



The space of invariant measures of Kan-like diffeomorphisms

by

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CHAPTER 1

Introduction

The study of the statistical behavior of dynamical systems is a central topic in modern smooth ergodic theory. A fundamental objective in this area is to understand how invariant probability measures describe the asymptotic dynamics observed for typical initial conditions. In many systems of interest, particularly beyond the uniformly hyperbolic setting, the coexistence of multiple invariant measures and the structure of their basins of attraction give rise to rich and complex phenomena.

Recall that the *basin* of an f -invariant probability measure μ is

$$\mathcal{B}(\mu) = \left\{ x \in M : \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \psi(f^j(x)) = \int \psi d\mu, \text{ for all } \psi: M \rightarrow \mathbb{R} \text{ continuous} \right\}.$$

An invariant probability measure is called *physical* if its basin has positive Lebesgue measure. Physical measures play an important role, as they capture the statistical behavior observed for a positive proportion of initial conditions with respect to a natural reference measure (Lebesgue) on the phase space. In many classical settings, physical measures are well understood: for transitive C^r Anosov diffeomorphisms with $r > 1$, a unique physical measure exists and its basin has full Lebesgue measure; more generally, for C^2 Axiom A attractors, physical measures are finite in number, the union of their basins covers Lebesgue-almost every point of the ambient manifold, and each basin is an open set modulo a Lebesgue-null set [35, 31, 8, 7].

Beyond the uniformly hyperbolic setting, this classical picture can break down. A striking example was introduced by Ittai Kan in [20]: he constructed a dynamical system exhibiting multiple physical measures whose basins are not separated by smooth or measurable boundaries, but are densely intertwined throughout the phase space.

Kan's construction takes the form of a skew-product map $K: \mathbb{S}^1 \times [0, 1] \rightarrow \mathbb{S}^1 \times [0, 1]$, where $\mathbb{S}^1$ plays the role of the base and $[0, 1]$ the role of the fiber, defined by

$$(1.1) \quad K(\theta, t) = \left(3\theta, t + \cos(2\pi\theta) \left(\frac{t}{32} \right) (1 - t) \right).$$

Despite the apparent simplicity of its definition, the map K exhibits a rich ergodic structure resting on three key properties: both boundary components $\mathbb{S}^1 \times \{0\}$ and $\mathbb{S}^1 \times \{1\}$ are invariant under K ; the restriction of K to each boundary is uniformly expanding; and the average fiber contraction near each boundary is negative. Together, these properties guarantee that the Lebesgue measure on each boundary component is invariant and physical, yielding two physical measures. Furthermore, the fiber maps at two special fixed points of the base — which we call *elevators* — exhibit contrasting behavior: at one elevator, $t = 0$ is attracting and $t = 1$ is repelling, while at the other the roles are reversed. This generates a heterodimensional cycle responsible for the intermingling of their basins: every nonempty open subset of $\mathbb{S}^1 \times [0, 1]$ intersects each basin in a set of positive Lebesgue measure, the union of these basins covers Lebesgue-almost every point in $\mathbb{S}^1 \times [0, 1]$, and this behavior persists under small perturbations [19]. These ingredients were later abstracted and systematized in [4, Chapter 11], leading to the class of *Kan-like maps*, which provide a natural framework for studying the robustness of intermingled basins and serve as a point of departure for the investigations developed in this thesis.

Kan-like systems have since been studied from two complementary perspectives. From the topological viewpoint, Gan and Shi [16] proved that Kan’s original cylinder endomorphism is robustly topologically mixing within the class of C^2 boundary-preserving maps. In the invertible setting, Cheng, Gan, and Shi [12] constructed a robustly transitive diffeomorphism of Kan’s type on $\mathbb{T}^3$ by inserting a blender into Kan’s example, and Xia [38] proved that Kan’s original diffeomorphism on $\mathbb{T}^2 \times [0, 1]$ is itself topologically transitive, under a mild irrationality condition on the derivatives of the elevator maps. These results reveal a fundamental contrast: despite admitting two distinct ergodic behaviors, Kan-like systems are topologically indecomposable.

From the measure-theoretic viewpoint, the interior ergodic structure has been studied primarily in the endomorphism setting. Besides the measures supported on the boundary of the cylinder, Bonifant and Milnor [6] detected, for Kan’s original cylinder endomorphism on $\mathbb{S}^1 \times [0, 1]$, a third hyperbolic invariant measure supported in the interior of the cylinder, projecting onto Lebesgue measure on the circle, though not physical in the usual sense.

In [26], Kan’s construction was generalized by replacing the multiplication-by-3 map with a uniformly expanding endomorphism E of the base. In this endomorphism setting, the phenomena observed by Kan and Bonifant–Milnor were shown to persist for a wide class of E -invariant probability measures ν , including measures of maximal entropy and

equilibrium states for Hölder potentials (see Section 2.1). Considering the natural reference measure $m_\nu = \nu \times \text{Leb}_I$ on the ambient manifold, an invariant measure μ is called *observable* with respect to m_ν if its basin $\mathcal{B}(\mu)$ has positive m_ν -measure. It is established that the boundary measures are observable and intermingled with respect to m_ν , their basins cover m_ν -almost every point of the ambient manifold, a unique interior ergodic measure exists, and all these measures are hyperbolic.

In the same pioneering work [20], Kan also introduced a diffeomorphism version of his construction, considering the map $K: \mathbb{T}^2 \times [0, 1] \rightarrow \mathbb{T}^2 \times [0, 1]$ defined by

$$(1.2) \quad K(x, y, t) = \left(3x + y, 2x + y, t + \cos(2\pi x) \left(\frac{t}{32} \right) (1 - t) \right),$$

and established that the Lebesgue measures on the boundary components are physical measures with intermingled basins, showing that the phenomenon first observed in the endomorphism setting persists in the invertible case.

Motivated by Kan's construction, this thesis studies a general class of diffeomorphisms sharing these features. Let $\mathbb{T}^d$ denote the d -dimensional torus, $I = [0, 1]$, and $M = \mathbb{T}^d \times I$. We consider a C^r -diffeomorphism $K: M \rightarrow M$ defined by

$$(1.3) \quad K(x, t) \stackrel{\text{def}}{=} (A(x), \varphi(x, t)),$$

where the base diffeomorphism $A: \mathbb{T}^d \rightarrow \mathbb{T}^d$ is a C^r -Anosov diffeomorphism and the fiber map $\varphi: M \rightarrow I$ is a C^r map, $r \geq 1$. Throughout this thesis, A is always assumed to be transitive.

We call K defined in (1.3) a *Kan-like diffeomorphism* if it satisfies the following conditions:

- [KD1] K *preserves the boundary*: For every $x \in \mathbb{T}^d$ we have $\varphi(x, j) = j$, with $j \in \{0, 1\}$.
- [KD2] K *has two “elevators”*: The Anosov diffeomorphism A has two fixed points $p, q \in \mathbb{T}^d$. The map $\varphi(p, \cdot): I \rightarrow I$ has exactly two fixed points: a hyperbolic source at $t = 0$ and a hyperbolic sink at $t = 1$. Analogously, the map $\varphi(q, \cdot): I \rightarrow I$ has exactly two fixed points: a hyperbolic sink at $t = 0$ and a hyperbolic source at $t = 1$.
- [KD3] K is a *partially hyperbolic diffeomorphism*: There exists a Riemannian metric on M (adapted to K) and a constant $\lambda < 1$ such that for every $(x, t) \in M$,

$$\|DK|_{E_K^s(x,t)}\| < \lambda, \quad \|(DK|_{E_K^u(x,t)})^{-1}\| < \lambda,$$

$$\|DK|_{E_K^s(x,t)}\| \cdot \|(DK|_{E_K^c(K(x,t))})^{-1}\| < 1,$$

$$\|DK|_{E_K^c(x,t)}\| \cdot \|(DK|_{E_K^u(K(x,t))})^{-1}\| < 1.$$

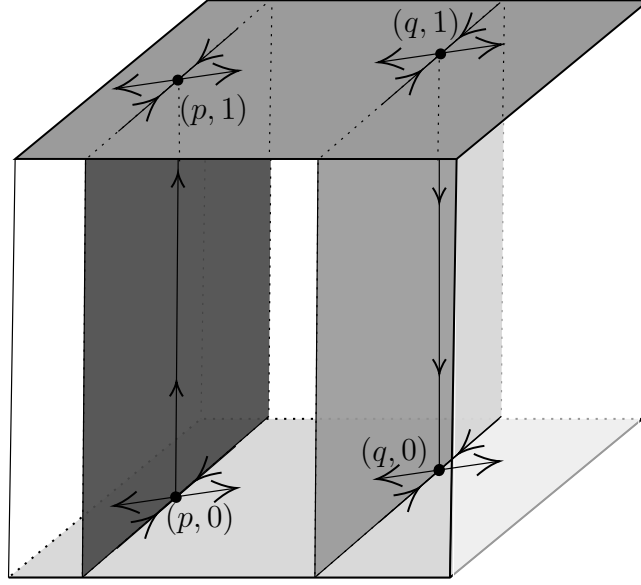


FIGURE 1. Kan-like diffeomorphisms

The question of which ergodic measures can occur for a Kan-like diffeomorphism, and how their basins are organized, has remained open. This thesis addresses these questions.

The goal of this work is to understand how the asymptotic behavior along the fibers, as captured by the central Lyapunov exponent, determines both the structure of invariant measures and the geometry of their basins of attraction. Our main results establish that every ergodic K -invariant measure is a graph measure supported on a K -invariant graph, and that the sign of its central Lyapunov exponent determines the geometry of its basin along the fibers; these conclusions are made precise in Theorems [A](#), [B](#), and [C](#).

The relevant terminology is as follows. For each $x \in \mathbb{T}^d$, we denote by $I_x \stackrel{\text{def}}{=} \{x\} \times I$ the fiber over x . Given an ergodic K -invariant probability measure μ , its *central Lyapunov exponent* (or *fiber Lyapunov exponent*) is

$$\chi^c(\mu) = \int_M \log |\partial_t \varphi(x, t)| d\mu(x, t).$$

Since K is partially hyperbolic by [\[KD3\]](#), the fiber direction ∂_t spans the central bundle E_K^c , so $\chi^c(\mu)$ measures the average asymptotic expansion in this central (fiber) direction. Denote by $\pi: M \rightarrow \mathbb{T}^d$ the natural projection. Given a measurable function $\phi: \mathbb{T}^d \rightarrow I$, we call $\Gamma(\phi) \stackrel{\text{def}}{=} \{(x, \phi(x)): x \in \mathbb{T}^d\}$ an *invariant graph* for K if $\Gamma(\phi)$ is K -invariant; given

an A -invariant probability measure ν on $\mathbb{T}^d$, the associated *graph measure* is defined by $\mu_\phi(E) \stackrel{\text{def}}{=} \nu(\pi(E \cap \Gamma(\phi)))$ for every measurable $E \subseteq M$.

THEOREM A (Geometry of basins). *Let $K: M \rightarrow M$ be a Kan-like C^r -diffeomorphism, $r > 1$. Then, for every ergodic K -invariant probability measure μ ,*

- (1) *the pushforward measure $\nu \stackrel{\text{def}}{=} \pi_*\mu$ is ergodic;*
- (2) *there exists a measurable function $\phi: \mathbb{T}^d \rightarrow I$ whose graph $\Gamma(\phi)$ is K -invariant and $\mu = \mu_\phi$;*
- (3) *for ν -almost every $x \in \mathbb{T}^d$ (where ν is as in (1)), the intersection of the basin $\mathcal{B}(\mu)$ with the fiber I_x is an interval (possibly degenerate). In addition,*
 - (a) *if the central Lyapunov exponent of μ is positive, then $\mathcal{B}(\mu) \cap I_x$ is a single point; and*
 - (b) *if the central Lyapunov exponent of μ is negative, then the set $\mathcal{B}(\mu) \cap I_x$ contains an open interval.*

While Theorem A describes the geometric structure of the basin of each ergodic K -invariant measure, it does not characterize which such measures actually occur, nor how many there are.

An ergodic A -invariant probability measure ν on $\mathbb{T}^d$ is called *geometric* if it has local product structure, uniformly bounded Radon–Nikodym derivatives, and full support; see Section 2.3 for the precise definition and further properties.

By condition [KD1], the boundary components $\mathbb{T}^d \times \{0\}$ and $\mathbb{T}^d \times \{1\}$ are K -invariant. Given an A -invariant probability measure ν on $\mathbb{T}^d$, the *boundary measures* are $\nu^0 \stackrel{\text{def}}{=} \nu \times \delta_0$ and $\nu^1 \stackrel{\text{def}}{=} \nu \times \delta_1$, that is, the measures supported on the invariant boundary components $\mathbb{T}^d \times \{0\}$ and $\mathbb{T}^d \times \{1\}$.

We call an ergodic K -invariant measure *interior* if its support is not contained in the boundary $\mathbb{T}^d \times \{0, 1\}$.

The next result provides a complete description of the ergodic measures projecting onto a given geometric measure on the base.

THEOREM B (Structure of ergodic measures). *Let $K: M \rightarrow M$ be a Kan-like C^r -diffeomorphism, $r > 1$. Let ν be a geometric measure on the base $\mathbb{T}^d$. Among all ergodic K -invariant probability measures μ satisfying $\pi_*\mu = \nu$, exactly one of the following alternatives holds:*

- (1) *No interior measures: the set of interior ergodic measures is empty, and the boundary measures satisfy $\chi^c(\nu^0) \chi^c(\nu^1) \leq 0$.*

- (2) Lyapunov alternating sequence: *there exists a finite or infinite sequence of interior ergodic graph measures $\{\mu_{\phi_i}\}$ whose associated graphs satisfy $\phi_i(x) < \phi_{i+1}(x)$ for ν -almost every x , and whose central Lyapunov exponents alternate in sign. If the sequence is infinite in some direction, it accumulates on the corresponding boundary measure, whose central Lyapunov exponent is equal to zero.*

A more precise statement, including the exact structure in the finite case, is given in Theorem 6.10.

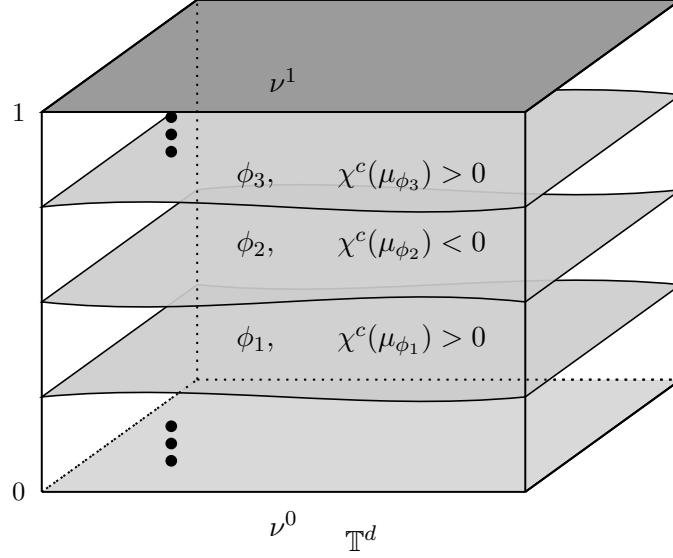


FIGURE 2. Schematic illustration of Theorem B, Case (2): the Lyapunov alternating sequence. Each interior ergodic graph measure μ_{ϕ_i} is supported on the invariant graph $\Gamma(\phi_i)$; the strict ordering $\phi_1(x) < \phi_2(x) < \phi_3(x)$ holds for ν -almost every x , and the central Lyapunov exponents alternate in sign.

We now study the intermingling phenomenon for boundary measures that are m_ν -observable. A geometric measure ν is called *strongly geometric* if it additionally satisfies the uniformly bounded Jacobian (UBJ) condition (see Section 2.3 for details). The following result shows that, when both boundary measures ν^0 and ν^1 are m_ν -observable, their basins are intermingled.

THEOREM C. *Let $K: M \rightarrow M$ be a Kan-like C^r -diffeomorphism, $r > 1$. Let ν be a strongly geometric measure on the base $\mathbb{T}^d$. Assume that ν^j is m_ν -observable, $j = 0, 1$. Then,*

- (1) *The basins $\mathcal{B}(\nu^0)$ and $\mathcal{B}(\nu^1)$ are dense and intermingled with respect to m_ν , their union covers m_ν -almost every point of M , and ν^0, ν^1 are the only m_ν -observable K -invariant measures projecting onto ν .*
- (2) *There exists a unique ergodic K -invariant probability measure $\mu \neq \nu^0, \nu^1$ projecting onto ν ; such a measure has positive central Lyapunov exponent.*

We note that strongly geometric measures always exist for Kan-like diffeomorphisms; see Corollary 7.11 (Section 7.6).

The thesis is organized as follows. Chapter 2 recalls background notions from smooth ergodic theory that will be used throughout the work, including entropy, equilibrium states, Gibbs measures, geometric measures, and basic properties of Anosov diffeomorphisms.

Chapter 3 studies the ergodic decomposition of invariant measures on the boundary components and the topological structure of the corresponding space of measures. It provides the tools needed to produce geometric and strongly geometric measures with prescribed signs of the central Lyapunov exponent on the boundary.

Chapter 4 establishes the fundamental dynamical properties of Kan-like diffeomorphisms used throughout the thesis: their partial hyperbolicity, the projection of the dynamics onto the boundary components, the elevator structure connecting these components, and the resulting decomposition of the space of invariant measures.

Chapter 5 develops the main tools for ergodic basins in the skew-product setting: the correspondence between ergodic measures and invariant graphs, the interval structure of basins along fibers, and the basin geometry of hyperbolic measures. These results yield the proof of Theorem A.

Chapter 6 treats the case of a geometric base measure. Using the Avila–Viana Invariance Principle, we show that interior ergodic measures are necessarily hyperbolic and obtain a complete structural description of the set of ergodic measures projecting onto the base. This yields the proof of Theorem B.

Chapter 7 is devoted to the proof of Theorem C. We show that, when both boundary measures are m_ν -observable, their basins are dense, intermingled, and together cover almost every point of the manifold, and establish the existence and uniqueness of the interior measure.

CHAPTER 2

Preliminaries

2.1. Entropy, pressure, invariant measures, and equilibrium states

Topological pressure and entropy. Let (X, dist_X) be a compact metric space and let $F: X \rightarrow X$ be a continuous map. Given $\epsilon > 0$ and $n \in \mathbb{N}$, the n -th *Bowen ball* of size ϵ centered at $x \in X$ is the set

$$B_n(x, \epsilon) \stackrel{\text{def}}{=} \{y \in X : \text{dist}_X(F^k(x), F^k(y)) \leq \epsilon, \text{ for every } k = 0, \dots, n-1\}.$$

A set $B \subseteq X$ is said to be (n, ϵ) -separated if for every distinct $x, y \in B$, there exists $k \in \{0, \dots, n-1\}$ such that $\text{dist}_X(F^k(x), F^k(y)) \geq \epsilon$.

Consider a continuous function $\psi: X \rightarrow \mathbb{R}$. The n -th *Birkhoff sum* of ψ along the orbit of x is defined by

$$(2.1) \quad S_n\psi(x) \stackrel{\text{def}}{=} \sum_{k=0}^{n-1} \psi(F^k(x)).$$

We denote by

$$P(\psi, n, \epsilon) \stackrel{\text{def}}{=} \sup \sum_{x \in B} \exp(S_n\psi(x)),$$

where the supremum is taken over all the sets $B \subset X$ being (n, ϵ) -separated. For $\epsilon > 0$, we denote

$$(2.2) \quad P_\epsilon(\psi) \stackrel{\text{def}}{=} \limsup_{n \rightarrow \infty} \frac{1}{n} \log P(\psi, n, \epsilon).$$

The *topological pressure* is defined as

$$(2.3) \quad P(\psi) \stackrel{\text{def}}{=} \lim_{\epsilon \rightarrow 0} P_\epsilon(\psi).$$

In particular, the *topological entropy of F* is the pressure of the null potential,

$$(2.4) \quad h_{\text{top}}(F) \stackrel{\text{def}}{=} P(0).$$

Recall that, given a measure η , its *push forward* is the measure $F_*\eta \stackrel{\text{def}}{=} \eta \circ F^{-1}$.

Let η be an F -invariant Borel probability measure (that means $F_*\eta = \eta$). A (measurable) *partition* $\mathcal{P}$ on X is a countable (finite or infinite) family of pairwise disjoint

measurable subsets of X whose union has full η measure. The *entropy of the partition* $\mathcal{P}$ is the real number

$$H_\eta(\mathcal{P}) = \sum_{P \in \mathcal{P}} -\eta(P) \log \eta(P).$$

Given two partitions $\mathcal{Q}_1, \mathcal{Q}_2$ of X , their *join* is the partition $\mathcal{Q}_1 \vee \mathcal{Q}_2 \stackrel{\text{def}}{=} \{Q_1 \cap Q_2 : Q_1 \in \mathcal{Q}_1, Q_2 \in \mathcal{Q}_2\}$. More generally, given a countable family of partitions $\{\mathcal{Q}_k\}_{k \in I}$, we denote by $\bigvee_{k \in I} \mathcal{Q}_k$ the partition whose elements are the nonempty intersections $\bigcap_{k \in I} Q_k$, with $Q_k \in \mathcal{Q}_k$.

Given a partition $\mathcal{P}$ on X , with finite entropy, denote

$$\mathcal{P}^n = \bigvee_{k=0}^{n-1} F^{-k}(\mathcal{P}) \quad \text{for each } n \geq 1.$$

The *metric entropy* of F with respect to the measure η is defined by

$$h_\eta(F) = \sup_{\mathcal{P}} h_\eta(F, \mathcal{P}),$$

where the supremum is taken over all the partitions with finite entropy and $h_\eta(F, \mathcal{P})$ denotes the entropy of F with respect to the partition $\mathcal{P}$, which is defined by

$$h_\eta(F, \mathcal{P}) = \lim_n \frac{1}{n} H_\eta(\mathcal{P}^n).$$

Equilibrium states and Gibbs measures. The *variational principle* establishes that for every continuous function $\psi : X \rightarrow \mathbb{R}$, we have

$$P(\psi) = \sup_{\eta} \{h_\eta(F) + \eta(\psi)\},$$

where the supremum is taken over all the invariant probability measures of F and $\eta(\psi)$ denotes the integral $\int \psi d\eta$.

The invariant measure η is an *equilibrium state* for the potential ψ if

$$P(\psi) = h_\eta(F) + \eta(\psi).$$

In particular, any equilibrium state for the potential $\psi \equiv 0$ is called a *measure of maximal entropy*.

An invariant measure η is a *Gibbs measure* or a *Gibbs state* with respect to ψ if it satisfies the *Gibbs property*: for every $\epsilon > 0$ small enough, there is $Q_\psi = Q_\psi(\epsilon) > 0$ such that for all $x \in X$ and $n \in \mathbb{N}$, we have

$$Q_\psi^{-1} \leq \frac{\eta(B_n(x, \epsilon))}{e^{-nP(\psi) + S_n\psi(x)}} \leq Q_\psi.$$

In particular, any Gibbs measure in a transitive Anosov system is ergodic [7].

Every equilibrium state is a Gibbs state for the potential ψ [32, Chapter 4]. The converse need not be true in general, but a *summability condition* on the potential is indeed sufficient for the coincidence of Gibbs and equilibrium states [22, 11]. Such a summability condition is satisfied if ψ is, for instance, Hölder continuous.

Space of invariant measures. Denote by $\mathcal{M}_1(X)$ the space of Borel probability measures on X . The following defines a metric on $\mathcal{M}_1(X)$.

For $B \subseteq X$, denote by $B^\epsilon \stackrel{\text{def}}{=} \{x \in X : \text{dist}_X(x, B) < \epsilon\}$ the ϵ -neighborhood of B . Given $\eta_1, \eta_2 \in \mathcal{M}_1(X)$,

$$\text{dist}_{\mathcal{M}_1(X)}(\eta_1, \eta_2) \stackrel{\text{def}}{=} \inf \{ \epsilon > 0 : \eta_2(B) \leq \eta_1(B^\epsilon) + \epsilon \text{ for every Borel set } B \subset X \}.$$

Although the definition is apparently not symmetric in η_1 and η_2 , one verifies that the condition above holds with η_1, η_2 interchanged for the same ϵ , so $\text{dist}_{\mathcal{M}_1(X)}$ is indeed a metric.

This metric, called the Lévy–Prokhorov metric, induces the weak* topology and therefore $\mathcal{M}_1(X)$ is a compact metrizable convex topological space [36, Section 2.1.3] (see also [37, Theorem 6.4]).

We denote by dist_H the induced *Hausdorff distance* on compact sets in $\mathcal{M}_1(X)$.

Disintegration and ergodic decomposition. If $\mathcal{P}$ is a partition of X into measurable subsets, there is a canonical projection $\varpi : X \rightarrow \mathcal{P}$ that assigns to each point $x \in X$ the element $\mathcal{P}(x) = \varpi(x)$ of the partition that contains it. We say that a subset $\mathcal{Q} \subseteq \mathcal{P}$ is *measurable* if there is a measurable subset $E \subset X$ such that

$$\varpi^{-1}(\mathcal{Q}) = \bigcup_{x \in E} \mathcal{P}(x);$$

in this case we write $\mathcal{Q} = E^\mathcal{P}$. The collection $\mathcal{B}^\mathcal{P}$ of all such measurable subsets of $\mathcal{P}$ is a σ -algebra. The inverse image of the projection ϖ thus endows $\mathcal{P}$ with the structure of a probability space $(\mathcal{P}, \mathcal{B}^\mathcal{P}, \eta^\mathcal{P})$, where the measure $\eta^\mathcal{P}$ is defined by (see [25, 36])

$$(2.5) \quad \eta^\mathcal{P}(E^\mathcal{P}) = \eta(E).$$

A *disintegration* of η with respect to the partition $\mathcal{P}$ is a family $\{\eta_P^\mathcal{P} : P \in \mathcal{P}\}$ of probability measures on X such that $\eta_P^\mathcal{P}(P) = 1$ for $\eta^\mathcal{P}$ -almost every $P \in \mathcal{P}$; for every measurable set $E \subset X$ the map $P \mapsto \eta_P^\mathcal{P}(E)$ is measurable, and

$$(2.6) \quad \eta(E) = \int \eta_{\mathcal{P}(x)}^{\mathcal{P}}(E) d\eta^{\mathcal{P}}(\mathcal{P}(x)).$$

The measures $\{\eta_P^{\mathcal{P}} : P \in \mathcal{P}\}$ are called *conditional probabilities* of η with respect to $\mathcal{P}$. Given $x \in X$, we write $\eta_x^{\mathcal{P}} \stackrel{\text{def}}{=} \eta_{\mathcal{P}(x)}^{\mathcal{P}}$, so that (2.6) reads

$$(2.7) \quad \eta(E) = \int_X \eta_x^{\mathcal{P}}(E) d\eta^{\mathcal{P}}(x).$$

If X is a complete separable metric space, Rokhlin's Theorem [36, Theorem 5.1.11] states that for any measurable partition $\mathcal{P}$, there is a unique family of *conditional probability measures* $\{\eta_x^{\mathcal{P}} : x \in X\}$ which is the disintegration of η with respect to $\mathcal{P}$.

We denote by $\mathcal{M}(F)$ the space of F -invariant Borel probability measures and by $\mathcal{E}(F)$ the space of ergodic measures of F .

The *Ergodic Decomposition Theorem* states that for every $\eta \in \mathcal{M}(F)$, there exists a partition $\mathcal{P}$ of X into measurable subsets and a family $\{\eta_x : x \in X\}$ of conditional probability measures with respect to $\mathcal{P}$, where $\eta_x \in \mathcal{E}(F)$ for η -almost every $x \in X$ ¹.

In particular, the ergodic decomposition $\{\eta_x\}_{x \in X}$ implies that, for every $\psi : X \rightarrow \mathbb{R}$ measurable

$$(2.8) \quad \int \psi d\eta = \int \left(\int \psi d\eta_x \right) d\eta^{\mathcal{P}}(x)$$

and

$$(2.9) \quad h_{\eta}(F) = \int h_{\eta_x}(F) d\eta^{\mathcal{P}}(x).$$

In Section 2.3 we apply the previous setting for $X = \mathbb{T}^d$ and $F = A$.

2.2. Invariant manifolds for Anosov diffeomorphisms

Recall that a C^r -diffeomorphism $A : \mathbb{T}^d \rightarrow \mathbb{T}^d$ is an *Anosov diffeomorphism* if there exists a splitting of the tangent bundle $T\mathbb{T}^d = E_A^s \oplus E_A^u$, invariant for the derivative of A , and constants $C > 0$, $\lambda \in (0, 1)$ such that, for every $x \in \mathbb{T}^d$ and $n \geq 1$,

$$\|DA^n|_{E_A^s(x)}\| \leq C\lambda^n \quad \text{and} \quad \|DA^{-n}|_{E_A^u(A^n x)}\| \leq C\lambda^n.$$

Replacing the Riemannian metric on $\mathbb{T}^d$ by an adapted metric, we may and do assume throughout that $C = 1$, so that for every $x \in \mathbb{T}^d$,

$$\|DA|_{E_A^s(x)}\| < 1 < \|DA^{-1}|_{E_A^u(A(x))}\|^{-1}.$$

¹In this particular case, we omit the dependence on $\mathcal{P}$ in the notation.

Given $r_0 > 0$, the *local unstable manifold of size r_0* of a point $x_0 \in \mathbb{T}^d$ is the set $W_A^u(x_0, r_0)$ of points $z \in \mathbb{T}^d$ such that

$$\lim_{n \rightarrow +\infty} \text{dist}_{\mathbb{T}^d}(A^{-n}x_0, A^{-n}z) = 0 \quad \text{and} \quad \text{dist}_{\mathbb{T}^d}(A^{-n}x_0, A^{-n}z) \leq r_0 \quad \text{for all } n \geq 0.$$

According to the Unstable Manifold Theorem [33], for every $r_0 > 0$ and every $x_0 \in \mathbb{T}^d$, $W_A^u(x_0, r_0)$ is a disk C^r -embedded in $\mathbb{T}^d$ with $T_{x_0}W_A^u(x_0, r_0) = E_A^u(x_0)$.

The *unstable manifold* $W_A^u(x_0)$ of $x_0 \in \mathbb{T}^d$ is the set of points $z \in \mathbb{T}^d$ whose backward orbit is asymptotic to that of x_0 :

$$\lim_{n \rightarrow +\infty} \text{dist}_{\mathbb{T}^d}(A^{-n}x_0, A^{-n}z) = 0.$$

It follows immediately from the definitions that

$$W_A^u(x_0) = \bigcup_{n \geq 0} A^n(W_A^u(A^{-n}x_0, r_0)).$$

Therefore, $W_A^u(x_0)$ is a C^r immersed submanifold.

Let us denote by dist^u the *intrinsic distance* defined on a fixed unstable manifold. More precisely, given points $x \in \mathbb{T}^d$, and $z, y \in W_A^u(x)$, $\text{dist}^u(z, y) \stackrel{\text{def}}{=} \inf_{\gamma} \text{length}(\gamma)$ where the infimum is taken over all the C^1 -curves γ in $W_A^u(x)$ connecting z to y . Since the derivative of A is uniformly expanding along the tangent bundle E_A^u , this distance is uniformly contracted by the backward iterates of A . Indeed, given any piecewise C^1 curve γ connecting z to y inside $W_A^u(x)$, the length of $A^{-n}(\gamma)$ is less than $\lambda_u^{-n} \text{length}(\gamma)$, for every $n \geq 1$, where $\lambda_u = \|DA^{-1}|_{E_A^u}\|^{-1} > 1$. Therefore,

$$\text{dist}^u(A^{-n}(z), A^{-n}(y)) \leq \lambda_u^{-n} \text{dist}^u(z, y) \quad \text{for every } n \geq 1.$$

Fix $x \in \mathbb{T}^d$. Given $z \in W_A^u(x)$, and $r_0 > 0$, the *u-ball* centered at x of radius r_0 , is the set

$$B^u(x, r_0) = \{z \in W_A^u(x) : \text{dist}^u(x, z) < r_0\}.$$

If r_0 is small enough, there is a constant $0 < \alpha < 1$ such that

$$W_A^u(x, \alpha \cdot r_0) \subseteq B^u(x, r_0) \subseteq W_A^u(x, r_0).$$

A set $\gamma \subseteq \mathbb{T}^d$ is a *u-disk* if it is a subset of some unstable manifold $W_A^u(x)$ that is diffeomorphic to a *u-ball*.

The Unstable Manifold Theorem also states that $W_A^u(x, r_0)$ varies continuously with the point $x \in \mathbb{T}^d$. More precisely, for every $x_0 \in \mathbb{T}^d$ and $r_0 > 0$, there exists a neighborhood $U_{x_0, r_0} \subseteq \mathbb{T}^d$ of x_0 , and a continuous map

$$\Phi_{x_0, r_0} : U_{x_0, r_0} \rightarrow \text{Emb}^r(W_A^u(x_0, r_0), \mathbb{T}^d),$$

such that $\Phi_{x_0, r_0}(x_0)$ is the inclusion of $W_A^u(x_0, r_0)$ in $\mathbb{T}^d$, and for every $z \in U_{x_0, r_0}$, $W_A^u(z, r_0) = \Phi_{x_0, r_0}(z)(W_A^u(x_0, r_0))$. Recall that $r \geq 1$ is the class of differentiability of A .

Let $x_0 \in \mathbb{T}^d$, $r_0 > 0$, and $U_{x_0, r_0} \subseteq \mathbb{T}^d$ be the neighborhood of x_0 given by the Unstable Manifold Theorem. Let $z \in U_{x_0, r_0}$ be a point such that $\Phi_{x_0, r_0}(z) = W_A^u(z, r_0)$. Given $\epsilon > 0$, we say that two closed u -disks $\gamma \subseteq W_A^u(x_0, r_0)$ and $\tilde{\gamma} \subseteq W_A^u(z, r_0)$ are ϵ -close in the C^r -topology if $\Phi_{x_0, r_0}(z)$ is ϵ -close to $\Phi_{x_0, r_0}(x_0)$ in the C^r topology of $\text{Emb}^r(W_A^u(x_0, r_0), \mathbb{T}^d)$, and $\Phi_{x_0, r_0}(\gamma)$ and $\tilde{\gamma}$ are close in the intrinsic metric of $W_A^u(z, r_0)$ and vice versa.

Let $G : \gamma \rightarrow [0, 1]$, $\tilde{G} : \tilde{\gamma} \rightarrow [0, 1]$ be C^r -functions defined on the u -disks $\gamma \subseteq W_A^u(x_0, r_0)$ and $\tilde{\gamma} \subseteq W_A^u(z, r_0)$. We say that G and $\tilde{G}$ are ϵ -close in the C^r -topology if the disks γ and $\tilde{\gamma}$ and the functions G and $\tilde{G} \circ \Phi_{x_0, r_0}(z)$ are ϵ -close in the C^r -topology.

The *local stable manifold of size $r_0 > 0$* , denoted $W_A^s(x_0, r_0)$, and the *stable manifold*, $W_A^s(x_0)$, of a point $x \in \mathbb{T}^d$ are defined in the same way as the local unstable manifold and the unstable manifold, respectively, replacing A^{-1} by A . Again, local stable manifolds of points are C^r embedded disks, and the stable manifolds are C^r immersed.

2.3. Geometric measures for Anosov diffeomorphisms

Recall that A has a *local product structure*, that is, there exists $\delta > 0$ such that for every $x, y \in \mathbb{T}^d$ with $\text{dist}_{\mathbb{T}^d}(x, y) < \delta$ the intersection $W_A^s(x, r_0) \cap W_A^u(y, r_0)$ is transverse and consists of a single point z denoted by $[x, y]$. A closed subset $R \subseteq \mathbb{T}^d$ is a *rectangle* if for all $x, y \in R$ the point $[x, y]$ is well-defined and belongs to R . For every $x \in R$, we let

$$W_A^u(x, R) \stackrel{\text{def}}{=} W_A^u(x, r_0) \cap R.$$

Consider a rectangle R , a point $x \in R$, and a pair of probability measures ξ^u and ξ^s defined on $W_A^u(x, R)$ and $W_A^s(x, R)$, respectively. Their product measure, denoted $\xi^u \otimes \xi^s$ and defined on R , is given by

$$\xi^u \otimes \xi^s([R^u, R^s]) \stackrel{\text{def}}{=} \xi^u(R^u)\xi^s(R^s);$$

where $R^u \subseteq W_A^u(x, R)$, and $R^s \subseteq W_A^s(x, R)$ are Borel subsets and

$$[R^u, R^s] \stackrel{\text{def}}{=} \{[y, z] : y \in R^u, z \in R^s\}.$$

Given a rectangle R and points $y, z \in R$, consider the *stable holonomy map*

$$(2.10) \quad h_{yz}^s : W_A^u(y, R) \rightarrow W_A^u(z, R), \quad h_{yz}^s(x) \stackrel{\text{def}}{=} [x, z],$$

which maps one unstable leaf to another one by sliding along stable manifolds. Analogously, one defines the *unstable holonomy map*

$$h_{yz}^u: W_A^s(y, R) \rightarrow W_A^s(z, R), \quad h_{yz}^u(x) \stackrel{\text{def}}{=} [z, x],$$

between $W_A^s(y, R)$ and $W_A^s(z, R)$.

Given any rectangle R , we can consider the partition of R by local unstable sets

$$\mathcal{W}_R^u \stackrel{\text{def}}{=} \{W_A^u(x, R): x \in R\}.$$

Since $\mathcal{W}_R^u$ is a measurable partition, Rokhlin's Theorem (see Section 2.1) states that, for every $\eta \in \mathcal{M}_1(\mathbb{T}^d)$ there is a unique family $\{\eta_x^u: x \in R\}$ of conditional probability measures defined on $W_A^u(x, R) \in \mathcal{W}_R^u$ which is a disintegration of $\eta|_R$ with respect to the partition $\mathcal{W}_R^u$.

Fix $x \in R$. Due to the local product structure, each element $W_A^u(y, R) \in \mathcal{W}_R^u$ is identified with the point $h_x^u(y) \in W_A^s(x, R)$, where $h_x^u: R \rightarrow W_A^s(x, R)$ is defined by

$$(2.11) \quad h_x^u: R \rightarrow W_A^s(x, R), \quad h_x^u(y) \stackrel{\text{def}}{=} h_{yx}^u(y).$$

Moreover, the measure $\eta^{\mathcal{W}_R^u}$ defined in (2.5) can be identified with a measure $\hat{\eta}_x^s$ on $W_A^s(x, R)$ by

$$(2.12) \quad \hat{\eta}_x^s \stackrel{\text{def}}{=} \eta^{\mathcal{W}_R^u} \circ (h_x^u)^{-1}.$$

and (2.6) can be written in this case as

$$(2.13) \quad \eta(E) = \int_{W_A^s(x, R)} \eta_y^u(E) d\hat{\eta}_x^s(y).$$

Analogously, we can consider the partition of R by local stable sets

$$\mathcal{W}_R^s \stackrel{\text{def}}{=} \{W_A^s(x, R): x \in R\},$$

and obtain a unique family $\{\eta_x^s: x \in R\}$ of conditional probability measures defined on $W_A^s(x, R) \in \mathcal{W}_R^s$ which is also a disintegration of $\eta|_R$. In this case, the partition space $\mathcal{W}_R^s$ can be identified through the holonomy map with the local unstable manifold $W_A^u(x, R)$ and the measure $\eta^{\mathcal{W}_R^s}$ can be viewed as a measure $\hat{\eta}_x^u$ on $W_A^u(x, R)$, and (2.6) rewrites as

$$(2.14) \quad \eta(E) = \int_{W_A^u(x, R)} \eta_y^s(E) d\hat{\eta}_x^u(y).$$

When η has local product structure (see below), the disintegrations in (2.13) and (2.14) are compatible in the sense that $\eta_x^u \sim \hat{\eta}_x^u$ and $\eta_x^s \sim \hat{\eta}_x^s$ for η -a.e. $x \in R$.

Recall that two measures η, ξ are *equivalent* if $\eta \ll \xi$ and $\xi \ll \eta$; in this case, we write $\eta \sim \xi$.

A measure η on $\mathbb{T}^d$ has *local product structure* if there is $\delta > 0$ such that for every rectangle $R \subseteq \mathbb{T}^d$ with $\eta(R) > 0$ and $\text{diam}(R) < \delta$, it holds

$$\eta|_R \sim \eta_x^u \otimes \eta_x^s \quad \text{for } \eta\text{-a.e. } x \in R.$$

The next lemma characterizes measures with local product structure.

LEMMA 2.1 ([13, Lemma 2.12]). *Let R be a rectangle and η a measure with $\eta(R) > 0$. Then the following conditions are equivalent:*

- (1) $(h_{yz}^s)_* \eta_y^u \ll \eta_z^u$ for η -a.e. $y, z \in R$,
- (2) $(h_{yz}^s)_* \eta_y^u \sim \eta_z^u$ for η -a.e. $y, z \in R$,
- (3) $(h_{yz}^u)_* \eta_y^s \ll \eta_z^s$ for η -a.e. $y, z \in R$,
- (4) $(h_{yz}^u)_* \eta_y^s \sim \eta_z^s$ for η -a.e. $y, z \in R$,
- (5) *there exist $x \in R$ and measures $\tilde{\eta}_x^u, \tilde{\eta}_x^s$ on $W_A^u(x, R), W_A^s(x, R)$ such that $\eta|_R \sim \tilde{\eta}_x^u \otimes \tilde{\eta}_x^s$,*
- (6) $\eta|_R \sim \eta_x^u \otimes \eta_x^s$ for η -a.e. $x \in R$.

A probability measure η with local product structure has *uniformly bounded Radon–Nikodym derivatives (UBD)* if there are constants $C_1, \delta > 0$ such that for every rectangle R with $\text{diam}(R) < \delta$ and for every $y, z \in R$ the Radon–Nikodym derivative of the stable holonomy h_{yz}^s (defined in (2.10)) satisfies

$$(2.15) \quad C_1^{-1} \leq \frac{d(h_{yz}^s)_* \eta_y^u}{d\eta_z^u} \leq C_1.$$

REMARK 2.2. The conditional measures η_x^u depend on the choice of the rectangle R (although this is not reflected in the notation). In fact, the ambiguity only consists of a normalizing constant: if R_1, R_2 are two rectangles and $\{\eta_x^1\}_{x \in R_1}$ and $\{\eta_x^2\}_{x \in R_2}$ the corresponding systems of conditional measures on the sets $W_A^u(x, R_j)$, $j \in \{1, 2\}$, then for η -almost every $x \in R_1 \cap R_2$, the restrictions of the measures η_x^1 and η_x^2 to $W_A^u(x, R_1) \cap W_A^u(x, R_2)$ are scalar multiples of each other (see [13, Lemma 2.10]).

Given a finite covering $R_1, \dots, R_n$ of $\mathbb{T}^d$ by rectangles with $\text{diam}(R_j) < \delta$, for each $x \in \mathbb{T}^d$ we fix some index $j \in \{1, \dots, n\}$ with $x \in R_j$ and define η_x^u as the conditional measure of $\eta|_{R_j}$ on $W_A^u(x, R_j)$. The choice of j is immaterial: by the above, conditional measures corresponding to different rectangles containing x agree up to a scalar multiple on the intersection of the corresponding unstable leaves.

An ergodic measure of A is called *geometric* if it has local product structure, satisfies the UBD property, and has full support. We denote by $\mathcal{G} \stackrel{\text{def}}{=} \mathcal{G}(A) \subset \mathcal{E}(A)$ the set of geometric measures of A . We call every measure in $\mathcal{E}(A) \setminus \mathcal{G}$ *non-geometric*.

2.4. Strongly Geometric measures for Anosov diffeomorphisms

Let η be a probability measure on $\mathbb{T}^d$ that is not necessarily invariant under A . A measurable function $J_\eta : \mathbb{T}^d \rightarrow [0, \infty)$ is a *Jacobian* of A with respect to η if the restriction of J_η to any measurable subset $E \subseteq \mathbb{T}^d$ is integrable with respect to η and satisfies

$$\eta(A(E)) = \int_E J_\eta d\eta.$$

We denote by J_η^n , $n \geq 1$, the Jacobian of A^n with respect to η . We say that the Jacobian J_η has *uniformly bounded distortion* if there exists $C_2 > 0$ such that for every integer $n \geq 1$, every $x \in \mathbb{T}^d$ and every $y \in B_{n+1}(x, r_0)$,

$$(2.16) \quad \frac{1}{C_2} \leq \frac{J_\eta^n(A^n x)}{J_\eta^n(A^n y)} \leq C_2.$$

A standard argument shows that a Hölder-continuous Jacobian has bounded distortion.

A geometric probability measure η has a *uniformly bounded (unstable) Jacobian (UBJ)* if for every $x \in \mathbb{T}^d$, $r_0 > 0$ and $n \geq 1$ it holds that $A_*^n \eta_x^u \ll \eta_{A^n x}^u$ and the Radon–Nikodym derivative $J_{\eta_x^u}^n = \frac{d\eta_{A^n x}^u}{d(A_*^n \eta_x^u)}$ satisfies

$$(2.17) \quad C_2^{-1} \leq J_{\eta_x^u}^n(A^n z) \leq C_2,$$

for every $z \in A^{-n}(W_A^u(A^n x, r_0))$.

In particular, this implies that,

$$(2.18) \quad \frac{1}{C_2^2} \leq \frac{J_{\eta_x^u}^n(A^n z)}{J_{\eta_x^u}^n(A^n y)} \leq C_2^2,$$

for every $y, z \in A^{-n}(W_A^u(A^n x, r_0))$. Therefore, for any u -disk $E \subseteq W_A^u(A^n x, r_0)$ and any measurable subset $E' \subseteq E$, we have

$$(2.19) \quad \frac{1}{C_2^2} \frac{\eta_{A^n x}^u(E')}{\eta_{A^n x}^u(E)} \leq \frac{A_*^n \eta_x^u(E')}{A_*^n \eta_x^u(E)} \leq C_2^2 \frac{\eta_{A^n x}^u(E')}{\eta_{A^n x}^u(E)}.$$

We denote by $\mathcal{S} \subset \mathcal{G}$ the set of *strongly geometric measures*, consisting of those $\eta \in \mathcal{G}$ that satisfy the UBJ property.

If A is transitive, then $\mathcal{S} \neq \emptyset$. Indeed, since A is a transitive Anosov diffeomorphism on the connected manifold $\mathbb{T}^d$, A is topologically mixing. Hence, for any Hölder potential $\psi : \mathbb{T}^d \rightarrow \mathbb{R}$ there is a unique equilibrium state which is a Gibbs measure [7].

LEMMA 2.3 ([7, 13]). *Every Gibbs measure of A associated with a Hölder continuous potential is a strongly geometric measure.*

In order to prove that the union of the basins of $\nu \times \delta_0$ and $\nu \times \delta_1$ is a full $\nu \times \text{Leb}_I$ -measure subset (see Proposition 7.9 below for more details), we will require the following lemmas, which are a slight modification of Lemma 4.3 in [5]. In our case, we need to adapt the original statement for the Lebesgue measure to any strongly geometric probability measure η .

Now we state two lemmas, needed in Chapter 7.

Let η be a Borel probability measure on M and $Z \subseteq M$ be a measurable set. A point $z \in M$ is an η -density point of the set Z if

$$\lim_{\epsilon \rightarrow 0} \frac{\eta(Z \cap B(z, \epsilon))}{\eta(B(z, \epsilon))} = 1.$$

When η is other than Lebesgue, it follows from [15, Section 2.9] (see also [23, section 4]) that η -almost every point in Z is an η -density point.

LEMMA 2.4. *Let $A : \mathbb{T}^d \rightarrow \mathbb{T}^d$ be an Anosov diffeomorphism and η be any strongly geometric measure. Let $r_0 > 0$ be fixed. Then, for any $x \in \mathbb{T}^d$ and any u -disk $\gamma \subseteq W_A^u(x)$ such that $\eta_x^u(\gamma) > 0$ and having a boundary of zero η_x^u -measure, and for any integer n sufficiently large, there exist u -disks $\vartheta_i \subseteq \omega_i \subseteq \gamma$, $i = 1, \dots, j(n)$, such that*

- (i) *the ϑ_i are pairwise disjoint;*
- (ii) *$\eta_x^u(\cup_{i=1}^{j(n)} \omega_i)$ converges to $\eta_x^u(\gamma)$ as $n \rightarrow \infty$;*
- (iii) *each $A^n(\vartheta_i)$, $i = 1, \dots, j(n)$, is a u -ball of radius r_0 inside $A^n(\gamma)$; and*
- (iv) *each $A^n(\omega_i)$, $i = 1, \dots, j(n)$, is a u -ball of radius $2r_0$ inside $A^n(\gamma)$.*

PROOF. Given any large enough n , let $B^u(x_i, r_0)$, $i = 1, \dots, j(n)$, be a maximal family of disjoint u -balls of radius r_0 contained in $A^n(\gamma)$. This means that for any other $x \in A^n(\gamma)$, the u -ball of radius r_0 around x intersects either the boundary of γ or $B^u(x_i, r_0)$ for some $i = 1, \dots, j(n)$. In particular, the family $B^u(x_i, 2r_0)$ covers the set of points in $A^n(\gamma)$ whose distance to the boundary is larger than r_0 . We take

$$(2.20) \quad \vartheta_i = A^{-n}(B^u(x_i, r_0)) \quad \text{and} \quad \omega_i = A^{-n}(B^u(x_i, 2r_0)).$$

We are left to prove part (ii) of the statement. For this, note that the union of the ω_i , $i = 1, \dots, j(n)$ contains the set of points of γ whose distance to the boundary of γ is larger than $\lambda_u^{-n} r_0$. Since the boundary of γ has zero η_x^u -measure, the η_x^u -measure of the complement of this set approaches zero as n approaches infinity. $\square$

LEMMA 2.5. *Let $A : \mathbb{T}^d \rightarrow \mathbb{T}^d$ be an Anosov diffeomorphism, and let η be any strongly geometric measure. Let $\gamma \subseteq \mathbb{T}^d$ be a u -disk, and $\gamma' \subseteq \gamma$ such that $\eta_x^u(\gamma') > 0$, $x \in \gamma$. Then, for every $r_0 > 0$, there exists a sequence of u -balls of radius r_0 , $\gamma^n \subseteq A^n(\gamma)$, $n \geq 1$, such that*

$$\lim_{n \rightarrow \infty} \frac{\eta_{A^n x}^u(\gamma^n \cap A^n(\gamma'))}{\eta_{A^n x}^u(\gamma^n)} = 1.$$

PROOF. Let $z_0 \in \gamma'$ be an η_x^u -density point of γ' . Then, there exists a decreasing sequence of u -balls $\omega_k \subseteq \gamma$ around z_0 with a radius approaching 0, such that

$$(2.21) \quad \lim_{k \rightarrow \infty} \frac{\eta_x^u(\omega_k \cap \gamma')}{\eta_x^u(\omega_k)} = 1.$$

For each sufficiently large and fixed k , we choose $n = n(k)$ large enough such that Lemma 2.4 can be applied to $\gamma = \omega_k$. We may assume that $n(k) \rightarrow \infty$ as $k \rightarrow \infty$. Let ϑ_i^k and ω_i^k , $i = 1, \dots, j(k)$, be the u -disks in γ obtained from this lemma.

Claim: There exists a constant $0 < \tau \leq 1$, independent of $k \geq 1$, such that

$$(2.22) \quad \eta_x^u \left(\bigcup_{i=1}^{j(k)} \vartheta_i^k \right) \geq \tau \eta_x^u(\omega_k), \quad \text{for every } k \geq 1,$$

that is, the union of the ϑ_i^k 's covers a fixed fraction of ω_k for every k . Assuming the claim above, Lemma 2.4 part (ii) completes the proof of the lemma. Indeed, since the ϑ_i^k are pairwise disjoint, it follows from (2.22) combined with (2.21) that we can choose some $i_k \in \{1, \dots, j(k)\}$ such that the sequence $\vartheta_{i_k}^k$ satisfies

$$(2.23) \quad \lim_{k \rightarrow \infty} \frac{\eta_x^u(\vartheta_{i_k}^k \cap \gamma')}{\eta_x^u(\vartheta_{i_k}^k)} = 1.$$

We take

$$(2.24) \quad \gamma^{n(k)} = A^{n(k)}(\vartheta_{i_k}^k).$$

Using (2.19) and (2.23), we have

$$\begin{aligned} \frac{\eta_{A^{n(k)}x}^u(\gamma^{n(k)} \cap A^{n(k)}(\gamma'))}{\eta_{A^{n(k)}x}^u(\gamma^{n(k)})} &= \frac{\eta_{A^{n(k)}x}^u(A^{n(k)}(\vartheta_{i_k}^k \cap \gamma'))}{\eta_{A^{n(k)}x}^u(A^{n(k)}(\vartheta_{i_k}^k))} \\ &= 1 - \frac{\eta_{A^{n(k)}x}^u(A^{n(k)}(\vartheta_{i_k}^k \setminus \gamma'))}{\eta_{A^{n(k)}x}^u(A^{n(k)}(\vartheta_{i_k}^k))} \\ &\geq 1 - C_2^2 \frac{\eta_x^u(\vartheta_{i_k}^k \setminus \gamma')}{\eta_x^u(\vartheta_{i_k}^k)} \rightarrow 1, \end{aligned}$$

and therefore, the proof concludes.

PROOF OF CLAIM. Recall that ϑ_i^k and ω_i^k , $i = 1, \dots, j(k)$, were chosen satisfying

$$A^{n(k)}(\vartheta_i^k) = B^u(z_i^k, r_0) \text{ and } A^{n(k)}(\omega_i^k) = B^u(z_i^k, 2r_0),$$

for certain $z_i^k \in \mathbb{T}^d$. Equivalently, if $\tilde{z}_i^k \in \vartheta_i^k$ is such that $A^{n(k)}(\tilde{z}_i^k) = z_i^k$, then

$$\vartheta_i^k = B^u(\tilde{z}_i^k, n(k), r_0) \text{ and } \omega_i^k = B^u(\tilde{z}_i^k, n(k), 2r_0).$$

In particular, since η_x^u has uniformly bounded unstable Jacobian, from (2.18), we obtain

$$\frac{1}{C_2^2 J_{\eta_x^u}^{n(k)}(\tilde{z}_i^k)} \eta_{A^{n(k)}x}^u(B^u(z_i^k, r_0)) \leq \eta_x^u(\vartheta_i^k) \leq \frac{C_2^2}{J_{\eta_x^u}^{n(k)}(\tilde{z}_i^k)} \eta_{A^{n(k)}x}^u(B^u(z_i^k, r_0)),$$

analogously for ω_i^k . Then,

$$(2.25) \quad \frac{\eta_x^u\left(\bigcup_{i=1}^{j(k)} \vartheta_i^k\right)}{\eta_x^u\left(\bigcup_{i=1}^{j(k)} \omega_i^k\right)} \geq \frac{\sum_{i=1}^{j(k)} \eta_x^u(\vartheta_i^k)}{\sum_{i=1}^{j(k)} \eta_x^u(\omega_i^k)}$$

$$(2.26) \quad \geq \frac{\sum_{i=1}^{j(k)} (C_2^2 J_{\eta_x^u}^{n(k)}(\tilde{z}_i^k))^{-1} \eta_x^u(B^u(z_i^k, r_0))}{\sum_{i=1}^{j(k)} C_2^2 (J_{\eta_x^u}^{n(k)}(\tilde{z}_i^k))^{-1} \eta_x^u(B^u(z_i^k, 2r_0))}.$$

On the other hand, since η_x^u is fully supported along $W_A^u(x)$ and $\mathbb{T}^d$ is compact, for all $r_0 > 0$ there exist $C_{r_0}^+, C_{r_0}^- > 0$ such that

$$(2.27) \quad C_{r_0}^- \leq \eta_x^u(B^u(z, r_0)) \leq C_{r_0}^+,$$

for all $z \in \mathbb{T}^d$. Now, bounding each term in (2.26) by means of (2.27) and cancelling the common factor $\sum_{i=1}^{j(k)} (C_2^2 J_{\eta_x^u}^{n(k)}(\tilde{z}_i^k))^{-1}$, we obtain

$$\frac{\eta_x^u\left(\bigcup_{i=1}^{j(k)} \vartheta_i^k\right)}{\eta_x^u\left(\bigcup_{i=1}^{j(k)} \omega_i^k\right)} \geq \frac{C_{r_0}^- \sum_{i=1}^{j(k)} (C_2^2 J_{\eta_x^u}^{n(k)}(\tilde{z}_i^k))^{-1}}{C_{2r_0}^+ \sum_{i=1}^{j(k)} C_2^2 (J_{\eta_x^u}^{n(k)}(\tilde{z}_i^k))^{-1}} = \frac{C_{r_0}^-}{C_2^4 C_{2r_0}^+}.$$

Taking $\tau = C_{r_0}^- / (C_2^4 C_{2r_0}^+)$, we obtain the claim. □

□

CHAPTER 3

The space of measures of the boundary dynamics

3.1. Decompositions

Throughout this section, we assume that $A : \mathbb{T}^d \rightarrow \mathbb{T}^d$ is a transitive C^r Anosov diffeomorphism. We now recall some well-known properties of the space of A -invariant measures.

LEMMA 3.1. *For every $\nu \in \mathcal{M}(A)$ there exists a sequence $(\nu_n)_n \subset \mathcal{E}(A)$ such that $h_{\nu_n}(A) \rightarrow h_\nu(A)$ and $\nu_n \rightarrow \nu$, in the weak* topology, as $n \rightarrow \infty$.*

PROOF. Recall that the Anosov diffeomorphism A has the specification property. Hence, the density of ergodic measures follows from [34, Theorem 1]. Weak* and entropy density follow from [14, Theorem B]. $\square$

A set $\Omega \subseteq \mathbb{T}^d$ is a *basic set* of A if it is locally maximal, compact, invariant, and $A|_\Omega$ is topologically transitive.

LEMMA 3.2. *For every basic set $\Omega \subset \mathbb{T}^d$ of A , the space $\mathcal{E}(A|_\Omega)$ is arc-connected.*

PROOF. By Lemma 3.1 (which applies to $A|_\Omega$ since, being a transitive locally maximal hyperbolic set, it also has the specification property), the extreme points of $\mathcal{M}(A|_\Omega)$ are dense in this space; hence it is a Poulsen simplex. Therefore, by [24], $\mathcal{E}(A|_\Omega)$ is arc-connected. $\square$

The *convex hull* of a finite set of measures $\{\mu_1, \dots, \mu_k\} \subset \mathcal{M}(A)$ is

$$\text{conv}(\{\mu_1, \dots, \mu_k\}) \stackrel{\text{def}}{=} \left\{ \sum_{j=1}^k t_j \mu_j : t_j \geq 0, \sum_{j=1}^k t_j = 1 \right\}.$$

We state the following slightly strengthened version of the results in [21, Supplement S.5.12], which is contained in its proof¹. For this purpose, recall that for every pair of basic sets for the transitive Anosov diffeomorphism A there exists another one containing both sets.

¹Indeed, [21] deals with ergodic hyperbolic measures; here we are in a (uniformly) hyperbolic context and hence all considered measures are hyperbolic.

PROPOSITION 3.3. *For every $\epsilon > 0$ and every finite collection of ergodic measures $\nu_1, \dots, \nu_k \in \mathcal{E}(A)$:*

(1) *There exists a basic set $\Omega \subset \mathbb{T}^d$ such that*

$$\text{dist}_H(\mathcal{M}(A|_\Omega), \text{conv}(\{\nu_1, \dots, \nu_k\})) < \epsilon.$$

(2) *For each ν_j , $j = 1, \dots, k$, there exists a measure $\nu'_j \in \mathcal{E}(A|_\Omega)$ which is ϵ -close to ν_j and satisfies $|h_{\nu'_j}(A) - h_{\nu_j}(A)| < \epsilon$.*

Given $\psi: \mathbb{T}^d \rightarrow \mathbb{R}$ continuous and $c \in \mathbb{R}$, we consider the set

$$(3.1) \quad \mathcal{E}_{\psi < c}(A) = \{\eta \in \mathcal{E}(A) : \eta(\psi) < c\}.$$

Analogously, we can define spaces replacing $<$ by $>$, $\leq$, $\geq$, or $=$.

From Lemma 3.1, every $\nu \in \mathcal{M}(A)$ is approximated weak* and in entropy by ergodic measures. We can improve such approximation on ν by ergodic measures *inside* either of the spaces $\mathcal{E}_{\psi < c}(A)$ or $\mathcal{E}_{\psi > c}(A)$, for every continuous $\psi: \mathbb{T}^d \rightarrow \mathbb{R}$ and any c belonging to the open interval $(\psi^{\min}, \psi^{\max})$, where

$$(3.2) \quad \psi^{\min} \stackrel{\text{def}}{=} \min_{\nu \in \mathcal{E}(A)} \nu(\psi) \quad \text{and} \quad \psi^{\max} \stackrel{\text{def}}{=} \max_{\nu \in \mathcal{E}(A)} \nu(\psi).$$

It follows from the Ergodic Decomposition Theorem that min and max in (3.2) are indeed attained.

LEMMA 3.4. *Let $\nu \in \mathcal{M}(A)$ and let $\psi: \mathbb{T}^d \rightarrow \mathbb{R}$ be continuous such that*

$$(3.3) \quad \psi^{\min} < \nu(\psi) < \psi^{\max}.$$

Then, for every $\epsilon > 0$, there are $\nu^+, \nu^- \in \mathcal{E}(A)$ satisfying:

- (1) $\text{dist}_{\mathcal{M}(A)}(\nu^\pm, \nu) < \epsilon$;
- (2) $|h_{\nu^\pm}(A) - h_\nu(A)| < \epsilon$;
- (3) $\nu^-(\psi) < \nu(\psi) < \nu^+(\psi)$.

PROOF. Without loss of generality, assume that $\nu(\psi) = 0$. By hypothesis, there exist $\eta^+, \eta^- \in \mathcal{E}(A)$ such that $\psi^{\min} \leq \eta^-(\psi) < 0 < \eta^+(\psi) \leq \psi^{\max}$. For sufficiently small $t > 0$, the measures $\xi_t^\pm \stackrel{\text{def}}{=} t\eta^\pm + (1-t)\nu \in \mathcal{M}(A)$ satisfy (1)–(3) with $\epsilon/3$ in place of ϵ , and $\xi_t^-(\psi) < 0 < \xi_t^+(\psi)$. By Lemma 3.1, choose $\nu^\pm \in \mathcal{E}(A)$ with $\text{dist}_{\mathcal{M}(A)}(\xi_t^\pm, \nu^\pm) < \epsilon/3$ and $|h_{\xi_t^\pm}(A) - h_{\nu^\pm}(A)| < \epsilon/3$. The triangle inequality gives (1) and (2). For (3), note that $\xi_t^-(\psi) < 0 < \xi_t^+(\psi)$, with $\xi_t^\pm(\psi)$ bounded away from 0; since ψ is continuous, $\eta \mapsto \eta(\psi)$

is weak*-continuous, so, choosing the approximation $\nu^\pm$ close enough (which Lemma 3.1 permits), we also obtain $\nu^-(\psi) < \nu(\psi) < \nu^+(\psi)$, that is, (3). $\square$

PROPOSITION 3.5. *Consider a continuous map $\psi: \mathbb{T}^d \rightarrow \mathbb{R}$ and $c \in (\psi^{\min}, \psi^{\max})$. Then $\mathcal{E}_{\psi=c}(A)$ contains a Gibbs measure relative to some Hölder continuous function Ψ .*

The proof relies on the following intermediate step, which establishes the result in the Hölder setting.

LEMMA 3.6. *Assume that $\Psi: \mathbb{T}^d \rightarrow \mathbb{R}$ is Hölder continuous. For every $\beta \in (\Psi^{\min}, \Psi^{\max})$ there is a unique equilibrium state $\nu^{t_\beta \Psi}$ of the potential $t_\beta \Psi$ such that $\nu^{t_\beta \Psi}(\Psi) = \beta$. Moreover, the map $\beta \mapsto \nu^{t_\beta \Psi}$ is continuous in the weak* topology and in entropy.*

PROOF. Recall that if Ψ is a Hölder continuous function, then there exists a unique equilibrium state $\nu^\Psi \in \mathcal{E}(A)$ for Ψ , which is an ergodic Gibbs measure (see [7] and Lemma 2.3). In particular, as the map $t\Psi$ is Hölder continuous for every $t \in \mathbb{R}$, there exists a unique equilibrium state $\nu^{t\Psi}$ for the potential $t\Psi$, which is an ergodic Gibbs measure.

Recall that the pressure $t \mapsto P(t\Psi)$ is Lipschitz continuous and the entropy $t \mapsto h_{\nu^{t\Psi}}(A)$ is upper semicontinuous due to the specification property of A (see [37, Chapter 9]). If $t_n \rightarrow t$ is such that $\nu^{t_n \Psi} \rightarrow \eta$ in the weak* topology, then

$$\begin{aligned}
 h_{\nu^{t\Psi}}(A) + \int t\Psi d\nu^{t\Psi} = P(t\Psi) &= \lim_{n \rightarrow \infty} P(t_n \Psi) \\
 (3.4) \qquad \qquad \qquad &= \lim_{n \rightarrow \infty} h_{\nu^{t_n \Psi}}(A) + \int t_n \Psi d\nu^{t_n \Psi} \\
 &\leq h_\eta(A) + \int t\Psi d\eta.
 \end{aligned}$$

Since $P(t\Psi) \geq h_\eta(A) + \int t\Psi d\eta$ by the variational principle, equality holds throughout, and thus η is also an equilibrium state for $t\Psi$. By uniqueness, $\eta = \nu^{t\Psi}$. Since $\mathcal{M}(A)$ is weak*-compact and every convergent subsequence of $(\nu^{t_n \Psi})_n$ has limit $\nu^{t\Psi}$, the whole sequence converges to $\nu^{t\Psi}$; therefore $t \mapsto \nu^{t\Psi}$ is continuous in the weak* topology. As a consequence, the map $t \mapsto \nu^{t\Psi}(t\Psi)$ is continuous. Since

$$h_{\nu^{t\Psi}}(A) - h_{\nu^{t_n \Psi}}(A) = (P(t\Psi) - P(t_n \Psi)) + \left(\int t_n \Psi d\nu^{t_n \Psi} - \int t\Psi d\nu^{t\Psi} \right),$$

we conclude the continuity of $t \mapsto h_{\nu^{t\Psi}}(A)$.

As a consequence, for every $\beta \in (\Psi^{\min}, \Psi^{\max})$, there is $t_\beta \in \mathbb{R}$ such that $\nu^{t_\beta \Psi}(\Psi) = \beta$. $\square$

We can now prove Proposition 3.5.

PROOF OF PROPOSITION 3.5. Let

$$0 < \epsilon < \frac{1}{4} \min\{|\psi^{\min} - c|, |\psi^{\max} - c|\}$$

and choose some Hölder continuous function Ψ satisfying $\|\Psi - \psi\|_\infty < \epsilon$.

Note that

$$(3.5) \quad |\Psi^{\min} - \psi^{\min}| < \epsilon \quad \text{and} \quad |\Psi^{\max} - \psi^{\max}| < \epsilon.$$

By Lemma 3.6, for each $\beta \in (\Psi^{\min}, \Psi^{\max})$, there is a unique equilibrium state $\nu^{t_\beta \Psi}$, such that $\nu^{t_\beta \Psi}(\Psi) = \beta$ and the map $\beta \mapsto \nu^{t_\beta \Psi}$ is continuous in the weak* topology. Note that, since $\nu^{t_\beta \Psi}(\Psi) = \beta$ and $\|\psi - \Psi\|_\infty < \epsilon$,

$$\nu^{t_\beta \Psi}(\psi) = \nu^{t_\beta \Psi}(\Psi) + \nu^{t_\beta \Psi}(\psi - \Psi) \leq \beta + \|\psi - \Psi\|_\infty < \beta + \epsilon,$$

and analogously $\nu^{t_\beta \Psi}(\psi) \geq \beta - \|\psi - \Psi\|_\infty > \beta - \epsilon$.

Hence, by continuity, it follows that

$$(\Psi^{\min} + \epsilon, \Psi^{\max} - \epsilon) \subset \{\nu^{t_\beta \Psi}(\psi) : \beta \in (\Psi^{\min}, \Psi^{\max})\}$$

and therefore, by (3.5),

$$(\Psi^{\min} + 2\epsilon, \Psi^{\max} - 2\epsilon) \subset \{\nu^{t_\beta \Psi}(\psi) : \beta \in (\Psi^{\min}, \Psi^{\max})\}.$$

As, by hypothesis, c belongs to the latter interval, there exists $t_c \in \mathbb{R}$ such that $\nu^{t_c \Psi}(\psi) = c$. Since $t_c \Psi$ is Hölder continuous, $\nu^{t_c \Psi}$ is the required Gibbs measure. This proves the proposition. $\square$

REMARK 3.7. Note that Proposition 3.5 immediately implies that for every $c \in (\psi^{\min}, \psi^{\max})$ each of $\mathcal{E}_{\psi < c}(A)$ and $\mathcal{E}_{\psi > c}(A)$ also contains a Gibbs measure — in fact, infinitely many.

CHAPTER 4

Dynamics of the Kan-like diffeomorphisms

Throughout this chapter we describe the partially hyperbolic structure, invariant foliations, and decomposition of the space of invariant measures of $K: M \rightarrow M$, a Kan-like C^r -diffeomorphism, $r > 1$.

4.1. Partial hyperbolicity

As in Chapter 1, condition [KD3] asserts that K is partially hyperbolic: there is a DK -invariant splitting of the tangent bundle

$$(4.1) \quad TM = E_K^s \oplus E_K^c \oplus E_K^u,$$

where DK uniformly contracts E_K^s , uniformly expands E_K^u , and E_K^c is the center subbundle dominated by both in the sense that the contraction on E_K^s is stronger than any contraction on E_K^c , and the expansion on E_K^u is stronger than any expansion on E_K^c . A Riemannian metric on M is *adapted* to this splitting if the uniform contraction on E_K^s , the uniform expansion on E_K^u , and the domination of E_K^u over E_K^c and of E_K^c over E_K^s are all exhibited by DK itself, without requiring iteration. Such a metric always exists for any partially hyperbolic splitting [17, Theorem 3]. Working with this adapted metric, there exists a constant $\lambda < 1$ such that, for every $(x, t) \in M$,

$$(4.2) \quad \|DK|_{E_K^s(x,t)}\| < \lambda, \quad \|(DK|_{E_K^u(x,t)})^{-1}\| < \lambda,$$

$$(4.3) \quad \|DK|_{E_K^s(x,t)}\| \cdot \|(DK|_{E_K^c(K(x,t))})^{-1}\| < 1,$$

and

$$(4.4) \quad \|DK|_{E_K^c(x,t)}\| \cdot \|(DK|_{E_K^u(K(x,t))})^{-1}\| < 1.$$

Since K is a skew product over A , $DK(x, t)$ maps the vertical direction $(0, 1) \in T_{(x,t)}M$ to a vertical vector at $K(x, t)$. Denoting $I_x \stackrel{\text{def}}{=} \{x\} \times I$, K maps each fiber I_x to $I_{A(x)}$. The vertical direction is thus a DK -invariant one-dimensional subbundle; condition [KD3] excludes it from E_K^s and E_K^u , giving $E_K^c(x, t) = \langle (0, 1) \rangle$.

For $\alpha > 0$, define the *unstable and stable cone fields of size α* of A by

$$\begin{aligned} C_A^u(x) &= \{v = v^s + v^u \in T_x \mathbb{T}^d : \|v^s\| \leq \alpha \|v^u\|\}, \\ C_A^s(x) &= \{v = v^s + v^u \in T_x \mathbb{T}^d : \|v^u\| \leq \alpha \|v^s\|\}, \end{aligned}$$

with $v^s \in E_A^s(x)$ and $v^u \in E_A^u(x)$.

In the same way, we define the *unstable and stable cone fields of size α* of K : for each $(x, t) \in M$,

$$\begin{aligned} C_K^u(x, t) &= \{w = w^{cs} + w^u \in T_{(x,t)} M : \|w^{cs}\| \leq \alpha \|w^u\|\}, \\ C_K^s(x, t) &= \{w = w^{cu} + w^s \in T_{(x,t)} M : \|w^{cu}\| \leq \alpha \|w^s\|\}, \end{aligned}$$

where $w^{cs} \in E_K^s \oplus E_K^c$, $w^{cu} \in E_K^c \oplus E_K^u$, $w^s \in E_K^s$, and $w^u \in E_K^u$.

If α is chosen sufficiently small, then these cone fields satisfy the following strict invariance properties (see [28, Section 2.2]). The cones of A satisfy

$$\begin{aligned} (4.5) \quad DA(x) C_A^u(x) &\subset \text{int } C_A^u(Ax), \\ DA^{-1}(x) C_A^s(x) &\subset \text{int } C_A^s(A^{-1}x), \end{aligned}$$

and the cones of K , by the partial hyperbolicity conditions (4.2)–(4.4), satisfy

$$\begin{aligned} (4.6) \quad DK(x, t) C_K^u(x, t) &\subset \text{int } C_K^u(K(x, t)), \\ DK^{-1}(x, t) C_K^s(x, t) &\subset \text{int } C_K^s(K^{-1}(x, t)). \end{aligned}$$

In other words, DA maps every vector $v \in C_A^u(x)$ to a vector in $\text{int } C_A^u(Ax)$ with a strictly larger ratio $\|(DAv)^u\|/\|(DAv)^s\|$; that is, DA expands the E_A^u component relative to E_A^s . Analogously, DK expands the E_K^u component of vectors in C_K^u relative to $E_K^s \oplus E_K^c$. The symmetric statements hold for DA^{-1} on C_A^s and DK^{-1} on C_K^s . Henceforth, we fix α satisfying these conditions.

As in the Anosov case (see Section 2.3), it is known that there are unique foliations $\mathcal{W}_K^s$ and $\mathcal{W}_K^u$ tangent to E_K^s and E_K^u , respectively [9, 18]. More precisely, there is $r_0 > 0$ such that, for every $(x, t) \in M$, the set $W_K^s((x, t), r_0)$ is a C^r -embedded disk in M with $T_{(x,t)} W_K^s((x, t), r_0) = E_K^s(x, t)$. Moreover, $W_K^s((x, t), r_0)$ varies continuously with the point $(x, t) \in M$.

The stable manifold $W_K^s(x, t)$ of $(x, t) \in M$ is given by

$$W_K^s(x, t) = \bigcup_{n \geq 0} K^{-n} (W_K^s(K^n(x, t), r_0)),$$

and the *stable foliation* of K is the family

$$\mathcal{W}_K^s = \{W_K^s(x, t) : (x, t) \in M\}.$$

The local unstable manifold of size r_0 , denoted $W_K^u((x, t), r_0)$, and the unstable manifold, $W_K^u(x, t)$, of a point $(x, t) \in M$ are defined in the same way as the local stable manifold and the stable manifold, respectively, replacing K by its inverse. Therefore, we obtain the *unstable foliation* of K , $\mathcal{W}_K^u$, as before.

Moreover, a Kan-like diffeomorphism is *dynamically coherent*: for every $(x, t) \in M$, the manifolds $W_K^c(x, t) = I_x$, $W_K^{cs}(x, t) = W_A^s(x) \times I$ and $W_K^{cu}(x, t) = W_A^u(x) \times I$ are tangent to the directions $E_K^c(x, t)$, $E_K^{cs}(x, t) \stackrel{\text{def}}{=} E_K^s(x, t) \oplus E_K^c(x, t)$ and $E_K^{cu}(x, t) \stackrel{\text{def}}{=} E_K^c(x, t) \oplus E_K^u(x, t)$, respectively.

4.2. Projection of the dynamics onto the boundary

We denote by $\pi : M \rightarrow \mathbb{T}^d$ the natural projection defined by $\pi(x, t) \stackrel{\text{def}}{=} x$, $x \in \mathbb{T}^d$. Since K is a skew product over A , we have the semiconjugacy

$$(4.7) \quad \pi \circ K = A \circ \pi.$$

Condition [\[KD1\]](#) means that K preserves each boundary component of M and $K(x, j) = (A(x), j)$, for every $x \in \mathbb{T}^d$ and $j \in \{0, 1\}$. Therefore,

$$E_K^s(x, j) = E_A^s(x) \times \{0\}, \quad \text{and} \quad E_K^u(x, j) = E_A^u(x) \times \{0\}.$$

Similarly, the cone fields of K restrict on the boundary to those of A :

$$C_K^u(x, j) \cap T_x \mathbb{T}^d = C_A^u(x), \quad C_K^s(x, j) \cap T_x \mathbb{T}^d = C_A^s(x),$$

where $T_x \mathbb{T}^d$ is identified with the horizontal subspace of $T_{(x, j)} M$.

Moreover,

$$W_K^s(x, j) = W_A^s(x) \times \{j\} \quad \text{and} \quad W_K^u(x, j) = W_A^u(x) \times \{j\},$$

and $\pi(W_K^s(x, t)) = W_A^s(x)$, $\pi(W_K^u(x, t)) = W_A^u(x)$, for every $(x, t) \in M$, which follows from [\(4.7\)](#). In particular, the constant λ in [\(4.2\)](#)–[\(4.4\)](#) and the cone size α in [\(4.5\)](#)–[\(4.6\)](#) can be taken the same for K on M and for A on ∂M .

4.3. Elevators

The center-stable manifolds of the saddle points guaranteed by condition [\[KD2\]](#) play a key role in the dynamics of K : they connect the two boundary components through the interior of M , acting as *elevators* that allow ergodic information to be transported between $\mathbb{T}^d \times \{0\}$ and $\mathbb{T}^d \times \{1\}$. Their precise structure is as follows.

By condition [\[KD2\]](#), there are hyperbolic (fixed) saddles $(p, 0)$ and $(q, 1)$ with stability index $d^s \stackrel{\text{def}}{=} \dim E_A^s$. Similarly, $(p, 1)$ and $(q, 0)$ are hyperbolic saddles with stability index

$d^s + 1$. Since $(p, 1)$ has stability index $d^s + 1$, the center direction at $(p, 1)$ is also contracting, so its stable manifold coincides with its center-stable manifold:

$$W_K^s(p, 1) = W_K^{cs}(p, 1) = W_A^s(p) \times (0, 1].$$

Note that the boundary of $W_K^{cs}(p, 1)$ contains $W_A^s(p) \times \{1\}$. A similar situation occurs with the stable manifold $W_K^{cs}(q, 0)$ of $(q, 0)$. Moreover, for any fixed $L > 0$, there exists $r_L > 0$ such that the $(d^s + 1)$ -dimensional manifolds

$$(4.8) \quad \begin{aligned} \Sigma_p &\stackrel{\text{def}}{=} W_A^s(p, r_L) \times [0, 1) \subseteq W_K^{cs}(p, 0), \text{ and} \\ \Sigma_q &\stackrel{\text{def}}{=} W_A^s(q, r_L) \times (0, 1] \subseteq W_K^{cs}(q, 1), \end{aligned}$$

are such that, for every $(x, t) \in \mathbb{T}^d \times (0, 1)$, the local unstable manifold $W_K^u((x, t), L)$ of size L transversely intersects Σ_p and Σ_q (as guaranteed by condition [\[KD2\]](#)). The endpoint $t = 1$ (resp. $t = 0$) is excluded from Σ_p (resp. Σ_q) because $(p, 1)$ (resp. $(q, 0)$) is a distinct elevator fixed point, of stability index $d^s + 1$, with its own center-stable manifold.

4.4. Decomposition of the space of invariant measures

The Kan-like map K satisfies $K(\partial M) = \partial M$ and $K(\text{int}(M)) = \text{int}(M)$. Therefore, the space of K -invariant Borel probability measures $\mathcal{M}(K)$ splits as follows

$$\mathcal{M}(K) = \mathcal{M}^\partial(K) \cup \mathcal{M}^{\text{int}}(K),$$

where $\mathcal{M}^\partial(K)$ is the set of K -invariant measures supported on ∂M , and $\mathcal{M}^{\text{int}}(K)$ is the set of K -invariant measures such that their supports intersect $\text{int}(M)$. Note that the support of a measure in $\mathcal{M}^{\text{int}}(K)$ may intersect ∂M as well.

Similarly, one obtains a decomposition of the ergodic measures of K :

$$\mathcal{E}(K) = \mathcal{E}^\partial(K) \cup \mathcal{E}^{\text{int}}(K),$$

where $\mathcal{E}^\partial(K) \stackrel{\text{def}}{=} \mathcal{E}(K) \cap \mathcal{M}^\partial(K)$ and $\mathcal{E}^{\text{int}}(K) \stackrel{\text{def}}{=} \mathcal{E}(K) \cap \mathcal{M}^{\text{int}}(K)$.

Note that

$$\mathcal{M}^\partial(K) = \mathcal{M}^0(K) \cup \mathcal{M}^1(K) \quad \text{and} \quad \mathcal{E}^\partial(K) = \mathcal{E}^0(K) \cup \mathcal{E}^1(K),$$

where $\mathcal{M}^j(K)$ (resp. $\mathcal{E}^j(K)$) is the set of invariant (resp. ergodic) measures supported on $\mathbb{T}^d \times \{j\}$, and

$$\mathcal{M}^j(K) = \{\nu \times \delta_j : \nu \in \mathcal{M}(A)\} = \mathcal{M}(A) \times \{\delta_j\}$$

and

$$(4.9) \quad \mathcal{E}^j(K) = \{\nu \times \delta_j : \nu \in \mathcal{E}(A)\} = \mathcal{E}(A) \times \{\delta_j\}.$$

Consider the map $J: M \rightarrow \mathbb{R}$ defined by

$$J(x, t) = \log |\partial_t \varphi(x, t)|, \quad (x, t) \in M.$$

For any $\mu \in \mathcal{E}(K)$, the *central Lyapunov exponent* of μ is

$$\chi^c(\mu) = \int_M J(x, t) d\mu(x, t).$$

Recalling (3.1), given a potential ψ and a number c , we define the sets $\mathcal{E}_{\psi < c}(K)$, $\mathcal{E}_{\psi < c}^\partial(K)$, $\mathcal{E}_{\psi < c}^j(K)$, and $\mathcal{E}_{\psi < c}^{\text{int}}(K)$.¹

We are particularly interested in the case $\psi = J$ and $c = 0$, where J is the potential defined above. In such a case, $\mathcal{E}_{J < 0}(K)$ (or $\mathcal{E}_{<}(K)$) is the set of ergodic measures of K having a negative central Lyapunov exponent².

As mentioned above, the set $\mathcal{E}_{<}(K)$ can be split as

$$\mathcal{E}_{<}(K) = \mathcal{E}_{<}^\partial(K) \cup \mathcal{E}_{<}^{\text{int}}(K)$$

and

$$\mathcal{E}_{<}^\partial(K) = \mathcal{E}_{<}^0(K) \cup \mathcal{E}_{<}^1(K).$$

Fixing $j \in \{0, 1\}$, we can induce on $\mathbb{T}^d$ the potential J^j defined by

$$J^j(x) \stackrel{\text{def}}{=} J(x, j), \quad x \in \mathbb{T}^d.$$

Let $\mu \in \mathcal{E}^j(K)$. In view of (4.9), there exists a unique $\nu \in \mathcal{E}(A)$ such that $\mu = \nu \times \delta_j$. Then we have,

$$\begin{aligned} \chi^c(\mu) &= \int_M J(x, t) d(\nu \times \delta_j)(x, t) \\ &= \int_{\mathbb{T}^d} J(x, j) d\nu(x) \\ &= \int_{\mathbb{T}^d} J^j(x) d\nu(x). \end{aligned}$$

Hence, $\mathcal{E}_{<}^j(K)$ can be identified with $\mathcal{E}_{J^j < 0}(A)$.

For notational simplicity, we write

$$\mathcal{E}_\dagger(K) = \mathcal{E}_{J^\dagger 0}(K), \quad \dagger \in \{<, >, \leq, \geq, =\}$$

and

$$\mathcal{E}_\dagger^j(A) = \mathcal{E}_{J^j \dagger 0}(A), \quad \dagger \in \{<, >, \leq, \geq, =\}.$$

¹Analogously, we can define spaces by replacing $<$ with $>, \leq, \geq$, or $=$.

²For simplicity, $\psi = J$ and $c = 0$ are omitted in the notation. For instance, $\mathcal{E}_{J \geq 0}(K)$ becomes $\mathcal{E}_{\geq}(K)$.

Note that we can similarly define the set $\mathcal{E}_{\psi < c}(A)$ and its variations for $>$, $\leq$, $\geq$, or $=$.

CHAPTER 5

Ergodic Basins

In this chapter, we introduce some general terminology and properties. Consider a diffeomorphism $g: M \rightarrow M$ defined on the Riemannian manifold M . The *basin* of a measure $\mu \in \mathcal{M}(g)$ is the set

$$\mathcal{B}_g(\mu) = \left\{ x \in M : \frac{1}{n} \sum_{k=0}^{n-1} \delta_{g^k(x)} \xrightarrow{*} \mu, n \rightarrow \infty \right\}.$$

The points in $\mathcal{B}_g(\mu)$ are called μ -generic. Note that $\mathcal{B}_g(\mu)$ is g -invariant.

A point $x \in M$ is called μ -Birkhoff regular (or simply μ -regular) if it has well-defined Birkhoff averages for the future and the past and these averages are equal. As a consequence of Birkhoff Theorem, the set of μ -regular points is g -invariant, has full μ -measure, and μ -almost every point is regular and belongs to $\mathcal{B}_g(\mu) \cap \mathcal{B}_{g^{-1}}(\mu)$.

Note that for every $x \in \mathcal{B}(\mu)$ it holds $W_g^s(x) \subseteq \mathcal{B}(\mu)$, where $W_g^s(x)$ is the *stable set* of x defined by

$$W_g^s(x) \stackrel{\text{def}}{=} \{x' \in M : \text{dist}_M(g^n(x), g^n(x')) \rightarrow 0, n \rightarrow \infty\}.$$

Therefore $\mathcal{B}(\mu)$ is an *s-saturated set*, that is,

$$\mathcal{B}(\mu) = \bigcup_{x \in \mathcal{B}(\mu)} W_g^s(x).$$

Given a reference Borel probability measure m (not necessarily g -invariant), we say the measure $\mu \in \mathcal{M}(g)$ is *observable with respect to m* (or simply μ is *m -observable*) if $m(\mathcal{B}(\mu)) > 0$.

Two measures $\mu_0, \mu_1 \in \mathcal{E}(K)$ are *m -intermingled* if for every open set $U \subset M$, it holds $m(U \cap \mathcal{B}(\mu_0)) > 0$ and $m(U \cap \mathcal{B}(\mu_1)) > 0$.

5.1. Graph measures and a natural order

Let $K: M \rightarrow M$ be a *Kan-like* map, defined by (1.3). We begin by recalling how invariant measures disintegrate along the fibers and how ergodic measures of K project onto ergodic measures for the base dynamics.

For any $\mu \in \mathcal{E}(K)$, we can consider its disintegration $\{\mu_x^{\mathcal{I}} : x \in \mathbb{T}^d\}$ with respect to the partition $\mathcal{I} = \{I_x = \{x\} \times I : x \in \mathbb{T}^d\}$. In this case, the partition can be naturally identified with $\mathbb{T}^d$ and the quotient measure $\mu^{\mathcal{I}}$ can be identified with a measure $\nu \in \mathcal{E}(A)$. Hence, the conditional probabilities $\mu_x^{\mathcal{I}}$ are supported in $\{x\} \times I$ and (2.6) becomes

$$(5.1) \quad \mu(E) = \int_{\mathbb{T}^d} \mu_x^{\mathcal{I}}(E) d\nu(x).$$

In the opposite direction, given $\nu \in \mathcal{E}(A)$, define the set

$$\mathcal{E}_\nu(K) = \{\mu \in \mathcal{E}(K) : \pi_*\mu = \nu\}.$$

Then,

$$\mathcal{E}(K) = \bigcup_{\nu \in \mathcal{E}(A)} \mathcal{E}_\nu(K).$$

For every $\nu \in \mathcal{E}(A)$, the set $\mathcal{E}_\nu(K)$ always contains the measures

$$\nu^j \stackrel{\text{def}}{=} \nu \times \delta_j, \quad j \in \{0, 1\}.$$

We let $\mathcal{E}_\nu^{\text{int}}(K) \stackrel{\text{def}}{=} \mathcal{E}_\nu(K) \cap \mathcal{E}^{\text{int}}(K)$.

Let us now introduce a natural order in the set of measures in $\mathcal{E}_\nu(K)$. Given $\mu, \eta \in \mathcal{E}_\nu(K)$, consider the sets

$$(5.2) \quad \{x \in \mathbb{T}^d : \int_0^1 t d\mu_x^{\mathcal{I}}(t) < \int_0^1 t d\eta_x^{\mathcal{I}}(t)\},$$

$$(5.3) \quad \{x \in \mathbb{T}^d : \int_0^1 t d\mu_x^{\mathcal{I}}(t) = \int_0^1 t d\eta_x^{\mathcal{I}}(t)\},$$

$$(5.4) \quad \{x \in \mathbb{T}^d : \int_0^1 t d\mu_x^{\mathcal{I}}(t) > \int_0^1 t d\eta_x^{\mathcal{I}}(t)\}.$$

Each of these sets is measurable and A -invariant. By ergodicity of ν , just one of them has full ν -measure. We say that $\mu < \eta$ if (5.2) has full ν -measure, $\mu = \eta$ if (5.3) has full ν -measure, and $\eta < \mu$ if (5.4) has full ν -measure. Analogously, we can define inequalities with $\leq$ or $\geq$.

Note that $\mu < \eta$ if, and only if,

$$(5.5) \quad \int_M t d\mu(x, t) < \int_M t d\eta(x, t).$$

A measurable function $\phi : \mathbb{T}^d \rightarrow I$ is an *invariant graph* (with respect to K) if for ν -almost every $x \in \mathbb{T}^d$

$$(5.6) \quad K(x, \phi(x)) = (A(x), \phi(A(x))),$$

We will also call the corresponding set of points

$$(5.7) \quad \Gamma(\phi) \stackrel{\text{def}}{=} \{(x, \phi(x)) : x \in \mathbb{T}^d\},$$

an *invariant graph*.

Given $\nu \in \mathcal{E}(A)$ and some invariant graph $\phi : \mathbb{T}^d \rightarrow I$, we can define a K -ergodic Borel probability measure in $\mathcal{E}_\nu(K)$, by

$$(5.8) \quad \mu_\phi(E) \stackrel{\text{def}}{=} \nu(\pi(E \cap \Gamma(\phi))) \quad \text{for every } E \subset M \text{ measurable,}$$

where $\pi : M \rightarrow \mathbb{T}^d$ is the natural projection defined by $\pi(x, t) = x$. Following the arguments to prove [2, Theorem 1.8.4 (iv)], one can show that the converse is also true. More precisely,

LEMMA 5.1. *For every $\mu \in \mathcal{E}_\nu(K)$, there exists an invariant graph ϕ_μ such that $\mu = \mu_{\phi_\mu}$.*

PROOF. Let $\mu \in \mathcal{E}_\nu(K)$. For ν -almost every $x \in \mathbb{T}^d$, let

$$(5.9) \quad \phi_\mu(x) \stackrel{\text{def}}{=} \inf\{t \in I : \mu_x^\mathcal{I}(\{x\} \times [0, t]) \geq 1/2\},$$

and the sets $E_x^- = \{x\} \times [0, \phi_\mu(x)]$, $E_x^+ = \{x\} \times [\phi_\mu(x), 1]$, and $E^\pm = \bigcup_{x \in \mathbb{T}^d} E_x^\pm$.

Since μ is K -invariant, $K_*\mu_x^\mathcal{I} = \mu_{Ax}^\mathcal{I}$ for ν -almost every $x \in \mathbb{T}^d$; as $\varphi(x, \cdot)$ is a non-decreasing homeomorphism of I , it preserves the median of $\mu_x^\mathcal{I}$, so $\phi_\mu(Ax) = \varphi(x, \phi_\mu(x))$ and hence $K(E_x^\pm) = E_{Ax}^\pm$. Therefore $E^\pm$ are K -invariant, and since μ is ergodic, $\mu(E^\pm) \in \{0, 1\}$. It follows from (5.1) that

$$\mu(E^-) = \int_{\mathbb{T}^d} \mu_x^\mathcal{I}(E_x^-) d\nu(x) \geq \frac{1}{2},$$

and therefore $\mu(E^-) = 1$. Note that from (5.9),

$$\phi_\mu(x) = \sup\{t \in I : \mu_x^\mathcal{I}(\{x\} \times [t, 1]) \geq 1/2\},$$

and arguing as above, $\mu(E^+) = 1$. Hence, $\mu_x^\mathcal{I}(E_x^\pm) = 1$, for ν -almost every point $x \in \mathbb{T}^d$.

Since $E_x^- \cap E_x^+ = \{(x, \phi_\mu(x))\}$, for ν -almost every point $x \in \mathbb{T}^d$ we have

$$\mu_x^\mathcal{I}(\{(x, \phi_\mu(x))\}) = \mu_x^\mathcal{I}(E_x^- \cap E_x^+) = 1.$$

Therefore, the resulting function $\mathbb{T}^d \ni x \mapsto \phi_\mu(x) \in I$ is measurable, its graph is K -invariant and $\mu_x^\mathcal{I} = \delta_{(x, \phi_\mu(x))}$, for ν -almost every $x \in \mathbb{T}^d$. Then $\mu = \mu_{\phi_\mu}$. $\square$

5.2. Geometry of the basins

Let $K: M \rightarrow M$ be a *Kan-like* C^r -diffeomorphism, $r > 1$, defined by (1.3). The fiber map $\varphi: M \rightarrow I$ is a C^r map, $r > 1$, preserving the boundaries. Therefore, for every $x \in \mathbb{T}^d$, $\varphi(x, \cdot): I \rightarrow I$ is a nondecreasing C^r -diffeomorphism. We denote by $I_x \stackrel{\text{def}}{=} \{x\} \times I$.

For $n \geq 1$, the n -th iterate of K at $(x, t) \in M$ is

$$K^n(x, t) = (A^n x, \varphi^n(x, t)),$$

where

$$\varphi^n(x, t) = \varphi(A^{n-1}x, \varphi^{n-1}(x, t)).$$

The next statement says that for every invariant measure $\mu \in \mathcal{M}(K)$, its basin intersects each fiber I_x in an interval.

PROPOSITION 5.2. *Let $\mu \in \mathcal{M}(K)$. Suppose that $(x, t), (x, s) \in \mathcal{B}(\mu)$, where $x \in \mathbb{T}^d$ and $0 \leq s \leq t \leq 1$. Then, $(x, \xi) \in \mathcal{B}(\mu)$ for every $s \leq \xi \leq t$.*

We say that a continuous function $\psi: M \rightarrow \mathbb{R}$ is *nondecreasing along the fibers* if for every $x \in \mathbb{T}^d$ and $0 \leq s \leq t \leq 1$, we have

$$\psi(x, s) \leq \psi(x, t).$$

The proof of Proposition 5.2 relies on the following lemma.

LEMMA 5.3. *Let $\mu \in \mathcal{M}(K)$ and $\psi: M \rightarrow \mathbb{R}$ be a C^1 -function. If $(x, s), (x, t) \in \mathcal{B}(\mu)$, where $x \in \mathbb{T}^d$ and $0 \leq s \leq t \leq 1$, then*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \psi(K^j(x, \xi)) = \int \psi d\mu,$$

for every $s \leq \xi \leq t$.

PROOF. Claim: Let $\psi: M \rightarrow \mathbb{R}$ be a C^1 -function. Then there exist continuous functions $u, v: M \rightarrow \mathbb{R}$, nondecreasing along the fibers, such that $\psi = u - v$.

PROOF OF CLAIM. We adapt the proof of the Jordan Decomposition Theorem for bounded variation functions (see [30, Section 6.3]). For every $(x, t) \in M$, define $V_\psi: M \rightarrow \mathbb{R}$ by

$$V_\psi(x, t) = \int_0^t |\partial_t \psi(x, s)| ds.$$

Note that V_ψ is C^0 in the first variable and C^1 and nondecreasing in the fiber. We define

$$u(x, t) \stackrel{\text{def}}{=} \frac{V_\psi(x, t) + \psi(x, t)}{2}, \quad v(x, t) \stackrel{\text{def}}{=} \frac{V_\psi(x, t) - \psi(x, t)}{2}.$$

Since $\partial_t V_\psi(x, t) = |\partial_t \psi(x, t)|$, we have

$$\partial_t u(x, t) = \frac{|\partial_t \psi(x, t)| + \partial_t \psi(x, t)}{2} \geq 0 \quad \text{and} \quad \partial_t v(x, t) = \frac{|\partial_t \psi(x, t)| - \partial_t \psi(x, t)}{2} \geq 0,$$

so u and v are continuous and nondecreasing along the fibers. It is clear that

$$\psi(x, t) = u(x, t) - v(x, t), \quad \text{for every } (x, t) \in M. \quad \square$$

Claim: Let $\mu \in \mathcal{M}(K)$ and $\psi: M \rightarrow \mathbb{R}$ be continuous and nondecreasing along the fibers. If $(x, s), (x, t) \in \mathcal{B}(\mu)$, where $x \in \mathbb{T}^d$ and $0 \leq s \leq t \leq 1$, then

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \psi(K^j(x, \xi)) = \int \psi d\mu,$$

for every $s \leq \xi \leq t$.

PROOF OF CLAIM. Recall that $\varphi(x, \cdot)$ is a nondecreasing diffeomorphism of I , hence for $s \leq \xi \leq t$,

$$\varphi^n(x, s) \leq \varphi^n(x, \xi) \leq \varphi^n(x, t), \quad \text{for all } n \geq 1.$$

Since ψ is nondecreasing along the fibers, it follows that

$$\psi(K^n(x, s)) \leq \psi(K^n(x, \xi)) \leq \psi(K^n(x, t)), \quad \text{for every } n \geq 1,$$

and therefore

$$\frac{1}{n} \sum_{j=0}^{n-1} \psi(K^j(x, s)) \leq \frac{1}{n} \sum_{j=0}^{n-1} \psi(K^j(x, \xi)) \leq \frac{1}{n} \sum_{j=0}^{n-1} \psi(K^j(x, t)).$$

Since $(x, s), (x, t) \in \mathcal{B}(\mu)$, taking $n \rightarrow \infty$ we conclude that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \psi(K^j(x, \xi)) = \int \psi d\mu. \quad \square$$

By the first Claim, write $\psi = u - v$ where u and v are continuous and nondecreasing along the fibers. By the second Claim applied to u and v , for every $s \leq \xi \leq t$,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} u(K^j(x, \xi)) = \int u d\mu \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} v(K^j(x, \xi)) = \int v d\mu.$$

Hence

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \psi(K^j(x, \xi)) &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} u(K^j(x, \xi)) - \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} v(K^j(x, \xi)) \\ &= \int u d\mu - \int v d\mu = \int \psi d\mu. \quad \square \end{aligned}$$

We are now in a position to prove Proposition 5.2.

PROOF OF PROPOSITION 5.2. Let $(x, s), (x, t) \in \mathcal{B}(\mu)$ with $0 \leq s \leq t \leq 1$, and fix $s \leq \xi \leq t$. Let $\psi: M \rightarrow \mathbb{R}$ be continuous. Then, there exists a sequence $\psi_k: M \rightarrow \mathbb{R}$, $k \geq 1$, of C^1 -functions such that $\psi_k \rightarrow \psi$, $k \rightarrow \infty$, in the C^0 -topology. That means, for every $\epsilon > 0$, there exists $k_0 \geq 1$ such that, for every $y \in M$ and $k \geq k_0$,

$$(5.10) \quad |\psi(y) - \psi_k(y)| < \epsilon,$$

and so

$$(5.11) \quad \int_M |\psi - \psi_k| d\mu < \epsilon,$$

and then, for every $n \geq 1$

$$(5.12) \quad \frac{1}{n} \sum_{j=0}^{n-1} |\psi(K^j(x, \xi)) - \psi_k(K^j(x, \xi))| < \epsilon.$$

Fix $k \geq k_0$. Then, for every $n \geq 1$,

$$\begin{aligned} \frac{1}{n} \sum_{j=0}^{n-1} \psi(K^j(x, \xi)) - \int_M \psi d\mu &= \frac{1}{n} \sum_{j=0}^{n-1} \psi(K^j(x, \xi)) - \frac{1}{n} \sum_{j=0}^{n-1} \psi_k(K^j(x, \xi)) \\ &\quad + \frac{1}{n} \sum_{j=0}^{n-1} \psi_k(K^j(x, \xi)) - \int_M \psi_k d\mu \\ (5.13) \quad &\quad + \int_M \psi_k d\mu - \int_M \psi d\mu \end{aligned}$$

It follows from (5.11) and (5.12) that, for every $n \geq 1$,

$$\left| \frac{1}{n} \sum_{j=0}^{n-1} \psi(K^j(x, \xi)) - \int_M \psi d\mu \right| < 2\epsilon + \left| \frac{1}{n} \sum_{j=0}^{n-1} \psi_k(K^j(x, \xi)) - \int_M \psi_k d\mu \right|.$$

From Lemma 5.3, there exists $n_0 \geq 1$ such that, for every $n \geq n_0$,

$$\left| \frac{1}{n} \sum_{j=0}^{n-1} \psi_k(K^j(x, \xi)) - \int_M \psi_k d\mu \right| < \epsilon,$$

completing the proof. □

Having established that statistical basins are intervals along fibers, we now define the measurable graphs that capture the boundary of $\mathcal{B}(\mu)$. Given $\mu \in \mathcal{E}_\nu(K)$, we consider the *measurable graphs* $\phi_\mu^\pm: \mathbb{T}^d \rightarrow I$ defined by

$$(5.14) \quad \begin{aligned} \phi_\mu^+(x) &= \sup\{t \in [0, 1]: (x, t) \in \mathcal{B}(\mu)\}, \\ \phi_\mu^-(x) &= \inf\{t \in [0, 1]: (x, t) \in \mathcal{B}(\mu)\}. \end{aligned}$$

Note that both are invariant graphs as defined in (5.6). Then as in (5.8), we define the measures $\mu^\pm = \mu_{\phi_\mu^\pm} \in \mathcal{E}_\nu(K)$. The next lemma follows directly from the definitions.

LEMMA 5.4. $\mu^- \leq \mu \leq \mu^+$ for every $\mu \in \mathcal{E}_\nu(K)$.

The next lemma is a direct consequence of Lemma 5.1 and Proposition 5.2.

LEMMA 5.5. Let $\mu \in \mathcal{E}_\nu(K)$. For ν -almost every point $x \in \mathbb{T}^d$, where $\phi_\mu^\pm$ are as in (5.14), one of the following possibilities is satisfied:

- (1) If $\phi_\mu^-(x) < \phi_\mu(x) < \phi_\mu^+(x)$, then $\mathcal{B}(\mu) \cap I_x = \{x\} \times (\phi_\mu^-(x), \phi_\mu^+(x))$;
- (2) If $\phi_\mu^-(x) = \phi_\mu^+(x)$, then $\mathcal{B}(\mu) \cap I_x = \{(x, \phi_\mu(x))\}$;
- (3) If $\phi_\mu^-(x) = \phi_\mu(x) < \phi_\mu^+(x)$, then $\mathcal{B}(\mu) \cap I_x = \{x\} \times [\phi_\mu^-(x), \phi_\mu^+(x))$ and
- (4) If $\phi_\mu^-(x) < \phi_\mu(x) = \phi_\mu^+(x)$, then $\mathcal{B}(\mu) \cap I_x = \{x\} \times (\phi_\mu^-(x), \phi_\mu^+(x)]$.

The converse statements are also true.

PROOF. Let us prove the first item. Assume $\phi_\mu^-(x) < \phi_\mu(x) < \phi_\mu^+(x)$. It follows from Proposition 5.2 that $\{x\} \times (\phi_\mu^-(x), \phi_\mu^+(x)) \subseteq \mathcal{B}(\mu) \cap I_x$. We argue by contradiction. The set $S \stackrel{\text{def}}{=} \{x \in \mathbb{T}^d: (x, \phi_\mu^+(x)) \in \mathcal{B}(\mu)\}$ is A -invariant, because $\Gamma(\phi_\mu^+)$ is a K -invariant graph and $\mathcal{B}(\mu)$ is K -invariant; hence, by ergodicity of ν , $\nu(S) \in \{0, 1\}$. Suppose $\nu(S) = 1$, that is, $(x, \phi_\mu^+(x)) \in \mathcal{B}(\mu)$ for ν -almost every $x \in \mathbb{T}^d$. Then, the invariant graph $\Gamma(\phi_\mu^+) \subseteq \mathcal{B}(\mu)$. On the other hand, $\Gamma(\phi_\mu^+) \subset \mathcal{B}(\mu^+)$. Hence, we get $\mathcal{B}(\mu) \cap \mathcal{B}(\mu^+) \neq \emptyset$ which implies $\mu = \mu^+$; this contradicts the standing assumption $\phi_\mu(x) < \phi_\mu^+(x)$ (equivalently, $\mu < \mu^+$). Therefore $\nu(S) = 0$, that is, $(x, \phi_\mu^+(x)) \notin \mathcal{B}(\mu)$ for ν -almost every x , so the upper endpoint is excluded and $\mathcal{B}(\mu) \cap I_x = \{x\} \times (\phi_\mu^-(x), \phi_\mu^+(x))$. The other cases are analogous. $\square$

COROLLARY 5.6. Let $\mu \in \mathcal{E}_\nu(K)$. If $\phi_\mu(x) < \phi_\mu^+(x)$ for ν -almost every $x \in \mathbb{T}^d$, then there is no $\mu' \in \mathcal{E}_\nu(K)$ such that $\mu_\phi < \mu' < \mu^+$. A similar conclusion holds if $\phi_\mu^-(x) < \phi_\mu(x)$ for ν -almost every $x \in \mathbb{T}^d$.

These results show that every ergodic measure projecting onto ν admits a precise description in terms of invariant graphs and that the geometry of its basin along the fibers is completely encoded by the relative position of these graphs. This structure will be

crucial in the proof of Theorem B and in the analysis of intermingled basins in the next section.

5.3. Basins of hyperbolic measures

Let $\mu \in \mathcal{E}(K)$. By the Multiplicative Ergodic Theorem of Oseledets, for μ -almost every $(x, t) \in M$ there exist Lyapunov exponents $\chi_1 > \dots > \chi_k$ and a DK -invariant decomposition

$$TM = E_{(x,t)}^1 \oplus \dots \oplus E_{(x,t)}^k,$$

such that $\chi_i = \lim_{n \rightarrow \pm\infty} \frac{1}{n} \log \|DK^n(x, t)v\|$ for every $v \in E_{(x,t)}^i \setminus \{0\}$. Since μ is ergodic, these exponents are μ -almost everywhere constant. Moreover, this decomposition is compatible with the partially hyperbolic splitting (4.1): there exists $1 < \ell < k$ such that

$$E_K^u = E_{(x,t)}^1 \oplus \dots \oplus E_{(x,t)}^{\ell-1}, \quad E_K^c = E_{(x,t)}^\ell, \quad E_K^s = E_{(x,t)}^{\ell+1} \oplus \dots \oplus E_{(x,t)}^k,$$

with $\chi_1 > \dots > \chi_{\ell-1} > 0$ and $0 > \chi_{\ell+1} > \dots > \chi_k$. Since μ is ergodic, the central exponent χ_ℓ is μ -almost everywhere constant and coincides with $\chi^c(\mu) = \int_M J d\mu$; its sign is a priori unknown. We say that μ is *hyperbolic* if $\chi^c(\mu) \neq 0$. In the following, we describe some geometric aspects of the basin $\mathcal{B}(\mu)$ when μ is hyperbolic.

LEMMA 5.7. *Let $\nu \in \mathcal{E}(A)$. If $\mu \in \mathcal{E}_\nu(K)$ has negative central Lyapunov exponent, then for ν -almost every $x \in \mathbb{T}^d$, the set $\mathcal{B}(\mu) \cap I_x$ contains an open interval.*

PROOF. Let $\mu \in \mathcal{E}(K)$ have negative central Lyapunov exponent. Then μ -almost every point $(x, t) \in M$ belongs to its basin $\mathcal{B}(\mu)$. Moreover, by Pesin theory [27, Theorem 4.1], μ -almost every point $(x, t) \in M$ admits a Pesin stable manifold $W_K^{cs}(x, t)$, which is an embedded disk tangent to $E_K^c(x, t) \oplus E_K^s(x, t)$. Since $\pi(W_K^{cs}(x, t)) = W_A^s(x)$, the local Pesin stable manifold $W_K^{cs}(x, t) \cap I_x$ contains an open interval.

On the other hand, $W_K^{cs}(x, t) \subseteq \mathcal{B}(\mu)$. Indeed, by definition of $W_K^{cs}(x, t)$, we have $d(K^n(x, s), K^n(x, t)) \rightarrow 0$ as $n \rightarrow \infty$ for every $(x, s) \in W_K^{cs}(x, t)$. Since ψ is uniformly continuous, $|\psi(K^n(x, s)) - \psi(K^n(x, t))| \rightarrow 0$, and therefore

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \psi(K^j(x, s)) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \psi(K^j(x, t)),$$

for every continuous map $\psi: M \rightarrow \mathbb{R}$. Then, for μ -almost every $(x, t) \in M$, $\mathcal{B}(\mu) \cap I_x \supset W_K^{cs}(x, t) \cap I_x$ contains an open interval. $\square$

Now we deal with ergodic measures $\mu \in \mathcal{E}(K)$ that have positive central Lyapunov exponent. Our goal is to prove that, in this case, the basin is trivial (see Lemma 5.11 below). For this purpose, following [1], we use hyperbolic times provided by the Pliss lemma [29].

Let $\sigma > 1$ be fixed. An integer $n \geq 1$ is a σ -hyperbolic time for the point $(x, t) \in M$ if

$$(5.15) \quad |\partial_t \varphi^{n-k}(K^k(x, t))| = \prod_{j=k}^{n-1} |\partial_t \varphi(K^j(x, t))| \geq \sigma^{n-k},$$

for all $0 \leq k \leq n-1$.

Since $\varphi(x, \cdot)$ is a diffeomorphism, $\partial_t \varphi(x, t) > 0$ for all $(x, t) \in M$. By uniform continuity of the derivative, we choose $\delta > 0$ small enough that for every $x \in \mathbb{T}^d$ and every $a, b \in I$, if $|\varphi(x, a) - \varphi(x, b)| < \delta$, then

$$(5.16) \quad \frac{1}{\sqrt{\sigma}} |\partial_t \varphi(x, b)| \leq |\partial_t \varphi(x, a)| \leq \sqrt{\sigma} |\partial_t \varphi(x, b)|.$$

LEMMA 5.8. *If $n \geq 1$ is a σ -hyperbolic time for the point $(x, t) \in M$, and $s \in I$ is such that $|\varphi^n(x, t) - \varphi^n(x, s)| < \delta$, then*

$$(5.17) \quad |\varphi^{n-k}(x, t) - \varphi^{n-k}(x, s)| \leq \sigma^{-k/2} |\varphi^n(x, t) - \varphi^n(x, s)|,$$

for all $1 \leq k \leq n$.

PROOF. We follow [1, Lemma 2.7], adapted to the one-dimensional central fibre.

Assume $0 \leq s \leq t \leq 1$; the case $t \leq s$ is analogous. We prove (5.17) by induction on k . The case $k = 0$ is the hypothesis $|\varphi^n(x, t) - \varphi^n(x, s)| < \delta$; assume the claim for every $0 \leq j \leq k-1$.

By the Fundamental Theorem of Calculus,

$$\begin{aligned} |\varphi^{n-k}(x, t) - \varphi^{n-k}(x, s)| &= \int_s^t \partial_t \varphi^{n-k}(x, \xi) d\xi \\ &= \int_s^t \prod_{j=n-k}^{n-1} \partial_t \varphi(K^j(x, \xi))^{-1} \partial_t \varphi^n(x, \xi) d\xi, \end{aligned}$$

where the second equality uses $\partial_t \varphi^n(x, \xi) = \partial_t \varphi^k(K^{n-k}(x, \xi)) \partial_t \varphi^{n-k}(x, \xi)$ and $\partial_t \varphi > 0$.

For $n-k \leq j \leq n-1$ and $\xi \in [s, t]$, monotonicity of $\varphi^{j+1}(x, \cdot)$ and the inductive hypothesis give

$$|\varphi^{j+1}(x, \xi) - \varphi^{j+1}(x, t)| \leq |\varphi^{j+1}(x, t) - \varphi^{j+1}(x, s)| \leq \sigma^{-(n-1-j)/2} |\varphi^n(x, t) - \varphi^n(x, s)| < \delta.$$

Hence (5.16) applies at each factor and, together with (5.15),

$$(5.18) \quad \prod_{j=n-k}^{n-1} |\partial_t \varphi(K^j(x, \xi))|^{-1} \leq \sigma^{k/2} |\partial_t \varphi^k(K^{n-k}(x, t))|^{-1} \leq \sigma^{k/2} \sigma^{-k} = \sigma^{-k/2}.$$

Since this holds for every $\xi \in [s, t]$, the identity above gives

$$|\varphi^{n-k}(x, t) - \varphi^{n-k}(x, s)| \leq \sigma^{-k/2} \int_s^t \partial_t \varphi^n(x, \xi) d\xi = \sigma^{-k/2} |\varphi^n(x, t) - \varphi^n(x, s)|,$$

which is (5.17). $\square$

Using the Pliss lemma, it is possible to establish the existence and abundance of hyperbolic times since the diffeomorphism satisfies a nonuniformly expanding condition along the center direction.

LEMMA 5.9 ([1, Lemma 3.1]). *Given $\Lambda \geq c_2 > c_1 > 0$, let $\theta_0 = (c_2 - c_1) / (\Lambda - c_1)$. Then, for any $N \geq 1$ and any real numbers $a_0, \dots, a_{N-1}$ such that*

$$\sum_{j=0}^{N-1} a_j \geq c_2 N \quad \text{and} \quad a_j \leq \Lambda \quad \text{for every } 0 \leq j \leq N-1,$$

there are $l > \theta_0 N$ and $1 < n_1 < \dots < n_l \leq N$ so that

$$\sum_{j=k}^{n_i-1} a_j \geq c_1 (n_i - k) \quad \text{for every } 0 \leq k < n_i \text{ and } i = 1, \dots, l.$$

LEMMA 5.10. *Assume $(x, t) \in M$ satisfies*

$$(5.19) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \log |\partial_t \varphi(K^j(x, t))| = \chi > 0.$$

Let $\sigma \stackrel{\text{def}}{=} e^{\chi/4}$ and

$$\Lambda \stackrel{\text{def}}{=} \sup_{(x,t) \in M} \log |\partial_t \varphi(x, t)|.$$

Then, given any sufficiently large $n \geq 1$, there exist σ -hyperbolic times $1 \leq n_1 < \dots < n_l \leq n$ for (x, t) , with $l \geq \left(\frac{\chi}{4\Lambda - \chi}\right) n$.

PROOF. It follows from (5.19), that there is an integer $N \geq 1$, such that for every $n \geq N$, we have

$$\sum_{j=0}^{n-1} \log |\partial_t \varphi(K^j(x, t))| \geq n \frac{\chi}{2}.$$

Applying Lemma 5.9 with Λ as above,

$$a_j = \log |\partial_t \varphi(K^j(x, t))|, \quad c_1 = \frac{\chi}{4}, \quad c_2 = \frac{\chi}{2},$$

for $0 \leq j \leq n-1$, we obtain that there are $1 < n_1 < \dots < n_l \leq n$ so that for every $0 \leq k < n_i$

$$(5.20) \quad \sum_{j=k}^{n_i-1} \log |\partial_t \varphi(K^j(x, t))| \geq (n_i - k) \frac{\chi}{4}.$$

Note that, as $\sigma = e^{\chi/4} > 1$, (5.20) is equivalent to (5.15). Moreover, from Lemma 5.9, it follows that the frequency of hyperbolic times is at least

$$\theta_0 = \frac{c_2 - c_1}{\Lambda - c_1} = \frac{\chi}{4\Lambda - \chi} > 0,$$

which does not depend on the point $(x, t) \in M$. $\square$

We remark that the nonuniform expansion along the center direction condition (5.19) is a direct consequence of the positivity of the central Lyapunov exponent of K . In fact, an ergodic measure $\mu \in \mathcal{E}_\nu(K)$ has a positive central Lyapunov exponent if for μ -almost every $(x, t) \in M$, we have

$$(5.21) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \log |\partial_t \varphi^n(x, t)| = \chi^c(\mu) > 0.$$

Since the central direction is one-dimensional, from the chain rule, (5.21) is equivalent to

$$(5.22) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \log |\partial_t \varphi(K^j(x, t))| = \chi^c(\mu) > 0.$$

Moreover, by Birkhoff Theorem, (5.22) is equivalent to

$$(5.23) \quad \int_M J(x, t) d\mu(x, t) = \chi^c(\mu) > 0,$$

where the map $J: M \rightarrow \mathbb{R}$ above is defined by

$$J(x, t) = \log |\partial_t \varphi(x, t)|, \quad (x, t) \in M.$$

LEMMA 5.11. *Let $\mu \in \mathcal{E}(K)$ with $\chi^c(\mu) > 0$. For μ -almost every $(x, t) \in M$,*

$$\mathcal{B}(\mu) \cap I_x = \{(x, t)\}.$$

PROOF. Arguing by contradiction, suppose that there exists a set of positive μ -measure of points (x, t) such that $\mathcal{B}(\mu) \cap I_x \neq \{(x, t)\}$. We may choose (x, t) in this set to be μ -generic. Since $\mathcal{B}(\mu) \cap I_x$ is an interval by Proposition 5.2, there exists $s \neq t$ with

$(x, s) \in \mathcal{B}(\mu)$; without loss of generality $s < t$. These x , s , and t remain fixed throughout the proof; in particular, $|t - s| > 0$ is a fixed constant.

Consider $H: M \rightarrow \mathbb{R}$, defined by $H(x, t) \stackrel{\text{def}}{=} t$, for all $(x, t) \in M$. Since μ is ergodic, we have

$$(5.24) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} H(K^j(x, t)) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} H(K^j(x, s)) = \int H d\mu.$$

Note that $H(K^j(x, t)) = \varphi^j(x, t)$, and therefore we have

$$(5.25) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} [\varphi^j(x, t) - \varphi^j(x, s)] = 0.$$

Since $\varphi(x, \cdot)$ is nondecreasing and $s < t$, the numbers $b_j \stackrel{\text{def}}{=} \varphi^j(x, t) - \varphi^j(x, s)$ satisfy $0 \leq b_j \leq 1$. As every index j with $b_j \geq \delta$ contributes at least δ to the nonnegative sum $\sum_{j < n} b_j$, for every $\delta > 0$ we have

$$(5.26) \quad \#\{0 \leq j < n : b_j \geq \delta\} \leq \frac{1}{\delta} \sum_{j=0}^{n-1} b_j = o(n),$$

where the last equality follows from (5.25). Consequently, for every fixed $\delta > 0$, the set $\{j : b_j < \delta\}$ has density 1.

Let $\sigma = e^{\chi^c(\mu)/4}$ and let $\delta > 0$ be as in Lemma 5.8. By Lemma 5.10, the σ -hyperbolic times of (x, t) occur with positive frequency, whereas by (5.26) the times j with $b_j \geq \delta$ form a set of density zero. Removing this negligible set from the σ -hyperbolic times still leaves arbitrarily large ones; hence there exist arbitrarily large σ -hyperbolic times m with $b_m < \delta$. For each such m , Lemma 5.8 with $k = m$ gives

$$(5.27) \quad |t - s| \leq \sigma^{-m/2} (\varphi^m(x, t) - \varphi^m(x, s)) < \sigma^{-m/2} \delta.$$

Letting $m \rightarrow \infty$ yields $|t - s| = 0$, contradicting $s < t$. This proves the lemma. $\square$

The previous results provide all the ingredients needed to prove Theorem A.

PROOF OF THEOREM A. Part (1) is established by the disintegration argument at the beginning of Section 5.1: since π semiconjugates K to A , the pushforward $\nu \stackrel{\text{def}}{=} \pi_* \mu$ is ergodic, that is, $\nu \in \mathcal{E}(A)$. Part (2) follows from Lemma 5.1. For part (3), Proposition 5.2 ensures that $\mathcal{B}(\mu) \cap I_x$ is an interval for ν -almost every x . The cases of negative and positive central Lyapunov exponent follow respectively from Lemmas 5.7 and 5.11. $\square$

5.4. Observable measures

Throughout this section, fix $\nu \in \mathcal{E}(A)$. We study the basin of a measure $\mu \in \mathcal{E}_\nu(K)$ with respect to the reference measure

$$(5.28) \quad m_\nu = \nu \otimes \text{Leb}_{[0,1]}.$$

Recall that μ is m_ν -observable if $m_\nu(\mathcal{B}(\mu)) > 0$.

The following is merely a restatement of Lemma 5.5.

LEMMA 5.12. *Let $\mu \in \mathcal{E}_\nu(K)$. Then μ is m_ν -observable if, and only if, $\phi_\mu^+(x) - \phi_\mu^-(x) > 0$ for ν -almost every $x \in \mathbb{T}^d$.*

We now discuss the case when $\mu \in \mathcal{E}_\nu(K)$ is not observable, meaning that $m_\nu(\mathcal{B}(\mu)) = 0$. In such a case, we say that $\mu \in \mathcal{E}_\nu(K)$ has a m_ν -trivial basin (or μ is m_ν -trivial). The following is an immediate consequence of Lemma 5.7.

LEMMA 5.13. *Every $\mu \in \mathcal{E}_\nu(K)$ with m_ν -trivial basin $\mathcal{B}(\mu)$ has nonnegative central Lyapunov exponent.*

PROOF. From Proposition 5.2, $\mathcal{B}(\mu) \cap I_x$ is an interval for ν -almost every $x \in \mathbb{T}^d$. Therefore μ is m_ν -trivial if and only if $\mathcal{B}(\mu) \cap I_x = \{(x, \phi_\mu(x))\}$. By Lemma 5.7, if μ has negative central Lyapunov exponent, then it is m_ν -observable. Hence, we conclude that μ has a nonnegative central Lyapunov exponent. $\square$

CHAPTER 6

Geometric measures

6.1. Basin of attraction of geometric measures

To get further information about the basins of measures, we assume that the measure ν is geometric. Recall that an ergodic invariant measure ν of A is called *geometric* if it has local product structure, satisfies the uniformly bounded Radon–Nikodym derivatives (UBD) property, and has full support. We denote by $\mathcal{G} = \mathcal{G}(A) \subset \mathcal{E}(A)$ the set of geometric measures of A (see Section 2.3 for more details).

For $x' \in W_A^s(x)$, the *stable holonomy of K* is the map

$$H_{x,x'}^s: I_x \rightarrow I_{x'}, \quad H_{x,x'}^s(t) \stackrel{\text{def}}{=} t',$$

where t' is the unique point such that $(x', t') \in W_K^s(x, t)$. It is a homeomorphism varying continuously with x' , and is distinct from the stable holonomy $h_{x,x'}^s$ of A defined in (2.10). Analogously, for $x' \in W_A^u(x)$, one defines the *unstable holonomy of K* , denoted $H_{x,x'}^u: I_x \rightarrow I_{x'}$, using $W_K^u(x, t)$ in place of $W_K^s(x, t)$.

For geometric measures, the Avila-Viana Invariance Principle [3, Theorem D] reads as follows:

LEMMA 6.1 (Invariance Principle). *Let $\nu \in \mathcal{G}$ and $\mu \in \mathcal{E}_\nu^{\text{int}}(K)$ with disintegration $\{\mu_x^{\mathcal{I}}: x \in \mathbb{T}^d\}$. If $\chi^c(\mu) = 0$, then the measures $\mu_x^{\mathcal{I}}$ depend continuously on x and are invariant under the strong stable and unstable holonomies.*

The next lemmas are direct consequences of the Invariance Principle.

LEMMA 6.2. *Let $\nu \in \mathcal{G}$.*

- (1) *Every $\mu \in \mathcal{E}_\nu^{\text{int}}(K)$ is hyperbolic.*
- (2) *A measure $\mu \in \mathcal{E}_\nu^{\text{int}}(K)$ has trivial basin if, and only if, $\chi^c(\mu) > 0$.*

PROOF. Let $\mu \in \mathcal{E}_\nu^{\text{int}}(K)$. To prove that μ is hyperbolic we argue by contradiction, assuming that $\chi^c(\mu) = 0$. We will show that this forces $\text{supp } \mu \subseteq \mathbb{T}^d \times \{0\}$ or $\text{supp } \mu \subseteq \mathbb{T}^d \times \{1\}$, contradicting $\mu \in \mathcal{E}_\nu^{\text{int}}(K)$. By Lemma 6.1, the disintegration $\{\mu_x^{\mathcal{I}}: x \in \mathbb{T}^d\}$ of μ is invariant under the strong stable and unstable holonomies, and depends continuously

on x . Let $K_x: I_x \rightarrow I_{Ax}$ be the function defined by $K_x(t) = K(x, t)$, $t \in I_x$. By invariance and continuity, this disintegration satisfies

$$(K_x)_* \mu_x^{\mathcal{I}} = \mu_{Ax}^{\mathcal{I}}.$$

In particular, considering the fixed point p of A given by [KD2], we have

$$(K_p)_* \mu_p^{\mathcal{I}} = \mu_p^{\mathcal{I}}.$$

This implies that the support of $\mu_p^{\mathcal{I}}$ is contained in the subset $\{(p, 0), (p, 1)\}$ of $\{p\} \times [0, 1]$. Since μ is ergodic, its fiber conditional $\mu_p^{\mathcal{I}}$ is a single Dirac mass (Theorem A); therefore either $\mu_p^{\mathcal{I}} = \delta_{(p,0)}$ or $\mu_p^{\mathcal{I}} = \delta_{(p,1)}$.

If $\mu_p^{\mathcal{I}} = \delta_{(p,0)}$, then $\mu_x^{\mathcal{I}} = \delta_{(x,0)}$ for every $(x, 0) \in W_K^u((p, 0))$, because the conditional measures are invariant under the strong unstable holonomy. As the Anosov diffeomorphism A is transitive, we have $\mu_x^{\mathcal{I}} = \delta_{(x,0)}$ for every $x \in \mathbb{T}^d$. Therefore $\text{supp } \mu = \mathbb{T}^d \times \{0\}$, contradicting that $\mu \in \mathcal{E}^{\text{int}}(K)$. Similarly, if $\mu_p^{\mathcal{I}} = \delta_{(p,1)}$, then $\text{supp } \mu = \mathbb{T}^d \times \{1\}$, contradicting that $\mu \in \mathcal{E}^{\text{int}}(K)$. Hence $\chi^c(\mu) \neq 0$, proving the first statement.

The second item follows directly from the hyperbolicity of μ (first item), Lemma 5.7, and Lemma 5.11. $\square$

In the following, let $\nu \in \mathcal{G}$ and $\mu \in \mathcal{E}_\nu(K)$, with $\phi_\mu: \mathbb{T}^d \rightarrow [0, 1]$ the measurable map such that μ is characterized by (5.8), $\phi_\mu^\pm$ defined in (5.14), and $\mu^\pm$ the corresponding graph measures defined by (5.8) taking $\phi_\mu^\pm$ in place of ϕ_μ .

LEMMA 6.3. *Assume that $\phi_\mu < \phi_\mu^+$. Then the basin of μ^+ is trivial, and only one of the following situations occurs:*

- (1) *either $\mu^+ \notin \mathcal{E}^{\text{int}}(K)$ and $\chi^c(\mu^+) \geq 0$,*
- (2) *or $\mu^+ \in \mathcal{E}^{\text{int}}(K)$ with $\chi^c(\mu^+) > 0$.*

PROOF. Since $\phi_\mu < \phi_\mu^+$ ν -a.e., either $\mu^+ \in \mathcal{E}^{\text{int}}(K)$ or $\mu^+ = \nu^1$, and we treat the two cases separately.

Case 1: $\mu^+ \in \mathcal{E}^{\text{int}}(K)$. Since $\phi_\mu < \phi_\mu^+$, items (1) and (3) of Lemma 5.5 show that, for ν -almost every $x \in \mathbb{T}^d$, the fiber $\mathcal{B}(\mu) \cap I_x$ is an interval whose upper endpoint is $\phi_\mu^+(x)$ and satisfies $(x, \phi_\mu^+(x)) \notin \mathcal{B}(\mu)$. If $\chi^c(\mu^+) < 0$, then by Lemma 5.7 $\mathcal{B}(\mu^+) \cap I_x$ contains an open interval around $(x, \phi_\mu^+(x))$ for ν -almost every x , contradicting $\mathcal{B}(\mu) \cap \mathcal{B}(\mu^+) = \emptyset$. Hence $\chi^c(\mu^+) \geq 0$. From Lemma 6.2 item (1), μ^+ is then hyperbolic and thus $\chi^c(\mu^+) > 0$, and Lemma 5.11 implies that the basin of μ^+ is trivial.

Case 2: $\mu^+ = \nu^1$. Since $\phi_\mu < \phi_\mu^+$ forces $\phi_\mu^+ = 1$ ν -a.e., we have $\mu^+ = \nu^1 = \nu \times \delta_1$. As in the first case, Lemma 5.5 gives $\{x\} \times (\phi_\mu^-(x), 1) \subseteq \mathcal{B}(\mu) \cap I_x$ for ν -almost every x . By Proposition 5.2, $\mathcal{B}(\nu^1) \cap I_x$ is an interval which contains $(x, 1)$, since $\nu^1 = \nu \times \delta_1$ is supported on $\{t = 1\}$. As it is disjoint from $\mathcal{B}(\mu)$ because the basins of distinct ergodic measures are disjoint, we get $\mathcal{B}(\nu^1) \cap I_x = \{(x, 1)\}$. Then $\mathcal{B}(\nu^1)$ must be m_ν -trivial, and finally Lemma 5.13 yields $\chi^c(\nu^1) \geq 0$. $\square$

The lemma above is applicable, for instance, when $\chi^c(\mu) < 0$. To handle the case $\chi^c(\mu) > 0$, we use that K^{-1} is also a Kan-like diffeomorphism with $\mathcal{E}(K) = \mathcal{E}(K^{-1})$ and $\chi_K^c(\mu) = -\chi_{K^{-1}}^c(\mu)$.

As before, we can define the measurable function $\check{\phi}_\mu^+ : \mathbb{T}^d \rightarrow I$, the K^{-1} -analogue of ϕ_μ^+ , by

$$(6.1) \quad \check{\phi}_\mu^+(x) \stackrel{\text{def}}{=} \sup\{t \in [0, 1] : (x, t) \in \mathcal{B}_{K^{-1}}(\mu)\}.$$

Hence, by definition, $\phi_\mu \leq \check{\phi}_\mu^+$. Let $\check{\mu}^+ \in \mathcal{E}_\nu(K)$ be the graph measure defined through (5.8) with $\check{\phi}_\mu^+$. Applying Lemma 6.3 to K^{-1} (which is Kan-like with $\chi_{K^{-1}}^c = -\chi_K^c$) then yields the following.

LEMMA 6.4. *Assume that $\phi_\mu < \check{\phi}_\mu^+$. Then, $\check{\mu}^+$ is m_ν -observable, and only one of the following situations occurs:*

- (1) $\check{\mu}^+ = \nu^1$ with $\chi^c(\nu^1) \leq 0$; or
- (2) $\check{\mu}^+ \in \mathcal{E}^{\text{int}}(K)$ with $\chi^c(\check{\mu}^+) < 0$.

LEMMA 6.5. *Let $\nu \in \mathcal{G}$ and $\mu \in \mathcal{E}_\nu(K)$. If $\chi^c(\mu) > 0$, then $\phi_\mu(x) < \check{\phi}_\mu^+(x)$ for ν -almost every $x \in \mathbb{T}^d$.*

PROOF. By the discussion following Lemma 6.3, K^{-1} is a Kan-like diffeomorphism with $\chi_{K^{-1}}^c(\mu) = -\chi_K^c(\mu) < 0$. Therefore, Lemma 5.7 applied to K^{-1} implies that $\mathcal{B}_{K^{-1}}(\mu) \cap I_x$ contains an open interval for ν -almost every $x \in \mathbb{T}^d$. Hence $\phi_\mu(x) < \check{\phi}_\mu^+(x)$ for ν -almost every $x \in \mathbb{T}^d$. $\square$

As a consequence of the proof above, $\mathcal{B}_{K^{-1}}(\mu) \cap I_x$ contains an open interval, for ν -almost every $x \in \mathbb{T}^d$. Therefore,

COROLLARY 6.6. *If μ has positive central Lyapunov exponent, then there is no $\mu' \in \mathcal{E}_\nu(K)$ such that $\mu_\phi < \mu' < \check{\mu}^+$.*

6.2. Lifts of geometric measures

In this section, we characterize $\mathcal{E}_\nu(K)$ for $\nu \in \mathcal{G}$. A key feature of the interior measures, as the construction below reveals, is that their central Lyapunov exponents alternate in sign when the measures are ordered. We formalize this by saying that a finite or infinite sequence of graph measures $\{\mu_n\}$ is *Lyapunov alternating* if

$$\chi^c(\mu_n)\chi^c(\mu_{n+1}) < 0$$

for every n . We distinguish three cases according to whether $\mathcal{E}_\nu^{\text{int}}(K)$ is empty, finite, or countable; the uncountable case does not occur, as Proposition 6.8 below shows. The key step is Lemma 6.7, which describes how to construct each successive measure.

LEMMA 6.7. *Let $\nu \in \mathcal{G}$ and $\mu \in \mathcal{E}_\nu^{\text{int}}(K)$ be a hyperbolic graph measure. Then there exists $\mu' \in \mathcal{E}_\nu(K)$ with $\mu < \mu' \leq \nu^1$ and no measure $\mu'' \in \mathcal{E}_\nu(K)$ satisfying $\mu < \mu'' < \mu'$, for which exactly one of the following holds:*

- (1) *if $\chi^c(\mu) > 0$, then μ' is m_ν -observable, and either $\mu' = \nu^1$ with $\chi^c(\nu^1) \leq 0$, or $\mu' \in \mathcal{E}_\nu^{\text{int}}(K)$ with $\chi^c(\mu') < 0$;*
- (2) *if $\chi^c(\mu) < 0$, then μ' has trivial basin, and either $\mu' = \nu^1$ with $\chi^c(\nu^1) \geq 0$, or $\mu' \in \mathcal{E}_\nu^{\text{int}}(K)$ with $\chi^c(\mu') > 0$.*

PROOF. For (1): since $\chi^c(\mu) > 0$, Lemma 6.5 gives $\phi_\mu < \check{\phi}_\mu^+$ ν -a.e. Applying Lemma 6.4 to $\mu' \stackrel{\text{def}}{=} \mu_{\check{\phi}_\mu^+} \in \mathcal{E}_\nu(K)$, defined via (6.1) and (5.8), we conclude that μ' is m_ν -observable and exactly one of the two sub-cases holds. By Corollary 6.6, there is no measure in $\mathcal{E}_\nu(K)$ strictly between μ and μ' .

For (2): since $\chi^c(\mu) < 0$, μ is m_ν -observable and $\phi_\mu < \phi_\mu^+$ ν -a.e. Applying Lemma 6.3 to $\mu' \stackrel{\text{def}}{=} \mu_{\phi_\mu^+} \in \mathcal{E}_\nu(K)$, defined via (5.14) and (5.8), we conclude that μ' has trivial basin and exactly one of the two sub-cases holds. By Corollary 5.6, there is no measure in $\mathcal{E}_\nu(K)$ strictly between μ and μ' . $\square$

PROPOSITION 6.8. *Let $\nu \in \mathcal{G}$ and assume that ν^0 is m_ν -observable. Then exactly one of the following cases occurs:*

- (1) *The set $\mathcal{E}_\nu^{\text{int}}(K) = \emptyset$. Moreover, the measure ν^1 has trivial basin and $\chi^c(\nu^1) \geq 0$.*
- (2) *The set $\mathcal{E}_\nu^{\text{int}}(K) = \{\mu_1, \mu_2, \dots, \mu_n\}$, where*

$$\nu^0 < \mu_1 < \mu_2 < \dots < \mu_n < \nu^1$$

with

$$\chi^c(\mu_{2k-1}) > 0, \quad \chi^c(\mu_{2k}) < 0.$$

Moreover, if n is even then ν^1 has trivial basin and $\chi^c(\nu^1) \geq 0$, and if n is odd then ν^1 is m_ν -observable and $\chi^c(\nu^1) \leq 0$.

(3) The set $\mathcal{E}_\nu^{\text{int}}(K) = \{\mu_n : n \in \mathbb{N}\}$, where

$$\nu^0 < \mu_1 < \mu_2 < \cdots < \nu^1$$

with

$$\chi^c(\mu_{2k-1}) > 0 \quad \text{and} \quad \chi^c(\mu_{2k}) < 0.$$

Moreover, $\mu_n \rightarrow \nu^1$ weak* and $\chi^c(\nu^1) = 0$.

PROOF. The strategy is to apply Lemma 6.7 iteratively to construct measures in $\mathcal{E}_\nu(K)$ between ν^0 and ν^1 ; the boundary measures always exist, but the existence of interior ones is what we must determine. Since ν^0 is m_ν -observable, its invariant graph is $\phi_{\nu^0} = 0$ (Lemma 5.1 and (5.9)), so $\phi_{\nu^0}^+$ defined by (5.14) satisfies $0 < \phi_{\nu^0}^+$ ν -a.e. Define $\mu_1 \stackrel{\text{def}}{=} \mu_{\phi_{\nu^0}^+} \in \mathcal{E}_\nu(K)$ via (5.8); by Lemma 6.3, μ_1 has trivial basin and $\chi^c(\mu_1) \geq 0$. Since the only boundary measure above ν^0 is ν^1 , either $\mu_1 = \nu^1$, which proves case (1), or $\mu_1 \in \mathcal{E}_\nu^{\text{int}}(K)$ with $\chi^c(\mu_1) > 0$, which initiates the inductive construction below.

Now assume that $\mathcal{E}_\nu^{\text{int}}(K) \neq \emptyset$. Then $\mu_1 \in \mathcal{E}_\nu^{\text{int}}(K)$ with $\chi^c(\mu_1) > 0$, and by Corollary 5.6 there is no measure in $\mathcal{E}_\nu(K)$ strictly between ν^0 and μ_1 .

We apply Lemma 6.7 inductively. Given $\mu_k \in \mathcal{E}_\nu^{\text{int}}(K)$, the lemma yields $\mu_{k+1} \in \mathcal{E}_\nu(K)$ with $\mu_k < \mu_{k+1} \leq \nu^1$ and no measure in $\mathcal{E}_\nu(K)$ strictly between μ_k and μ_{k+1} . The central Lyapunov exponents alternate: $\chi^c(\mu_{2k-1}) > 0$ and $\chi^c(\mu_{2k}) < 0$. If the induction terminates at step n , that is, Lemma 6.7 yields $\mu_{n+1} = \nu^1$ instead of an interior measure, then $\mathcal{E}_\nu^{\text{int}}(K) = \{\mu_1, \dots, \mu_n\}$ and the sign conditions on $\chi^c(\nu^1)$ from the same lemma yield case (2).

If $\mu_k \in \mathcal{E}_\nu^{\text{int}}(K)$ for all $k \geq 1$, we obtain an infinite increasing sequence $\mu_1 < \mu_2 < \cdots$ in $\mathcal{E}_\nu^{\text{int}}(K)$. Note that this sequence cannot accumulate at any interior measure $\mu^* \in \mathcal{E}_\nu^{\text{int}}(K)$, since the alternating sign conditions force $\chi^c(\mu^*) = 0$, contradicting Lemma 6.2 (1). Hence the sequence can only accumulate at ν^1 . The corresponding invariant graphs $(\phi_k)_k$ are increasing and bounded in I , so $\phi_\infty \stackrel{\text{def}}{=} \lim_{k \rightarrow \infty} \phi_k$ exists ν -a.e. The limit measure $\mu = \mu_{\phi_\infty}$ satisfies $\pi_*\mu = \nu$, and $\mu_k \rightarrow \mu$ in the weak* topology. Since $\log |\partial_t \varphi|$ is continuous on the compact manifold M , the map $\eta \mapsto \chi^c(\eta)$ is continuous in the weak* topology; evaluating along the odd and even indices yields

$$\chi^c(\mu) = \lim_{k \rightarrow \infty} \chi^c(\mu_{2k-1}) \geq 0 \quad \text{and} \quad \chi^c(\mu) = \lim_{k \rightarrow \infty} \chi^c(\mu_{2k}) \leq 0,$$

so $\chi^c(\mu) = 0$. By Lemma 6.2 (1) this forces $\mu \notin \mathcal{E}_\nu^{\text{int}}(K)$, and therefore $\mu = \nu^1$. This proves case (3). $\square$

REMARK 6.9. We obtain a similar conclusion in the proposition above if we start with $\nu \in \mathcal{G}$ and ν^1 having nontrivial basin. If $\mathcal{E}_\nu^{\text{int}}(K) = \emptyset$, the measure ν^0 has trivial basin and $\chi^c(\nu^0) \geq 0$. If $\mathcal{E}_\nu^{\text{int}}(K) \neq \emptyset$, then

$$\mathcal{E}_\nu(K) = \{\nu^0 < \cdots < \mu_n < \cdots < \mu_2 < \mu_1 < \nu^1\},$$

with every $\mu_n \in \mathcal{E}_\nu^{\text{int}}(K)$ a hyperbolic graph measure and

$$\chi^c(\mu_{2k-1}) > 0, \quad \chi^c(\mu_{2k}) < 0.$$

If the sequence is infinite, it accumulates at ν^0 and $\chi^c(\nu^0) = 0$.

Note that Proposition 6.8 can be applied starting from any hyperbolic $\mu_0 \in \mathcal{E}_\nu^{\text{int}}(K)$, independently of the boundary measures. Moreover, if $\mathcal{E}_\nu^{\text{int}}(K) \neq \emptyset$, applying the construction in both directions yields a Lyapunov alternating sequence

$$\nu^0 < \cdots < \mu_m^- < \cdots < \mu_1^- < \mu_0 < \mu_1^+ < \cdots < \mu_n^+ < \cdots < \nu^1,$$

where all measures $\mu_k^\pm \in \mathcal{E}_\nu^{\text{int}}(K)$ are hyperbolic.

THEOREM 6.10. *Let $\nu \in \mathcal{G}$. Then, exactly one of the following cases occurs:*

- (1) *The set $\mathcal{E}_\nu^{\text{int}}(K) = \emptyset$ and the measures satisfy $\chi^c(\nu^0) \cdot \chi^c(\nu^1) \leq 0$.*
- (2) *There is an increasing sequence of Lyapunov alternating graph measures*

$$\mu_m^- < \cdots < \mu_1^- < \mu_0 < \mu_1^+ < \cdots < \mu_n^+,$$

where $0 \leq m, n \leq \infty$, such that $\{\mu_m^-, \dots, \mu_1^-, \mu_0, \mu_1^+, \dots, \mu_n^+\} = \mathcal{E}_\nu^{\text{int}}(K)$.

Moreover,

- (a) *If $n < \infty$, then $\chi^c(\mu_n^+) \cdot \chi^c(\nu^1) \leq 0$.*
- (b) *If $m < \infty$, then $\chi^c(\mu_m^-) \cdot \chi^c(\nu^0) \leq 0$.*
- (c) *If $n = \infty$, then $\mu_k^+ \rightarrow \nu^1$ as $k \rightarrow \infty$, and $\chi^c(\nu^1) = 0$.*
- (d) *If $m = \infty$, then $\mu_k^- \rightarrow \nu^0$ as $k \rightarrow \infty$, and $\chi^c(\nu^0) = 0$.*

In addition, if ν^0 (resp. ν^1) is hyperbolic, then $m < \infty$ (resp. $n < \infty$).

PROOF. We prove item (1) by contraposition: assume $\chi^c(\nu^0) \cdot \chi^c(\nu^1) > 0$, so that $\chi^c(\nu^0)$ and $\chi^c(\nu^1)$ are nonzero and of the same sign; we show that $\mathcal{E}_\nu^{\text{int}}(K) \neq \emptyset$.

Suppose first $\chi^c(\nu^0) < 0$ and $\chi^c(\nu^1) < 0$. By Lemma 5.7, $\chi^c(\nu^0) < 0$ gives $\phi_{\nu^0} = 0 < \phi_{\nu^0}^+$ ν -a.e., so Lemma 6.3 applies to $\mu^+ \stackrel{\text{def}}{=} \mu_{\phi_{\nu^0}^+}$: either $\mu^+ = \nu^1$ with $\chi^c(\nu^1) \geq 0$, or $\mu^+ \in \mathcal{E}_\nu^{\text{int}}(K)$. The former contradicts $\chi^c(\nu^1) < 0$; hence $\mu^+ \in \mathcal{E}_\nu^{\text{int}}(K)$.

If instead $\chi^c(\nu^0) > 0$ and $\chi^c(\nu^1) > 0$, then Lemma 6.5 gives $\phi_{\nu^0} = 0 < \check{\phi}_{\nu^0}^+$ ν -a.e., and Lemma 6.4 applied to $\check{\mu}^+ \stackrel{\text{def}}{=} \mu_{\check{\phi}_{\nu^0}^+}$ yields either $\check{\mu}^+ = \nu^1$ with $\chi^c(\nu^1) \leq 0$, or $\check{\mu}^+ \in \mathcal{E}_\nu^{\text{int}}(K)$; the former contradicts $\chi^c(\nu^1) > 0$, so again $\check{\mu}^+ \in \mathcal{E}_\nu^{\text{int}}(K)$.

In either case $\mathcal{E}_\nu^{\text{int}}(K) \neq \emptyset$. Equivalently, the interior measure is the graph measure supported on the graph of $\phi_{\nu^0}^+$ (resp. $\check{\phi}_{\nu^0}^+$), i.e. on the closure of the center-unstable Pesin manifolds emanating from the boundary. This proves item (1).

For item (2), by Lemma 6.2 (1) there exists a hyperbolic $\mu_0 \in \mathcal{E}_\nu^{\text{int}}(K)$; assume $\chi^c(\mu_0) < 0$, so μ_0 is m_ν -observable. Applying Lemma 6.7 iteratively from μ_0 upward yields the Lyapunov alternating sequence

$$\mu_0 < \mu_1^+ < \mu_2^+ < \cdots \leq \nu^1.$$

Applying Remark 6.9 symmetrically downward from μ_0 yields

$$\nu^0 \leq \cdots < \mu_2^- < \mu_1^- < \mu_0.$$

Joining both sequences gives the claimed structure. The sign conditions on $\chi^c(\nu^0)$ and $\chi^c(\nu^1)$ and the convergence properties follow from the conclusions of Lemma 6.7 at each terminal step.

Finally, if ν^0 (resp. ν^1) is hyperbolic, then $\chi^c(\nu^0) \neq 0$ (resp. $\chi^c(\nu^1) \neq 0$), so item (d) (resp. item (c)) rules out an infinite lower (resp. upper) sequence; hence $m < \infty$ (resp. $n < \infty$). $\square$

PROOF OF THEOREM B. Theorem 6.10 is a more precise version of Theorem B. Case (1) of Theorem 6.10 is exactly case (1) of Theorem B. Case (2) yields the sequence of interior ergodic graph measures of case (2) of Theorem B, with alternating central Lyapunov exponents; when this sequence is infinite in some direction, items (c)–(d) give its accumulation on the corresponding boundary measure, whose central Lyapunov exponent is zero. $\square$

CHAPTER 7

Intermingleness

In this chapter we prove Theorem C. Throughout, K , ν , and $\nu^j \stackrel{\text{def}}{=} \nu \times \delta_j$ are as in the statement of Theorem C. Strongly geometric measures always exist for Kan-like diffeomorphisms; see Section 7.6 and Corollary 7.11.

Recall that a closed subset $R \subseteq \mathbb{T}^d$ is a *rectangle* if for all $x, y \in R$ the point $[x, y]$ is well-defined and belongs to R . For every $x \in R$, we let $W_A^u(x, R) \stackrel{\text{def}}{=} W_A^u(x, r) \cap R$ (see Section 2.3 for more details). In the following, we consider only rectangles with non-empty interiors. Such rectangles have positive ν -measure because ν is a geometric measure and therefore has full support.

7.1. Preliminary results

Since ν^0 (resp. ν^1) is supported on $\mathbb{T}^d \times \{0\}$ (resp. $\mathbb{T}^d \times \{1\}$), we have $\phi_{\nu^0}^- \equiv 0$ (resp. $\phi_{\nu^1}^+ \equiv 1$). By Lemma 5.12, if ν^0 is m_ν -observable then $\phi_{\nu^0}^+(x) > 0$ for ν -a.e. x , and if ν^1 is m_ν -observable then $\phi_{\nu^1}^-(x) < 1$ for ν -a.e. x . Throughout, we abbreviate $\phi^+ = \phi_{\nu^0}^+$ and $\phi^- = \phi_{\nu^1}^-$.

LEMMA 7.1. *Assume ν^0 is m_ν -observable. For ν -a.e. $x \in \mathbb{T}^d$ and every $x' \in W_A^s(x)$,*

$$(x', \phi^+(x')) \in W_K^s(x, \phi^+(x)).$$

In particular, ϕ^+ is continuous along $W_A^s(x)$.

PROOF. Since $\mathcal{B}(\nu^0)$ is s -saturated and $\mathcal{B}(\nu^0) \cap I_x = \{x\} \times [0, \phi^+(x))$ (Lemma 5.5), the holonomy $H_{x,x'}^s$ maps $\mathcal{B}(\nu^0) \cap I_x$ onto $\mathcal{B}(\nu^0) \cap I_{x'}$. Taking $t \nearrow \phi^+(x)$ and using continuity of $H_{x,x'}^s$ gives $(x', \phi^+(x')) \in W_K^s(x, \phi^+(x))$. Continuity of $x' \mapsto \phi^+(x') = H_{x,x'}^s(\phi^+(x))$ follows from continuity of $H_{x,x'}^s$ in x' . $\square$

LEMMA 7.2. *Assume ν^j is m_ν -observable for some $j \in \{0, 1\}$. Let $R \subseteq \mathbb{T}^d$ be a rectangle and let $\epsilon > 0$. Then there exist a compact s -saturated subset $\tilde{R} \subset R$ with $\nu(R \setminus \tilde{R}) < \epsilon$ and whose points are ν -generic for A , and $\alpha = \alpha(R, \epsilon) > 0$, such that ϕ^+ (resp. ϕ^-) is continuous on $\tilde{R}$ with $\phi^+(x) \geq \alpha$ (resp. $\phi^-(x) \leq 1 - \alpha$), and $(z, \phi^+(z)) \in W_K^s(x, \phi^+(x))$ (resp. $(z, \phi^-(z)) \in W_K^s(x, \phi^-(x))$), for every $x \in \tilde{R}$ and $z \in W_A^s(x, R)$.*

Moreover, defining

$$\begin{aligned} B^0(R, \epsilon) &\stackrel{\text{def}}{=} \{(x, t) : x \in \tilde{R}, 0 \leq t < \phi^+(x)\}, \\ B^1(R, \epsilon) &\stackrel{\text{def}}{=} \{(x, t) : x \in \tilde{R}, \phi^-(x) < t \leq 1\}, \end{aligned}$$

we have $B^j(R, \epsilon) \subseteq \mathcal{B}(\nu^j)$ and $m_\nu(B^j(R, \epsilon)) > 0$.

PROOF. Assume ν^0 is m_ν -observable. By Lemma 7.1, for ν -a.e. $x \in \mathbb{T}^d$ and every $x' \in W_A^s(x)$, we have $(x', \phi^+(x')) \in W_K^s(x, \phi^+(x))$ and ϕ^+ is continuous along $W_A^s(x)$.

Since $\phi^+|_R$ is measurable, Lusin's Theorem implies that, for every $\epsilon > 0$, there is a compact subset $\tilde{R} \subset R$, with $\nu(R \setminus \tilde{R}) < \epsilon$, such that $\phi^+|_{\tilde{R}}$ is continuous. We may further assume that every $\tilde{x} \in \tilde{R}$ satisfies $(\tilde{x}, 0) \in \mathcal{B}(\nu^0)$. Since $\mathcal{B}(\nu^0)$ is s -saturated and $W_K^s(\tilde{x}, 0) = W_A^s(\tilde{x}) \times \{0\}$, it follows that $W_A^s(\tilde{x}, R) \subset \tilde{R}$, and hence $\tilde{R}$ is s -saturated.

It follows from the construction above that there is $\alpha = \alpha(R, \epsilon) > 0$ such that $\phi^+(x) \geq \alpha$ for every $x \in \tilde{R}$, and as a consequence $m_\nu(B^0(R, \epsilon)) > 0$. $\square$

Since $\nu \in \mathcal{S}$, ν has full support and local product structure (see Section 2.3); in particular, by Lemma 2.1 item (6), $\nu|_R \sim \nu_x^u \otimes \nu_x^s$ for ν -a.e. $x \in R$, for every rectangle R with $\text{diam}(R)$ small enough.

As a direct consequence of Lemma 7.2 and the fact that $\nu \in \mathcal{S}$, we have the following corollary.

COROLLARY 7.3. *Assume ν^j , $j = 0, 1$, are m_ν -observable. Let $R \subseteq \mathbb{T}^d$ be a rectangle and let $\epsilon > 0$ be fixed. Let $\tilde{R}$ and $B^j(R, \epsilon)$ be the sets provided by Lemma 7.2. Then, there is a constant $\tilde{\beta}_0 = \tilde{\beta}_0(R, \epsilon) > 0$ such that, for ν -almost every $x \in R$, we have,*

$$(7.1) \quad \nu_x^u \left(W_A^u(x, R) \cap \tilde{R} \right) \geq \tilde{\beta}_0 > 0.$$

PROOF. Since $\nu \in \mathcal{S}$, by Lemma 2.1 item (6), there is a constant $C_0 > 0$ such that

$$\frac{1}{C_0} \nu_x^u(\tilde{R}) \leq \nu(\tilde{R}) \leq C_0 \nu_x^u(\tilde{R}).$$

Since $\nu(R \setminus \tilde{R}) < \epsilon$, we have $\nu(\tilde{R}) > \beta_0 > 0$, and so

$$\nu_x^u(\tilde{R}) \geq \frac{\beta_0}{C_0} > 0.$$

Moreover, for ν -almost every $z \in R$, the stable holonomy

$$h_{xz}^s : W_A^u(x, R) \rightarrow W_A^u(z, R),$$

is well-defined and satisfies (2) in Lemma 2.1. In particular, there is a constant $C_1 > 0$ such that (2.15) is satisfied, and then,

$$\begin{aligned}\nu_z^u(\tilde{R}) &\geq \frac{1}{C_1}(h_{zx}^s)^*\nu_x^u(\tilde{R}) \\ &= \frac{1}{C_1}\nu_x^u(\tilde{R}) \\ &\geq \frac{\beta_0}{C_0C_1}.\end{aligned}$$

We denote $\tilde{\beta}_0 \stackrel{\text{def}}{=} \beta_0/(C_0C_1) > 0$, and conclude $\nu_z^u(\tilde{R}) \geq \tilde{\beta}_0 > 0$.

□

Recall that $\pi: M \rightarrow \mathbb{T}^d$ denotes the natural projection $\pi(x, t) \stackrel{\text{def}}{=} x$. As a direct consequence of Lemma 7.2, we have the following lemma.

LEMMA 7.4. *Assume ν^0 is m_ν -observable. Let $R \subseteq \mathbb{T}^d$ be a rectangle with $\text{diam}(R) < \delta$ and $\nu(R) > 0$. Let $\epsilon > 0$ be fixed, and $\alpha > 0$, $\tilde{R} \subseteq R$, and $B^0(R, \epsilon)$ be the sets provided by Lemma 7.2.*

Let $\gamma = W_A^u(x, R)$ and $\sigma \subseteq M$ be the graph of a C^1 function $G: \gamma \rightarrow [0, 1]$ such that $0 \leq G(y) < \alpha$, for every $y \in \gamma$. Then, there is a constant $\tilde{\beta} = \tilde{\beta}(R, \epsilon) > 0$ such that, for ν -almost every $z \in R$, we have,

$$\nu_z^u(h_{xz}^s(\pi(\sigma \cap \mathcal{B}(\nu^0)))) \geq \nu_z^u(h_{xz}^s(\gamma \cap \tilde{R})) = \nu_z^u(W_A^u(z, R) \cap \tilde{R}) > \tilde{\beta}.$$

PROOF. Since $0 \leq G(y) < \alpha$ for every $y \in \gamma$, $\text{dist}(\sigma, W_A^u(x, R) \times \{0\}) \leq \alpha$. By the construction of $B^0(R, \epsilon)$ in Lemma 7.2, we have that σ crosses $B^0(R, \epsilon)$. Moreover,

$$\sigma \cap B^0(R, \epsilon) = \{(y, G(y)) \in \sigma : y \in \gamma \cap \tilde{R} \text{ and } 0 < G(y) < \alpha\}.$$

Then, $\pi(\sigma \cap B^0(R, \epsilon)) = \gamma \cap \tilde{R}$.

Since $\tilde{R}$ is s -saturated, $h_{xz}^s(\gamma \cap \tilde{R}) = W_A^u(z, R) \cap \tilde{R}$. By Corollary 7.3,

$$(7.2) \quad \nu_x^u(\gamma \cap \tilde{R}) \geq \tilde{\beta}_0 > 0.$$

It follows from Lemma 2.1 that there is a constant $C_1 > 0$ such that (2.15) is satisfied, and then,

$$(7.3) \quad \nu_z^u(W_A^u(z, R) \cap \tilde{R}) = \nu_z^u(h_{xz}^s(\gamma \cap \tilde{R})) \geq \frac{1}{C_1}\nu_x^u(\gamma \cap \tilde{R}).$$

Combining (7.2) and (7.3),

$$\nu_z^u(W_A^u(z, R) \cap \tilde{R}) \geq \frac{\tilde{\beta}_0}{C_1} > 0.$$

It is enough to take $\tilde{\beta} = \tilde{\beta}_0/C_1$ to complete the proof. $\square$

An analogous result holds when ν^1 is m_ν -observable.

LEMMA 7.5. *Assume ν^1 is m_ν -observable. Let $R \subseteq \mathbb{T}^d$ be a rectangle with $\text{diam}(R) < \delta$ and $\nu(R) > 0$. Let $\epsilon > 0$ be fixed, and $\alpha > 0$, $\tilde{R} \subseteq R$, and $B^1(R, \epsilon)$ be the sets provided by Lemma 7.2.*

Let $\gamma = W_A^u(x, R)$ and $\sigma \subseteq M$ be the graph of a C^1 function $G : \gamma \rightarrow [0, 1]$ such that $1 - \alpha < G(y) \leq 1$, for every $y \in \gamma$. Then, there is a constant $\tilde{\beta} = \tilde{\beta}(R, \epsilon) > 0$ such that, for ν -almost every $z \in R$, we have,

$$\nu_z^u(h_{xz}^s(\pi(\sigma \cap \mathcal{B}(\nu^1)))) \geq \nu_z^u(h_{xz}^s(\gamma \cap \tilde{R})) = \nu_z^u(W_A^u(z, R) \cap \tilde{R}) > \tilde{\beta}.$$

PROOF. The proof is analogous to that of Lemma 7.4. $\square$

7.2. Density of nontrivial basins

Our first goal is to prove Lemma 7.6: if ν^j is m_ν -observable, then its basin is dense ($j = 0, 1$). We give the proof for $j = 0$; the case $j = 1$ is analogous.

LEMMA 7.6. *Let $\nu \in \mathcal{G}$, $j \in \{0, 1\}$, such that ν^j is m_ν -observable. Then, any strong unstable disk in $\mathbb{T}^d \times (0, 1)$ contains ν^j -generic points. In particular, $\overline{\mathcal{B}(\nu^j)} = \mathbb{T}^d \times [0, 1]$.*

PROOF. We prove the case $j = 0$; the case $j = 1$ follows by an analogous argument.

Given any $(y, t) \in \mathbb{T}^d \times (0, 1)$, consider an unstable disk $\sigma \subset W_K^u(y, t)$ such that $(y, t) \in \sigma$. We will prove that $\sigma \cap \mathcal{B}(\nu^0) \neq \emptyset$.

Let $\gamma \stackrel{\text{def}}{=} \pi(\sigma) \subset \mathbb{T}^d$. Then, $y \in \gamma$ and $\gamma \subseteq W_A^u(y)$. Since A is a transitive Anosov diffeomorphism, its unstable foliation is dense (see [10, Theorem 5.10.3]) and therefore, $A^n(\gamma)$ intersects $W_A^s(p, r_L)$ for every n large enough, where $p \in \mathbb{T}^d$ is the elevator fixed point of A with a sink at $t = 0$ (see [KD2]). Hence, $K^n(\sigma)$ intersects the stable disk Σ_p (see (4.8)).

By the λ -Lemma, for n large enough, $K^n(\sigma)$ contains disks arbitrarily C^1 -close to $W_K^u(p, 0) = W_A^u(p) \times \{0\}$.

Fix $\epsilon > 0$ and $R \subseteq \mathbb{T}^d$ a rectangle such that $\nu(R) > 0$. Let $\tilde{R} \subseteq R$ and $B^0(R, \epsilon) \subseteq \mathbb{T}^d \times I$ satisfy the conclusion of Lemma 7.2. Since $W_A^u(p)$ is dense in $\mathbb{T}^d$, for any sufficiently large $m \geq 1$, $A^{n+m}(\gamma)$ intersects R transversally. Since $\tilde{R}$ is s -saturated, $A^{n+m}(\gamma)$ also intersects $\tilde{R}$. Hence, since $K^{n+m}(\sigma)$ is close to $W_K^u(p, 0)$, $K^{n+m}(\sigma)$ intersects the set $B^0(R, \epsilon)$. Since $B^0(R, \epsilon) \subseteq \mathcal{B}(\nu^0)$ (see Lemma 7.2) such intersection is ν^0 -generic. Recall

that any preimage of a ν^0 -generic point by K is also ν^0 -generic. This implies that σ contains ν^0 -generic points. $\square$

7.3. Intermingled basins

Since ν has local product structure (see Section 2.3), for every rectangle $R \subseteq \mathbb{T}^d$ and $x \in R$, there exists $\rho_x: R \rightarrow \mathbb{R}$ integrable with $0 < \rho_x(z) \leq 1$, such that for every measurable $E \subseteq R$,

$$(7.4) \quad \nu(E) = \int_{W_A^s(x,R)} \int_{W_A^u(x,R)} \mathbb{1}_E(z) \rho_x(z) d\nu_x^u(z) d\nu_x^s(z).$$

Let $R \subseteq \mathbb{T}^d$ be a rectangle. For every measurable $E \subseteq R \times I$, the reference measure satisfies

$$(7.5) \quad \begin{aligned} m_\nu(E) &= \int_I \int_R \mathbb{1}_E(z, t) d\nu dt \\ &= \int_I \int_{W_A^s(x,R)} \int_{W_A^u(x,R)} \mathbb{1}_E(z, t) \rho_x(z) d\nu_x^u(z) d\nu_x^s(z) dt, \end{aligned}$$

for ν -almost every $x \in R$.

PROPOSITION 7.7. *Let $\nu \in \mathcal{G}$ such that ν^0 and ν^1 are m_ν -observable. Then, their basins are intermingled with respect to m_ν .*

PROOF. Take any unstable disk $\gamma \subset \mathbb{T}^d$ and $z \in \gamma$ such that $\nu_z^u(\gamma) > 0$. Fix $t \in (0, 1)$ and denote $\gamma_t \stackrel{\text{def}}{=} \gamma \times \{t\}$. Note that γ_t is not necessarily an unstable curve, but we can choose γ small enough so that the tangent vectors of γ_t belong to the unstable cone field and are transverse to the E^{cs} -direction. Since K is partially hyperbolic (see [KD3]) with E_K^u uniformly expanding, under iteration by K the size of γ_t grows exponentially and its angle with E^{cs} remains bounded away from 0. Then, proceeding as in the proof of Lemma 7.6, for $n \geq 1$ large enough we have that $K^n(\gamma_t)$ crosses (transversely) the $(d^s + 1)$ -dimensional manifolds Σ_p and Σ_q , and so the orbit of γ_t accumulates on both boundary components of the cylinder $\mathbb{T}^d \times [0, 1]$. Moreover, by the λ -Lemma, for n large enough, $K^n(\gamma_t)$ contains disks arbitrarily C^1 -close to $W_K^u(p, 0) = W_A^u(p) \times \{0\}$.

Fix $\epsilon > 0$ and $R \subseteq \mathbb{T}^d$ a rectangle such that $\nu(R) > 0$. Let $\tilde{R} \subseteq R$ and $B^j(R, \epsilon) \subseteq \mathbb{T}^d \times I$ satisfy the conclusion of Lemma 7.2 for ν^j , $j = 0, 1$. With a slight modification to the argument in the proof of Lemma 7.6, we can assume that, for each $j \in \{0, 1\}$, some iterate $K^{n(j)}(\gamma_t)$ crosses $B^j(R, \epsilon)$ transversally.

Claim: For ν -almost every $z \in R$ and $j \in \{0, 1\}$,

$$(7.6) \quad \nu_z^u(\pi(\gamma_t \cap \mathcal{B}(\nu^j))) \geq \frac{\tilde{\beta}}{C_2^2} > 0.$$

Assuming the Claim, we complete the proof. Let $U \subseteq M$ be an open set. Let $R_0 \subseteq \mathbb{T}^d$ be a rectangle such that $\nu(R_0) > 0$, and let $J_0 \subseteq (0, 1)$ be a closed interval such that $R_0 \times J_0 \subseteq U$. Let $x \in R_0$ be such that the characterization of m_ν in (7.5) holds, and $(x, t) \in R_0 \times J_0$ for every $t \in J_0$.

To prove that ν^0 and ν^1 are intermingled, it is enough to check that, for $j = 0, 1$ the following integrals are positive

$$m_\nu(\mathcal{B}(\nu^j) \cap (R_0 \times J_0)) = \int_{J_0} \int_{W_A^s(x, R_0)} \int_{W_A^u(x, R_0)} \mathbf{1}_{\mathcal{B}(\nu^j)}(z, t) \rho_x(z) d\nu_x^u(z) d\nu_x^s(z) dt > 0.$$

Now, it follows from (7.6) that for every $t \in J_0$ and ν_x^s -almost every $z \in W_A^s(x, R)$,

$$\nu_z^u(W_A^u(z, R_0) \cap \mathcal{B}(\nu^j)) \geq \frac{\tilde{\beta}}{C_2^2} > 0,$$

and, again by Lemma 2.1 and (2.15), we have

$$\nu_x^u(W_A^u(x, R_0) \cap \mathcal{B}(\nu^j)) \geq \frac{\tilde{\beta}}{C_1 C_2^2} > 0.$$

Because the reference measure m_ν is a product measure and $t \in J_0$ was chosen arbitrarily, Fubini's theorem completes the argument.

PROOF OF CLAIM. Fix $j \in \{0, 1\}$ and write $n = n(j)$. From Lemmas 7.4 and 7.5, we conclude that

$$\nu_{A^n z}^u(\pi(K^n(\gamma_t) \cap \mathcal{B}(\nu^j))) \geq \nu_{A^n z}^u(\pi(K^n(\gamma_t) \cap B^j(R, \epsilon))) = \nu_{A^n z}^u(\pi(K^n(\gamma_t)) \cap \tilde{R}) > \tilde{\beta} > 0.$$

The measure ν has the uniformly bounded (unstable) Jacobian property (see Section 2.3); then, for every $x \in \mathbb{T}^d$ and $n \geq 1$ it holds $A_*^n \nu_x^u \ll \nu_{A^n x}^u$ and there is $C_2 > 0$ such that the Radon–Nikodym derivative satisfies (2.18). Denoting $E_n \stackrel{\text{def}}{=} \pi(K^n(\gamma_t) \cap B^j(R, \epsilon)) = \pi(K^n(\gamma_t)) \cap \tilde{R}$ and applying (2.18) to A^n :

$$\begin{aligned}
(7.7) \quad \nu_z^u \left(\pi \left(K^{-n} (K^n(\gamma_t) \cap B^j(R, \epsilon)) \right) \right) &\stackrel{\text{def}}{=} \nu_z^u \left(A^{-n}(E_n) \right) \\
&\stackrel{\text{def}}{=} A_*^n \nu_z^u(E_n) \\
&\geq \frac{1}{C_2^2} \nu_{A^n z}^u(E_n) \\
&\geq \frac{\tilde{\beta}}{C_2^2} > 0.
\end{aligned}$$

Here, the last inequality uses that $\nu_{A^n z}^u(E_n) > \tilde{\beta}$, which is the estimate for $\pi(K^n(\gamma_t)) \cap \tilde{R}$ obtained above.

Since the basin is an invariant set, we have

$$(7.8) \quad K^{-n}(K^n(\gamma_t) \cap B^j(R, \epsilon)) \subseteq \gamma_t \cap \mathcal{B}(\nu^j),$$

and hence from (7.8) and (7.7), (7.6) holds. $\square$

$\square$

As a consequence of the proof of Proposition 7.7, we have the following.

COROLLARY 7.8. *Let $\gamma_A \subseteq \mathbb{T}^d$ be a closed u -disk for A with $\nu_x^u(\gamma_A) > 0$ for every $x \in \gamma_A$, and let $\sigma_K \subseteq M$ be a closed u -disk for K with $\pi(\sigma_K) = \gamma_A$. Then there exist $\tilde{\beta} > 0$ and $\epsilon > 0$ such that, for every $j \in \{0, 1\}$: if $\gamma \subseteq \mathbb{T}^d$ is a closed u -disk that is ϵ - C^1 -close to γ_A with $\nu_z^u(\gamma) > 0$ for $z \in \gamma$, and $\sigma \subseteq M$ is the graph of a C^1 function $G: \gamma \rightarrow [0, 1]$, which is ϵ - C^1 -close to σ_K , then*

$$\nu_z^u(\pi(\sigma \cap \mathcal{B}(\nu^j))) \geq \frac{\tilde{\beta}}{C_1 C_2^2} > 0.$$

PROOF. Fix γ_A and σ_K as in the statement. By the proof of Proposition 7.7, there exist $N \geq 1$ and a rectangle $R \subseteq \mathbb{T}^d$ with $\nu(R) > 0$ such that $K^N(\sigma_K)$ crosses $B^j(R, \epsilon)$ for $j = 0, 1$. By the Claim in that proof,

$$\nu_z^u(\pi(\sigma_K \cap \mathcal{B}(\nu^j))) \geq \frac{\tilde{\beta}}{C_2^2} > 0.$$

By the Unstable Manifold Theorem, there exists $\epsilon > 0$ such that for every u -disk γ that is ϵ - C^1 -close to γ_A with $\nu_z^u(\gamma) > 0$, and every σ as in the statement, $K^N(\sigma)$ also crosses $B^j(R, \epsilon)$. Applying the Claim and Lemma 7.4, we obtain

$$\nu_z^u(\pi(\sigma \cap \mathcal{B}(\nu^j))) \geq \frac{\tilde{\beta}}{C_1 C_2^2} > 0. \quad \square$$

7.4. The union of the basins has full m_ν -measure

PROPOSITION 7.9. *Let ν be a strongly geometric measure on the base $\mathbb{T}^d$. Assume that ν^j are m_ν -observable, $j = 0, 1$. Then, the set $\mathcal{B}(\nu^0) \cup \mathcal{B}(\nu^1)$ has full m_ν -measure.*

PROOF. Assume, by contradiction, that the set $\mathcal{B}(\nu^0) \cup \mathcal{B}(\nu^1)$ does not have full m_ν -measure, and hence the set

$$(7.9) \quad Z \stackrel{\text{def}}{=} M \setminus (\mathcal{B}(\nu^0) \cup \mathcal{B}(\nu^1))$$

has positive m_ν -measure. Let $\hat{Z} = \pi(Z) \subseteq \mathbb{T}^d$. Note that the set Z is K -invariant and the set $\hat{Z}$ is A -invariant. Moreover, since $m_\nu(Z) > 0$, by Fubini's theorem, $\hat{Z}$ has positive ν -measure, and since ν is ergodic, $\hat{Z} \cap R$ has positive ν -measure for every rectangle $R \subset \mathbb{T}^d$ with $\nu(R) > 0$.

For any rectangle $R \subseteq \mathbb{T}^d$ and every $t \in I$, we denote $Z_t \stackrel{\text{def}}{=} Z \times \{t\}$, $\hat{Z}_t \stackrel{\text{def}}{=} \pi(Z_t)$, $Z_t^R \stackrel{\text{def}}{=} Z_t \cap (\tilde{R} \times \{t\})$ and $\hat{Z}_t^R \stackrel{\text{def}}{=} \pi(Z_t^R)$.

Then, there exist a rectangle $R \subseteq \mathbb{T}^d$ and an interval $J \subseteq I$, such that

$$m_\nu(Z \cap (R \times J)) > 0.$$

Define

$$J^R = \{t \in J : \nu(\hat{Z}_t^R) > 0\}.$$

By Fubini's Theorem, we have that $\text{Leb}_I(J^R) > 0$. Fix R as above and $t \in J^R$. Since ν has local product structure (Lemma 2.1), for ν -almost every $x \in R$ we have $\nu \sim \nu_x^u \otimes \nu_x^s$; fix such an x . Since $\nu(\hat{Z}_t^R) > 0$, for ν_x^s -almost every point $z \in W_A^s(x, R)$ we have

$$\nu_z^u(W_A^u(z, R) \cap \hat{Z}_t^R) > 0.$$

Let us denote $\gamma = W_A^u(z, R)$ and $\gamma_t \stackrel{\text{def}}{=} \gamma \times \{t\}$, so that $\nu_z^u(\gamma \cap \hat{Z}_t^R) > 0$; without loss of generality, assume that z is a ν_z^u -density point of $\gamma \cap \hat{Z}_t^R$ inside γ .

Claim: Suppose that $m_\nu(Z) > 0$. Then

$$(7.10) \quad \lim_{n \rightarrow \infty} \nu_{A^n z}^u \left(\pi(\sigma_t^n \cap [\mathcal{B}(\nu^0) \cup \mathcal{B}(\nu^1)]) \right) = 0.$$

Assuming the Claim, we complete the proof as follows. For every $n \geq 1$, $\sigma_t^n = K^n(\vartheta_t^n)$ is the graph of a C^1 function $G^n: \gamma^n \rightarrow [0, 1]$. The tangent vectors to $\vartheta_t^n = \vartheta^n \times \{t\}$ lie in the unstable cone field C_K^u of K . By partial hyperbolicity (4.4), DK maps C_K^u into itself, so the tangent vectors to $\sigma_t^n = K^n(\vartheta_t^n)$ remain in C_K^u for all $n \geq 1$. It follows that $\|DG^n\| \leq C_0$ uniformly; in particular, $\{G^n\}$ is equicontinuous. Since the γ^n are u -disks of fixed radius r_0 in the compact space $\mathbb{T}^d$, by compactness there is a subsequence such that

$\gamma^{n_k} \rightarrow \gamma_\infty$ in the Hausdorff metric and $A^{n_k}z \rightarrow z_\infty \in \gamma_\infty$. By Arzelà-Ascoli, the graphs $\sigma_t^{n_k}$ converge in the C^1 -topology to a C^1 -graph σ_t^∞ over γ_∞ .

Since ν is a strongly geometric measure, its unstable conditional measures ν_x^u vary continuously (in the weak topology) along unstable leaves. Thus $\nu_{A^{n_k}z}^u \rightarrow \nu_{z_\infty}^u$ weakly, so $\nu_{z_\infty}^u$ is well-defined. By Corollary 7.8, there exists $\tilde{\beta} = \tilde{\beta}(\sigma_t^\infty) > 0$ such that

$$\nu_{z_\infty}^u \left(\pi(\sigma_t^\infty \cap [\mathcal{B}(\nu^0) \cup \mathcal{B}(\nu^1)]) \right) \geq \tilde{\beta}.$$

Moreover, Corollary 7.8 states that there exists $\epsilon > 0$ such that for every sufficiently large $k \geq 1$, $\sigma_t^{n_k}$ is ϵ -close to σ_t^∞ in the C^1 -topology,

$$\nu_{A^{n_k}(z)}^u \left(\pi(\sigma_t^{n_k} \cap [\mathcal{B}(\nu^0) \cup \mathcal{B}(\nu^1)]) \right) \geq \tilde{\beta},$$

contradicting (7.10).

It remains to prove the Claim.

PROOF OF CLAIM. Since z is a ν_z^u -density point of $\gamma \cap \hat{Z}_t^R$ inside γ , for every $r_0 > 0$, it follows from Lemma 2.5 that there exists a sequence of unstable balls $\gamma^n \subseteq A^n(\gamma)$, $n \geq 1$, of radius r_0 such that

$$\lim_{n \rightarrow \infty} \frac{\nu_{A^n z}^u \left(\gamma^n \cap A^n(\gamma \cap \hat{Z}_t) \right)}{\nu_{A^n z}^u(\gamma^n)} = 1.$$

Recall that by construction (Lemma 2.4–2.5, see (2.20) and (2.24)), for each $n \geq 1$ there is a u -disk $\vartheta^n \subseteq \gamma$ such that $A^n(\vartheta^n) = \gamma^n$. Let $\vartheta_t^n = \vartheta^n \times \{t\} \subseteq \gamma_t$ and denote $\sigma_t^n = K^n(\vartheta_t^n)$. Note that $\pi(\sigma_t^n) = \gamma^n$. With the notation above, we have

$$(7.11) \quad \lim_{n \rightarrow \infty} \frac{\nu_{A^n z}^u \left(\pi(\sigma_t^n) \cap A^n(\pi(\gamma_t \cap Z_t)) \right)}{\nu_{A^n z}^u(\gamma^n)} = 1.$$

Since Z is K -invariant and the projection $\pi: M \rightarrow \mathbb{T}^d$ is a semiconjugacy between K and A (see (4.7) and Section 4.2), then for every $n \geq 1$, we have

$$A^n(\pi(\gamma_t \cap Z_t)) = \pi(K^n(\gamma_t \cap Z_t)) = \pi(K^n(\gamma_t) \cap Z),$$

and since $\sigma_t^n \subseteq K^n(\gamma_t)$, then

$$\pi(\sigma_t^n) \cap A^n(\pi(\gamma_t \cap Z_t)) = \pi(\sigma_t^n \cap Z).$$

Therefore, by (7.9), the limit (7.11) implies (7.10). □

□

7.5. Hyperbolic measures with intermingled basins: lifts of strongly geometric measures

PROPOSITION 7.10. *Let ν be a strongly geometric measure, and let ν^0 and ν^1 be m_ν -observable. Then, there exists exactly one $\mu \in \mathcal{E}_{>}^{\text{int}}(K)$ such that $\pi_*\mu = \nu$.*

PROOF. Since ν^0 and ν^1 are m_ν -observable, part (1) of Proposition 6.8 is excluded (it requires ν^1 to have trivial basin); hence $\mathcal{E}^{\text{int}}(K) \neq \emptyset$. Moreover, since ν^1 is m_ν -observable, Proposition 6.8 part (3) cannot occur. Indeed, by Remark 6.9, the construction downward from ν^1 yields a measure $\mu_* \in \mathcal{E}^{\text{int}}(K)$ that is the immediate predecessor of ν^1 in $\mathcal{E}_\nu(K)$, with $\chi^c(\mu_*) > 0$; in particular, there is no measure strictly between μ_* and ν^1 . But in part (3) the sequence μ_n increases to ν^1 , so $\mu_* < \mu_n < \nu^1$ for large n — a contradiction. Then, let us take $\mu = \mu_1$ in Proposition 6.8 part (2), so that $\chi^c(\mu) > 0$.

Claim: $\mathcal{E}^{\text{int}}(K) = \mathcal{E}_{>}^{\text{int}}(K) = \{\mu\}$.

PROOF OF CLAIM. Otherwise, if there is $\tilde{\mu} \in \mathcal{E}_{>}^{\text{int}}(K)$, $\mu < \tilde{\mu}$, by Proposition 6.8 part (2), there should be a third one $\hat{\mu} \in \mathcal{E}_{<}^{\text{int}}(K)$ such that $\mu < \hat{\mu} < \tilde{\mu}$. Since $\chi^c(\hat{\mu}) < 0$, $\hat{\mu}$ should be m_ν -observable (Lemma 6.2 part (2)). Therefore $m_\nu(\mathcal{B}(\hat{\mu})) > 0$, contradicting the fact that $\mathcal{B}(\nu^0) \cup \mathcal{B}(\nu^1)$ has full m_ν -measure. $\square$

$\square$

PROOF OF THEOREM C. The proof was discussed throughout Chapter 7. Indeed, the assertions of Theorem C part (1) are treated as follows: the density of the basins is established in Lemma 7.6, the intermingling in Proposition 7.7, and Proposition 7.9 establishes that the union of the basins has full m_ν -measure. The assertion that there is no other m_ν -observable, K -invariant measure μ on M such that $\pi_*\mu = \nu$ follows from Proposition 7.10. Part (2) of Theorem C follows from Proposition 7.10. $\square$

7.6. Existence of geometric measures in the boundaries

We finish this section by establishing that geometric, and even strongly geometric, measures $\nu \in \mathcal{E}(A)$ always exist; moreover, each of the spaces $\mathcal{E}_{<}^j(K)$, $\mathcal{E}_{=}^j(K)$, $\mathcal{E}_{>}^j(K)$, $j \in \{0, 1\}$, is nonempty and contains strongly geometric measures.

COROLLARY 7.11. *Each of the spaces $\mathcal{E}_{<}^j(K)$, $\mathcal{E}_{=}^j(K)$, $\mathcal{E}_{>}^j(K)$, $j \in \{0, 1\}$, contains a strongly geometric measure.*

PROOF. Under our general hypotheses [KD2], for $j \in \{0, 1\}$, each of the spaces $\mathcal{E}_{<}^j(K)$, $\mathcal{E}_{=}^j(K)$, and $\mathcal{E}_{>}^j(K)$ is nonempty. Indeed, the Dirac measures $\delta_{(p,0)}$ and $\delta_{(q,0)}$ are in $\mathcal{E}_{>}^0(K)$

and $\mathcal{E}_{<}^0(K)$, respectively. Analogously for $\delta_{(p,1)}$ and $\delta_{(q,1)}$. Moreover, since $\mathcal{E}^j(K)$ is arc-connected (Lemma 3.2), there are continuous maps $\gamma_j: [0, 1] \rightarrow \mathcal{E}^j(K)$ such that $\gamma_j(0) = \delta_{(p,j)}$ and $\gamma_j(1) = \delta_{(q,j)}$ and hence $\gamma_j(t_j) \in \mathcal{E}_{=}^j(K)$ for some $t_j \in (0, 1)$.

The result is now an immediate consequence of Proposition 3.5 and Remark 3.7 taking $\psi = J^j$ together with Lemma 2.3. $\square$

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