also worth stressing that in the case in which we are considering (a symmetric σ-model) there are always more symmetries than scalar fields. Therefore, there are more 2-forms $\mathbf{Q}_A[k]$ satisfying a Gauss law than scalars. Obviously, not all of them will be independent. In any case, the conservation laws of those currents can be used to reconstruct the equations of motion of the scalars using the identity

$$\delta_x{}^y = g^{AB} k_{A\,x} k_B{}^y \,, \tag{3.53}$$

in which g^{AB} is the Killing metric of the duality group G.

It also follows that there are more scalar charges than scalars, but we are going to see that the conventionally-defined scalar charges Σ^x can be expressed in terms of the charges $Q_{A,k}$ that we have just defined.

It is worth mentioning that there is a slightly different procedure that allows us to obtain the same expression Eq. (3.49) and that was used in the case of dilaton black holes in Ref. [61]. In that case there is only one scalar and one target space Killing vector $k = 1$ that generates the constant shifts of the scalar which are compensated by rescalings of the vector field (see Section 3.4). In our case, we have to project the scalar equations with the different Killing vectors $k_A{}^x$ first, as in the first line of Eq. (3.36). Then, we take the inner product of the resulting equation with $\imath_k$

$$\imath_k k_A{}^x \mathbf{E}_x = -\imath_k d \star \hat{k}_A - \Omega_{MP} T_A{}^P{}_N \imath_k F^M \wedge F^N \,. \tag{3.54}$$

If all the fields are invariant under the diffeomorphism generated by k

$$\begin{aligned} -\imath_k d \star \hat{k}_A &= d\imath_k \star \hat{k}_A \,, \\ \imath_k F^M &= -dP_k{}^M \,, \end{aligned} \tag{3.55}$$

and, integrating by parts we arrive to $d\mathbf{Q}_A[k] = 0$.

3.3 First law and scalar charges

Taking into account the results obtained in Refs. [56, 57, 130, 131] for the inclusion of matter fields, in Wald's formalism [66–68], the first law of black hole thermodynamics for a non-extremal black hole whose bifurcate horizon coincides with the Killing horizon of the Killing vector field $k = \partial_t + \Omega\partial_\varphi$, can be derived by integrating the on-shell identity

$$d\mathbf{W}[k] \doteq 0 \,, \tag{3.56}$$

where

$$\mathbf{W}[k] \equiv \delta\mathbf{Q}[k] + \imath_k\mathbf{\Theta}(\varphi, \delta\varphi) - \varpi_k\,, \tag{3.57}$$

over a spacelike hypersurface with boundaries at spatial infinity (S^2_∞) and at the bifurcation sphere $\mathcal{BH}$ and applying the Stokes theorem.

In the above identity $\mathbf{Q}[k]$ is the Noether-Wald charge for the Killing vector k, $\mathbf{\Theta}(\varphi, \delta\varphi)$ is the *presymplectic* $(d-1)$*-form* defined in Ref. [66] and $\varpi_k$is defined by[14]

$$\delta_{\Lambda_k}\mathbf{\Theta}(\varphi, \delta\varphi) \equiv d\varpi_k\,. \tag{3.58}$$

Furthermore, it is assumed that the variations of the fields $\delta\varphi$ satisfy the linearized equations of motion in the black-hole's background.

The first law, thus, follows from the identity

$$\int_{S^2{}_\infty}\mathbf{W}[k] = \int_{\mathcal{BH}}\mathbf{W}[k]\,. \tag{3.59}$$

In previous works, following Ref. [68], we assumed that, almost by definition, the first integral simply gives the variation of the conserved charges associated to the Killing vector k, that is,

$$\delta M - \Omega\delta J\,. \tag{3.60}$$

A closer look reveals that, in presence of matter fields, it contains additional terms that contribute to the first law [57]. In particular, as we are going to see, it contains terms related to the scalar charges that we have just defined.

A standard calculation along the lines of Refs. [56, 57, 130, 131] gives

$$\mathbf{Q}[k] = \star(e^a \wedge e^b)P_{k\,ab} - P_k{}^\Lambda F_\Lambda\,, \tag{3.61}$$

where $P_{k\,ab}$ is the *Lorentz momentum map* defined in Ref. [56] and coincides with the *Killing bivector*

$$P_{k\,ab} = \nabla_a k_b\,, \tag{3.62}$$

and $P_k{}^\Lambda$ is the Maxwell momentum map defined in Eq. (3.46). A quick calculation gives

$$\delta\mathbf{Q}[k] = P_{k\,ab}\delta\star(e^a \wedge e^b) + \star(e^a \wedge e^b)\delta P_{k\,ab} - F_\Lambda\delta P_k{}^\Lambda - P_k{}^\Lambda\delta F_\Lambda\,. \tag{3.63}$$

The presymplectic 3-form is given in Eq. (3.6) and another short calculation gives

$$\begin{aligned}\imath_k\mathbf{\Theta} = {}& -\imath_k\star(e^a \wedge e^b)\wedge\delta\omega_{ab} - \star(e^a \wedge e^b)\wedge\delta\imath_k\omega_{ab} + g_{xy}\imath_k\star d\phi^x\delta\phi^y \\ & -\tfrac{1}{2}\imath_k F_\Lambda \wedge \delta A^\Lambda - \tfrac{1}{2}F_\Lambda\wedge\delta\imath_k A^\Lambda\,.\end{aligned} \tag{3.64}$$

[14]This term arises when the effect of the induced gauge transformations are correctly taken into account as in Ref. [57]. In Eq. (3.58) δ_{Λ_k} stands for all the gauge transformations induced by the isometry generated by k.

Since, on-shell, the dual 1-forms obey the same equations as the original ones, we can define the *dual (magnetic) momentum maps* $P_{k\,\Lambda}$ through the equation

$$\imath_k F_\Lambda + dP_{k\,\Lambda} = 0\,, \tag{3.65}$$

and, substituting this definition in the above expression and integrating by parts, we get

$$\begin{aligned}\imath_k\mathbf{\Theta} = &-\imath_k \star (e^a \wedge e^b) \wedge \delta\omega_{ab} - \star(e^a \wedge e^b) \wedge \delta\imath_k\omega_{ab} + g_{xy}\imath_k \star d\phi^x \delta\phi^y \\ &+ P_{k\,\Lambda} \wedge \delta F^\Lambda - F_\Lambda \wedge \delta\imath_k A^\Lambda\,,\end{aligned} \tag{3.66}$$

up to an irrelevant total derivative.

Another simple calculation gives [57]

$$\begin{aligned}\delta_{\Lambda_k}\mathbf{\Theta} &= (\delta_{\sigma_k} + \delta_{\chi_k})\mathbf{\Theta} \\ &= -\delta_{\sigma_k}\left[\star(e^a \wedge e^b) \wedge \delta\omega_{ab}\right] - F_\Lambda \wedge \delta_{\chi_k}\delta A^\Lambda \\ &= -\star(e^a \wedge e^b) \wedge \mathcal{D}\delta\sigma_{k\,ab} - F_\Lambda \wedge d\delta{\chi_k}^\Lambda \\ &= d\left\{-\star(e^a \wedge e^b) \wedge \delta\sigma_{k\,ab} - F_\Lambda\delta{\chi_k}^\Lambda\right\}\,,\end{aligned} \tag{3.67}$$

where the parameters of the induced Lorentz and Maxwell gauge transformations are, respectively

$${\sigma_k}^{ab} = \imath_k\omega^{ab} - {P_k}^{ab}\,, \tag{3.68a}$$

$${\chi_k}^\Lambda = \imath_k A^\Lambda - {P_k}^\Lambda\,. \tag{3.68b}$$

Therefore,

$$-\varpi_k = \star(e^a \wedge e^b) \wedge \delta\sigma_{k\,ab} + F_\Lambda\delta{\chi_k}^\Lambda\,. \tag{3.69}$$

Combining all these partial results, we arrive at

$$\begin{aligned}\mathbf{W}[k] = &P_{k\,ab}\delta \star (e^a \wedge e^b) - \imath_k \star (e^a \wedge e^b) \wedge \delta\omega_{ab} \\ &- {P_k}^\Lambda\delta F_\Lambda + P_{k\,\Lambda}\delta F^\Lambda + g_{xy}\imath_k \star d\phi^x\delta\phi^y\,.\end{aligned} \tag{3.70}$$

Let us consider the integral of $\mathbf{W}[k]$ at spatial infinity first, restoring the global factor $1/(16\pi G_N^{(4)})$. The first two terms give the gravitational contribution

$$\frac{1}{16\pi G_N^{(4)}}\int_{S^2_\infty}\left\{P_{k\,ab}\delta\star(e^a\wedge e^b)-\imath_k\star(e^a\wedge e^b)\wedge\delta\omega_{ab}\right\}=\delta M-\Omega\delta J\,, \tag{3.71}$$

while the third and fourth give[15]

$$\frac{1}{16\pi G_N^{(4)}}\int_{S^2_\infty}\left\{-P_k{}^{\Lambda}\delta F_{\Lambda}+P_{k\,\Lambda}\delta F^{\Lambda}\right\}=-\Phi^{\Lambda}_{\infty}\delta q_{\Lambda}+\Phi_{\Lambda\,\infty}\delta p^{\Lambda}=-\Omega_{MN}\Phi^{M}_{\infty}\delta q^{N}\,, \tag{3.72}$$

where Φ^{Λ}_{∞} and $\Phi_{\Lambda\,\infty}$ are the values of the electrostatic and magnetostatic potentials at spatial infinity.

Let us consider the last term. In the previous cases only conserved charges are involved and it is natural to use the definition of scalar charges we have proposed here to rewrite that term. Using the identity $g_{xy}=g^{AB}k_{A\,x}k_{B\,y}$

$$g_{xy}\imath_k\star d\phi^x\delta\phi^y=g^{AB}\imath_k\star\hat{k}_A k_{B\,y}\delta\phi^y=\left(\mathbf{Q}_A[k]-\Omega_{MP}T_A{}^{P}{}_{N}P_k{}^{M}F^{N}\right)\delta^A\,. \tag{3.73}$$

where we have defined

$$\delta^A\equiv g^{AB}k_{B\,y}\delta\phi^y\,, \tag{3.74}$$

Restoring the global factor $1/(16\pi G_N^{(4)})$, we find

$$\int_{S^2_\infty}\left(\mathbf{Q}_A[k]-\frac{1}{16\pi G_N^{(4)}}\Omega_{MP}T_A{}^{P}{}_{N}P_k{}^{M}F^{N}\right)\delta^A=\left(\mathcal{Q}_{A\,k}-\Omega_{MP}T_A{}^{P}{}_{N}\Phi^{M}_{\infty}q^{N}\right)\delta^A_{\infty}\,. \tag{3.75}$$

Then,

$$\int_{S^2_\infty}\mathbf{W}[k]=\delta M-\Omega\delta J-\Omega_{MN}\Phi^{M}_{\infty}\delta q^{N}+\left(\mathcal{Q}_A-\Omega_{MP}T_A{}^{P}{}_{N}\Phi^{M}_{\infty}q^{N}\right)\delta^A_{\infty}\,. \tag{3.76}$$

The bifurcation surface is defined by the property $k=0$ and, on it,

$$P_{k\,ab}\overset{\mathcal{BH}}{=}\kappa n_{ab}\,, \tag{3.77}$$

where n^{ab} is the binormal to the horizon wit the normalization $n^{ab}n_{ab}=-2$ and κ is the surface gravity. Therefore,

$$\begin{aligned}\int_{\mathcal{BH}}\mathbf{W}[k]&=\frac{1}{16\pi G_N^{(4)}}\int_{\mathcal{BH}}\left\{P_{k\,ab}\delta\star(e^a\wedge e^b)-P_k{}^{\Lambda}\delta F_{\Lambda}+P_{k\,\Lambda}\delta F^{\Lambda}\right\}\\&=\frac{\kappa\delta A_{\mathcal{H}}}{8\pi G_N^{(4)}}-\Phi^{\Lambda}_{\mathcal{H}}\delta q_{\Lambda}+\Phi_{\Lambda\,\mathcal{H}}\delta p^{\Lambda}\,,\end{aligned} \tag{3.78}$$

[15] The electric and magnetic Maxwell momentum maps can be identified with the electrostatic and magnetostatic potentials Φ^{Λ} and Φ_{Λ}, respectively.

where $A_{\mathcal{H}}$ is the area of the horizon and $\Phi^{\Lambda}_{\mathcal{H}}$ and $\Phi_{\Lambda\,\mathcal{H}}$ are the values of the electrostatic and magnetostatic potentials over the horizon (constant according to the generalized zeroth law).

We arrive at our main result:[16]

$$\delta M = \frac{\kappa \delta A_{\mathcal{H}}}{8\pi G_N^{(4)}} + \Omega \delta J - \Omega_{MN}\left(\Phi^M_{\mathcal{H}} - \Phi^M_{\infty}\right)\delta q^N - \left(\mathcal{Q}_{A\,k} - \Omega_{MP} T_A{}^P{}_N \Phi^M_{\infty} q^N\right)\delta^A_{\infty}\,. \tag{3.79}$$

In this expression the object δ^A_{∞} is unusual, but it just reflects the different forms in which the dualities of the theory can modify the values of the moduli at infinity, which are also naturally associated to the charges that we have defined.

The last term involving Φ^M_{∞} is also unusual, but it has to be there if we are going to allow for potentials which do not vanish at infinity. In the examples that we are going to study explicitly, $\Phi^M_{\infty} = 0$ and the scalar charges take the expected value. Furthermore, in that case, the scalar term can be brought to the form found in Ref. [59] (up to the normalization of the charges):

$$-\mathcal{Q}_{A\,k}\delta^A_{\infty} = -\mathcal{Q}_{A\,k} g^{AB} k_B{}^x_{\infty} g_{xy\,\infty}\delta\phi^y_{\infty} = -\tfrac{1}{4}\Sigma^x g_{xy\,\infty}\delta\phi^y_{\infty}\,, \tag{3.80}$$

where the scalar charges defined through the asymptotic expansions, Σ^x are related to the ones associated to the duality symmetries Q_A by

$$\Sigma^x = 4\mathcal{Q}_A g^{AB} k_B{}^x_{\infty}\,. \tag{3.81}$$

Finally, observe that, on the bifurcation surface

$$\mathbf{Q}_A[k] \stackrel{\mathcal{BH}}{=} \Omega_{MP} T_A{}^P{}_N P_{k\,\mathcal{H}}^M F^N\,, \tag{3.82}$$

and, therefore

$$\mathcal{Q}_{A\,k} = -\Omega_{MP} T_A{}^P{}_N \Phi^M_{\mathcal{H}} q^N\,. \tag{3.83}$$

This formula, which is our second main result, gives a universal relation between the scalar charges of a black hole and the electric and magnetic charges and potentials evaluated on the horizon generalizing the result found in Ref. [61] in a gauge-invariant way. Observe that The existence of a bifurcate Killing horizon is crucial: in other spacetime backgrounds the scalar charges may take arbitrary values in agreement with the no-hair "theorem" and the interpretation of the non-trivial scalar fields of these black-hole solutions as secondary scalar hair [127].[17]

If we plug that formula back into the first law we arrive at

[16] The overall sign of the electric and magnetic terms is unconventional. It is due to the definition of F_{Λ} with a negative-definite kinetic matrix $I_{\Lambda\Sigma}$. It can be easily be changed, but the relative sign between the electric and magnetic terms can only be changed at the expense of losing explicit symplectic invariance.

[17] Static, spherically-symmetric solutions of pure gravity and dilaton gravity with primary scalar hair (*i.e.* scalar fields with charges which are independent parameters of the solutions) can be found in Refs. [128, 129] (see also the higher-dimensional generalizations in Chapter 16 of Ref. [126]) and are singular.

$$\delta M = \frac{\kappa \delta A_{\mathcal{H}}}{8\pi G_N^{(4)}} + \Omega \delta J - \Omega_{MN} \left(\Phi^M_{\mathcal{H}} - \Phi^M_{\infty}\right) \delta q^N - \Omega_{MP} T_A{}^P{}_N \left(\Phi^M_{\mathcal{H}} - \Phi^M_{\infty}\right) q^N \delta^A_{\infty} \,, \quad (3.84)$$

which is manifestly independent of the choice of asymptotic value of the potentials.

Notice that the right-hand side of this expression only contains the variations of quantities which are independent physical parameters of the black-hole solutions. The variations of the scalar charges cannot and do not appear. The scalar charges actually pleay the roles of thermodynamical potentials.

In the next two sections we are going to compare the scalar charges we have defined with those obtained through the asymptotic expansion and the first law that we have obtained with the first law obtained through the variation of the entropy with respect to the physical parameters in two sets of solutions: static, electrically-charged black holes and static axion-dilaton black holes.

3.4 Static dilaton black hole solutions

Dilaton black holes are solutions of the family of models defined by the action[18]

$$S[e, A, \phi] = \frac{1}{16\pi} \int \left\{ -\star (e^a \wedge e^b) \wedge R_{ab} + \tfrac{1}{2} d\phi \wedge \star d\phi + \tfrac{1}{2} e^{-a\phi} F \wedge \star F \right\} \,, \quad (3.85)$$

which depends on the real parameter a and determines the strength of the coupling of the dilaton and the Maxwell field. The static black-hole solutions of this model were found in Refs. [150–152] and can be written in the form

$$\begin{aligned} ds^2 &= H^{-\frac{2}{1+a^2}} W dt^2 - H^{\frac{2}{1+a^2}} \left[W^{-1} dr^2 + r^2 d\Omega^2_{(2)} \right] \,, \\ A_t &= \alpha e^{a\phi_\infty/2} (H^{-1} - 1) \,, \\ e^{-\phi} &= e^{-\phi_\infty} H^{\frac{2a}{1+a^2}} \,, \end{aligned} \quad (3.86)$$

where the functions H and W take the form

$$H = 1 + \frac{h}{r} \,, \qquad W = 1 + \frac{\omega}{r} \,, \quad (3.87)$$

and the integration constants h, ω, α satisfy the following relation

$$\omega = h \big[1 - (1 + a^2)(\alpha/2)^2 \big] \,. \quad (3.88)$$

In terms of the physical parameters M, q, ϕ_∞ (ADM mass, electric charge and modulus) and the coupling constant a, the integration constants h, ω and α are given by

[18] Sometimes they are called Einstein-Maxwell-Dilaton (EMD) actions. We set $G_N^{(4)} = 1$ throughout all this section.

$$
\begin{aligned}
h &= -\frac{a^2+1}{a^2-1}\left\{M - \sqrt{M^2+4(a^2-1)e^{a\phi_\infty}q^2}\right\}, \\
\omega &= -\frac{2}{a^2-1}\left\{a^2 M - \sqrt{M^2+4(a^2-1)e^{a\phi_\infty}q^2}\right\}, \\
\alpha &= -4qe^{a\phi_\infty/2}/h\,,
\end{aligned}
\tag{3.89}
$$

for $a \neq 1$ and

$$
\begin{aligned}
h &= \frac{4e^{\phi_\infty}q^2}{M}\,, \\
\omega &= -2\frac{M^2-2e^{\phi_\infty}q^2}{M}\,, \\
\alpha &= -e^{-\phi_\infty/2}M/q\,.
\end{aligned}
\tag{3.90}
$$

The scalar charge Σ, computed using the conventional asymptotic definition

$$
\phi \sim \phi_\infty + \frac{\Sigma}{r}\,, \tag{3.91}
$$

takes the value

$$
\Sigma = -\frac{2ah}{a^2+1}\,. \tag{3.92}
$$

We have chosen the sign of the square roots in h and ω so as to always have $h > 0$ and ω negative if certain non-extremality conditions are met: for all values of a

$$
M^2 > \frac{4}{a^2+1}e^{a\phi_\infty}q^2\,. \tag{3.93}
$$

In that case, there is an event horizon at

$$
r = -\omega \equiv r_0\,, \tag{3.94}
$$

with Bekenstein-Hawking entropy

$$
S = \pi r_0^{\frac{2a^2}{a^2+1}}(r_0+h)^{\frac{2}{a^2+1}}\,, \tag{3.95}
$$

and Hawking temperature

$$
T = \frac{r_0}{4S}\,. \tag{3.96}
$$

We can derive the first law for these families of black holes by varying the entropy with respect to all the independent physical parameters, including the modulus ϕ_∞:

$$\delta S = \frac{1}{T}\left[\delta M + \frac{4}{(a^2+1)\alpha}e^{a\phi_\infty/2}\delta q + \tfrac{1}{4}\Sigma\delta\phi_\infty\right], \tag{3.97}$$

for $a^2 \neq 1$ and

$$\delta S = \frac{1}{T}\left[\delta M + \frac{2}{\alpha}e^{\phi_\infty/2}\delta q + \tfrac{1}{4}\Sigma\delta\phi_\infty\right], \tag{3.98}$$

for $a^2 = 1$.

In the above expressions Σ is the scalar charge defined through the asymptotic expansion Eq. (3.91).

These theories are invariant under the global transformations generated by

$$\delta\phi = -1\,, \qquad \delta A = -\frac{a}{2}A\,, \tag{3.99}$$

and Eq. (3.49) takes the form

$$\mathbf{Q}[k] = -\frac{1}{16\pi}\left\{\imath_k \star d\phi + \frac{a}{2}P_k e^{-a\phi}\star F\right\} = -\frac{ah}{8\pi(a^2+1)}\omega_{(2)}\,, \tag{3.100}$$

where $\omega_{(2)}$ is the volume form of the round 2-sphere of unit radius. It is evident that these 2-forms satisfy a Gauss law and they give the same value when they are integrated over 2-spheres of any radius:

$$\mathcal{Q}_k = -\frac{ah}{2(a^2+1)} = -\tfrac{1}{4}\Sigma\,, \tag{3.101}$$

as expected according to our general arguments. This is, essentially, the result obtained by Pacilio in Ref. [61].

3.5 Static axion-dilaton black hole solutions

The so-called axion-dilaton model is just a generalization to an arbitrary number of vector fields n_V of pure, ungauged, $\mathcal{N}=4, d=4$ supergravity [153], although this model can also be embedded in $\mathcal{N}=2, d=4$ supergravity for $n_V=2$.

We can introduce it as a model with two real scalars $\phi^1 = a$ (the axion) and $\phi^2 = \phi$ (the dilaton) which are naturally combined into the complex scalar (*axidilaton*)

$$\lambda = a + ie^{-2\phi}\,, \tag{3.102}$$

and where the σ-model metric and the period matrix are given by

$$(g_{xy}) = \begin{pmatrix} e^{4\phi} & 0 \\ 0 & 4 \end{pmatrix}, \qquad \mathcal{N}_{\Lambda\Sigma} = -\lambda\delta_{\Lambda\Sigma}\,. \tag{3.103}$$

The most general non-extremal, static, black-hole solution of the axion-dilaton model was presented in Ref. [154] and it is a generalization of the solutions presented in Refs. [150, 155–159].[19] A very useful feature of this solution is that it is written in terms of its physical parameters only: the ADM mass M, the asymptotic value of the axidilaton $\lambda_\infty = a_\infty + ie^{-2\phi_\infty}$, the complex electromagnetic charges Γ^Λ (a combination of the real electric charges q_Λ, the real magnetic charges p_Λ and the moduli λ_∞) and the complex axidilaton charge $\Upsilon = \Sigma + i\Delta$. All these parameters are defined by the asymptotic expansions ($G_N^{(4)} = 1$)

$$g_{tt} \sim 1 - \frac{2M}{r}\,, \tag{3.104a}$$

$$\lambda \sim \lambda_\infty - ie^{-2\phi_\infty}\frac{2\Upsilon}{r}\,, \tag{3.104b}$$

$$\tfrac{1}{2}\left[F^\Lambda{}_{tr} + i\star F^\Lambda{}_{tr}\right] \sim \frac{e^{+\phi_\infty}\Gamma^\Lambda}{r^2} = \frac{e^{+2\phi_\infty}(q_\Lambda - \lambda^*_\infty p^\Lambda)}{r^2}\,. \tag{3.104c}$$

The asymptotic behavior of λ implies for those of a and ϕ

$$a \sim a_\infty + \frac{2e^{-2\phi_\infty}\Im\mathfrak{m}\Upsilon}{r}\,, \tag{3.105a}$$

$$\phi \sim \phi_\infty + \frac{\Re\mathfrak{e}\Upsilon}{r}\,, \tag{3.105b}$$

so

$$\Sigma^1 = 2e^{-2\phi_\infty}\Im\text{m}(\Upsilon)\,, \qquad \Sigma^2 = \Re\text{e}(\Upsilon)\,. \tag{3.106}$$

The axidilaton charge is a function of the rest of the physical parameters:

$$\Upsilon = -\frac{2}{M}\Gamma^{\Lambda\,*}\Gamma^{\Lambda\,*}\,. \tag{3.107}$$

The ADM mass can be defined more rigorously as a conserved quantity through the ADM [161], the Abbott-Deser [162] or many other formalisms. The electric and magnetic charges can also be defined as conserved charges by standard methods Refs. [60, 163] as

$$p^\Lambda \equiv \frac{1}{16\pi G_N^{(4)}}\int F^\Lambda\,, \tag{3.108a}$$

$$q_\Lambda \equiv \frac{1}{16\pi G_N^{(4)}}\int F_\Lambda\,. \tag{3.108b}$$

In contrast, as we have stressed, the scalar charges are conventionally defined through the above asymptotic expansion which is not based on any conservation (*Gauss*) law. Our

[19] The most general stationary, non-extremal black-hole solution of this theory was presented in Ref. [160].

goal in this section will be to show that the definition of scalar charges that we have proposed in Section 3.2 gives exactly the same result for the static solutions of the axion-dilaton model.

The most economical way of presenting this kind of solutions is through the time components of the original and dual 1-form fields $A^{\Lambda}{}_t$ and $A_{\Lambda\, t}$, respectively. They contain enough information to recover the rest of the components of each of them.[20] The solution is, then, [154]

$$ds^2 = e^{2U}dt^2 - e^{-2U}dr^2 - R^2 d\Omega^2_{(2)}\,,$$

$$\lambda = \frac{\lambda_\infty r + \lambda^*_\infty \Upsilon}{r+\Upsilon}\,,$$

$$A^{\Lambda}{}_t = 2e^{\phi_\infty}R^{-2}[\Gamma^\Lambda(r+\Upsilon) + \text{c.c.}]\,,$$

$$A_{\Lambda\, t} = -2e^{\phi_\infty}R^{-2}[\Gamma^\Lambda(\lambda_\infty r + \lambda^*_\infty \Upsilon) + \text{c.c.}]\,.$$

The functions e^{2U} and R are given by

$$\begin{aligned} e^{2U} &= R^{-2}(r-r_+)(r-r_-)\,, \\ R^2 &= r^2 - |\Upsilon|^2\,, \end{aligned} \tag{3.109}$$

and the parameters $r_\pm$ that appear in e^{2U} (actually, the positions of the outer and inner horizons when they take real values, *i.e.* when $r_0^2 > 0$) are given by

$$r_\pm = M \pm r_0\,, \quad \text{with} \quad r_0^2 = M^2 + |\Upsilon|^2 - 4\Gamma^\Lambda\Gamma^{\Lambda\,*}\,. \tag{3.110}$$

Since we are just interested in the thermodynamics of these black holes, we only need their Hawking temperature and Bekenstein-Hawking entropy, which are given by

$$T = \frac{r_0}{2S}\,, \tag{3.111a}$$

$$S = 2\pi\left\{M^2 + Mr_0 - 2\Gamma^{\Lambda\,*}\Gamma^{\Lambda\,*}\right\}\,. \tag{3.111b}$$

Varying S with respect to the physical charges M, q_Λ, p^Λ and the moduli λ_∞ we get the first law:

$$\delta M = T\delta S + \Phi^\Lambda \delta q_\Lambda - \Phi_\Lambda \delta p^\Lambda - \tfrac{1}{2}\Im\text{m}(\Upsilon)e^{2\phi_\infty}\delta a_\infty - \Re\text{e}(\Upsilon)\delta\phi_\infty\,. \tag{3.112}$$

The last two terms can be rewritten in two different fashions:

[20] They are computed explicitly in Ref. [139].

$$-\tfrac{1}{2}\Im\text{m}(\Upsilon)e^{2\phi_\infty}\delta a_\infty - \Re\text{e}(\Upsilon)\delta\phi_\infty = -\tfrac{1}{2}\Im\text{m}\left(\Upsilon^*\frac{\delta\lambda_\infty}{e^{-2\phi_\infty}}\right) \tag{3.113}$$
$$= -\tfrac{1}{4}g_{xy}(\phi_\infty)\Sigma^x\delta\phi^y_\infty\,,$$

where Σ^1, Σ^2 are the asymptotic scalar charges defined in Eqs. (3.106). Both expressions are manifestly duality-invariant.[21]

We are now going to see how the scalar charges Σ^x are related to those defined in Section 3.2 and how the scalar term in the first law agrees with the one in Eq. (3.79).

The Killing vectors of the target-space metric are

$$k_1 = a\partial_a - \tfrac{1}{2}\partial_\phi\,, \quad k_2 = \tfrac{1}{2}(1-a^2+e^{-4\phi})\partial_a + \tfrac{1}{2}a\partial_\phi\,, \quad k_3 = \tfrac{1}{2}(1+a^2-e^{-4\phi})\partial_a - \tfrac{1}{2}a\partial_\phi\,, \tag{3.114}$$

and their Lie brackets satisfy the sl$(2,\mathbb{R})$ $\sim$so$(2,1)$ algebra

$$[k_A, k_B] = \varepsilon_{ABD}\eta^{DC}k_C\,, \tag{3.115}$$

where $(\eta_{AB}) = (\eta^{AB})$ =diag$(++-)$ is the SO$(2,1)$ invariant metric.

The SL$(2,\mathbb{R})$ matrices which act on the 1-form fields are tensor products

$$S = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \otimes \mathbb{1}_{n_V\times n_V}\,, \quad \text{with} \quad AD - BC = 1\,. \tag{3.116}$$

The generators (always $\otimes\mathbb{1}_{n_V\times n_V}$) are

$$T_1 = -\tfrac{1}{2}\sigma^3\,, \qquad T_2 = -\tfrac{1}{2}\sigma^1\,, \qquad T_3 = \tfrac{i}{2}\sigma^2\,, \tag{3.117}$$

and their commutation relations are

$$[T_A, T_B] = -\varepsilon_{ABD}\eta^{DC}k_C\,. \tag{3.118}$$

It is somewhat simpler to work with the bases $k_1, k_\pm = k_2 \pm k_3$ and $T_1, T_\pm = T_2 \pm T_3$. We compute separately the $\imath_k \star \hat{k}_A$ and $\Omega_{MP}T_A{}^P{}_N P^M_k F^N$ contributions, which in this case correspond to

$$\begin{aligned} \Omega_{MP}T_1{}^P{}_N P^M_k F^N &= \tfrac{1}{2}\left(P^\Lambda_k F_\Lambda + P_{k\,\Lambda}F^\Lambda\right)\,, \\ \Omega_{MP}T_+{}^P{}_N P^M_k F^N &= -P^\Lambda_k F^\Lambda\,, \\ \Omega_{MP}T_-{}^P{}_N P^M_k F^N &= P_{k\,\Lambda}F_\Lambda\,. \end{aligned} \tag{3.119}$$

In this case we can use as potentials the time components of the original and dual vector fields given in Eqs. (3.109)

[21] $e^{2\phi}\delta\lambda$ and Υ are multiplied by the same phase under SL$(2,\mathbb{R})$ transformations.

$$P_k^M = A^M{}_t\,, \tag{3.120}$$

which vanish identically at infinity.

We are only interested in the pullback of these 2-forms over 2-spheres.[22] The results, after a long calculation are ($G_N^{(4)}=1$)

$$\begin{aligned}
\mathbf{Q}_{1\,k} &= -\frac{1}{8\pi}e^{2\phi_\infty}\Im\mathfrak{m}(\lambda_\infty^*\Upsilon)\omega_{(2)}\,,\\
\mathbf{Q}_{+\,k} &= -\frac{1}{8\pi}e^{2\phi_\infty}\Im\mathfrak{m}(\Upsilon)\omega_{(2)}\,,\\
\mathbf{Q}_{-\,k} &= \frac{1}{8\pi}e^{2\phi_\infty}\Im\mathfrak{m}(\lambda_\infty^{*\,2}\Upsilon)\omega_{(2)}\,,
\end{aligned} \tag{3.121}$$

Again, it is evident that these 2-forms satisfy a Gauss law and they give the same value when they are integrated over 2-spheres of any radius, namely

$$\begin{aligned}
\mathcal{Q}_{1\,k} &= -\tfrac{1}{2}e^{2\phi_\infty}\Im\mathfrak{m}(\lambda_\infty^*\Upsilon)\,,\\
\mathcal{Q}_{+\,k} &= -\tfrac{1}{2}e^{2\phi_\infty}\Im\mathfrak{m}(\Upsilon)\,,\\
\mathcal{Q}_{-\,k} &= \tfrac{1}{2}e^{2\phi_\infty}\Im\mathfrak{m}(\lambda_\infty^{*\,2}\Upsilon)\,.
\end{aligned} \tag{3.122}$$

It is now trivial to see that the asymptotic charges Eqs. (3.106) are recovered using Eq. (3.81) with the Killing vectors given above and

$$\left(g^{AB}\right) = \begin{pmatrix} 1 & 0 & 0\\ 0 & 0 & 1/2\\ 1/2 & 0 & 0\end{pmatrix}, \tag{3.123}$$

for the $1,+,-$ basis.

The first law Eq. (3.79) is recovered with

$$\delta_{1\,\infty} = \tfrac{1}{2}e^{4\phi_\infty}\delta|\lambda|^2_\infty\,, \qquad \delta_{+\,\infty} = e^{4\phi_\infty}\delta a_\infty\,, \qquad \delta_{-\,\infty} = -\tfrac{1}{2}e^{4\phi_\infty}(\lambda_\infty^{*\,2}\delta\lambda_\infty + \text{c.c.})\,. \tag{3.124}$$

3.6 Discussion

Some final comments on our results are in order.

First of all, it is unclear how to give coordinate-independent definitions of scalar charges satisfying a Gauss law in absence of global symmetries. This limitation led us

[22] There are additional tr components that we ignore since they do not contribute to the integrals.

to focus on theories with enough global symmetries to account for all the possible scalar charges. On the other hand, there are not many examples of black-hole solutions in theories with no or very few global isometries. Most of the general recipes elaborated to construct black holes in $\mathcal{N} = 2, d = 4$ theories, for instance, [164] are only valid for extremal black holes, which lie outside the scope of our methods. Working with non-extremal black holes is much more difficult [165] although some general methods have been developed [166] and they should be revisited to study this problem.

In general, a Gauss law is not equivalent to a full conservation law. In our case, the restriction to backgrounds with timelike Killing vectors makes it trivially equivalent to a conservation law in those particular backgrounds, but not in general. We expect, however, that the existence of a rigorous definition can be used to study the evolution of scalar charge or at least its behavior under perturbations.

It is worth stressing the relation between the value of the scalar charge and the existence of a regular bifurcate Killing horizon. In absence of such a horizon there does not seem to be a restriction on that value. It is because of this relation that it can be understood as secondary black-hole hair.

The general procedure that has allowed us to define a $(d-2)$-form satisfying a Gauss law starting from the $(d-1)$-form (Noether current) associated to a global symmetry can probably used in more general settings (fermionic matter, for instance).

As we mentioned before, it should be stressed that these results can be generalized to higher-rank fields and higher dimensions. The NGZ currents have been determined in Ref. [145] and one simply has to follow the same steps. It also seems that it should also be possible to find $(d-2)$-forms satisfying Gauss laws starting from any standard Noether current $(d-1)$-forms associated to a global symmetry.

Concerning the first law, in order to recover the GKK scalar term it has been essential to realize that the integral of $\mathbf{W}[k]$ at spatial infinity gives more than just the variations of the gravitational charges at infinity. Often, these contributions have been ignored or set to zero via convenient boundary conditions at spatial infinity. Often, the integral on the bifurcation surface has been also identified with the $T\delta S$ term of the firm law ignoring other contributions (work terms). We think it is now clear that there are different contributions to the first law coming from that integral as well and that the only one which is associate to the entropy is the one that takes the form of a conserved Lorentz charge, as we have pointed out in Refs. [57,130,131]. Actually, the title of Ref. [67] should be replaced by "Black hole entropy is the (Lorentz) Noether charge."

4

Hairy black holes, scalar charges and extended thermodynamics

This chapter is based on:
Hairy black holes, scalar charges and extended thermodynamics
R. Ballesteros and T. Ortín
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One of the most remarkable aspects of black holes is the fact that all their properties are determined by their conserved charges, irrespectively of their formation history. This fact is referred to as *uniqueness* (there is only one black hole for a given set of conserved charges) or absence of *hair* (there are no other parameters apart from the conserved charges characterizing different black holes) and it can be argued that it lies at the very heart of the thermodynamic interpretation of the dynamics of black holes.

This property has been tested in theories in which matter fields giving rise to different conserved charges are coupled to gravity. We can distinguish two broad types:

1. Matter fields with gauge symmetries, such as 1-forms in 4 dimensions. The conserved charges associated to these symmetries (electric and magnetic charges) are defined through surface integrals and are believed to be preserved by quantum gravity.

2. Matter fields with global symmetries, such as scalar fields.[1] The conserved charges associated to these symmetries are defined through volume integrals and are believed not to be preserved by quantum gravity. Therefore, black holes should not be characterized by this kind of charge, and possible non-trivial fields of this kind in black-hole spacetimes would, then, be understood as "hair" violating uniqueness.

It goes without saying that it is the second type of matter and the possible violations of black-hole uniqueness that it may induce that has attracted most interest. It is also the subject of this work.

In order to discuss black holes with scalar hair we first have to characterize scalar hair more precisely. The most naive way to do it would be to use the conserved charge associated to the global symmetries of the theory that act on the scalars. There are several reasons why this is not possible, even though the contrary is sometimes assumed in the literature:

1. There may not be any global symmetry acting on the scalars at all and, therefore, there may not be an associated conserved charge. This is what actually happens in

[1] The symmetries may act on other fields as well.

the theories considered in this paper in which the shift symmetry of a real scalar is broken by a scalar potential. In some works, the charge that would be conserved in absence of the potential is used, even though it is obviously not conserved and does not satisfy a Gauss law. The main problem with this kind of definitions comes from the next point, though.

2. In static black-hole spacetimes the volume integral that gives the globally conserved charge usually vanishes when integrated over a spacelike hypersurface [49].

For these reasons, in most of the literature it has been customary to use a definition of scalar charge based on the asymptotic expansion of the scalar: the scalar charge would be given, up to normalization, by the coefficient of the $1/r$ term in that expansion (see, for instance, Ref. [59]). This definition can be used in simple settings but it is clear that a coordinate-dependent definition is necessary to study the properties of this charge and establish general results.

In Refs. [49, 61] a covariant definition of scalar charge of a stationary black hole as the integral of an on-shell closed 2-form was proposed. In the cases considered so far, this definition gives the same value as the conventional definition based on the asymptotic expansion, but with the new definition one can go farther: the closedness of the 2-form charge implies that this scalar charge satisfies a Gauss law[2] and the covariant definition can be used to recover the scalar term in the first law of black-hole mechanics found in Ref. [59].

One of the empirical properties of this scalar charge is that, in black-hole spacetimes, it is usually completely determined by the conserved charges and asymptotic values of the scalars. In the language of Ref. [127] this kind of scalar charge corresponds to "secondary hair" and the black hole is still completely determined by the values of its truly conserved charges (plus the asymptotic values of the scalars). Whenever there are solutions with the same conserved charges but the scalar charge does not have that value (a particular function of the conserved charges and asymptotic values of the scalars) but is a free parameter that describes "primary hair" in the language of Ref. [127], the solution does not have a regular horizon and does not describe a black hole. This is illustrated by the solutions in Refs. [128, 129]. The covariant definition of scalar charge of Refs. [49, 61] can be used to determine the particular value of the scalar charge allowed in presence of a bifurcate black-hole horizon, which is, as a matter of fact, equivalent to a "no-hair theorem".

In this paper we want to study the extensions of the results obtained in Refs. [49, 61] to the case in which a real scalar is coupled to itself via a scalar potential instead of being coupled to vector fields. This is a very simple case which has been very much studied in the past and several "no-hair theorems" have been proven for more or less general classes of scalar potentials in Refs. [47, 167–171].[3]

One of the main assumptions in the proofs of these theorems is the positivity of the scalar potential, related to the energy conditions and, not surprisingly, asymptotically-flat black-hole solutions with scalar hair have been found in theories whose scalar potential violates that condition [118, 172–175]. These solutions and their thermodynamics have not

[2] It is worth stressing that by no means this implies that this scalar charge is conserved.

[3] See Ref. [46] for a review on the topic of hairy black holes with many references.

been studied from the point of view of their scalar charges[4] and our main goal is to do so, which, first of all requires a generalization of the definition of scalar charge of Refs. [49,61]: the theories considered in Refs. [49,61] have global symmetries and the scalar charges are related to them. As we are going to see, the covariant definition can be extended to the theories that we are going to consider, which have no global symmetries and it can be used to determine which values are allowed in the presence of a bifurcate black-hole horizon. As a byproduct we are going to see that the potentials that allow for asymptotically-flat black-hole solutions with well-defined scalar charges must satisfy a quite restrictive set of conditions previously found in Ref. [175].

This paper is organized as follows: in Section 4.1 we are going to describe the kind of theories that we are going to study. In Section 4.2 we are going to give a covariant definition of scalar charge for static solutions of these theories which satisfies a Gauss law and we are going to see which scalar potentials allow for well-defined scalar charges in the presence of a bifurcate horizon. In Section 4.3 we are going to derive a general Smarr formula for the black-hole solutions of these theories and in Section 4.4 we will derive the first law. in Section 4.5 we are going to test the general results obtained in the previous sections using the asymptotically-flat Anabalón-Oliva black hole [174]. The results obtained are discussed in Section 3.6, which also contains pointers to further research.

4.1 The theory

The theory we are going to work with consists of gravity, described by the Vielbein $e^a = e^a{}_\mu dx^\mu$, coupled minimally to a real scalar field ϕ which couples to itself via a scalar potential $V(\phi)$. The action, in differential-form language, is simply given by[5]

$$\begin{aligned} S[e,\phi] &= \frac{1}{16\pi G_N^{(4)}} \int \Big\{ -\star(e^a\wedge e^b)\wedge R_{ab} + \tfrac{1}{2}d\phi\wedge\star d\phi + \star V(\phi)\Big\} \\ &\equiv \int \mathbf{L}\,, \end{aligned} \tag{4.1}$$

where $\mathbf{L}$ is the Lagrangian 4-form.

Under a general variation of the fields, the action transforms as

$$\delta S = \int \{\mathbf{E}_a\wedge\delta e^a + \mathbf{E}\delta\phi + d\mathbf{\Theta}(e,\phi,\delta e,\delta\phi)\}\ , \tag{4.2}$$

where, ignoring the normalization factor $(16\pi G_N^{(4)})^{-1}$ for the time being (we will recover it when necessary), the Einstein equations $\mathbf{E}_a$ and the scalar equation $\mathbf{E}$ are given by

$$\mathbf{E}_a = \imath_a\star(e^b\wedge e^c)\wedge R_{bc} + \tfrac{1}{2}\left(\imath_a d\phi\star d\phi + d\phi\wedge\imath_a\star d\phi\right) - \imath_a\star V\ , \tag{4.3a}$$

$$\mathbf{E} = -d\star d\phi + \star V'\,, \tag{4.3b}$$

[4]In Ref. [176] the extended thermodynamics of some of these solutions has been studied, but not from the point of view of their scalar charges.

[5]Our conventions are those of Ref. [49,126].

where $V' \equiv dV/d\phi$, and where

$$\mathbf{\Theta}(e, \phi, \delta e, \delta\phi) = -\star(e^a \wedge e^b) \wedge \delta\omega_{ab} + \star d\phi\delta\phi\,. \tag{4.4}$$

Several no-hair theorems for this system have been proven in the literature [47, 169–171, 177] with the positivity of the scalar potential as one of the main assumptions. Several asymptotically-flat black-hole solutions with regular horizon and scalar hair have been found in systems that violate this particular assumption [118, 172–175][6]. We will only study in detail the asymptotically-flat Anabalón-Oliva black hole Ref. [174].

Our goal in this paper is to study how the concept of scalar charge can be used to study these solutions and what allows them to exist at all. Thus, we first study the definition of scalar charge in this system.

4.2 Scalar charge

Following Refs. [49, 61] we take the inner product of the Killing vector k with the scalar equation of motion, getting

$$\imath_k \mathbf{E} = d\left[\imath_k \star d\phi + \mathcal{W}_k\right]\,, \tag{4.5}$$

where we have defined

$$d\mathcal{W}_k = \imath_k \star V'\,. \tag{4.6}$$

The existence of the 2-form $\mathcal{W}_k$ is (locally) guaranteed by the assumptions concerning the symmetry of the system: if the diffeomorphism generated by k leaves invariant all the fields of the configuration,

$$\pounds_k \star V' = d\imath_k \star V' = 0\,. \tag{4.7}$$

Then, we define the scalar charge 2-form associated to the Killing vector k, $\mathbf{Q}_\phi[k]$, by[7]

$$\mathbf{Q}_\phi[k] \equiv -\frac{1}{4\pi G_N^{(4)}}\left[\imath_k \star d\phi + \mathcal{W}_k\right]\,. \tag{4.8}$$

We have just shown that $\mathbf{Q}_\phi[k]$ is closed on-shell or, in other words, that it satisfies a Gauss law on-shell.

We are going to consider stationary black-hole spacetimes and k will be the timelike Killing vector that generates their Killing horizon. As we will show later, for spherically-symmetric, static, asymptotically-flat black holes, this choice gives the scalar charge Σ defined in Eq. (4.17) as the integral over any closed 2-dimensional surface Σ^2 enclosing the black hole horizon

$$\Sigma \equiv \int_{\Sigma^2} \mathbf{Q}_\phi[k]\,. \tag{4.9}$$

[6]See Ref. [46] for a review on this topic with many references.

[7]The sign and normalization have been chosen so as to reproduce the conventional value of the scalar charge in absence of a potential. This is defined by the asymptotic expansion Eq. (4.17).

The on-shell closedness of $\mathbf{Q}_\phi[k]$ ensures that this definition does not depend on the integration surface chosen as long as they are homologically equivalent.

$\mathcal{W}_k$ is defined up to closed forms. We can use that freedom to make it vanish at spatial infinity:

$$\mathcal{W}_k(\infty) = 0\,. \tag{4.10}$$

Then, if we integrate $\mathbf{Q}_\phi[k]$ over the 2-sphere at spatial infinity, S^2_∞, we find that

$$\Sigma = -\frac{1}{4\pi G_N^{(4)}} \int_{S^2_\infty} \imath_k \star d\phi\,, \tag{4.11}$$

which recovers the conventional definition of scalar charge, as we are going to see.

If the black hole has a bifurcate horizon and we choose to integrate $\mathbf{Q}_\phi[k]$ over the bifurcation surface $\mathcal{BH}$ in which $k = 0$, we get

$$\Sigma = -\frac{1}{4\pi G_N^{(4)}} \int_{\mathcal{BH}} \mathcal{W}_k\,, \tag{4.12}$$

which provides an interesting relation between the scalar potential and the scalar charge of a black hole with bifurcate horizon. If we use the boundary condition Eq. (4.10) and the definition of $\mathcal{W}_k$ Eq. (4.6), applying Stokes' theorem we can rewrite the above formula in the form

$$\Sigma = -\frac{1}{4\pi G_N^{(4)}} \int_{\Sigma^3} \imath_k \star V'\,, \tag{4.13}$$

where Σ^3 is a hypersurface with boundaries at the bifurcation surface and spatial infinity.

Black holes with regular bifurcate horizons and scalar hair corresponding to the scalar charge Σ will only exist if the integral in the right-hand side is finite, which imposes strong conditions on the scalar potentials that allow for hairy black hole solutions. In order to find these conditions, we are going to focus on static, asymptotically-flat, spherically-symmetric black holes with metrics of the form

$$ds^2 = \lambda(\rho)dt^2 - \lambda^{-1}(\rho)dr^2 - R^2(\rho)d\Omega^2_{(2)}\,. \tag{4.14}$$

Since the integral of $\mathbf{Q}_\phi[k]$ over any 2-sphere of constant radius ρ should give the same result, Σ,

$$\mathbf{Q}_\phi[k] = \frac{\Sigma}{16\pi}\omega_{(2)}\,, \tag{4.15}$$

where $\omega_{(2)}$ is the volume 2-form of the unit sphere.

On the other hand,

$$-\imath_k \star d\phi = -\lambda R^2 \partial_\rho \phi\, \omega_{(2)}\,, \tag{4.16}$$

and, if ϕ behaves at spatial infinity as[8]

[8]This is the conventional definition of the scalar charge Σ.

$$\phi \sim \phi_\infty + \frac{G_N^{(4)}\Sigma}{\rho} + \mathcal{O}(\rho^{-2})\,, \tag{4.17}$$

where ϕ_∞ is the constant value of the scalar at spatial infinity, we find that, in that limit,

$$-\frac{1}{4\pi G_N^{(4)}} \imath_k \star d\phi \sim \frac{\Sigma}{4\pi}\omega_{(2)} + \mathcal{O}(\rho^{-1})\,. \tag{4.18}$$

which implies that, in the same limit,

$$\mathcal{W}_k \sim \mathcal{O}(\rho^{-1})\,, \tag{4.19}$$

which is consistent with the boundary condition we had chosen for $\mathcal{W}_k$, Eq. (4.10), and, in its turn, implies that

$$-\frac{1}{4\pi G_N^{(4)}} \mathcal{W}_k(\mathcal{BH}) = \frac{\Sigma}{4\pi}\omega_{(2)}\,. \tag{4.20}$$

Observe that, since λ must vanish on the horizon, $\imath_k \star d\phi$ vanishes everywhere on the horizon and not just on the bifurcation surface.

Let us find an explicit expression for $\mathcal{W}_k$. For the metrics we are dealing with,

$$d\mathcal{W}_k = d\left[\mathcal{W}_k(\rho)\omega_{(2)}\right]\,, \tag{4.21}$$

where the function $\mathcal{W}_k(\rho)$ is defined by

$$d\mathcal{W}_k(\rho) = -R^2(\rho)V'[\phi(\rho)]d\rho\,. \tag{4.22}$$

From Eq. (4.15) and the definition of $\mathbf{Q}_\phi[k]$

$$-\frac{1}{4\pi G_N^{(4)}} \left[\imath_k \star d\phi + \mathcal{W}_k\right] = \frac{\Sigma}{4\pi}\omega_{(2)}\,, \tag{4.23}$$

and, using Eqs. (4.16) and (4.21) we have

$$\mathcal{W}_k(\rho) = -\Sigma G_N^{(4)} - R^2\partial_\rho\phi = \int_\rho^\infty R^2(\rho)V'[\phi(\rho)]\,. \tag{4.24}$$

If we expand asymptotically the right-hand side of the definition of $\mathcal{W}_k(\rho)$, Eq. (4.22), assuming

$$\phi \sim \phi_\infty + \frac{G_N^{(4)}\Sigma}{r} + \frac{\Delta}{r^2} + \mathcal{O}(r^{-3})\,, \tag{4.25}$$

we get

$$
\begin{aligned}
d\mathcal{W}_k(r) = &\left(1 - \frac{2(G_N^{(4)}M)^2}{r^2} + \mathcal{O}(r^{-3})\right) \times \\
&\times \left\{V'[\phi_\infty]r^2 + V''[\phi_\infty]\Sigma G_N^{(4)} r + \tfrac{1}{2}\left[V'''[\phi_\infty]\left(\Sigma G_N^{(4)}\right)^2 + 2V''[\phi_\infty]\Delta\right] + \mathcal{O}(r^{-3})\right\} dr \\
= &\left(1 - \frac{2(G_N^{(4)}M)^2}{r^2} + \mathcal{O}(r^{-3})\right) \times \\
&\times \left\{V'[\phi_\infty]r^2 + V''[\phi_\infty]\Sigma G_N^{(4)} r + \tfrac{1}{2}V'''[\phi_\infty]\left(\Sigma G_N^{(4)}\right)^2 + V''[\phi_\infty]\Delta \right. \\
&\left. -2(G_N^{(4)}M)^2 V'[\phi_\infty] + \mathcal{O}(r^{-1})\right\} dr\,,
\end{aligned}
\tag{4.26}
$$

and, comparing with an asymptotic expansion of $\mathcal{W}_k(r)$ that takes into account that $\mathcal{W}_k(\infty) = 0$

$$
\mathcal{W}_k(r) = \mathcal{O}(r^{-1})\,, \quad \Rightarrow \quad d\mathcal{W}_k(r) = \mathcal{O}(r^{-2})dr\,, \tag{4.27}
$$

we find that the potential and its first four derivatives must vanish at the asymptotic value of the scalar:

$$
V(\phi_\infty) = V'[\phi_\infty] = V''[\phi_\infty] = V'''[\phi_\infty] = V''''[\phi_\infty] = 0\,, \tag{4.28}
$$

where we have taken into account that we are considering asymptotically-flat black holes only and where we have assumed that $\Sigma \neq 0$.[9]

These conditions are satisfied by the potential of the theory of Ref. [174] for the asymptotic value of the asymptotically-flat (Anabalón-Oliva) black hole $\phi_\infty = 0$.[10] They are not satisfied for a massive scalar, though, because $V''[\phi_\infty] = m^2 \neq 0$. Therefore, the result that we have obtained, based on a definition of scalar charge that satisfies a Gauss law is equivalent to Bekenstein's no-hair theorem of Ref. [167] and also discards many other scalar potentials.

4.3 Smarr formula

Our next step in the study of these theories is the derivation of a Smarr formula using the techniques developed in Refs. [64, 65, 69, 138–140, 178].

[9] These conditions have been previously derived in a different but equivalent way in Ref. [175] in which the requirement of a well-defined scalar charge has been implicitly used.

[10] They are also satisfied by the first scalar potential of Ref. [173]. The scalar potential of Ref. [172] was not given in full. On the other hand, the scalar potential in Ref. [118] manifestly violates those conditions, but the asymptotic behaviour of the scalar field is exponential, $\phi \sim e^{-\alpha\rho}/\rho$ so that $\Sigma = 0$.

The Smarr formula follows from the integration of the generalized Komar charge [63] on the hypersurface Σ^3 that interpolates between the bifurcation sphere and the sphere at spatial infinity (its two boundaries). The generalized Komar charge is given by

$$\mathbf{K}[k] \equiv -\left(\mathbf{Q}[k] - \omega_k\right) , \tag{4.29}$$

where $\mathbf{Q}[k]$ is the Noether-Wald charge associated to the Killing vector k and ω_k is the 2-form implicitly defined by[11]

$$\imath_k \mathbf{L} \doteq d\omega_k \, . \tag{4.30}$$

The (local) existence of ω_k, as that of $\mathcal{W}_k$, is guaranteed by the assumption that k generates a symmetry of all the field of the solution:

$$\pounds_k \mathbf{L} = d\imath_k \mathbf{L} = 0 \, . \tag{4.31}$$

As explained in Ref. [56], in order to compute the Noether-Wald charge one must properly take into account the gauge freedoms of the fields of the theory. In this case, the only gauge symmetry of the theory (apart from diffeomorphisms), is the local Lorentz symmetry acting on the Vielbein and the right way to deal with it is to replace the Lie derivative of the Vielbein by the Lorentz-covariant (or Lie-Lorentz) derivative [58] (see also Ref. [179]), so that

$$\delta_\xi e^a = -\left(\mathcal{D}\xi^a + P_\xi{}^a{}_b e^b\right) , \qquad \delta_\xi \phi = -\imath_\xi d\phi \, , \tag{4.32}$$

where $P_\xi{}^{ab}$, the *Lorentz momentum map*, is defined by the equation

$$\mathcal{D}P_k{}^{ab} + \imath_k R^{ab} = 0 \, , \tag{4.33}$$

for Killing vectors k. This equation is satisfied by the *Killing bivector*

$$P_k{}^{ab} = \nabla^a k^b \, . \tag{4.34}$$

Substituting the transformations Eqs. (4.32) into Eq. (4.2) and using the Noether identity associated to the invariance under local Lorentz transformations of the action

$$P_\xi{}^a{}_b \mathbf{E}_a \wedge e^b = 0 \, , \tag{4.35}$$

and the Noether identity associated to the invariance under diffeomorphisms of the action

$$\mathcal{D}\mathbf{E}_a + \mathbf{E}\imath_a d\phi = 0 \, , \tag{4.36}$$

we find that the variation of the action under the transformations Eqs. (4.32)

$$\begin{aligned} \delta_\xi S = -\int \Big\{ & \left(\mathcal{D}\mathbf{E}_a + \mathbf{E}\imath_a d\phi\right)\xi^a + P_\xi{}^a{}_b \mathbf{E}_a \wedge e^b \\ & + d\left[-\mathbf{E}_a \xi^a - \star(e^a \wedge e^b) \wedge (\imath_\xi R_{ab} + \mathcal{D}P_{\xi\, ab}) + \star d\phi \imath_\xi d\phi\right]\Big\} \, , \end{aligned} \tag{4.37}$$

[11] We indicate relations which only hold on-shell with $\doteq$.

is just a total derivative.

Massaging this total derivative a bit, we arrive to

$$\delta_\xi S = -\int d\left\{\imath_\xi \mathbf{L} - d\left[\star(e^a \wedge e^b)P_{\xi\, ab}\right]\right\} . \tag{4.38}$$

Since the action is only invariant under a total derivative under these transformations, we arrive to the Noether-Wald charge of pure Einstein gravity [67], which is nothing but the Komar charge of pure Einstein gravity

$$\mathbf{Q}[\xi] = \frac{1}{16\pi G_N^{(4)}} \star (e^a \wedge e^b)P_{\xi\, ab} . \tag{4.39}$$

Now, in order to find ω_k we need to evaluate the on-shell Lagrangian. We first take the trace of the Einstein equation

$$e^a \wedge \mathbf{E}_a = -2\left[\mathbf{L} + \star V\right] . \tag{4.40}$$

Then,

$$\mathbf{L} = -\tfrac{1}{2} e^a \wedge \mathbf{E}_a - \star V \doteq - \star V , \tag{4.41}$$

and, for a Killing vector k that leaves all the fields invariant

$$\imath_k \mathbf{L} \doteq -\imath_k \star V \equiv d\mathcal{V}_k . \tag{4.42}$$

Again, the (local) existence of the 2-form $\mathcal{V}_k$ is guaranteed by the assumptions on the symmetry of the configurations.

The Komar charge of this theory is finally given by

$$\mathbf{K}[k] = -\frac{1}{16\pi G_N^{(4)}}\left[\star(e^a \wedge e^b)P_{k\, ab} - \mathcal{V}_k\right] , \tag{4.43}$$

and it is not difficult to check that it is closed on-shell:

$$d\mathbf{K}[k] = \tfrac{1}{2} e^a \wedge \mathbf{E}_a - k^a \mathbf{E}_a + \frac{1}{16\pi G_N^{(4)}} \imath_k d\phi \star d\phi \doteq 0 , \tag{4.44}$$

recalling that, by assumption, $\imath_k d\phi = 0$.

Let us consider asymptotically-flat ($V = 0$ at spatial infinity), static black holes with bifurcate Killing horizons $\mathcal{H}$ associated to k and let us integrate $d\mathbf{K}[k]$ over a hypersurface Σ^3 whose boundaries are the bifurcation sphere $\mathcal{BH}$ where $k = 0$ and the 2-sphere at spatial infinity S^2_∞. Applying Stokes theorem

$$0 \doteq \int_{\mathrm{S}^2_\infty} \mathbf{K}[k] - \int_{\mathcal{BH}} \mathbf{K}[k] . \tag{4.45}$$

At infinity, by assumption

$$\int_{\mathrm{S}^2_\infty} \mathbf{K}[k] = \tfrac{1}{2}M + \frac{1}{16\pi G_N^{(4)}} \int_{\mathrm{S}^2_\infty} \mathcal{V}_k \,. \tag{4.46}$$

Over the bifurcation sphere

$$\int_{\mathcal{BH}} \mathbf{K}[k] = \frac{\kappa A}{4G_N^{(4)}} + \frac{1}{16\pi G_N^{(4)}} \int_{\mathcal{BH}} \mathcal{V}_k \,. \tag{4.47}$$

Thus, using again Stokes' theorem and the definition of $\mathcal{V}_k$ Eq. (4.42), we arrive at the Smarr formula

$$\begin{aligned} M &= 2ST - \frac{1}{8\pi G_N^{(4)}} \left[\int_{\mathrm{S}^2_\infty} \mathcal{V}_k - \int_{\mathcal{BH}} \mathcal{V}_k \right] \\ &= 2ST - \frac{1}{8\pi G_N^{(4)}} \int_{\Sigma^3} \imath_k \star V \,. \end{aligned} \tag{4.48}$$

This is not the form in which the Smarr formula is usually presented. The scalar potential must be proportional to one or several dimensionful coupling constants. Let α be that constant and let it have dimensions of inverse length squared so that

$$V = \alpha \frac{\partial V}{\partial \alpha} \,. \tag{4.49}$$

Then, the Smarr formula can be written in the more standard form

$$M = 2ST + 2\alpha\Phi_\alpha \,, \quad \text{where} \quad \Phi_\alpha \equiv -\frac{1}{16\pi G_N^{(4)}} \int_{\Sigma^3} \imath_k \star \frac{\partial V}{\partial \alpha} \,, \tag{4.50}$$

which, as proposed in Refs. [69, 138, 140], must be interpreted in terms of extended thermodynamics: α plays the role of a new thermodynamic variable and Φ_α plays the role of its conjugate potential. The validity of this Smarr formula can be tested directly in the existing hairy solutions and we will do so for the Anabalón-Oliva black hole.

For the static, spherically symmetric metrics we are considering

$$\imath_k \star V = -R^2(\rho) V(\phi(\rho)) d\rho \wedge \omega_{(2)} \,, \tag{4.51}$$

and

$$\Phi_\alpha = \frac{1}{4G_N^{(4)}} \int_{\rho_h}^{\infty} R^2 \frac{\partial V}{\partial \alpha} d\rho \,. \tag{4.52}$$

For large values of ρ

$$\begin{aligned}\imath_k \star V = &-\left(1-\frac{2(G_N^{(4)}M)^2}{\rho^2}+\mathcal{O}(\rho^{-3})\right)\times \\ &\times \rho^2\left\{V(\phi_\infty)+V'(\phi_\infty)\frac{\Sigma G_N^{(4)}}{\rho}+\cdots\right\}d\rho\wedge\omega_{(2)}\,.\end{aligned}\tag{4.53}$$

Asymptotic flatness implies $V(\phi_\infty)=0$ and the convergence of the integral demands, again, Eqs. (4.28) to hold.

4.4 The first law of black hole mechanics

As shown in Ref. [138], in order to derive the first law of black-hole mechanics using Wald's formalism in theories with dimensionful parameters such as those that must necessarily occur in the scalar potential (α in the case discussed in Section 4.3), it is necessary to dualize those parameters into $(d-1)$-form potentials which have a gauge symmetry generated by $(d-2)$-form parameters and work with the dual formulation of the theory. In particular, we have to rederive the Noether-Wald charge, which will have an additional term associated to the new gauge symmetry.

4.4.1 The dual theory

We can directly draw from the results of Ref. [138] and write an action which contains two additional dynamical fields: the scalar ϑ and the 3-form C. ϑ is the square root of the dimensionful constant α discussed in Section 4.3, which in this setting is promoted to a scalar field,

$$\alpha=\vartheta^2\,,\tag{4.54}$$

so that

$$\vartheta\frac{\partial V}{\partial\vartheta}=2V\,.\tag{4.55}$$

The 3-form C is the dual of ϑ and it is introduced in the action as a Lagrange multiplier enforcing the constraint $d\vartheta=0$. The action is that in Eq. (3.2) supplemented by the Lagrange multiplier term, which is topological and does not modify the Einstein equations:

$$\begin{aligned}S[e,\phi,\vartheta,C]&=\frac{1}{16\pi G_N^{(4)}}\int\left\{-\star(e^a\wedge e^b)\wedge R_{ab}+\tfrac{1}{2}d\phi\wedge\star d\phi-C\wedge d\vartheta+\star V(\phi)\right\}\\ &\equiv\int\mathbf{L}\,.\end{aligned}\tag{4.56}$$

Under a general variation of the fields,

$$\delta S = \int \{\mathbf{E}_a \wedge \delta e^a + \mathbf{E}\delta\phi + \mathbf{E}_C \wedge \delta C + \mathbf{E}_\vartheta \wedge \delta\vartheta + d\mathbf{\Theta}(\varphi, \delta\varphi)\} \,, \tag{4.57}$$

where the Einstein and scalar equations $\mathbf{E}_a, \mathbf{E}$ are identical to those of the original action Eqs. (4.3a) and (4.3b), and the equations of the 3-form C and of the dimensionful "constant" ϑ are given by

$$\mathbf{E}_\vartheta = -G + \star\frac{\partial V}{\partial\vartheta} \,, \tag{4.58a}$$

$$\mathbf{E}_C = d\vartheta \,, \tag{4.58b}$$

where

$$G \equiv dC \,, \tag{4.59}$$

is the field strength of the 3-form C, invariant under the gauge transformations

$$\delta_\chi C = d\chi \,, \tag{4.60}$$

where χ is an arbitrary 2-form.

On-shell, the equation of motion of ϑ is a duality relation between the 3-form and ϑ [180]

$$G = \star\frac{\partial V}{\partial\vartheta} \,, \tag{4.61}$$

and, as expected, the equation of motion of C just says that ϑ is constant.[12]

Finally, $\mathbf{\Theta}$ contains a new term

$$\mathbf{\Theta}(\varphi, \delta\varphi) = - \star (e^a \wedge e^b) \wedge \delta\omega_{ab} + \star d\phi\delta\phi + C\delta\vartheta \,. \tag{4.62}$$

Observe that the action Eq. (4.56) is only invariant under the gauge transformations of C up to a total derivative (defined itself up to total derivatives), which we will have to take into account:

$$\delta_\chi S = \int d\left(\vartheta d\chi\right) \,. \tag{4.63}$$

4.4.2 The Noether-Wald charge of the dual theory

We just have to use the general form of the variation of the dual action Eq. (4.57) in the particular case of the variations Eqs. (4.32) and

$$\delta_\xi\vartheta = -\imath_\xi\vartheta \,, \qquad \delta_\xi C = -\left(\imath_\xi G + dP_\xi\right) \,, \tag{4.64}$$

[12]In presence of sources of 3-form C, 2-branes (domain walls) charged with respect to C, it can be piecewise constant, see for instance, [181]. This action, thus, is slightly more general than the original one.

where, following the general rules, we have defined the momentum map 2-form P_ξ to satisfy the equation

$$\imath_k G + dP_k = 0\,, \tag{4.65}$$

for a Killing vector k that leaves invariant all the fields of the theory. This assumption guarantees the existence of P_k.

The Noether-Wald and Komar charges can be simply read from Ref. [138]

$$\mathbf{Q}[\xi] = \frac{1}{16\pi G_N^{(4)}}\left[\star(e^a \wedge e^b)P_{\xi\, ab} + \vartheta P_\xi\right]\,, \tag{4.66a}$$

$$\mathbf{K}[\xi] = -\frac{1}{16\pi G_N^{(4)}}\left[\star(e^a \wedge e^b)P_{\xi\, ab} - \tfrac{1}{2}\vartheta P_\xi\right]\,, \tag{4.66b}$$

but we have to take into account that they have been computed using the particular choice of total derivative Eq. (4.63)

$$\delta_\xi S = \int \{-d\vartheta \wedge \imath_\xi C - \vartheta dP_\xi\}\,. \tag{4.67}$$

Before deriving the Smarr formula and the first law for the dual theory, it is necessary to consider the generalized, restricted, zeroth law [56]. If k is the Killing vector that generates the horizon, in the bifurcation surface where $k = 0$ Eq. (4.65) implies that

$$dP_k \stackrel{\mathcal{BH}}{=} 0\,. \tag{4.68}$$

In the static case in which we are interested here, this just means that

$$P_k \stackrel{\mathcal{BH}}{=} 16\pi G_N^{(4)}\Phi_G \omega_{(2)}\,, \tag{4.69}$$

where Φ_G is a constant and $\omega_{(2)}$ is the volume form of the unit 2-sphere.

Observe that, on-shell and using the homogeneity of V in ϑ, the equation of motion of C, the momentum map equation (4.65) and the equation of motion of ϑ we have the following relation:

$$\imath_k \star V = \frac{1}{2}\vartheta \imath_k \star \frac{\partial V}{\partial \vartheta} \doteq \frac{1}{2}\vartheta \imath_k G = -\frac{1}{2}\vartheta dP_k \doteq d\left(-\frac{1}{2}\vartheta P_k\right)\,, \tag{4.70}$$

which means that $\mathcal{V}_k$ defined in Eq. (4.42) is given by

$$\tfrac{1}{2}\vartheta P_k \doteq \mathcal{V}_k\,. \tag{4.71}$$

Replacing this result in the Komar charge Eq. (4.66b) we recover that of the original theory Eq. (4.43), which means that we get the same Smarr formula Eq. (4.50).

Let us now consider the derivation of the first law in full detail, improving the derivations made in Refs. [49,56,57,130,131,138,139] in which either scalar charges had not been considered or the action was exactly invariant under gauge transformations (which is not the case here).

We start by defining the symplectic 3-form [66–68]

$$\omega(\varphi, \delta_1\varphi, \delta_2\varphi) \equiv \delta_1 \mathbf{\Theta}(\varphi, \delta_2\varphi) - \delta_2 \mathbf{\Theta}(\varphi, \delta_1\varphi)\,, \tag{4.72}$$

where φ denotes collectively all the fields of the theory. In this case we have to choose in Eq. (4.72) $\delta_1\varphi = \delta\varphi$, variations of the fields which satisfy the linearized equations of motion but which are, otherwise, arbitrary, and $\delta_2\varphi = \delta_\xi\varphi$, the transformations under diffeomorphisms given in Eqs. (4.32) and (4.64) which, we must recall, include induced gauge transformations $\delta_{\sigma_\xi}, \delta_{\chi_\xi}$ which we will denote, collectively, by δ_{Λ_ξ}, so that

$$\delta_\xi = -\pounds_\xi + \delta_{\Lambda_\xi}\,. \tag{4.73}$$

Under these transformations, using the general expression for the variation of the action, we get

$$\delta_\xi S = \int d\mathbf{\Theta}'(\varphi, \delta_\xi\varphi)\,, \tag{4.74}$$

where $\mathbf{\Theta}'$ is a combination of $\mathbf{\Theta}$ and equations of motion, so that, on-shell, $\mathbf{\Theta} = \mathbf{\Theta}'$.

On the other hand, varying directly the action we find that

$$\delta_\xi S = \int d\left\{-\imath_\xi \mathbf{L} + \mathbf{X}(\delta_{\Lambda_\xi}\varphi)\right\}\,, \quad \text{where} \quad d\mathbf{X}(\delta_{\Lambda_\xi}\varphi) \equiv \delta_{\Lambda_\xi}\mathbf{L}\,. \tag{4.75}$$

In the theory we are considering we can read $X(\Lambda_\xi)$ from Eq. (4.67):

$$\mathbf{X}(\delta_{\Lambda_\xi}\varphi) = -d\vartheta \wedge \imath_\xi C - \vartheta dP_\xi \doteq d\left(-\vartheta P_\xi\right)\,. \tag{4.76}$$

Equating Eqs. (4.74) and (4.75) we arrive at

$$d\mathbf{J}[\xi] = 0\,, \quad \text{where} \quad \mathbf{J}[\xi] \equiv \mathbf{\Theta}'(\varphi, \delta_\xi\varphi) + \imath_\xi \mathbf{L} - \mathbf{X}(\delta_{\Lambda_\xi}\varphi)\,. \tag{4.77}$$

Then,

$$\begin{aligned}
\omega(\varphi, \delta\varphi, \delta_\xi\varphi) &\doteq \delta\mathbf{\Theta}'(\varphi, \delta_\xi\varphi) - \delta_\xi\mathbf{\Theta}'(\varphi, \delta\varphi) \\
&= \delta\left(\mathbf{J}[\xi] - \imath_\xi\mathbf{L} + \mathbf{X}(\delta_{\Lambda_\xi}\varphi)\right) - \left(-\pounds_\xi + \delta_{\Lambda_\xi}\right)\mathbf{\Theta}'(\varphi, \delta\varphi) \\
&= \delta\mathbf{J}[\xi] - \imath_\xi\delta\mathbf{L} + \delta\mathbf{X}(\delta_{\Lambda_\xi}\varphi) + \left(\imath_\xi d + d\imath_\xi - \delta_{\Lambda_\xi}\right)\mathbf{\Theta}'(\varphi, \delta\varphi) \\
&= \delta d\mathbf{Q}[\xi] - \imath_\xi\left(\mathbf{E}_\varphi \wedge \delta\varphi + d\mathbf{\Theta}(\varphi, \delta\varphi)\right) \\
&\quad + \left(\imath_\xi d + d\imath_\xi - \delta_{\Lambda_\xi}\right)\mathbf{\Theta}'(\varphi, \delta\varphi) + \delta\mathbf{X}(\delta_{\Lambda_\xi}\varphi) \\
&\doteq d\left[\delta\mathbf{Q}[\xi] + \imath_\xi\mathbf{\Theta}'(\varphi, \delta\varphi)\right] - \delta_{\Lambda_\xi}\mathbf{\Theta}'(\varphi, \delta\varphi) + \delta\mathbf{X}(\delta_{\Lambda_\xi}\varphi)\,.
\end{aligned} \tag{4.78}$$

The last two terms must also be total derivatives:

$$\delta_{\Lambda_\xi}\mathbf{\Theta}(\varphi,\delta\varphi) \equiv d\varpi_\xi\,, \tag{4.79a}$$

$$\delta\mathbf{X}(\delta_{\Lambda_\xi}\varphi) \equiv d\pi_\xi\,. \tag{4.79b}$$

Let us consider ϖ_ξ first. In the case at hands, there are two kinds of gauge transformations:

1. Local Lorentz transformations, δ_σ. The parameter of the local Lorentz transformation induced by the diffeomorphism generated by ξ is

$$\sigma_\xi^{ab} = \imath_\xi\omega^{ab} - P_\xi{}^{ab}\,. \tag{4.80}$$

2. Gauge transformations of the 3-form C, δ_χ. The parameter of the gauge transformation induced by the diffeomorphism generated by ξ is

$$\chi_\xi = \imath_\xi C - P_\xi\,. \tag{4.81}$$

We find that

$$\delta_{\Lambda_\xi}\mathbf{\Theta}(\varphi,\delta\varphi) \doteq d\left[-\star(e^a\wedge e^b)\delta\sigma_{\xi\,ab} + \chi_\xi\delta\vartheta\right] \equiv d\varpi_\xi\,. \tag{4.82}$$

As for π_ξ, we find

$$\delta\mathbf{X}(\delta_{\Lambda_\xi}\varphi) \doteq d\delta\left(-\vartheta P_\xi\right) \equiv d\pi_\xi\,. \tag{4.83}$$

Thus, we find that

$$\omega(\varphi,\delta\varphi,\delta_\xi\varphi) \doteq d\mathbf{W}[\xi]\,, \tag{4.84}$$

with

$$\mathbf{W}[\xi] \equiv -\delta\mathbf{Q}[\xi] - \imath_\xi\mathbf{\Theta}(\varphi,\delta\varphi) + \varpi_\xi - \pi_\xi\,, \tag{4.85}$$

and we also find that for vector fields k that generate symmetries of all the fields $\delta_k\varphi = 0$

$$d\mathbf{W}[k] \doteq 0\,. \tag{4.86}$$

Integrating this identity over the same hypersurface Σ^3 we considered for the Smarr formula and using again the Stokes theorem we will derive the first law, but we must compute $\mathbf{W}[k]$ first. We find

$$\imath_k\mathbf{\Theta}(\varphi,\delta\varphi) = -\imath_k\star(e^a\wedge e^b)\wedge\delta\omega_{ab} - \star(e^a\wedge e^b)\wedge\delta\imath_k\omega_{ab} + \imath_k\star d\phi\delta\phi + \imath_k C\delta\vartheta\,, \tag{4.87a}$$

$$\delta\mathbf{Q}[k] = \delta\star(e^a\wedge e^b)P_{k\,ab} + \star(e^a\wedge e^b)\delta P_{k\,ab} + \delta\left(\vartheta P_k\right)\,, \tag{4.87b}$$

and, therefore,

$$\mathbf{W}[k] = -\delta \star (e^a \wedge e^b) P_{k\,ab} + \imath_k \star (e^a \wedge e^b) \wedge \delta\omega_{ab} - P_k \delta\vartheta - \imath_k \star d\phi\delta\phi\,, \tag{4.88}$$

where we have used Eq. (4.71).

As a non-trivial test of all the manipulations we have performed, it can be checked by an explicit and direct calculation that, indeed, $\mathbf{W}[k]$ is closed on-shell when the variations of the fields satisfy the linearized equations of motion and k leaves invariant the background solution. This calculation can be found in Appendix D.1.

Let us proceed to derive the first law:

$$\begin{aligned}
0 &\doteq \int_{\Sigma^3} d\mathbf{W}[k] \\
&= \frac{1}{16\pi G_N^{(4)}} \int_{\Sigma^3} d\left\{-\delta \star (e^a \wedge e^b) P_{k\,ab} + \imath_k \star (e^a \wedge e^b) \wedge \delta\omega_{ab} - \imath_k \star d\phi\delta\phi\right\} \\
&\quad - \frac{\delta\vartheta}{16\pi G_N^{(4)}} \int_{\Sigma^3} dP_k \\
&\doteq \frac{1}{16\pi G_N^{(4)}} \left\{\int_{S^2_\infty} - \int_{\mathcal{BH}}\right\} \left\{-\delta \star (e^a \wedge e^b) P_{k\,ab} + \imath_k \star (e^a \wedge e^b) \wedge \delta\omega_{ab} - \imath_k \star d\phi\delta\phi\right\} \\
&\quad + \frac{\delta\vartheta}{16\pi G_N^{(4)}} \int_{\Sigma^3} \imath_k G \\
&\doteq \delta M + \tfrac{1}{4}\Sigma\delta\phi_\infty - T\delta S + \frac{\delta\vartheta}{16\pi G_N^{(4)}} \int_{\Sigma^3} \imath_k \star \frac{\partial V}{\partial\vartheta} \\
&= \delta M + \tfrac{1}{4}\Sigma\delta\phi_\infty - T\delta S - \Phi_\vartheta\delta\vartheta\,,
\end{aligned} \tag{4.89}$$

where we have defined

$$\Phi_\vartheta \equiv -\frac{1}{16\pi G_N^{(4)}} \int_{\Sigma^3} \imath_k \star \frac{\partial V}{\partial\vartheta}\,. \tag{4.90}$$

Observe that

$$\Phi_\vartheta\delta\vartheta = \Phi_\alpha\delta\alpha\,, \tag{4.91}$$

where Φ_α is the potential that occurs in the Smarr formula Eq. (4.50).

In the above derivation we have used Eq. (4.11) and the vanishing of k over the bifurcation sphere.

Thus, we arrive to our final expression for the first law

$$\delta M = T\delta S - \tfrac{1}{4}\Sigma\delta\phi_{\infty} + \Phi_{\alpha}\delta\alpha\,. \tag{4.92}$$

4.5 The Anabalón-Oliva hairy black hole

In this section we study, as an illustration of our general results, the asymptotically-flat Anabalón-Oliva (AO) hairy black hole constructed in Ref. [174] as a solution of the system we are considering Eq. (3.2) for the particular scalar potential[13]

$$\begin{aligned} V(\phi) = \frac{2\alpha}{\nu^2}\Bigg\{ & \frac{\nu-1}{\nu+2}\sinh\left[\sqrt{\frac{\nu+1}{\nu-1}}\phi\right] - \frac{\nu+1}{\nu-2}\sinh\left[\sqrt{\frac{\nu-1}{\nu+1}}\phi\right] \\ & +4\frac{\nu^2-1}{\nu^2-4}\sinh\left(\frac{1}{\sqrt{\nu^2-1}}\phi\right)\Bigg\}\,, \end{aligned} \tag{4.93}$$

where α is a constant with dimensions of inverse length squared and ν is a dimensionless parameter (which means that we are, actually, dealing with a family of theories) such that $\nu > 1$.

The metric and the scalar field of the AO solution are given in Ref. [174] in the form

$$ds^2 = \Omega(r)\left[F(r)dt^2 - \frac{dr^2}{F(r)} - d\Omega^2_{(2)}\right]\,, \tag{4.94a}$$

$$\phi = \sqrt{\nu^2-1}\,\ln\left(r/\eta\right)\,, \tag{4.94b}$$

where the functions Ω and F are given by

$$\Omega(r) = \frac{\nu^2\eta^{\nu-1}r^{\nu-1}}{\left(r^{\nu}-\eta^{\nu}\right)^2}\,, \tag{4.95a}$$

$$F(r) = \frac{r}{\eta\Omega(r)} + \alpha\left[\frac{1}{(\nu^2-4)} - \left(1 + \frac{\eta^{\nu}r^{-\nu}}{\nu-2} - \frac{\eta^{-\nu}r^{\nu}}{\nu+2}\right)\frac{r^2}{\eta^2\nu^2}\right]\,, \tag{4.95b}$$

but it is convenient to define a new coordinate ρ, related to r by

$$r = \eta W^{1/\nu}(\rho)\,, \quad \text{where} \quad W(\rho) \equiv \left(1 - \frac{\nu/\eta}{\rho}\right)\,, \tag{4.96}$$

which satisfies

[13] Apart from the change in the signature of the metric, we have set $\kappa = 1/2$ in the results of Ref. [174].

$$dr = \frac{d\rho}{\Omega(\rho)} \,. \tag{4.97}$$

Then, defining

$$\lambda \equiv F\Omega \,, \quad \text{and} \quad R^2 \equiv \Omega \,, \tag{4.98}$$

the metric and scalar field of the solution take the form

$$ds^2 = \lambda(\rho)dt^2 - \lambda^{-1}(\rho)d\rho^2 - R^2(\rho)d\Omega^2_{(2)} \,, \tag{4.99a}$$

$$\phi = \frac{\sqrt{\nu^2 - 1}}{\nu} \ln W \,, \tag{4.99b}$$

with the functions λ and R given by

$$R^2(\rho) = W^{1-1/\nu}\rho^2 \,, \tag{4.100a}$$

$$\lambda(\rho) = W^{1/\nu} + \alpha \left\{ \frac{W^{1-1/\nu}}{(\nu^2-4)} - \frac{W^{1+1/\nu}}{\nu^2} - \frac{W^{1/\nu}}{\nu^2(\nu-2)} + \frac{W^{2+1/\nu}}{\nu^2(\nu+2)} \right\} \rho^2 \,. \tag{4.100b}$$

In this form the metric is asymptotically flat with the standard normalization for $\rho \to \infty$: for large ρ

$$\lambda \sim 1 - \frac{3\eta^2 + \alpha}{3\eta^3 \rho} + \mathcal{O}(1/\rho^2) \,, \tag{4.101a}$$

$$R^2 \sim \rho^2 + \mathcal{O}(\rho) \,, \tag{4.101b}$$

$$\phi \sim -\frac{\sqrt{\nu^2 - 1}}{\eta\rho} + \mathcal{O}(1/\rho^2) \,, \tag{4.101c}$$

so that the ADM mass M, the scalar charge Σ and the asymptotic value of the scalar ϕ_∞ are given by

$$M = \frac{3\eta^2 + \alpha}{6\eta^3} \,, \tag{4.102a}$$

$$\Sigma = -\frac{\sqrt{\nu^2 - 1}}{\eta} \,, \tag{4.102b}$$

$$\phi_\infty = 0 \,. \tag{4.102c}$$

For this asymptotic value of the dilaton, the conditions Eqs. (4.28) are, indeed, satisfied.

Observe that, since α and ν are parameters of the theory, the solution contains only one free parameter, η, which determines the ADM mass. The scalar charge Σ can, then, be written in terms of the ADM mass and the parameters that define the theory. Therefore, it describes secondary hair.

It is not difficult to recover some known metrics: for $|\nu| = 1$, the "hairless limit"[14] the scalar vanishes identically and

$$R^2(\rho) = \rho^2\,, \tag{4.103a}$$

$$\lambda(\rho) = W + \frac{\alpha}{3}\,(W-1)^3\,\rho^2 = 1 - \left(\frac{1}{\eta} + \frac{\alpha}{3\eta^3}\right)\frac{1}{\rho}\,, \tag{4.103b}$$

which correspond to the Schwarzschild metric with

$$M = \frac{3\eta^2 + \alpha}{6\eta^3}\,. \tag{4.104}$$

When $\alpha = 0$ the potential vanishes and one recovers the Janis-Newman-Winicour solutions [128]

$$R^2(\rho) = W^{1-1/\nu}\rho^2\,, \tag{4.105a}$$

$$\lambda(\rho) = W^{1/\nu}\,, \tag{4.105b}$$

$$\phi = \frac{\sqrt{\nu^2-1}}{\nu}\ln W\,. \tag{4.105c}$$

The parameter ν does not occur in the action and it is just an integration constant related to the mass and scalar charge by

$$\nu = \frac{\sqrt{M^2 + \Sigma^2/4}}{M}\,, \tag{4.106}$$

and the metric is singular except when the scalar charge vanishes, ($\Sigma = 0$, $|\nu| = 1$)

$\lambda(\rho)$ vanishes for $\rho = \nu/\eta$, but so does $R^2(\rho)$ (except for $|\nu| = 1$, Schwarzschild), which indicates that there is a singularity there. It can be shown numerically that, for $\nu < 3$ λ has another zero at some $\rho_h > \nu/\eta$ which converges towards the singularity at $\rho = \rho_{sing} = \nu/\eta$. Finding an analytical expression for ρ_h in terms of α, ν and η is too complicated, but we can check the Smarr formula and the results concerning the scalar charge that we have obtained in the previous sections using the property $\lambda(\rho_h) = 0$ and $R^2(\rho_h) \neq 0$ only. In the calculations it is often convenient to use a coordinate[15]

[14] The potential diverges for $|\nu| = 1$ and the theory is not well defined for this value of the parameter.

[15] This coordinate x is different from the one defined in Ref. [174].

$$x \equiv \frac{\nu}{\eta\rho}\,, \qquad x_h \equiv \frac{\nu}{\eta\rho_h}\,. \tag{4.107}$$

Using these properties, we find the surface gravity

$$\begin{aligned}\kappa = &\frac{\left[-\alpha\rho_h^2 + \nu^2(\nu-2)\right] W^{-1+1/\nu}(\rho_h)}{2\eta\nu^2(\nu-2)\rho_h^2} + \frac{\alpha\rho_h W^{2+1/\nu}(\rho_h)}{\nu^2(\nu+2)} \\ &- \frac{\alpha\left[2\eta\left(\nu^2-4\right)\rho_h - 2\,\nu^2 + 3\,\nu + 2\right] W^{1+1/\nu}(\rho_h)}{2\eta\,\nu^2\left(\nu^2-4\right)} + \frac{\alpha\rho_h W^{1-1/\nu}(\rho_h)}{\left(\nu^2-4\right)} \\ &+ \frac{\alpha(\nu-1)W^{-1/\nu}(\rho_h)}{2\eta\left(\nu^2-4\right)} - \frac{\alpha\left(2\,\eta\,\rho_h + \nu^2 - \nu - 2\right) W^{1/\nu}(\rho_h)}{2\eta\nu^2\left(\nu-2\right)}\,,\end{aligned} \tag{4.108}$$

and the area of the horizon

$$A = 4\pi R^2(\rho_h) = 4\pi W^{1-1/\nu}(\rho_h)\rho_h^2\,. \tag{4.109}$$

We can, then, compute

$$M - 2ST = \frac{3\eta^2+\alpha}{6\eta^3} - R^2(\rho_h)\kappa\,, \tag{4.110}$$

which, according to the Smarr formula Eq. (4.50) should equal to

$$\frac{1}{8\pi G_N^{(4)}}\int_{\Sigma^3} \imath_k \star V = -\frac{1}{2G_N^{(4)}}\int_{\rho_h}^{\infty} R^2 V d\rho\,, \tag{4.111}$$

which is proportional to α. We have checked that the Smarr formula holds in this form.

We have also checked that Eq. (4.24) which follows from the coordinate-independent definition of scalar charge Eq. (4.9) holds.

Finally, checking the first law Eq. (4.92) in this solution is difficult. In order to test the term proportional to the scalar charge one must have a family of solutions in which the asymptotic value of the scalar is a free parameter, which is not the case here.

4.6 Discussion

In this paper we have derived a Smarr formula and a first law for the extended thermodynamics of the black-hole solutions of the theories described by the action Eq. (3.2) using Wald's formalism and the results of Refs. [56, 138]. Our results coincide with those of Ref. [176] except for the inclusion of the term proportional to the scalar charge and the variation of the asymptotic value of the scalar. This term is somewhat mysterious since there are no asymptotically flat black-hole solutions for asymptotic values of the scalar other than zero, but it may make sense in a wider class of not asymptotically-flat solutions. In any case, the term is clearly there since it arises exactly in the same way as in all the theories considered in Refs. [49, 59].

Using an extension of the covariant definition of scalar charge given in Refs. [49, 61] we have shown that, in the presence of a bifurcate horizon, the scalar charge is determined by the parameters of the theory, the particular scalar potential, and the value of the scalar on the horizon, and should be considered as "secondary hair". On the other hand, a well-defined scalar charge is possible in the presence of a bifurcate black-hole horizon only if the scalar potential satisfies a set of quite restrictive conditions given in Eq. (4.28), previously found in Ref. [175] following other considerations. These conditions are equivalent to a "no-hair theorem" for all the theories whose potentials do not satisfy them. Some scalar profiles such as those considered in Ref. [118] may evade these constraints, though.

Our results still leave some questions unanswered: are there hairy black-hole solutions in all the theories whose scalar potentials satisfy all the right conditions? What happens in theories with more scalar fields or in theories in which the scalars couple to curvature scalars? Clearly, much more work is necessary to find a general pattern of behaviour of scalar fields in black-hole spacetimes and some work in this direction is already in progress.

Part III

Conclusions and appendices

A
Conclusions

A.1 English version

The main objective of this thesis has been to understand the role played by fields and scalar charges in the thermodynamics of black holes.

For the first part, regions of stability were identified for both black hole solutions in configurations close to extremality. The use of the Brown-York formalism supplemented with counterterms allowed the entropy and the quantum-statistical relation to be established for both solutions. For the black hole with Gauss-Bonnet corrections, it was observed that higher-order terms stabilized the solution in a similar way to how the scalar field stabilized the black hole solution with secondary hair in an asymptotically flat spacetime.

The definition of a scalar charge for theories with and without global symmetries proved crucial for an extended version of Wald's formalism, which explicitly includes scalar charges in the first law of thermodynamics. This definition confirmed that the scalar charge must be secondary hair and that the restrictions on the scalar potential are consistent with the no-hair theorems.

The results obtained in both parts of the work allow for a complementary understanding of the role of scalars in the thermodynamics of black holes. While the first part explores thermodynamic stability in specific configurations and demonstrates the viability of stable regions in theories with and without additional corrections, the second part establishes a formal framework for incorporating scalar charges into thermodynamic laws. Together, they provide a broader perspective on how the thermodynamic relations of black holes depend on scalar fields.

All these works opens several lines of research for the future, as the role of scalar fields is not still clear. One of these is to extend the analysis of thermodynamic stability to theories with massive scalar fields and to explore a definition of scalar charge that allows for conditions of no-hair in these theories. It is also possible to investigate how the restrictions on the scalar potential affect the existence of hairy black hole solutions in asymptotically Anti-de Sitter spacetimes. With the methodology for constructing charges presented in this thesis, we can explore the thermodynamics of Taub-NUT spacetimes by defining the NUT charge and understanding how it is incorporated into thermodynamic relations, for a physical interpretation of the NUT charge within Wald's formalism. Finally, we could extend Wald's formalism to extremal black holes using plausible integration limits when defining the charges.

A.2 Spanish version

El objetivo principal de esta tesis ha sido el de comprender el rol que juegan los campos y las cargas escalares en la termodinámica de agujeros negros.

Para la primera parte, se identificaron regiones de estabilidad para ambas soluciones de agujero negro en configuraciones cercanas a la extremalidad. El uso del formalismo de Brown-York suplementado con contratérminos permitió encontrar la entropía y la relación estadístico-cuántica para ambas soluciones. Para el agujero negro con correcciones de Gauss-Bonnet, se observó que los términos de orden superior estabilizan la solución de manera similar a cómo el campo escalar estabiliza la solución del agujero negro con pelo asintóticamente plano.

La definición de una carga escalar para teorías con y sin simetrías globales resultó clave para una versión extendida del formalismo de Wald, que incluyera explícitamente las cargas escalares en la primera ley de la termodinámica. Esta definición confirmó que la carga escalar debe ser pelo secundario y que las restricciones del potencial escalar son consistentes con los teoremas de no-pelo.

Los resultados obtenidos en ambas partes del trabajo permiten comprender de manera complementaria el rol de los escalares en la termodinámica de los agujeros negros. Mientras que la primera parte explora la estabilidad termodinámica en configuraciones específicas y demuestra la viabilidad de regiones estables en teorías con y sin correcciones adicionales, la segunda parte establece un marco formal para incorporar las cargas escalares en las leyes termodinámicas. En conjunto, permiten una visión más amplia de cómo las relaciones termodinámicas de los agujeros negros dependen de los escalares.

Este trabajo abre varias líneas de investigación para el futuro, pues el rol de los escalares aún no está claro. Una de ellas es extender el análisis de la estabilidad termodinámica a teorías con campos escalares masivos y explorar una definición de carga escalar que permita obtener condiciones de no-pelo en estas teorías. También se puede investigar cómo las restricciones sobre el potencial escalar afectan a la existencia de soluciones de agujeros negros con pelo asintóticamente Anti-de Sitter. Con la metodología para la construcción de cargas expuestas en esta tesis, podemos explorar la termodinámica de los espaciotiempos Taub-NUT definiendo la carga NUT y entendiendo cómo se incorpora en las relaciones termodinámicas, para una interpretación física de la carga de NUT en el formalismo de Wald. Por último, podríamos extender el formalismo de Wald para agujeros negros extremos utilizando límites de integración pausibles al integrar las cargas definidas.

B

Charges, symmetries and conservation laws

B.1 Noether-Wald charge for Einstein-Maxwell theory (p-forms)

Consider the generalization of the d-dimensional Einstein-Maxwell theory given by,

$$S[e^a, A] = \frac{1}{16\pi G_N^{(d)}} \int \left[(-1)^{d-1} \star (e^a \wedge e^b) \wedge R_{ab} + \frac{(-1)^{d(p-1)}}{2} F \wedge \star F \right] \equiv \mathbf{L}. \tag{B.1}$$

where the $(p+1)$-form A satisfies $F = dA$, with F a $(p+2)$-form. Under a general variation of the fields, omitting at the moment $(16\pi G_N{}^{(d)})^{-1}$

$$\delta S = \int \left\{ \mathbf{E}_a \wedge \delta e^a + \mathbf{E} \wedge \delta A + d\mathbf{\Theta}(e^a, A, \delta e^a, \delta A) \right\}, \tag{B.2}$$

where $\mathbf{E}_a$ are the Einstein equations of motion, $\mathbf{E}$ are the Maxwell equations of motion and $\mathbf{\Theta}(e^a, A, \delta e^a, \delta A)$ is the symplectic prepotential $(d-1)$-form. These are given by

$$\mathbf{E}_a = \imath_a \star (e^c \wedge e^d) \wedge R_{cd} + \frac{(-1)^{dp}}{2} \left(\imath_a F \wedge G + (-1)^{p+1} F \wedge \imath_a G \right) \tag{B.3}$$

$$\mathbf{E} = -dG, \quad G = \star F \tag{B.4}$$

$$\mathbf{\Theta}(e^a, A, \delta e^a, \delta A) = - \star (e^a \wedge e^b) \wedge \delta\omega_{ab} + G \wedge \delta A. \tag{B.5}$$

The action in Eq.(B.1) is invariant under infinitesimal transformations up to a total derivative,

$$\delta_\xi S = - \int d\imath_\xi \mathbf{L}. \tag{B.6}$$

Applying the Lie derivative plus the compensating gauge transformation to the general variation of the fields,

$$\delta_\xi = -\mathcal{L}_\xi + \delta_{\Lambda_\xi}, \tag{B.7}$$

and note that,

$$\delta_\xi \omega^{ab} = - (d\imath_\xi + \imath_\xi d)\, \omega^{ab} + D\sigma_\xi^{ab} = - \left(DP_\xi^{ab} + \imath_\xi R^{ab} \right). \tag{B.8}$$

$$\delta_\xi A = -\mathcal{L}_\xi A + \delta_{\chi_\xi} A = - (d\imath_\xi + \imath_\xi d)\, A + d\chi_\xi = - (dP_\xi + \imath_\xi F), \tag{B.9}$$

$$\delta_\xi e^a = -\left(d\imath_\xi + \imath_\xi d\right) e^a + \sigma_{\xi}{}^{a}{}_{b}\, e^b = D\xi^a + P_{\xi}{}^{a}{}_{b}\, e^b. \tag{B.10}$$

Replacing in Eq.(B.2) with Eq.(B.7)

$$\delta_\xi S = \int \left\{ -\mathbf{E}_a \wedge \left(D\xi^a + P_{\xi}{}^{a}{}_{b}\, e^b \right) - \mathbf{E} \wedge \left(dP_\xi + \imath_\xi F \right) + d\mathbf{\Theta}(e^a, A, \delta_\xi e^a, \delta_\xi A) \right\}. \tag{B.11}$$

$$\mathbf{\Theta}(e^a, A, \delta_\xi e^a, \delta_\xi A) = \star(e^a \wedge e^b) \wedge \left(DP_\xi^{ab} + \imath_\xi R^{ab} \right) + G \wedge \left(dP_\xi + \imath_\xi F \right). \tag{B.12}$$

We find,

$$\delta_\xi S = \int \left\{ (-1)^{d-1} \mathbf{E}_a \xi^a - \mathbf{E} \wedge \imath_\xi F + (-1)^{d-p-1} d\mathbf{E} \wedge P_\xi + d\mathbf{\Theta}'(e^a, A, \delta_\xi e^a, \delta_\xi A) \right\}, \tag{B.13}$$

where we defined

$$\mathbf{\Theta}'(e^a, A, \delta_\xi e^a, \delta_\xi A) = \mathbf{\Theta}(e^a, A, \delta_\xi e^a, \delta_\xi A) + (-1)^d \mathbf{E}_a \xi^a + (-1)^{d-p} \mathbf{E} \wedge P_\xi. \tag{B.14}$$

Using the Noether identities associated to gauge transformations $d\mathbf{E} = 0$ and diffeomorphisms, we arrive at

$$D\mathbf{E}_a \xi^a + (-1)^d \mathbf{E} \wedge \imath_\xi F = 0, \tag{B.15}$$

then

$$\delta_\xi S = \int d\mathbf{\Theta}'(e^a, A, \delta_\xi e^a, \delta_\xi A), \tag{B.16}$$

which, combined with the total derivative, leads to

$$d\mathbf{J}[\xi] = 0, \quad \mathbf{J}[\xi] \equiv \mathbf{\Theta}'(e^a, A, \delta_\xi e^a, \delta_\xi A) + \imath_\xi \mathbf{L}. \tag{B.17}$$

As usual, this implies the local existence of the $(d-2)$-form $\mathbf{Q}[\xi]$ such that

$$\mathbf{J}[\xi] = d\mathbf{Q}[\xi], \tag{B.18}$$

$$\mathbf{Q}[\xi] = \frac{1}{16\pi G_N^{(d)}} \left\{ (-1)^{d-1} \star (e^a \wedge e^b) P_\xi^{ab} - (-1)^{d(p-1)} P_\xi \wedge G \right\}. \tag{B.19}$$

which is the Noether-Wald charge for the Einstein-Maxwell theory.

C

Toolkit for the existence of thermodynamically stable asymptotically flat black holes

C.1 Local stability conditions

The local stability of equilibrium configurations follows from demanding that heat capacity $C_Q \equiv T(\partial S/\partial T)_Q$ and isothermal permittivity $\epsilon_T \equiv (\partial Q/\partial\Phi)_T$ to be simultaneously positive defined. The proof follows from considering that at stable equilibrium, the entropy has a maximum value with respect to the entropy of the system when considering the fluctuations. We can use the argument (see, for instance, [51]) that, if $S(E,Q)$ represents the entropy before some fluctuations in E and Q, given by δE and δQ, respectively, then, for the configuration $S(E,Q)$ to be stable, the average entropy $\frac{1}{2}\left[S(E+\delta E, Q+\delta Q)+S(E-\delta E, Q-\delta Q)\right]$ after the perturbation cannot be greater than the initial one. The same argument is valid in terms of the energy: for a given state $E=E(S,Q)$ to be locally stable, the average energy after the perturbation can not be lower than the initial one. This gives rise to three mathematical conditions for local stability, namely,

$$\left(\frac{\partial^2 E}{\partial S^2}\right)_Q\left(\frac{\partial^2 E}{\partial Q^2}\right)_S-\left[\left(\frac{\partial}{\partial S}\right)_Q\left(\frac{\partial E}{\partial Q}\right)_S\right]^2>0\,,\qquad \left(\frac{\partial^2 E}{\partial S^2}\right)_Q>0\,,\qquad \left(\frac{\partial^2 E}{\partial Q^2}\right)_S>0 \tag{C.1}$$

Consider the canonical ensemble, for which the thermodynamic potential is $\mathcal{F}=E-TS$. For infinitesimal changes in its thermodynamics, we can describe the system by $d\mathcal{F}(T,Q)=-SdT+\Phi dQ$, and it can be shown, using the standard thermodynamic relations, that the conditions (C.1) can be written in terms of $\mathcal{F}$ and, indeed, can be reduced to only two independent requirements:

$$C_Q=T\left(\frac{\partial S}{\partial T}\right)_Q=-T\left(\frac{\partial^2\mathcal{F}}{\partial T^2}\right)_Q>0 \tag{C.2}$$

$$\epsilon_T=\left(\frac{\partial Q}{\partial\Phi}\right)_T=\left(\frac{\partial^2\mathcal{F}}{\partial Q^2}\right)_T^{-1}>0 \tag{C.3}$$

where we have used $S=-\left(\partial\mathcal{F}/\partial T\right)_Q$ and $\Phi=\left(\partial\mathcal{F}/\partial Q\right)_T$.

In the grand canonical ensemble, it can be shown that stability also follows from analyzing the concavity of the thermodynamic potential $\mathcal{G}$. In this case, $\mathcal{G}=E-TS-\Phi Q$, the first law takes the form $d\mathcal{G}(T,\Phi)=-SdT-Qd\Phi$, and the relevant response function for local stability in this case is

$$C_\Phi \equiv T\left(\frac{\partial S}{\partial T}\right)_\Phi = -T\left(\frac{\partial^2 \mathcal{G}}{\partial T^2}\right)_\Phi > 0 \tag{C.4}$$

where $S = -\left(\partial \mathcal{G}/\partial T\right)_\Phi$.

C.2 Dilaton counterterms in AdS vs flat spacetime

Let us consider the Einstein's equation derived from Einstein-Maxwell-scalar gravity (2.1), but now with the general potential (see, e.g., [182] where the extremal limit was also discussed)

$$V(\phi) = \frac{2\Lambda}{3}(2+\cosh\phi) + 2\Upsilon\left(2\phi + \phi\cosh\phi - 3\sinh\phi\right) \tag{C.5}$$

that contains the comological constant Λ. We now consider the asymptotically AdS solution, for which the limit $\Lambda \to 0$ leads to the asymptotically flat solution,

$$ds^2 = \Omega(x)\left[-f(x)dt^2 + \frac{\eta^2 dx^2}{x^2 f(x)} + d\theta^2 + \sin^2\theta d\phi^2\right], \quad A_\mu = \left[\Phi - \frac{q(x-1)}{x}\right]\delta^t_\mu \tag{C.6}$$

where

$$\Omega(x) = \frac{x}{\eta^2(x-1)^2}, \qquad f(x) = -\frac{\Lambda}{3} + \Upsilon\left(\frac{x^2-1}{2x} - \ln x\right) + \frac{\eta^2(x-1)^2}{x}\left[1 - \frac{2(x-1)q^2}{x}\right] \tag{C.7}$$

By taking the trace of the Einstein's equation, we can use the relation $R = \frac{1}{2}(\partial\phi)^2 + 2V$ to simplify the bulk part of the action that yields

$$I_{bulk} = \frac{1}{2\kappa}\int_{\mathcal{M}} d^4x\sqrt{-g}\left[R - \frac{1}{2}(\partial\phi)^2 - e^\phi F^2 - V(\phi)\right] = \frac{1}{2\kappa}\int_{\mathcal{M}} d^4x\sqrt{-g}\left[V(\phi) - e^\phi F^2\right] \tag{C.8}$$

The on-shell Euclidean action is

$$I^E_{bulk} = \frac{1}{4}\beta\int_{x_+}^{x_b} dx\left[\frac{2\eta\Omega(x)}{x} - \frac{(xf(x)\Omega'(x))'}{\eta}\right] \tag{C.9}$$

Together, the bulk part and Gibbons-Hawking boundary term add up to

$$I^E_{bulk} + I^E_{GH} = \frac{\beta\left[(12\eta^2q^2 - \Upsilon)(x_+-1) - 6\eta^2(x_+-3)\right]}{24\eta^3(x_+-1)} - \frac{\beta\left(6\eta^2-\Lambda\right)}{6\eta^3(x_b-1)} + \frac{\beta\Lambda}{2\eta^3(x_b-1)^2} + \frac{\beta\Lambda}{3\eta^3(x_b-1)^3} \tag{C.10}$$

where $x_b \to 1$ is the boundary location. We note that there are three divergent terms proportional with Λ that are not going to survive for the asymptotically flat solution when the cosmological constant vanishes. The gravitational counterterm required to remove the divergences is specific to the flat and AdS spacetimes. For asymptotically flat spacetime, the gravitational counterterm is

$$I^E_{ct(flat)} = \frac{1}{\kappa}\int_{\partial\mathcal{M}} d^3x\sqrt{|h|}\sqrt{2\mathcal{R}^{(3)}} = \frac{\beta}{\eta(x_b-1)} - \frac{\beta\left(12\eta^2q^2 - 6\eta^2 - \Upsilon\right)}{12\eta^3} + \mathcal{O}(x_b-1) \tag{C.11}$$

It is clear that the counterterm for asymptotically flat spacetime perfectly cancels the only divergence coming from $I_{bulk} + I_{GH}$ when $\Lambda = 0$. The total action for flat spacetime satisfies the quantum-statistical relation $\beta^{-1} I^E = E - TS - \Phi Q$, as shown in Section 2.1.

Now, the gravitational counterterm for asymptotically AdS spacetime is

$$I^E_{ct,(AdS)} = \frac{1}{\kappa}\int_{\partial\mathcal{M}} d^3x\sqrt{h}\left[\frac{2}{\ell} + \frac{\ell}{2}\mathcal{R}^{(3)}\right] \tag{C.12}$$

$$= \frac{\beta\left(\Lambda + 4\Upsilon + 24\eta^2 - 48\eta^2 q^2\right)}{48\eta^3} + \frac{\beta(8\eta^2 - \Lambda)}{8\eta^3(x_b - 1)} - \frac{\beta\Lambda}{2\eta^3(x_b-1)^2} - \frac{\beta\Lambda}{3\eta^3(x_b-1)^3} \tag{C.13}$$

where $\Lambda = -3/\ell^2$. For the AdS case, the divergence coming from $I_{bulk} + I_{GH}$ are not completely cancelled by (C.13), and one of the terms $\propto (x_b - 1)^{-1}$ survives. Concretely,

$$I^E_{bulk} + I^E_{GH} + I^E_{ct,(AdS)} = \frac{\beta\left[12\eta^2(x_+ + 1) + (\Lambda + 2\Upsilon - 24\eta^2 q^2)(x_+ - 1)\right]}{48\eta^3(x_+ - 1)} + \frac{\beta\Lambda}{24\eta^3(x_b - 1)} \tag{C.14}$$

and the remaining divergent term is cancelled by the contribution from the scalar field counterterm [183, 184] (consistent with the Hamiltonian formalism [185, 186])

$$I^E_\phi = \frac{1}{2\kappa}\int_{\partial\mathcal{M}} d^3x\sqrt{h}\left[\frac{\phi^2}{2\ell} + \frac{W(\phi)}{\ell A}\phi^3\right] = -\frac{\beta\Lambda}{24\eta^3(x_b-1)} - \frac{\beta\Lambda}{48\eta^3} \tag{C.15}$$

where $\phi(r) = A/r + B/r^2 + \cdots$, $r = \sqrt{\Omega(x)}$, and $B = dW(A)/dB$ (in this case, $B = 0$ and $W = 0$). The total action also satisfies the quantum-statistical relation, $\beta^{-1} I^E = E - TS - \Phi Q$, where $I^E = I^E_{bulk} + I^E_{GH} + I^E_{ct,(AdS)} + I^E_\phi$.

C.3 ADM mass in $D = 5$ dimensions

Let us employ the ADM formalism, which is as follows: first, consider a generic equation of motion for GB gravity

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \frac{1}{4}\alpha H_{\mu\nu} = \kappa T_{\mu\nu} \tag{C.16}$$

where $T_{\mu\nu}$ is a generic energy-momentum tensor. It is convenient to rewrite it as

$$R_{\mu\nu} + \frac{1}{4}\alpha\left(H_{\mu\nu} - \frac{1}{3}Hg_{\mu\nu}\right) = \kappa\bar{T}_{\mu\nu} \tag{C.17}$$

where $H \equiv g^{\mu\nu}H_{\mu\nu}$ and $\bar{T}_{\mu\nu} = T_{\mu\nu} - \frac{1}{3}Tg_{\mu\nu}$, $T = g^{\mu\nu}T_{\mu\nu}$. Now, we expand the left hand side for a small perturbation, $g_{\mu\nu} = \mathring{g}_{\mu\nu} + h_{\mu\nu}$, where $|h_{\mu\nu}| \ll 1$ is the perturbation around a background spacetime, $\mathring{g}_{\mu\nu}$. The left hand side of (C.17) expands as $-\frac{1}{2}\Box h_{\mu\nu} + \mathcal{O}(h^2)$ when the harmonic gauge condition is considered,

$$\mathring{\nabla}_\mu\left(h^{\mu\nu} - \frac{1}{2}\mathring{g}^{\mu\nu}h\right) = 0 \tag{C.18}$$

where $h = \mathring{g}^{\mu\nu}h_{\mu\nu}$. Since $|h_{\mu\nu}| \ll 1$, we consider the non-relativistic limit, under which $\Box h_{\mu\nu} \approx \nabla^2 h_{\mu\nu}$, $T \approx -T_{00}$. In this limit, we have the Poisson equation

$$\nabla^2 h_{\mu\nu} = -2\kappa\bar{T}_{\mu\nu} \tag{C.19}$$

The solution to (C.19) can be expressed as

$$h_{\mu\nu}(x^i) = \frac{\kappa}{A}\int \frac{\bar{T}_{\mu\nu}(y^i)d^4y}{|x-y|^2} \tag{C.20}$$

where $A = \int_0^\pi \sin^2\theta d\theta \int_0^\pi \sin\phi d\phi \int_0^{2\pi} d\psi = 2\pi^2$ is the area of the unit 3-sphere. We obtain

$$\nabla_x^2 h_{\mu\nu}(x^i) = \frac{\kappa}{A}\int \bar{T}_{\mu\nu}(y^i)d^4y\nabla_x^2\left(\frac{1}{|x-y|^2}\right) \tag{C.21}$$

Notice that

$$\int_V \nabla^2\left(\frac{1}{r^2}\right)dV = \oint_S \nabla\left(\frac{1}{r^2}\right)dS = -\frac{2}{r^3}\oint_S dS = -2A \tag{C.22}$$

where $\oint_S dS = Ar^3$ is the area of the 3-sphere of radius r. Therefore,

$$\nabla_x^2\left(\frac{1}{|x-y|^2}\right) = -2A\delta^4(x-y) \quad \to \quad \nabla_x^2 h_{\mu\nu}(x^i) = -2\kappa\int \bar{T}_{\mu\nu}(y^i)\delta^4(x-y)d^4y = -2\kappa\bar{T}_{\mu\nu}(x^i) \tag{C.23}$$

that is consistent with (C.20).

Now, to get the ADM mass, M_{ADM}, one must identify $M_{ADM} = \int T_{00}d^4x$. By expanding h_{00} from (C.20) for the asymptotic region, $|x| \gg |y|$, we obtain

$$h_{00}(x^i) = \frac{\kappa}{A}\left[\frac{1}{r^2}\int \bar{T}_{00}d^4y + \mathcal{O}(r^{-4})\right] \tag{C.24}$$

$$= \frac{\kappa}{A}\left[\frac{1}{r^2}\int\left(T_{00} - \frac{1}{3}g_{00}T\right)d^4y + \mathcal{O}(r^{-4})\right] \tag{C.25}$$

$$= \frac{\kappa}{A}\left[\frac{2}{3r^2}\int T_{00}d^4y + \mathcal{O}(r^{-4})\right] \tag{C.26}$$

$$= \frac{2\kappa M_{ADM}}{3Ar^2} + \mathcal{O}(r^{-4}) \tag{C.27}$$

Therefore, the ADM mass can be directly read by obtaining h_{00} from expanding g_{00},

$$M_{ADM} = \frac{3}{8}\pi r^2 h_{00} \tag{C.28}$$

where we have replaced $\kappa = 8\pi$ and $A = 2\pi^2$. For the GB metric (2.30), we asymptotically expand g_{tt} to read h_{00} around the flat spacetime,

$$g_{00} = -1 + \frac{\mu}{r^2} + \mathcal{O}(r^{-4}), \qquad h_{00} = \frac{\mu}{r^2} \tag{C.29}$$

Therefore, only for $j = 1$ we consistently obtain

$$M_{ADM} = E_{quasi} = \frac{3}{8}\pi\mu \tag{C.30}$$

and so (2.34) with $j = 1$ is the correct counterterm that is going to regularize the Euclidean action for any solution with the same boundary conditions.

D

Proofs

D.1 Proving that $d\mathbf{W}[k] \doteq 0$

In this appendix we give a detailed proof of the on-shell closedness of the 2-form charge given in Eq.(4.88). We the sequence of steps in which we use basic geometric properties, equations of motion and symmetry properties is almost self-descriptive and we will not comment upon them in order not to extend too much this appendix.

$$
\begin{aligned}
d\mathbf{W}[k] = & -\mathcal{D}\delta\left[\star(e^a\wedge e^b)\right]P_{k\,ab} - \delta\left[\star(e^a\wedge e^b)\right]\wedge\mathcal{D}P_{k\,ab} + \mathcal{D}\left[\imath_k\star(e^a\wedge e^b)\right]\wedge\delta\omega_{ab} \\
& - \imath_k\star(e^a\wedge e^b)\wedge\mathcal{D}\delta\omega_{ab} - dP_k\delta\vartheta - d\imath_k\star d\phi\delta\phi - \imath_k\star d\phi\wedge\delta d\phi \\
= & -\mathcal{D}\delta\left[\star(e^a\wedge e^b)\right]P_{k\,ab} + \delta\left[\star(e^a\wedge e^b)\right]\wedge\imath_k R_{ab} + \mathcal{D}\left[\imath_k\star(e^a\wedge e^b)\right]\wedge\delta\omega_{ab} \\
& - \imath_k\star(e^a\wedge e^b)\wedge\delta R_{ab} + \imath_k G\delta\vartheta + \imath_k d\star d\phi\delta\phi - \imath_k\star d\phi\wedge\delta d\phi \\
= & -\delta d\left[\star(e^a\wedge e^b)\right]P_{k\,ab} + 2\omega^a{}_c\delta\left[\star(e^c\wedge e^b)\right]P_{k\,ab} \\
& + \delta\left[\star(e^a\wedge e^b)\right]\wedge\imath_k R_{ab} - 2P_k{}^a{}_c\star(e^c\wedge e^b)\wedge\delta\omega_{ab} \\
& - \imath_k\star(e^a\wedge e^b)\wedge\delta R_{ab} + \imath_k\star\frac{\partial V}{\partial\vartheta}\delta\vartheta + \imath_k\star\frac{\partial V}{\partial\phi}\delta\phi - \imath_k\star d\phi\wedge\delta d\phi
\end{aligned}
\tag{D.1}
$$

$$
\begin{aligned}
d\mathbf{W}[k] = &-2\delta\left[\omega^a{}_c \wedge \star(e^c \wedge e^b)\right] P_{k\,ab} + 2\omega^a{}_c\delta\left[\star(e^c \wedge e^b)\right] P_{k\,ab} \\
&+ \delta\left[\star(e^a \wedge e^b)\right] \wedge \imath_k R_{ab} - 2P_k{}^a{}_c \star (e^c \wedge e^b) \wedge \delta\omega_{ab} \\
&- \imath_k \star (e^a \wedge e^b) \wedge \delta R_{ab} + \imath_k \star \frac{\partial V}{\partial \vartheta}\delta\vartheta + \imath_k \star \frac{\partial V}{\partial \phi}\delta\phi - \imath_k \star d\phi \wedge \delta d\phi \qquad \text{(D.2)} \\
= &-\delta\left\{\imath_k \star (e^a \wedge e^b) \wedge R_{ab}\right\} + \imath_k\left[\delta \star (e^a \wedge e^b) \wedge R_{ab}\right] \\
&+ \imath_k \star \frac{\partial V}{\partial \vartheta}\delta\vartheta + \imath_k \star \frac{\partial V}{\partial \phi}\delta\phi - \imath_k \star d\phi \wedge \delta d\phi
\end{aligned}
$$

$$
\begin{aligned}
d\mathbf{W}[k] = &-\delta\left\{-\tfrac{1}{2}\imath_k \star d\phi \wedge d\phi + \imath_k \star V\right\} + \imath_k\left[\delta \star (e^a \wedge e^b) \wedge R_{ab}\right] \\
&+ \imath_k \star \frac{\partial V}{\partial \vartheta}\delta\vartheta + \imath_k \star \frac{\partial V}{\partial \phi}\delta\phi - \imath_k \star d\phi \wedge \delta d\phi \\
= &\tfrac{1}{2}\imath_k\delta \star d\phi \wedge d\phi + \tfrac{1}{2}\imath_k \star d\phi \wedge \delta d\phi - \imath_k\delta \star V + \imath_k\left[-\imath_c \star (e^a \wedge e^b) \wedge R_{ab} \wedge \delta e^c\right] \\
&+ \imath_k \star \frac{\partial V}{\partial \vartheta}\delta\vartheta + \imath_k \star \frac{\partial V}{\partial \phi}\delta\phi - \imath_k \star d\phi \wedge \delta d\phi\,, \qquad \text{(D.3)}
\end{aligned}
$$

$$
\begin{aligned}
d\mathbf{W}[k] &= \tfrac{1}{2}\imath_k\delta\star d\phi\wedge d\phi + \tfrac{1}{2}\imath_k\star d\phi\wedge\delta d\phi - \imath_k\delta\star V \\
&\quad + \imath_k\left\{\left[\tfrac{1}{2}\imath_a\star d\phi\wedge d\phi + \tfrac{1}{2}\star d\phi\imath_a d\phi - \imath_a\star V\right]\wedge\delta e^a\right\} \\
&\quad + \imath_k\star\frac{\partial V}{\partial\vartheta}\delta\vartheta + \imath_k\star\frac{\partial V}{\partial\phi}\delta\phi - \imath_k\star d\phi\wedge\delta d\phi \\
&= \tfrac{1}{2}\imath_k\delta\star d\phi\wedge d\phi - \tfrac{1}{2}\imath_k\star d\phi\wedge\delta d\phi - \imath_k\delta\star V \\
&\quad + \imath_k\left\{\left[\tfrac{1}{2}\imath_a\star d\phi\wedge d\phi + \tfrac{1}{2}\star d\phi\imath_a d\phi\right]\wedge\delta e^a\right\} \\
&\quad \imath_k\left\{-\imath_c\star V\wedge\delta e^c + \star\frac{\partial V}{\partial\vartheta}\delta\vartheta + \imath_k\star\frac{\partial V}{\partial\phi}\delta\phi\right\} \\
&= \tfrac{1}{2}\imath_k\delta\star d\phi\wedge d\phi - \tfrac{1}{2}\imath_k\star d\phi\wedge\delta d\phi - \imath_k\delta\star V \\
&\quad + \imath_k\left\{\left[\tfrac{1}{2}\imath_a\star d\phi\wedge d\phi + \tfrac{1}{2}\star d\phi\imath_a d\phi\right]\wedge\delta e^a\right\} \\
&\quad + \imath_k\delta\star V \\
&= \tfrac{1}{2}\imath_k\left[(-\imath_a\star d\phi\wedge d\phi - \star d\phi\imath_a d\phi)\wedge\delta e^a + \star d\delta\phi\wedge d\phi\right] - \tfrac{1}{2}\imath_k\star d\phi\wedge\delta d\phi \\
&\quad + \imath_k\left\{\left[\tfrac{1}{2}\imath_a\star d\phi\wedge d\phi + \tfrac{1}{2}\star d\phi\imath_a d\phi\right]\wedge\delta e^a\right\} \\
&= 0\,.
\end{aligned}
\tag{D.4}
$$

Bibliography

[1] I. Newton, *Philosophiæ Naturalis Principia Mathematica*. England, 1687.

[2] J. Michell, *On the Means of Discovering the Distance, Magnitude, &c. of the Fixed Stars, in Consequence of the Diminution of the Velocity of Their Light, in Case Such a Diminution Should be Found to Take Place in any of Them, and Such Other Data Should be Procured from Observations, as Would be Farther Necessary for That Purpose.*, *Phil. Trans. Roy. Soc. Lond.* **74** (1784) 35–57.

[3] R. S. Westfall, *Never at Rest: A Biography of Isaac Newton*. Cambridge University Press, Cambridge, UK, 1983.

[4] A. Einstein, *The Field Equations of Gravitation*, *Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.)* **1915** (1915) 844–847.

[5] S. G. Turyshev, *Experimental Tests of General Relativity*, *Ann. Rev. Nucl. Part. Sci.* **58** (2008) 207–248, [0806.1731].

[6] J. Droste, *The field of a single centre in Einstein's theory of gravitation and the motion of a particle in that field*, *KNAW, Proceedings* **19 I** (1917) 197–215.

[7] K. Schwarzschild, *On the gravitational field of a mass point according to Einstein's theory*, *Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.)* **1916** (1916) 189–196, [physics/9905030].

[8] G. Lemaitre, *The expanding universe*, *Annales Soc. Sci. Bruxelles A* **53** (1933) 51–85.

[9] J. R. Oppenheimer and H. Snyder, *On Continued gravitational contraction*, *Phys. Rev.* **56** (1939) 455–459.

[10] R. Penrose, *Gravitational collapse and space-time singularities*, *Phys. Rev. Lett.* **14** (1965) 57–59.

[11] S. W. Hawking and R. Penrose, *The Singularities of gravitational collapse and cosmology*, *Proc. Roy. Soc. Lond. A* **314** (1970) 529–548.

[12] A. Einstein, *Approximative Integration of the Field Equations of Gravitation*, *Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.)* **1916** (1916) 688–696.

[13] A. Einstein and N. Rosen, *On Gravitational waves*, *J. Franklin Inst.* **223** (1937) 43–54.

[14] LIGO Scientific, Virgo collaboration, B. P. Abbott et al., *Observation of Gravitational Waves from a Binary Black Hole Merger*, *Phys. Rev. Lett.* **116** (2016) 061102, [1602.03837].

[15] Event Horizon Telescope collaboration, K. Akiyama et al., *First Sagittarius A* Event Horizon Telescope Results. I. The Shadow of the Supermassive Black Hole in the Center of the Milky Way*, *Astrophys. J. Lett.* **930** (2022) L12, [2311.08680].

[16] R. P. Kerr, *Gravitational field of a spinning mass as an example of algebraically special metrics*, *Phys. Rev. Lett.* **11** (1963) 237–238.

[17] H. Reissner, *Über die Eigengravitation des elektrischen Feldes nach der Einsteinschen Theorie*, *Annalen Phys.* **355** (1916) 106–120.

[18] G. Nordström, *On the Energy of the Gravitation field in Einstein's Theory*, *Koninklijke Nederlandse Akademie van Wetenschappen Proceedings Series B Physical Sciences* **20** (Jan., 1918) 1238–1245.

[19] E. T. Newman, R. Couch, K. Chinnapared, A. Exton, A. Prakash and R. Torrence, *Metric of a Rotating, Charged Mass*, *J. Math. Phys.* **6** (1965) 918–919.

[20] I. D. Novikov and V. P. Frolov, *PHYSICS OF BLACK HOLES*. Kluwer Academic, Dordrecht, Netherlands, 1989, 10.1007/978-94-017-2651-1.

[21] V. P. Frolov and I. D. Novikov, eds., *Black hole physics: Basic concepts and new developments*. 1998, 10.1007/978-94-011-5139-9.

[22] E. Poisson, *A Relativist's Toolkit: The Mathematics of Black-Hole Mechanics*. Cambridge University Press, 12, 2009, 10.1017/CBO9780511606601.

[23] D. Grumiller and M. M. Sheikh-Jabbari, *Black Hole Physics: From Collapse to Evaporation*. Grad.Texts Math. Springer, 11, 2022, 10.1007/978-3-031-10343-8.

[24] S. Chandrasekhar, *The mathematical theory of black holes*. 1985.

[25] R. M. Wald, *General Relativity*. Chicago Univ. Pr., Chicago, USA, 1984, 10.7208/chicago/9780226870373.001.0001.

[26] S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space-Time*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 2, 2023, 10.1017/9781009253161.

[27] P. S. Joshi, *Global aspects in gravitation and cosmology*. 1987.

[28] M. D. Kruskal, *Maximal extension of Schwarzschild metric*, *Phys. Rev.* **119** (1960) 1743–1745.

[29] S. W. Hawking, *Black holes in general relativity*, *Commun. Math. Phys.* **25** (1972) 152–166.

[30] D. Christodoulou, *Reversible and irreversible transforations in black hole physics*, *Phys. Rev. Lett.* **25** (1970) 1596–1597.

[31] B. Carter, *Axisymmetric Black Hole Has Only Two Degrees of Freedom*, *Phys. Rev. Lett.* **26** (1971) 331–333.

[32] S. W. Hawking, *Gravitational radiation from colliding black holes*, *Phys. Rev. Lett.* **26** (1971) 1344–1346.

[33] J. M. Bardeen, B. Carter and S. W. Hawking, *The Four laws of black hole mechanics*, *Commun. Math. Phys.* **31** (1973) 161–170.

[34] S. W. Hawking, *Particle Creation by Black Holes*, *Commun. Math. Phys.* **43** (1975) 199–220.

[35] J. D. Bekenstein, *Black holes and entropy*, *Phys. Rev. D* **7** (1973) 2333–2346.

[36] D. Grumiller, R. McNees and J. Salzer, *Black holes and thermodynamics - The first half century*, *Fundam. Theor. Phys.* **178** (2015) 27–70, [1402.5127].

[37] R. Ruffini and J. A. Wheeler, *Introducing the black hole*, *Phys. Today* **24** (1971) 30.

[38] P. T. Chrusciel, J. Lopes Costa and M. Heusler, *Stationary Black Holes: Uniqueness and Beyond*, *Living Rev. Rel.* **15** (2012) 7, [1205.6112].

[39] M. S. Volkov and D. V. Galtsov, *NonAbelian Einstein Yang-Mills black holes*, *JETP Lett.* **50** (1989) 346–350.

[40] P. Bizon, *Colored black holes*, *Phys. Rev. Lett.* **64** (Jun, 1990) 2844–2847.

[41] H. P. Kuenzle and A. K. M. Masood-ul Alam, *Spherically symmetric static SU(2) Einstein Yang-Mills fields*, *J. Math. Phys.* **31** (1990) 928–935.

[42] D. V. Baltsov and M. S. Volkov, *Charged nonAbelian SU(3) Einstein Yang-Mills black holes*, *Phys. Lett. B* **274** (1992) 173–178.

[43] W. Israel, *Event horizons in static electrovac space-times*, *Commun. Math. Phys.* **8** (1968) 245–260.

[44] D. C. Robinson, *Uniqueness of the Kerr black hole*, *Phys. Rev. Lett.* **34** (1975) 905–906.

[45] P. O. Mazur, *PROOF OF UNIQUENESS OF THE KERR-NEWMAN BLACK HOLE SOLUTION*, *J. Phys. A* **15** (1982) 3173–3180.

[46] C. A. R. Herdeiro and E. Radu, *Asymptotically flat black holes with scalar hair: a review*, *Int. J. Mod. Phys. D* **24** (2015) 1542014, [1504.08209].

[47] J. D. Bekenstein, *Novel "no-scalar-hair" theorem for black holes*, *Phys. Rev. D* **51** (1995) R6608.

[48] D. Astefanesei, R. Ballesteros, P. Cabrera, G. Casanova and R. Rojas, *Existence of thermodynamically stable asymptotically flat black holes*, *Phys. Rev. D* **110** (2024) 024045, [2404.15566].

[49] R. Ballesteros, C. Gómez-Fayrén, T. Ortín and M. Zatti, *On scalar charges and black hole thermodynamics*, *JHEP* **05** (2023) 158, [2302.11630].

[50] R. Ballesteros and T. Ortín, *Hairy black holes, scalar charges and extended thermodynamics*, *Class. Quant. Grav.* **41** (2024) 055007, [2308.04994].

[51] H. B. Callen, *Thermodynamics and an introduction to thermostatistics; 2nd ed.* Wiley, New York, NY, 1985.

[52] D. Astefanesei, J. Luis Blázquez-Salcedo, F. Gómez and R. Rojas, *Thermodynamically stable asymptotically flat hairy black holes with a dilaton potential: the general case*, *JHEP* **02** (2021) 233, [2009.01854].

[53] D. Astefanesei, D. Choque, F. Gómez and R. Rojas, *Thermodynamically stable asymptotically flat hairy black holes with a dilaton potential*, *JHEP* **03** (2019) 205, [1901.01269].

[54] L. D. Landau and E. M. Lifshitz, *Statistical Physics, Part 1*, vol. 5 of *Course of Theoretical Physics*. Butterworth-Heinemann, Oxford, 1980.

[55] D. Astefanesei, J. L. Blázquez-Salcedo, C. Herdeiro, E. Radu and N. Sanchis-Gual, *Dynamically and thermodynamically stable black holes in Einstein-Maxwell-dilaton gravity*, *JHEP* **07** (2020) 063, [1912.02192].

[56] Z. Elgood, P. Meessen and T. Ortín, *The first law of black hole mechanics in the Einstein-Maxwell theory revisited*, *JHEP* **09** (2020) 026, [2006.02792].

[57] T. Ortin and D. Pereñiguez, *Magnetic charges and Wald entropy*, *JHEP* **11** (2022) 081, [2207.12008].

[58] T. Ortin, *A Note on Lie-Lorentz derivatives*, *Class. Quant. Grav.* **19** (2002) L143–L150, [hep-th/0206159].

[59] G. W. Gibbons, R. Kallosh and B. Kol, *Moduli, scalar charges, and the first law of black hole thermodynamics*, *Phys. Rev. Lett.* **77** (1996) 4992–4995, [hep-th/9607108].

[60] G. Barnich, *Boundary charges in gauge theories: Using Stokes theorem in the bulk*, *Class. Quant. Grav.* **20** (2003) 3685–3698, [hep-th/0301039].

[61] C. Pacilio, *Scalar charge of black holes in Einstein-Maxwell-dilaton theory*, *Phys. Rev. D* **98** (2018) 064055, [1806.10238].

[62] L. Smarr, *Mass formula for Kerr black holes*, *Phys. Rev. Lett.* **30** (1973) 71–73.

[63] A. Komar, *Covariant conservation laws in general relativity*, *Phys. Rev.* **113** (1959) 934–936.

[64] D. Kastor, S. Ray and J. Traschen, *Smarr Formula and an Extended First Law for Lovelock Gravity*, *Class. Quant. Grav.* **27** (2010) 235014, [1005.5053].

[65] S. Liberati and C. Pacilio, *Smarr Formula for Lovelock Black Holes: a Lagrangian approach*, *Phys. Rev. D* **93** (2016) 084044, [1511.05446].

[66] J. Lee and R. M. Wald, *Local symmetries and constraints*, *J. Math. Phys.* **31** (1990) 725–743.

[67] R. M. Wald, *Black hole entropy is the Noether charge*, *Phys. Rev. D* **48** (1993) R3427–R3431, [gr-qc/9307038].

[68] V. Iyer and R. M. Wald, *Some properties of Noether charge and a proposal for dynamical black hole entropy*, *Phys. Rev. D* **50** (1994) 846–864, [gr-qc/9403028].

[69] T. Ortín, *Komar integrals for theories of higher order in the Riemann curvature and black-hole chemistry*, *JHEP* **08** (2021) 023, [2104.10717].

[70] J. W. York, Jr., *Black hole thermodynamics and the Euclidean Einstein action*, *Phys. Rev. D* **33** (1986) 2092–2099.

[71] S. W. Hawking and D. N. Page, *Thermodynamics of Black Holes in anti-De Sitter Space*, *Commun. Math. Phys.* **87** (1983) 577.

[72] J. M. Maldacena, *The Large N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231–252, [hep-th/9711200].

[73] R. C. Myers and J. Z. Simon, *Black Hole Thermodynamics in Lovelock Gravity*, *Phys. Rev. D* **38** (1988) 2434–2444.

[74] P. Bueno and P. A. Cano, *Four-dimensional black holes in Einsteinian cubic gravity*, *Phys. Rev. D* **94** (2016) 124051, [1610.08019].

[75] P. Bueno and P. A. Cano, *Universal black hole stability in four dimensions*, *Phys. Rev. D* **96** (2017) 024034, [1704.02967].

[76] D. Astefanesei, C. Herdeiro, J. a. Oliveira and E. Radu, *Higher dimensional black hole scalarization*, *JHEP* **09** (2020) 186, [2007.04153].

[77] J. D. Brown and J. W. York, Jr., *Quasilocal energy and conserved charges derived from the gravitational action*, *Phys. Rev. D* **47** (1993) 1407–1419, [gr-qc/9209012].

[78] D. Astefanesei and E. Radu, *Quasilocal formalism and black ring thermodynamics*, *Phys. Rev. D* **73** (2006) 044014, [hep-th/0509144].

[79] R. Emparan, *Rotating circular strings, and infinite nonuniqueness of black rings*, *JHEP* **03** (2004) 064, [hep-th/0402149].

[80] R. B. Mann and D. Marolf, *Holographic renormalization of asymptotically flat spacetimes*, *Class. Quant. Grav.* **23** (2006) 2927–2950, [hep-th/0511096].

[81] D. Astefanesei, R. B. Mann and C. Stelea, *Note on counterterms in asymptotically flat spacetimes*, *Phys. Rev. D* **75** (2007) 024007, [hep-th/0608037].

[82] D. Astefanesei, R. B. Mann, M. J. Rodriguez and C. Stelea, *Quasilocal formalism and thermodynamics of asymptotically flat black objects*, *Class. Quant. Grav.* **27** (2010) 165004, [0909.3852].

[83] G. Compere, F. Dehouck and A. Virmani, *On Asymptotic Flatness and Lorentz Charges*, *Class. Quant. Grav.* **28** (2011) 145007, [1103.4078].

[84] G. Compere and F. Dehouck, *Relaxing the Parity Conditions of Asymptotically Flat Gravity*, *Class. Quant. Grav.* **28** (2011) 245016, [1106.4045].

[85] D. Astefanesei, M. J. Rodriguez and S. Theisen, *Thermodynamic instability of doubly spinning black objects*, *JHEP* **08** (2010) 046, [1003.2421].

[86] R. B. Mann, D. Marolf and A. Virmani, *Covariant Counterterms and Conserved Charges in Asymptotically Flat Spacetimes*, *Class. Quant. Grav.* **23** (2006) 6357–6378, [gr-qc/0607041].

[87] R. B. Mann, D. Marolf, R. McNees and A. Virmani, *On the Stress Tensor for Asymptotically Flat Gravity*, *Class. Quant. Grav.* **25** (2008) 225019, [0804.2079].

[88] A. Anabalon, *Exact Black Holes and Universality in the Backreaction of non-linear Sigma Models with a potential in (A)dS4*, *JHEP* **06** (2012) 127, [1204.2720].

[89] A. Anabalon and D. Astefanesei, *Black holes in ω-defomed gauged $N = 8$ supergravity*, *Phys. Lett. B* **732** (2014) 137–141, [1311.7459].

[90] A. Anabalon, D. Astefanesei and R. Mann, *Exact asymptotically flat charged hairy black holes with a dilaton potential*, *JHEP* **10** (2013) 184, [1308.1693].

[91] A. Anabalón, D. Astefanesei, A. Gallerati and M. Trigiante, *Hairy Black Holes and Duality in an Extended Supergravity Model*, *JHEP* **04** (2018) 058, [1712.06971].

[92] A. Anabalon, D. Astefanesei, A. Gallerati and M. Trigiante, *New non-extremal and BPS hairy black holes in gauged $\mathcal{N} = 2$ and $\mathcal{N} = 8$ supergravity*, *JHEP* **04** (2021) 047, [2012.09877].

[93] A. Gallerati, *New Black Hole Solutions in $N = 2$ and $N = 8$ Gauged Supergravity*, *Universe* **7** (2021) 187.

[94] A. Anabalon and S. F. Ross, *Supersymmetric solitons and a degeneracy of solutions in AdS/CFT*, *JHEP* **07** (2021) 015, [2104.14572].

[95] A. Anabalón, A. Gallerati, S. Ross and M. Trigiante, *Supersymmetric solitons in gauged $\mathcal{N} = 8$ supergravity*, *JHEP* **02** (2023) 055, [2210.06319].

[96] A. Anabalon, M. Cesaro, A. Gallerati, A. Giambrone and M. Trigiante, *A positive energy theorem for AdS solitons*, *Phys. Lett. B* **846** (2023) 138226, [2304.09201].

[97] A. Anabalón, D. Astefanesei, A. Gallerati and J. Oliva, *Supersymmetric smooth distributions of M2-branes as AdS solitons*, *JHEP* **05** (2024) 077, [2402.00880].

[98] D. Astefanesei, R. Ballesteros, D. Choque and R. Rojas, *Scalar charges and the first law of black hole thermodynamics*, *Phys. Lett. B* **782** (2018) 47–54, [1803.11317].

[99] K. Hajian and M. M. Sheikh-Jabbari, *Redundant and Physical Black Hole Parameters: Is there an independent physical dilaton charge?*, *Phys. Lett. B* **768** (2017) 228–234, [1612.09279].

[100] G. W. Gibbons and S. W. Hawking, *Action Integrals and Partition Functions in Quantum Gravity*, *Phys. Rev. D* **15** (1977) 2752–2756.

[101] S. R. Lau, *Light cone reference for total gravitational energy*, *Phys. Rev. D* **60** (1999) 104034, [gr-qc/9903038].

[102] R. B. Mann, *Misner string entropy*, *Phys. Rev. D* **60** (1999) 104047, [hep-th/9903229].

[103] P. Kraus, F. Larsen and R. Siebelink, *The gravitational action in asymptotically AdS and flat space-times*, *Nucl. Phys. B* **563** (1999) 259–278, [hep-th/9906127].

[104] A. Chamblin, R. Emparan, C. V. Johnson and R. C. Myers, *Charged AdS black holes and catastrophic holography*, Phys. Rev. D **60** (1999) 064018, [hep-th/9902170].

[105] A. Chamblin, R. Emparan, C. V. Johnson and R. C. Myers, *Holography, thermodynamics and fluctuations of charged AdS black holes*, Phys. Rev. D **60** (1999) 104026, [hep-th/9904197].

[106] D. L. Wiltshire, *Black Holes in String Generated Gravity Models*, Phys. Rev. D **38** (1988) 2445.

[107] R. L. Arnowitt, S. Deser and C. W. Misner, *Canonical variables for general relativity*, Phys. Rev. **117** (1960) 1595–1602.

[108] R. Arnowitt, S. Deser and C. W. Misner, *Energy and the Criteria for Radiation in General Relativity*, Phys. Rev. **118** (1960) 1100–1104.

[109] R. L. Arnowitt, S. Deser and C. W. Misner, *Coordinate invariance and energy expressions in general relativity*, Phys. Rev. **122** (1961) 997.

[110] R. L. Arnowitt, S. Deser and C. W. Misner, *The Dynamics of general relativity*, Gen. Rel. Grav. **40** (2008) 1997–2027, [gr-qc/0405109].

[111] R. C. Myers, *Higher Derivative Gravity, Surface Terms and String Theory*, Phys. Rev. D **36** (1987) 392.

[112] Y. Brihaye, B. Kleihaus, J. Kunz and E. Radu, *Rotating black holes with equal-magnitude angular momenta in d=5 Einstein-Gauss-Bonnet theory*, JHEP **11** (2010) 098, [1010.0860].

[113] D. Astefanesei, N. Banerjee and S. Dutta, *(Un)attractor black holes in higher derivative AdS gravity*, JHEP **11** (2008) 070, [0806.1334].

[114] Y. Brihaye and E. Radu, *Black objects in the Einstein-Gauss-Bonnet theory with negative cosmological constant and the boundary counterterm method*, JHEP **09** (2008) 006, [0806.1396].

[115] S. W. Hawking and S. F. Ross, *Duality between electric and magnetic black holes*, Phys. Rev. D **52** (1995) 5865–5876, [hep-th/9504019].

[116] T. Clunan, S. F. Ross and D. J. Smith, *On Gauss-Bonnet black hole entropy*, Class. Quant. Grav. **21** (2004) 3447–3458, [gr-qc/0402044].

[117] D. Astefanesei, M. J. Rodriguez and S. Theisen, *Quasilocal equilibrium condition for black ring*, JHEP **12** (2009) 040, [0909.0008].

[118] U. Nucamendi and M. Salgado, *Scalar hairy black holes and solitons in asymptotically flat space-times*, Phys. Rev. D **68** (2003) 044026, [gr-qc/0301062].

[119] D. Astefanesei, K. Goldstein, R. P. Jena, A. Sen and S. P. Trivedi, *Rotating attractors*, JHEP **10** (2006) 058, [hep-th/0606244].

[120] A. Sen, *Black hole entropy function and the attractor mechanism in higher derivative gravity*, JHEP **09** (2005) 038, [hep-th/0506177].

[121] A. Sen, *Entropy function for heterotic black holes*, *JHEP* **03** (2006) 008, [hep-th/0508042].

[122] M. Guica, T. Hartman, W. Song and A. Strominger, *The Kerr/CFT Correspondence*, *Phys. Rev. D* **80** (2009) 124008, [0809.4266].

[123] D. Astefanesei, C. Herdeiro, A. Pombo and E. Radu, *Einstein-Maxwell-scalar black holes: classes of solutions, dyons and extremality*, *JHEP* **10** (2019) 078, [1905.08304].

[124] T. P. Sotiriou, *f(R) gravity and scalar-tensor theory*, *Class. Quant. Grav.* **23** (2006) 5117–5128, [gr-qc/0604028].

[125] T. P. Sotiriou and V. Faraoni, *f(R) Theories Of Gravity*, *Rev. Mod. Phys.* **82** (2010) 451–497, [0805.1726].

[126] T. Ortin, *Gravity and Strings*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 2nd ed. ed., 7, 2015, 10.1017/CBO9781139019750.

[127] S. R. Coleman, J. Preskill and F. Wilczek, *Quantum hair on black holes*, *Nucl. Phys. B* **378** (1992) 175–246, [hep-th/9201059].

[128] A. I. Janis, E. T. Newman and J. Winicour, *Reality of the Schwarzschild Singularity*, *Phys. Rev. Lett.* **20** (1968) 878–880.

[129] A. G. Agnese and M. La Camera, *General spherically symmetric solutions in charged dilaton gravity*, *Phys. Rev. D* **49** (1994) 2126–2128.

[130] Z. Elgood, D. Mitsios, T. Ortín and D. Pereñíguez, *The first law of heterotic stringy black hole mechanics at zeroth order in α'*, *JHEP* **07** (2021) 007, [2012.13323].

[131] Z. Elgood, T. Ortín and D. Pereñíguez, *The first law and Wald entropy formula of heterotic stringy black holes at first order in* α', *JHEP* **05** (2021) 110, [2012.14892].

[132] D. Sudarsky and R. M. Wald, *Extrema of mass, stationarity, and staticity, and solutions to the Einstein Yang-Mills equations*, *Phys. Rev. D* **46** (1992) 1453–1474.

[133] D. Sudarsky and R. M. Wald, *Mass formulas for stationary Einstein Yang-Mills black holes and a simple proof of two staticity theorems*, *Phys. Rev. D* **47** (1993) R5209–R5213, [gr-qc/9305023].

[134] G. Barnich and G. Compere, *Surface charge algebra in gauge theories and thermodynamic integrability*, *J. Math. Phys.* **49** (2008) 042901, [0708.2378].

[135] S. McCormick, *The Phase Space for the Einstein-Yang-Mills Equations and the First Law of Black Hole Thermodynamics*, *Adv. Theor. Math. Phys.* **18** (2014) 799–825, [1302.1237].

[136] K. Hajian, A. Seraj and M. M. Sheikh-Jabbari, *NHEG Mechanics: Laws of Near Horizon Extremal Geometry (Thermo)Dynamics*, *JHEP* **03** (2014) 014, [1310.3727].

[137] K. Hajian and M. M. Sheikh-Jabbari, *Solution Phase Space and Conserved Charges: A General Formulation for Charges Associated with Exact Symmetries*, Phys. Rev. D **93** (2016) 044074, [1512.05584].

[138] P. Meessen, D. Mitsios and T. Ortín, *Black hole chemistry, the cosmological constant and the embedding tensor*, JHEP **12** (2022) 155, [2203.13588].

[139] D. Mitsios, T. Ortín and D. Pereñíguez, *Komar integral and Smarr formula for axion-dilaton black holes versus S duality*, JHEP **08** (2021) 019, [2106.07495].

[140] K. Hajian, H. Özşahin and B. Tekin, *First law of black hole thermodynamics and Smarr formula with a cosmological constant*, Phys. Rev. D **104** (2021) 044024, [2103.10983].

[141] P. A. Cano, T. Ortín, A. Ruipérez and M. Zatti, *Non-extremal, α'-corrected black holes in 5-dimensional heterotic superstring theory*, JHEP **12** (2022) 150, [2210.01861].

[142] P. G. O. Freund and M. A. Rubin, *Dynamics of Dimensional Reduction*, Phys. Lett. B **97** (1980) 233–235.

[143] A. Aurilia, H. Nicolai and P. K. Townsend, *Hidden Constants: The Theta Parameter of QCD and the Cosmological Constant of N=8 Supergravity*, Nucl. Phys. B **176** (1980) 509–522.

[144] D. Chernyavsky and K. Hajian, *Cosmological constant is a conserved charge*, Class. Quant. Grav. **35** (2018) 125012, [1710.07904].

[145] I. A. Bandos and T. Ortin, *On the dualization of scalars into $(d-2)$-forms in supergravity. Momentum maps, R-symmetry and gauged supergravity*, JHEP **08** (2016) 135, [1605.05559].

[146] M. K. Gaillard and B. Zumino, *Duality Rotations for Interacting Fields*, Nucl. Phys. B **193** (1981) 221–244.

[147] G. Compere, *Note on the First Law with p-form potentials*, Phys. Rev. D **75** (2007) 124020, [hep-th/0703004].

[148] K. Prabhu, *The First Law of Black Hole Mechanics for Fields with Internal Gauge Freedom*, Class. Quant. Grav. **34** (2017) 035011, [1511.00388].

[149] K. Hajian, M. M. Sheikh-Jabbari and B. Tekin, *Gauge invariant derivation of zeroth and first laws of black hole thermodynamics*, Phys. Rev. D **106** (2022) 104030, [2209.00563].

[150] G. W. Gibbons, *Antigravitating Black Hole Solitons with Scalar Hair in N=4 Supergravity*, Nucl. Phys. B **207** (1982) 337–349.

[151] G. W. Gibbons, *ASPECTS OF SUPERGRAVITY THEORIES*, in *XV GIFT Seminar on Supersymmetry and Supergravity*, 6, 1984.

[152] C. F. E. Holzhey and F. Wilczek, *Black holes as elementary particles*, Nucl. Phys. B **380** (1992) 447–477, [hep-th/9202014].

[153] E. Cremmer, J. Scherk and S. Ferrara, *SU(4) Invariant Supergravity Theory*, *Phys. Lett. B* **74** (1978) 61–64.

[154] R. Kallosh and T. Ortin, *Charge quantization of axion - dilaton black holes*, *Phys. Rev. D* **48** (1993) 742–747, [hep-th/9302109].

[155] G. W. Gibbons and K.-i. Maeda, *Black Holes and Membranes in Higher Dimensional Theories with Dilaton Fields*, *Nucl. Phys. B* **298** (1988) 741–775.

[156] D. Garfinkle, G. T. Horowitz and A. Strominger, *Charged black holes in string theory*, *Phys. Rev. D* **43** (1991) 3140.

[157] A. D. Shapere, S. Trivedi and F. Wilczek, *Dual dilaton dyons*, *Mod. Phys. Lett. A* **6** (1991) 2677–2686.

[158] R. Kallosh, A. D. Linde, T. Ortin, A. W. Peet and A. Van Proeyen, *Supersymmetry as a cosmic censor*, *Phys. Rev. D* **46** (1992) 5278–5302, [hep-th/9205027].

[159] T. Ortin, *Electric - magnetic duality and supersymmetry in stringy black holes*, *Phys. Rev. D* **47** (1993) 3136–3143, [hep-th/9208078].

[160] E. Lozano-Tellechea and T. Ortin, *The General, duality invariant family of nonBPS black hole solutions of N=4, D = 4 supergravity*, *Nucl. Phys. B* **569** (2000) 435–450, [hep-th/9910020].

[161] R. L. Arnowitt, S. Deser and C. W. Misner, *Dynamical Structure and Definition of Energy in General Relativity*, *Phys. Rev.* **116** (1959) 1322–1330.

[162] L. F. Abbott and S. Deser, *Stability of Gravity with a Cosmological Constant*, *Nucl. Phys. B* **195** (1982) 76–96.

[163] G. Barnich and F. Brandt, *Covariant theory of asymptotic symmetries, conservation laws and central charges*, *Nucl. Phys. B* **633** (2002) 3–82, [hep-th/0111246].

[164] P. Meessen and T. Ortin, *The Supersymmetric configurations of N=2, D=4 supergravity coupled to vector supermultiplets*, *Nucl. Phys. B* **749** (2006) 291–324, [hep-th/0603099].

[165] P. Galli, T. Ortin, J. Perz and C. S. Shahbazi, *Non-extremal black holes of N=2, d=4 supergravity*, *JHEP* **07** (2011) 041, [1105.3311].

[166] P. Meessen, T. Ortin, J. Perz and C. S. Shahbazi, *H-FGK formalism for black-hole solutions of N=2, d=4 and d=5 supergravity*, *Phys. Lett. B* **709** (2012) 260–265, [1112.3332].

[167] J. D. Bekenstein, *Transcendence of the law of baryon-number conservation in black hole physics*, *Phys. Rev. Lett.* **28** (1972) 452–455.

[168] M. Heusler and N. Straumann, *Scaling arguments for the existence of static, spherically symmetric solutions of self-gravitating systems*, *Classical and Quantum Gravity* **9** (1992) 2177–2184.

[169] M. Heusler, *A No hair theorem for selfgravitating nonlinear sigma models*, *J. Math. Phys.* **33** (1992) 3497–3502.

[170] D. Sudarsky, *A Simple proof of a no hair theorem in Einstein Higgs theory,*, *Class. Quant. Grav.* **12** (1995) 579–584.

[171] M. Heusler, *No hair theorems and black holes with hair*, *Helv. Phys. Acta* **69** (1996) 501–528, [gr-qc/9610019].

[172] O. Bechmann and O. Lechtenfeld, *Exact black hole solution with selfinteracting scalar field*, *Class. Quant. Grav.* **12** (1995) 1473–1482, [gr-qc/9502011].

[173] H. Dennhardt and O. Lechtenfeld, *Scalar deformations of Schwarzschild holes and their stability*, *Int. J. Mod. Phys. A* **13** (1998) 741–764, [gr-qc/9612062].

[174] A. Anabalon and J. Oliva, *Exact Hairy Black Holes and their Modification to the Universal Law of Gravitation*, *Phys. Rev. D* **86** (2012) 107501, [1205.6012].

[175] M. Cadoni and E. Franzin, *Asymptotically flat black holes sourced by a massless scalar field*, *Phys. Rev. D* **91** (2015) 104011, [1503.04734].

[176] D. Astefanesei, P. Cabrera, R. B. Mann and R. Rojas, *Extended phase space thermodynamics for hairy black holes*, *Phys. Rev. D* **108** (2023) 104047, [2304.09203].

[177] M. Heusler and N. Straumann, *Scaling arguments for the existence of static, spherically symmetric solutions of self-gravitating systems*, *Classical and Quantum Gravity* **9** (sep, 1992) 2177.

[178] D. Kastor, *Komar Integrals in Higher (and Lower) Derivative Gravity*, *Class. Quant. Grav.* **25** (2008) 175007, [0804.1832].

[179] T. Jacobson and A. Mohd, *Black hole entropy and Lorentz-diffeomorphism Noether charge*, *Phys. Rev. D* **92** (2015) 124010, [1507.01054].

[180] E. A. Bergshoeff, J. Hartong, O. Hohm, M. Huebscher and T. Ortin, *Gauge Theories, Duality Relations and the Tensor Hierarchy*, *JHEP* **04** (2009) 123, [0901.2054].

[181] E. Bergshoeff, R. Kallosh, T. Ortin, D. Roest and A. Van Proeyen, *New formulations of $D = 10$ supersymmetry and D8 - O8 domain walls*, *Class. Quant. Grav.* **18** (2001) 3359–3382, [hep-th/0103233].

[182] A. Anabalón and D. Astefanesei, *On attractor mechanism of AdS_4 black holes*, *Phys. Lett. B* **727** (2013) 568–572, [1309.5863].

[183] D. Marolf and S. F. Ross, *Boundary Conditions and New Dualities: Vector Fields in AdS/CFT*, *JHEP* **11** (2006) 085, [hep-th/0606113].

[184] A. Anabalon, D. Astefanesei, D. Choque and C. Martinez, *Trace Anomaly and Counterterms in Designer Gravity*, *JHEP* **03** (2016) 117, [1511.08759].

[185] T. Hertog and G. T. Horowitz, *Designer gravity and field theory effective potentials*, *Phys. Rev. Lett.* **94** (2005) 221301, [hep-th/0412169].

[186] A. Anabalon, D. Astefanesei and C. Martinez, *Mass of asymptotically anti–de Sitter hairy spacetimes*, *Phys. Rev. D* **91** (2015) 041501, [1407.3296].