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Quantum-critical Tilted Non-Hermitian Dirac semimetals

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Abstract

Dirac materials, hosting linearly dispersing quasiparticles at low energies, exhibit an emergent Lorentz symmetry close to a quantum critical point (QCP) separating semimetallic state from a strongly-coupled gapped insulator or superconductor. This feature appears to be quite robust even in the open Dirac systems coupled to an environment, featuring non-Hermitian (NH) Dirac fermions: close to a strongly coupled QCP, a Yukawa-Lorentz symmetry emerges in terms of a unique terminal velocity for both the fermion and the bosonic order parameter fluctuations, while the system can either retain non-Hermiticity or completely decouple from the environment thus recovering Hermiticity as an emergent phenomenon.

In this thesis, we propose a minimal model describing a non-Hermitian (NH) two-dimensional tilted Dirac semimetal (DSM), by introducing masslike anti-Hermitian Dirac operators to the tilted Dirac Hamiltonian. We calculate density of states (DOS) and optical conductivity (OC), where because of NH and tilt dependence on these observables, we show that tilt can be used as a probe for non-Hermiticity in the low-energy regime. Additionally, we show that such a Yukawa-Lorentz symmetry can emerge at the quantum criticality even when the NH Dirac Hamiltonian includes a tilt term at the lattice scale. As we demonstrate by performing a leading order $\epsilon = 3 - d$ expansion close to $d = 3$ upper critical dimension of the theory, a tilt term becomes irrelevant close to the QCP separating the NH Dirac semimetal and a gapped (insulating or superconducting) phase. Such a behavior also extends to the case of the linear-in-momentum non-tilt perturbation, introducing the velocity anisotropy for the Dirac quasiparticles, which also becomes irrelevant at the QCP.

These predictions can be numerically tested in quantum Monte Carlo lattice simulations of the NH Hubbard-like models hosting low-energy NH tilted Dirac fermions.

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Chapter 1

Introduction

Dirac materials, hosting linearly dispersing quasiparticles at low energies, exhibit an emergent relativistic-like Lorentz symmetry due to the quasiparticles linear energy-momentum dispersion, with the Fermi velocity, typically a few hundred times smaller than the velocity of light, playing the role of the velocity of light [17, 106, 2]. This symmetry is surprisingly robust, persisting even with electron-electron interactions, which break it at the lattice (UV) scale. For example, such a pseudorelativistic Lorentz symmetry emerges in the deep infrared regime for two-dimensional and three-dimensional Dirac fermions interacting via the long-range Coulomb interaction, featuring a common terminal velocity for both the Dirac fermions and the photons [34, 38]. Additionally, for Yukawa (short-range) interacting Dirac fermions, the quantum-critical point (QCP) that separates the semimetallic phase from a strongly-coupled gapped (insulating or superconducting) phase also exhibits this symmetry [52, 88, 89, 87]. We discuss the concept of quantum phase transitions in Section 1.1 and Dirac fermions in Section 1.2.

Recently Lorentz symmetry has been extended to the non-Hermitian (NH) Dirac systems [40, 70, 71], pertaining to Dirac fermions coupled to an environment, as discussed in 1.5, described by a minimal Dirac operator capturing non-Hermiticity, as given by Eq. (3.2). Such an NH Dirac operator, with its anti-Hermitian part linear in momentum, is constructed by invoking a Dirac mass term in the Hermitian Dirac Hamiltonian, represented by a Hermitian matrix M that anticommutes with the Hermitian Dirac Hamiltonian [40]. It features a purely real or imaginary spectrum, with a linearly vanishing density of states (DOS) for a weak non-Hermiticity, as discussed in Sec. 3.1.1, implying the stability of NH Dirac semimetal for weak short-range Hubbard-like electron-electron interactions. However, when such a short-range interaction, effectively described by the Yukawa coupling, is strong enough, the system undergoes a quantum-phase transition with the effective Fermi velocity of the NH Dirac fermions and the bosonic order parameter (OP) excitations reaching a common value at a QCP. The fate of the coupling with the environment, however, depends on the canonical commutation relations between the OP and the Dirac mass matrix

M [40]: when the OP commutes with the matrix M , the latter therefore representing the commuting class mass (CCM) OP, the system retains non-Hermiticity, while featuring a common velocity for all participating degrees of freedom, thus yielding an emergent NH Yukawa-Lorentz symmetry. On the other hand, in the case of an anticommuting mass (ACM) OP, the system completely decouples from the environment, thereby restoring the usual Lorentz symmetry, with concomitant emergence of the Hermiticity. Such behavior of NH (Hermitian) Dirac fermions suggests ubiquity of the emergent Yukawa-Lorentz (Lorentz) symmetry at QCPs, despite being absent at the UV scale, and irrespective of the symmetry-breaking perturbations in the bare (lattice) Hamiltonian. This is also further supported by the emergence of this symmetry for birefringent NH Dirac fermions, featuring different velocities, therefore breaking Lorentz symmetry at the bare level [70].

From the point of view of the Dirac Hamiltonian, adding an operator linear in a single component of the momentum represents its minimal extension that breaks the rotational, and therefore also Lorentz symmetry, which is, at the same time, marginal close to the semimetallic ground state. Details of these features are discussed in Section 1.3. A particular minimal non-Lorentz symmetric extension of the Dirac Hamiltonian is tilt of the Dirac Hamiltonian [95], represented by a matrix that commutes with it, which was shown to be irrelevant at the QCP in the Hermitian case [81]. Such a tilt term appears in many instances when considering the Hermitian Dirac fermions, e.g., as a platform for simulating curved spacetime in condensed matter systems [104, 73, 39, 55, 24, 8, 105, 91, 48, 49], it can exhibit nontrivial quantum transport [99, 13, 101, 84, 82, 68, 96, 107, 85], while the effects of the long- and short-range components of the Coulomb interaction [94, 83, 32, 33, 57], various disorders [100, 93, 109, 54], and the magnetic field [29, 69, 72] have also been explored in this context. The tilt term may also exhibit nontrivial transport effects for NH Dirac models [78]. However, the role of this Lorentz-symmetry breaking perturbation and the ultimate fate of the tilt for quantum-critical NH Dirac fermions, in relation to the emergence of Yukawa-Lorentz symmetry, remain unexplored.

To this end, we consider the simplest scenario: a Hermitian tilt applied to an NH Dirac fermion, and the focus of this thesis is to understand if the addition of these two mentioned effects, namely, non-Hermiticity and the tilt, on the Dirac Fermions have a relevance in both the observables and in the quantum critical theory. As to see if the introduction of this tilt makes a difference in an observable we first analyze the zero-temperature optical conductivity. Our result is that the addition of tilt makes the OC depend on the NH parameter (Eqs. (3.21)-(3.23) and Fig. 3.3), where we calculated the zero-temperature OC within the Kubo formalism, which implies that tilt can be used as a probe for NH effects on DSMs. Our second and main result is the emergence of the Yukawa-Lorentz symmetry in the vicinity of the strongly coupled QCP separating the NH Dirac semimetallic phase from an insulating CCM phase since the tilt term becomes *irrelevant* in the vicinity of this

QCP (Eqs. (4.18)-(4.21) and Fig. 4.4), while remaining coupled to the environment. To show it, we here perform a field-theoretical analysis of the strongly interacting NH tilted Dirac fermions within a leading-order renormalization group (RG) $\epsilon = 3 - d$ -expansion, close to $d = 3$ upper-critical dimension of the Gross-Neveu-Yukawa field theory, in which the Yukawa and the quartic OP couplings are marginal. For weak short-range electron interactions, the NH Dirac semimetal is stable, due to the linearly vanishing DOS close to the zero energy, analyzed in Sec. 3.1.1, see also Fig. 3.1. Furthermore, our calculation of the mean-field susceptibilities shows that the competition between the ACM and CCM OPs, in spite of the ACM OPs being favored at the mean-field level (Figs. 4.1 and 4.2), depends on the system's parameters at the corresponding QCP. This further motivates the analysis of the quantum-critical behavior using the RG formalism, which includes the effects of the fluctuations in the quantum-critical region, ultimately uncovering the Yukawa-Lorentz symmetry for the quantum-critical tilted NH Dirac fermions. Furthermore, the fate of the non-Hermiticity depends on the algebra of the NH Hamiltonian and the OP, more precisely, whether the OP belongs to the ACM or CCM class, as shown in Fig. 4.4. In the former case, the system completely decouples from the environment, therefore restoring the hermiticity, while for the latter, the system remains coupled to the environment while displaying an emergent Yukawa-Lorentz symmetry. For completeness, we also consider a term linear in momentum that neither commutes nor anticommutes with the NH Hamiltonian, which introduces a velocity anisotropy, and as we demonstrate, it also becomes irrelevant close to the QCP and eventually vanishes (Fig. A.1), while the system decouples from the environment, restoring Hermiticity and Lorentz symmetry.

Altogether, our results support a scenario of the Yukawa-Lorentz symmetry being a universal feature of quantum-critical NH Dirac fermions. This prediction should be tested in the numerical simulations, such as quantum Monte Carlo, for instance, in the NH Hubbard model with the asymmetric hoppings together with the tilt term implemented on the honeycomb lattice, analogously to the Hubbard model for the tilt-free NH Dirac fermions [112].

This study has already been posted in the preprint server in arXiv in [77], and has been submitted to JHEP. The chapters 1, 3, 4 and A are based on the preprint.

Organization

The rest of this thesis is organized as follows. In Sec.1.1 we introduce the concepts of phase transitions, its quantum version, and scaling. Then, in Sec. 1.2 we discuss the Dirac Fermions, tilt, and discuss basic characteristics of its self interactions. We review the notions of non-Hermiticity in Sec. 1.5, with the emphasis on the concepts associated with it. The theoretical framework of this thesis is discussed in Chapter 2, where we show the mathematical formalism of for field theory, renormalization group, and Gross-Neveu-

Yukawa model, all of which are used in this study. In Sec. 3.1, we introduce the model of the NH Dirac fermions with an extra term linear in momentum, which includes the tilt and the velocity anisotropy, and we calculate the DOS (Sec. 3.1.1) and OC (Sec. 3.2), showing the possibility of using tilt as a probe of non-Hermiticity. Sec. 4.1 is dedicated to the analysis of the mean-field susceptibilities in $d = 3$, while in Sec. 4.2 we discuss the features of the quantum-critical Gross-Neveu-Yukawa theory for the tilted Dirac fermions. In particular, we calculate the RG flows of the fermionic and bosonic velocities and the tilt parameter, establishing the emergence of the Yukawa-Lorentz (Lorentz) symmetry at the QCP controlling the transition into a CCM (ACM) order. The conclusions are presented in Sec. 5, while the various technical details are relegated to the appendices.

1.1 Quantum Phase Transitions

1.1.1 Concepts of phase transitions

Phase transitions play an essential role in nature, ranging from everyday examples like the melting of ice to liquid, or more exotic ones as the superconductor-insulator quantum phase transition.

Phase transitions occur upon variation of an external control parameter; their common characteristics is a qualitative change in the (macroscopic) system properties. At first, phase transitions were thought to be caused only by thermal fluctuations (temperature), but during recent decades, a different class of phase transitions has attracted the attention of physicists, namely transitions taking place at zero temperature, which are named quantum phase transitions. In this class of phase transitions, a non-thermal control parameter such as pressure, magnetic field, or chemical composition, is varied to access the transition point, where order is destroyed solely by quantum fluctuations [103].

Phase transitions are traditionally classified into first-order and continuous transitions. First-order transitions are characterized by two phases coexisting at the same temperature (or other parameter), and involve a quantity called latent heat, whereas continuous (second-order) transitions these two phases do not coexist.

We will concentrate on systems near a continuous phase transition. Such a transition can usually be characterized by an order parameter (OP), a thermodynamic quantity that is zero in one phase (the disordered) and non-zero and non-unique in the other (the ordered) phase.

While the thermodynamic average of the order parameter is zero in the disordered phase, its fluctuations are non-zero. When we approach the transition point (critical point), the spatial correlations of the order parameter become long ranged, and close to the critical point, the correlation length ξ , a measure of how “ordered” our system is, and

serves us as a characteristic scale, diverges as

$$\xi \propto |t|^\nu \quad (1.1)$$

where ν is the correlation length critical exponent and t is a dimensionless measure of the distance from the critical point. Non-trivial macroscopic or large-distance phenomena are associated with this characteristic scale.

In addition to the long-range correlations in space there are analogous long-range correlations of the order parameter fluctuations in time. The typical time scale for a decay of the fluctuations is the correlation (or equilibration) time τ_c . As the critical point is approached the correlation time diverges as:

$$\tau_c \propto \xi^z \propto |t|^{z\nu} \quad (1.2)$$

where z is the dynamic critical exponent, which determines the relative scaling of space and time.

These two divergences (1.1) and (1.2) are the responsible for the critical phenomena. At the critical point, correlation length and time are infinite, fluctuations occur on all length and time scales, and the system is said to be scale-invariant. As such, all observables depend on the external parameters via power laws. The set of corresponding exponents—called critical exponents—completely characterizes the critical behavior near a particular phase transition.

1.1.2 Quantum mechanics and the vicinity of the critical point

Now we could question ourselves, why is quantum mechanics important for critical phenomena?. First of all, quantum mechanics can be essential to understand the existence of the ordered phases, for example when analyzing Ising spin systems. Second of all, we may wonder if there is any dependence of the microscopic details of the system in the power laws behavior for critical phenomena. For this discussion we have to compare two energy scales, namely $\hbar\omega_c$, which is the typical energy of long-distance order parameter fluctuations, and the thermal energy $k_B T$.

Quantum mechanics will be important as long as the first typical energy scale is larger than the thermal energy $k_B T$; on the other hand, for $\hbar\omega_c \ll k_B T$ a purely classical description can be applied to the order parameter fluctuations. In other words, the character of the order parameter fluctuations crosses over from quantum to classical when $\hbar\omega_c$ falls below $k_B T$. This means that transitions at $T = 0$, as a function of a non-thermal parameter (or coupling) g like pressure or magnetic field always are dominated by the microscopic degrees of freedom. Consequently, transitions at zero temperature are called “quantum” phase transitions.

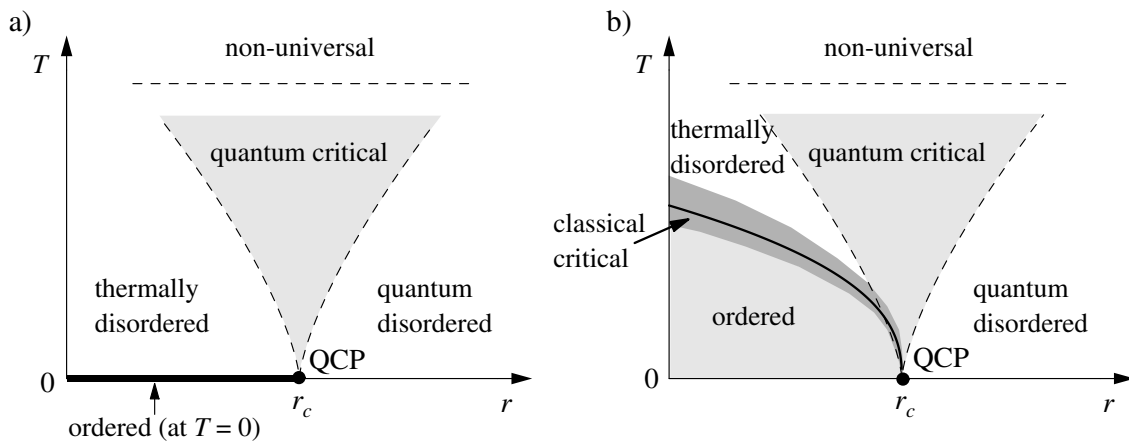


Figure 1.1: Schematic phase diagrams in the vicinity of a quantum critical point (QCP). The horizontal axis represents the control parameter r used to tune the system through the quantum phase transition, the vertical axis is the temperature T . a) Order is only present at zero temperature. These crossover lines are given by $k_B T \propto |r - r_c|^{\nu z}$. b) Order can also exist at finite temperature. The solid line marks the finite-temperature boundary between the ordered and disordered phases [92].

Let the Δ represent a energy scale characterizing fluctuations at $T = 0$ for $g \neq g_c$, e.g., Δ could be the energy of the lowest excitation above the ground state (energy gap), or if the energy spectrum is gapless, Δ is the scale at which there is a qualitative change in the nature of the frequency spectrum from its lowest frequency to its higher frequency behavior. In most cases, we will find that as g approaches g_c , Δ vanishes as:

$$\Delta \sim J|g - g_c|^{z\nu}, \quad (1.3)$$

where J is the energy scale of a characteristic microscopic coupling, z and ν is a critical exponent.

1.1.3 Universality Classes and Critical Exponents

One of the most remarkable features of continuous phase transitions is universality, i.e., the fact that the critical exponents are the same for entire classes of phase transitions which may occur in very different physical systems. These universality classes are determined only by the symmetries of the order parameter, by the space dimensionality of the system and whether or not the forces are short-ranged. This implies that the critical exponents of a phase transition occurring in nature can be determined exactly (at least in principle) by investigating any simple model system belonging to the same universality class. The mechanism behind universality is again the divergence of the correlation length. Close to the critical point the system effectively averages over large volumes rendering the microscopic details of the Hamiltonian unimportant [31].

Now, the default framework to study these universality classes lies in the conceptual framework of the Renormalization group (RG), which will be explained later in section 2.2, but the main idea of it is the scaling behaviour of observables under scaling transformations.

1.2 Dirac materials

During the last decade, there has been great interest in researching Dirac fermions systems in condensed matter physics, which originally were studied in high energy physics [30]. This interest started when studying the electrons in graphene (two-dimensional graphite in a honeycomb lattice), where in the low-energy band structure (near the touching points of the valence and conducting bands, in the so called Dirac points) and the Hamiltonian has a massless fermion-like form. This structure causes the graphene to have a number of exotic properties, that motivated the study of many systems that have this behavior, be it in two or three spatial dimensions, such as surfaces of three-dimensional topological insulators, quasi-2D organic materials, artificial graphenes, etc [35, 43, 98].

The observed similarity of these systems in the low-energy spectrum points to universal characteristics. There are symmetries that control the formation of the Dirac points (or possibly lines) in the energy spectrum. These symmetries are different for different materials, yet their net effect is to preserve Dirac nodes. Examples include time-reversal symmetry in topological insulators and sublattice symmetry in graphene [106]. Once Dirac nodes are established in the spectrum, the universality of response functions and susceptibilities naturally follows.

1.2.1 Dirac Fermions

We mentioned how these systems were originally studied in high-energy physics, where they describe relativistic spin- $\frac{1}{2}$ particles [114]. On the other hand, in condensed matter physics, the Lorentz symmetry emerges in a low energies, different dimensionalities (two or three) and with a velocity scale v_F , which is roughly two orders of magnitude smaller than the speed of light. These systems are described by effective low energy Hamiltonian of the form

$$H(p) = \gamma_0(iv_F\boldsymbol{\gamma} \cdot \mathbf{p} + mv_F^2), \quad (1.4)$$

which has eigenvalues:

$$H^2(p) = p^2 + m^2, \quad (1.5)$$

the $d \times d$ γ matrices, γ_0 and $\boldsymbol{\gamma} \equiv \{\gamma^1, \dots, \gamma^{d-1}\}$, are Hermitian. They satisfy

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\delta^{\mu\nu} \mathbf{1} \quad (1.6)$$

The description of these spin- $\frac{1}{2}$ particles is given by the Dirac equation:

$$(i\hbar v_F \gamma^\mu \partial_\mu - m v_F^2) \psi = 0. \quad (1.7)$$

As we do field theoretical calculations for the quantum phase transition phenomena, we define a Dirac fermion Minkowski (real time) action for a $(d-1)$ dimensional space [114]

$$S_f[\bar{\psi}, \psi] = \int d\tau \int d^d \mathbf{x} \bar{\psi}(\tau, \mathbf{x}) (i\hbar v_F \gamma^\mu \partial_\mu - m v_F^2) \psi(\tau, \mathbf{x}), \quad (1.8)$$

where $\bar{\psi} = \psi^\dagger \gamma^0$ is the Dirac adjoint field, which with ψ are the satisfy Grassmann algebra. This action is Lorentz invariant. A *Grassmann algebra* $\mathfrak{U}$ over $\mathbb{R}$ or $\mathbb{C}$ is an associative algebra constructed from a unit 1 and a set of generators θ_i with anticommuting products

$$\theta_i \theta_j + \theta_j \theta_i = 0, \quad \forall i, j. \quad (1.9)$$

For purposes of calculations, we do an analytic continuation to imaginary time ($t \rightarrow -i\tau$), also called Wick rotation, and we define an Euclidean action for a d -dimensional space as:

$$S_f[\bar{\psi}, \psi] = \int_0^\beta d\tau \int d^d \mathbf{x} \bar{\psi}(\tau, \mathbf{x}) (\gamma^\mu \partial_\mu - m) \psi(\tau, \mathbf{x}), \quad (1.10)$$

where β the inverse temperature.

For the RG calculations adressing the quantum critical phenomena, we define the partition function for self-interacting massive Dirac fermions is given by an integral over Grassmann fields of the form:

$$\mathcal{Z}(\beta) = \int [d\psi d\bar{\psi}] \exp \left[-S_f[\bar{\psi}, \psi] + \mu \int_0^\beta d\tau \int d^d \mathbf{x} \bar{\psi}(\tau, \mathbf{x}) \gamma_0 \psi(\tau, \mathbf{x}) \right] \quad (1.11)$$

where μ the chemical potential. The term coupled to the chemical potential μ is Hermitian, corresponds to a quantum operator which commutes with the Hamiltonian.

The fermion fields must satisfy anti-periodic boundary conditions in the time direction:

$$\psi(\tau = 0, \mathbf{x}) = -\psi(\tau = \beta, \mathbf{x}), \quad \bar{\psi}(\tau = 0, \mathbf{x}) = -\bar{\psi}(\tau = \beta, \mathbf{x}) \quad (1.12)$$

For this work, we set zero temperature and zero chemical pontential. In this limit, the boundary conditions play no role, and we can no longer distinguish between space and

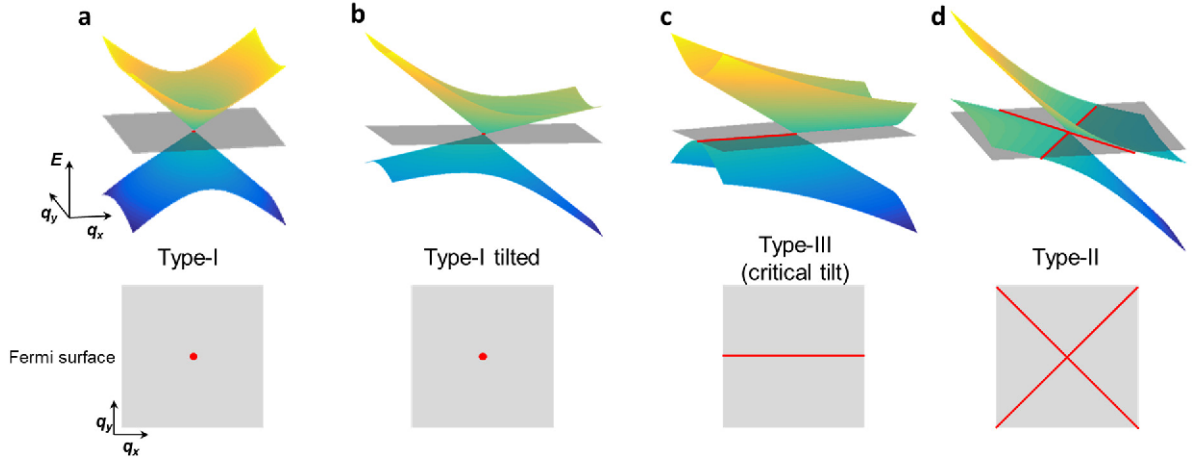


Figure 1.2: Types of Dirac dispersions in two dimensions. (top) Dispersions together with the zero energy plane (grey); (bottom, in red) zero-energy Fermi surface: a Standard, type-I Dirac cones characterised by a linear dispersion in all directions in k -space and a point-like Fermi surface. b Type-I tilted Dirac cone. c Type-III Dirac point (critically tilted), combining a flat band along a line and linear dispersions. Its Fermi surface is a line. d Type-II Dirac cone [66].

time.

1.3 Tilted Dirac semimetals

In graphene, the conduction band exhibits an almost perfect cone shape near the Dirac point, however, in other DSM materials, the Dirac cone is distorted, and we focus on the tilted Dirac cones, which can be realized in a number of materials, including properly deformed graphene, organic compound $\alpha - (\text{BEDTTTF})_2\text{I}_3$ [29], 8-Pmmn borophene [113], and partially hydrogenated graphene [58]. In addition, the surface of topological crystalline insulators, such as the (001) surface of SnSe, may also host a tilted Dirac cone [97].

These materials can near the Dirac points can be modeled as a free Hamiltonian of the form:

$$H(p) = \gamma_0(i\boldsymbol{\gamma} \cdot \mathbf{p} + m) + \gamma_0(\boldsymbol{\alpha} \cdot \mathbf{p}) . \quad (1.13)$$

where $\boldsymbol{\alpha} = \alpha \hat{\mathbf{n}}$ is the tilt parameter.

For $|\alpha| < 1$, the system is identified as type-I DSM, in which the Fermi surface is made out of discrete band-touching points. For $|\alpha| > 1$, the system corresponds to type-II DSM, whose Fermi surface is a finite open line. The fermion density of states vanishes at the Fermi level in type-I DSM, but takes a finite value in type-II DSM. Physically, $\alpha < 0$ or $\alpha > 0$ are equivalent [66].

Now, tilt can have many effects on these Dirac systems, such as a change in the

conductance and noise for quantum transport of Dirac cones [99], or anisotropic conductivity in the parallel and perpendicular direction with respect to the tilt direction [84]. They also serve as a platform to study other phenomena like anisotropic optics such as birefringence and gravitational lensing [49]. We can use these effects to experimentally characterize these tilted systems.

1.4 Field theory of self-interacting Dirac Fermions

Now, following the study of Dirac fermions at the criticality, we carry out a mean-field analysis of self-interacting massless Dirac fermions, which will serve us as a platform to study the phenomena associated with interacting tilted Dirac fermions we are interested in.

First, the massless Euclidean Dirac action in the momentum is of the form

$$S_f = \int \frac{d\omega}{2\pi} \int \frac{d^d \mathbf{k}}{(2\pi)^d} \bar{\psi}(\omega, \mathbf{k}) (-i\gamma^\mu k_\mu) \psi(\omega, \mathbf{k}), \quad (1.14)$$

with $k_\mu \equiv (\omega, \mathbf{k})$. In this section, we set $v_F = 1$

We now consider a local density-density interaction of the form

$$S_{\text{int}} = g \int d\tau \int d^2 \mathbf{x} (\bar{\psi}(\tau, \mathbf{x}) \psi(\tau, \mathbf{x}))^2. \quad (1.15)$$

The general form of the local density-density fermion interaction is given by $S_{\text{int}} = g_N \int d\tau \int d^d \mathbf{x} (\bar{\psi}(\tau, \mathbf{x}) \hat{N} \psi(\tau, \mathbf{x}))^2$, with $\hat{N}$ as a 4×4 Hermitian matrix. Let us compute engineering dimension of the coupling constant g . The engineering dimension of the Dirac field in units in which $\dim[k_i] = +1$ and $\dim[\omega] = z$, with z as the dynamical exponent, is $\dim[\psi] = d/2$, which is obtained from the term in the action containing the time derivative of the fermion field. This then implies that the dimension of the short-range coupling is $\dim[g] = z - d$. Therefore, for $z = 1$ and $d = 2$, and then $\dim[g] = -1$, so the short-range interaction is therefore irrelevant in this case. Irrelevance of the short-range interaction may be seen as a consequence of the linearly vanishing density of states in this case. The instability of the semimetal into an insulator may be therefore expected to occur at a finite strength of the interaction, which is indeed the case, as we show below [92].

We now formulate mean-field theory for the transition into an insulating state induced by the short-range interaction in two spatial dimensions, $d = 2$. We consider the action $S = S_f + S_{\text{int}}$, given by Eqs.(1.10) and (1.15). We then perform Hubbard-Stratonovich transformation to decouple this interaction term by introducing a new one $m = \langle \bar{\psi} \psi \rangle$. This is done in the channel which in the case of graphene corresponds to the charge-density wave instability with the sublattice symmetry broken by the difference of the chemical

potentials on the two sublattices (We could have chosen another channel, but it can be that the other channels are nonphysical, The action is then

$$S = \int d\tau \int d^2\mathbf{x} \left\{ \bar{\psi}(\tau, \mathbf{x}) (\gamma^\mu \partial_\mu + m) \psi - \frac{1}{2g} m^2 \right\} \quad (1.16)$$

The corresponding ground-state free energy is then

$$f = -\frac{1}{2g} m^2 + \int \frac{d^2\mathbf{q}}{(2\pi)^2} \sqrt{q^2 + m^2} \quad (1.17)$$

Minimizing above free energy with respect to the order parameter m , we obtain

$$-\frac{1}{g} + \int \frac{d^2\mathbf{q}}{(2\pi)^2} \frac{1}{\sqrt{q^2 + m^2}} = 0 \quad (1.18)$$

and therefore the order parameter m acquires a non-zero expectation value for $g > g_c = \frac{2\pi}{\Lambda}$, with Λ as the momentum cutoff. The value of the critical interaction involves the momentum cutoff and is therefore non-universal, as expected on general grounds. Then

$$\frac{1}{g} - \frac{1}{g_c} = \int \frac{d^2\mathbf{q}}{(2\pi)^2} \left(\frac{1}{\sqrt{q^2 + m^2}} - \frac{1}{q} \right) \quad (1.19)$$

yielding, after performing the integral and keeping the leading term in m/Λ ,

$$\frac{1}{g} - \frac{1}{g_c} = \frac{m}{2\pi} \quad (1.20)$$

The last equation then implies

$$m \sim (g - g_c) \quad (1.21)$$

yielding the correlation-length exponent $\nu = 1$ for the phase transition of the Dirac fermions in the mean-field theory. The last statement follows from $m \sim \xi^{-z} \sim (g - g_c)^{z\nu}$, and $z = 1$ for the phase transition we consider since the order-parameter respects pseudo-relativistic invariance of the Dirac action (1.10). The transition from the semimetallic ground state to the insulating one takes place at the quantum-critical point at a finite strength of the interaction. Since the system is two-dimensional and the Dirac fermions already in the semimetallic phase are gapless, we expect the fluctuations to play the crucial role at the transition, which will be studied later in the thesis.

1.5 Non-Hermitian physics

One of the basic axioms of quantum mechanics requires observables, such as the Hamiltonian of a closed system, to be Hermitian operators. Real physical systems, however, are at least

to some extent coupled to their environment, where the presence of dissipative processes makes their description more complicated[4]: In general, the known Schrödinger equation where we evaluate the evolution of the Hermitian Hamiltonian is replaced by a Liouvillian superoperator governing the time evolution of the density matrix[10]. In some cases, e.g. open systems in contact with an environment, we can use some easier formalisms, such as Lindblad quantum master equations [56], Feynman-Vernon theory [25], and the Keldysh formalism [46], still they are very complex and this limits the range of systems that can be efficiently studied. In this respect, effective non-Hermitian (NH) Hamiltonians provide a conceptually simpler and intuitive alternative to fully microscopic approaches.

These NH hamiltonians have been used in a myriad of physical platforms, which ranges from classical systems, including optical settings, electrical circuits, and mechanical systems, which may be mapped to an effective NH Schrödinger equation, all the way to quantum materials [3, 18, 21, 67, 75, 86]. In a wider historical context, effective NH concepts have been ubiquitous for many decades [6, 7, 11, 20, 19, 36, 9].

One could think that we cannot work with NH Hamiltonians because they may lead to complex energy eigenvalues and imply a non-unitary S matrix, which does not conserve probability, but in reality it turned out that it is possible to describe natural processes by means of $\mathcal{PT}$ -symmetric non-Hermitian Hamiltonians, where we can construct a time-independent inner product with a positive-definite norm, which has positive probabilities and unitary time evolution[3].

Salient examples include an anomalous bulk-boundary correspondence accompanied by the non-Hermitian skin effect[51, 53, 62, 110], the ubiquitous occurrence of exceptional nodal phases[12, 50, 74] with open Fermi-Seifert surfaces[14, 15], and a new system of generic symmetries [5] forming the basis for the topological classification of both gapless [12, 44] and gapped NH topological phases [45, 22].

Recent research has demonstrated classical analog of topological phases in a variety of settings ranging from photonics [80, 75] to electric circuits [1, 79] and mechanical systems [37, 42]. In these types of systems, non-Hermiticity appears naturally, in the role of dissipation.

non-Hermiticity also plays a role in quantum mechanical many-particle systems. The study of NH Hamiltonians in this context has been historically applied, as nuclear and atomic physics [9, 23, 60] More recently, the relevance of these Hamiltonians to topological phases has been investigated in several quantum many-body platforms such as:

Open systems where Non-Hermiticity surges in quantum many-body systems as quantum dissipation induced by coupling the system to its environment. A realm of direct relevance involves quantum optical setups and ultracold atomic gases, where experiments are often carried out in the regime of a weak coupling to a Markovian reservoir represented by the continuum of surrounding electromagnetic field modes. In such situations, the

relevant equation of motion for the reduced density matrix ρ of the open system is the Lindblad master equation [56]

$$\partial_t \rho = i[\rho, H] + \sum_n \left(L_n \rho L_n^\dagger - \frac{1}{2} \{L_n^\dagger L_n, \rho\} \right), \quad (1.22)$$

where the jump operators L_n account for the coupling to the environment. However, a different approach is desirable for obtaining an intuitive understanding of the interplay between coherent quantum dynamics, dissipation, and topology in complex quantum many-body systems. To this end, one useful approach is to note that the Lindblad equation can conveniently be written as $\partial_t = i(\rho H_{eff}^\dagger - H_{eff} \rho) + \sum_n L_n^\dagger \rho L_n$, where the effective NH Hamiltonian

$$H_{eff} = H - \frac{i}{2} \sum_n L_n^\dagger L_n \quad (1.23)$$

describes the dynamics at short times[16]. At longer times the so-called jump (or recycling) term $\sum_n L_n^\dagger \rho L_n$ accounting for the actual occurrence of quantum jumps can typically no longer be neglected, this makes decoherence appear, meanwhile the effective non-Hermitian description is by construction in terms of less general pure states. The relevance of NH Hamiltonians for Lindblad systems is not only at short lifetimes, its easy to construct intriguing examples where the steady state of the Lindblad equation is identical to the $t \rightarrow \infty$ state resulting from the time evolution of an effective NH Hamiltonian. It is also worth noting that the effective Hamiltonian (31) has eigenvalues in the lower complex half-plane $\text{Im}[E] \leq 0$. This highlights the fact that the Lindblad equation, even in the regime accurately captured by Eq.(1.23), imposes a fundamental constraint on eligible NH Hamiltonians relative to the fully generic case; see Section II.D.

Emergent dissipation in closed systems, here local observables in interacting quantum many-body systems obey nonlinear equations of motion, thus effectively leading to dissipative dynamics. In this context, it has been proposed that dissipation in the form of emergent non-Hermiticity can have a profound impact on the low-energy description of interacting and disordered quantum matter [50]. In these systems, quasiparticles acquire a finite lifetime by scattering off each other or at impurities, and the corresponding self-energy is non-Hermitian.

Along these lines, suggestions about emergent topological NH phenomena have been put forward in heavy fermion systems[111], in nodal semimetals [47], in strongly correlated Kondo materials [64], and for magnons[63]. Furthermore, the interplay between non-Hermiticity and superconductivity in toy models has been investigated[28]. Finally, we note that even when starting from entirely Hermitian systems, physical insights can be gained by formally extending a given model into the NH realm, as has been shown for Majorana wires [61] and interacting spin systems [59].

In this thesis, will use mass terms corresponding to our Hamiltonian. In general, there are several distinct ways to introduce a mass (or gaps) in these Dirac systems, which correspond to electronic instabilities done by, for example, breaking the sublattice symmetry in graphene systems, or introducing large enough magnetic field[90]. To include them in our description of Dirac systems, we introducing a mass term that anticommutes with the original Dirac Hamiltonian, which makes the Hamiltonian non-Hermitian. A more detailed description of this we will do in Chapter 3 of this thesis.

Chapter 2

Theoretical Framework

2.1 Field Theory remarks

One powerful tool for the calculations made in this thesis is Field theory. To understand better some details of the formalism we will give some examples, specially regarding correlation functions, regularization and renormalization. For this, we will discuss the ϕ^4 theory, which has been extensively studied, and is a classic example of how to do these procedures.

Many problems in condensed matter physics can be described in terms of a Hamiltonian H , or, equivalently, in terms of the action S , which is a function of some macroscopic number of degrees of freedom, which we will represent in terms of a field $\phi(x)$ and its derivatives (in space and time). The field ϕ may obey Fermi (Bose) statistics, in which case it will be represented by a set of Grassmann (scalar or vector) fields. Both in classical statistical mechanics and in a quantum field theory we can formally represent the system in terms of a path integral

$$\mathcal{Z} = \int \mathcal{D}\phi e^{-S(\phi)}, \quad (2.1)$$

where the action is written in an Euclidean metric.

The field $\phi(x, t) \equiv \phi(x)$ has the Fourier components

$$\phi(x) = \int \frac{d^D k}{(2\pi)^D} e^{ik \cdot x} \phi(k). \quad (2.2)$$

In the case of a scalar ϕ^4 theory, the action has the following form $S[\phi] = S_{\text{free}}[\phi] + S_{\text{int}}[\phi]$, where the free part

$$S_{\text{free}}[\phi] = \frac{1}{2} \int d^D \mathbf{r} [\phi(-\nabla^2 + m^2)\phi], \quad (2.3)$$

and the interaction term ,

$$S_{\text{int}}[\phi] = \int d^D \mathbf{r} \frac{\lambda}{4!} \phi^4 \quad (2.4)$$

2.1.1 Correlation functions

The core calculations of the critical phenomena come from the analysis of the n -point correlation functions, which are defined as:

$$\langle \phi(x_1) \dots \phi(x_n) \rangle \equiv \frac{1}{Z} \int [d\phi] \phi(x_1) \dots \phi(x_n) e^{-S[\phi]}, \quad (2.5)$$

where $Z = \int [d\phi] e^{-S[\phi]}$ is the partition function. In general, this calculation is very difficult or impossible, but we can make use of perturbation theory to expand these correlation functions in terms of the interaction parameter λ , and make use of Wick's theorem (and Feynmann diagrams) to get results that are proportional to the 2-point correlation function of the free theory, which is:

$$\langle \phi(x_1) \phi(x_2) \rangle = \int \frac{d^D \mathbf{k}}{(2\pi)^D} e^{i\mathbf{k} \cdot (\mathbf{x}_1 - \mathbf{x}_2)} G(k) \quad (2.6)$$

where

$$G(k) = \frac{1}{k^2 + m^2}, \quad (2.7)$$

is called the propagator. In general, due to translational invariance of these theories, we only focus on the correlation functions in momentum space.

2.1.2 Regularization

One problems of this approach is that, depending on the dimensions we are working with (in this case, in $D = 4$), these integrals in momentum may not converge due to their integrands falling off too slowly at large momentum. These divergences at large momenta (short-length) are called *ultraviolet (UV) divergences*, and can be worked around mathematically to find which is the divergent contribution, remove it and get physically sensible results. In particular, the treating these divergences properly, within a renormalization procedure, leads to meaningful results regarding the quantum-critical phenomena, as for instance, critical exponents.

There are many ways to regularize these integrals, but we focus on only two, which are used in this thesis to explain phenomena, or do calculations:

1.- Momentum cutoff Λ regularization: It consists of limiting the momentum integrals up to a region $|\mathbf{k}| < \Lambda$, and then the results of the integrals depends on this cutoff, which diverges as $\Lambda \rightarrow \infty$. The cutoff Λ has a direct physical interpretation in Dirac systems as

the momentum (or energy) up to which the dispersion of the Dirac fermions is linear in momentum.

2.-Dimensional regularization: We allow the dimension D of these integrals to be any number [102], and this has some features that make it attractive: First, it preserves all the symmetries of the original theory. Second, it allows an easy identification of the divergences. Third, it makes natural the *minimal subtraction renormalization scheme*. Fourth, it regularizes IR divergences (divergences at low momenta) of massless theories.

2.1.3 Renormalization

The idea behind renormalization is that, after calculating how these Feynman integrals diverge, be it by cutoff regularization or dimensional regularization, we have to deal with these infinities, and one way to do it is by absorbing these infinities in some (renormalization) constants of the action, be it mass, couplings, field renormalizations, etc. by adding what are called *counterterms* to the action. Now, these quantities, now called *renormalized quantities* (to differentiate them from the so called bare or microscopic quantities) have all the details required to analyze the critical phenomena of our theory.

There are many renormalization schemes, and we work with the *minimal subtraction scheme* (MS scheme), where we perform simultaneous expansions of our correlation functions and Feynman diagrams in powers of $\varepsilon = 4 - D$, then we subtract the ε -poles from the action. Then, the renormalization constants have the form

$$Z = 1 + \sum_{k=1}^L g^k \sum_{i=1}^k \frac{c_i^k}{\varepsilon^i} \quad (2.8)$$

where g is a coupling, and c_i are expansion coefficients.

One thing to consider is that not all theories are renormalizable, and to analyze it we have to check the vertex (also called one-particle irreducible or 1PI) functions dimensions,

$$\Gamma^{(n)}(x_1, \dots, x_n) = \langle \phi(x_1) \dots \phi(x_n) \rangle . \quad (2.9)$$

If at least one vertex has positive dimension, $\dim[V] > 0$, the theory is called non-renormalizable. When all vertices of the theory have strictly negative dimensions, only a finite number of Feynman diagrams are superficially divergent. The corresponding field theory is called super-renormalizable. Renormalizable theories are characterized by the property that at least one vertex has dimension 0, and no vertex has a positive dimension. For the case of the ϕ^4

$$\dim[\phi^4] = 4 - D , \quad (2.10)$$

implying that the vertex is renormalizable in $D = 4$.

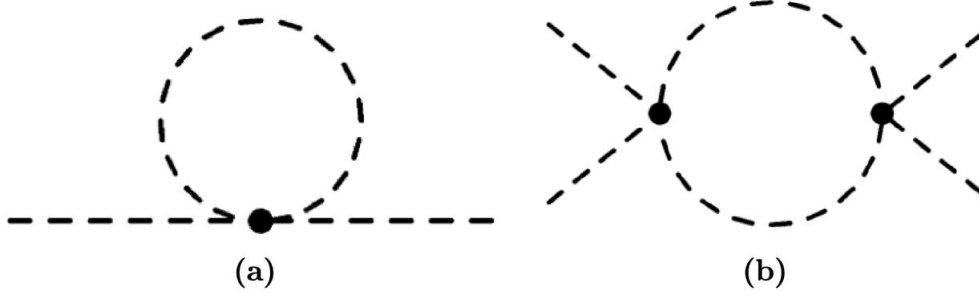


Figure 2.1: Diagrams contributing to the ϕ^4 renormalization at one loop order. The bosonic self-energy (a) only renormalizes the mass m and the four-point Feynman diagram (b) contributes to the renormalization of λ

Now, how we follow is:

From our original action, which now is the *bare action* and contains the bare quantities, and this is has the details of the physics at the microscopic scale.

Now, to this corresponds the *renormalized action*, in which we introduce the renormalization constants for each coupling, mass, and field. Then, the action is:

$$S_R = \int d^D \mathbf{r} \left[\frac{1}{2} Z_\phi (\nabla \phi)^2 + \frac{1}{2} Z_{m^2} m^2 \phi^2 + Z_\lambda \frac{\lambda \mu^\epsilon}{4!} \phi^4 \right], \quad (2.11)$$

where μ is the momentum (RG) scale, Z_ϕ , Z_{m^2} , Z_λ are the renormalizations constants of the field, mass, and coupling, respectively. We recover the bare action from:

$$m_0^2 \equiv \frac{Z_{m^2}}{Z_\phi} m^2 \mu^2 \quad (2.12)$$

$$\lambda_0 \equiv \frac{Z_\lambda}{Z_\phi^2} \mu^\epsilon \lambda, \quad (2.13)$$

where, the objects with (without) the subscript are the bare (renormalized) ones. The physical description of the critical phenomena is contained in the renormalization constants.

For this theory, the diagrams contributing to the renormalization are shown in Fig.2.1. At the one-loop order, the the counterterm needed to make the two-point vertex function $\Gamma_r^{(2)}$ finite is given by the bosonic self-energy

$$\Gamma_r^{(2)}(0, 0) = Z_{m^2} m^2 + \frac{\lambda}{2} (2N_B + 2) \int [d^D \mathbf{q}] \frac{1}{q^2 + m^2} \quad (2.14)$$

The correction to the four-point vertex function for λ is

$$\Gamma_r^{(4)}(\mathbf{k}) = Z_\lambda \frac{\lambda \mu^\epsilon}{2} - \left(\frac{\lambda}{2} \right)^2 \frac{1}{2} (4N_B + 1) \int \int [d^D \mathbf{q}] \frac{1}{q^2 + m^2} \frac{1}{(\mathbf{q} + \mathbf{k})^2 + m^2} \quad (2.15)$$

where $\mathbf{k}$ is the external momenta. We solved these integrals using the dimensional regularization and in the critical plane ($m = 0$), and setting the external momentum and frequency to zero to obtain the divergent part, which we use to calculate the renormalization constants

$$Z_{m^2} = 1 + (N_B + 1) \frac{\lambda}{8\pi^2} \frac{1}{\varepsilon} \quad (2.16)$$

$$Z_\lambda = 1 + (4N_B + 1) \frac{\lambda}{8\pi^2} \frac{1}{\varepsilon} \quad (2.17)$$

$$Z_\phi = 1 \quad (2.18)$$

The last term is since there is no counterterm proportional to $\mathbf{k}$.

2.2 The Renormalization Group approach to critical phenomena

2.2.1 Scale invariance

The renormalization group is a powerful tool to analyze strongly coupled systems and critical phenomena [26]. It was originally formulated in the context of perturbative quantum electrodynamics [27], to analyze the scaling of then electromagnetic coupling. Then, it was used in statistical physics to explain the critical phenomena [41], which is one of its more powerful uses. The most important ideas extracted from the renormalization group are the concepts of a fixed point and universality, which the latest was already named. Many of the ideas used in RG were developed by Wilson, Fisher, Kadanoff [41, 108], which is the reason that some applications of RG are called the Kadanoff, Fisher and Wilson's renormalization group (And to differentiate it from QFT RG, which uses the formalism introduced earlier).

In its simplest representation the renormalization group is a transformation that maps a system with a set of coupling constants and a scale (representing a cutoff) to another equivalent system with a different set of renormalized coupling constants and a different scale. This is done by a procedure in which the short-distance degrees of freedom are integrated out, and all the other degrees of freedom are rescaled to the original scale. This procedure in spin systems is called the "block-spin" transformation.

In momentum space, the procedure corresponds to integrate out the large-momentum $k \equiv (\omega, \vec{k})$ components, which are represented by $\phi_>(x)$, then we can find an effective action (or Hamiltonian) for the configurations $\phi_<(x)$ with lower k . In real space-time we are eliminating degrees of freedom at short distances, or rather we are defining configurations

averaged over a short distance scale ba (a is the lattice constant) or with wave vectors $|k| < \Lambda/b$ (where Λ is a high-energy cutoff), where $b \geq 1$.

In most cases $S(\phi)$ will change upon a process of this sort

$$\mathcal{Z} = \int \mathcal{D}\phi e^{-S(\phi)} = \int \mathcal{D}\phi_{<} \mathcal{D}\phi_{>} e^{-S(\phi_{<} + \phi_{>})}, \quad (2.19)$$

where we separated the field over its slow and fast parts

$$\phi = \phi_{<} + \phi_{>}. \quad (2.20)$$

where $\phi_{<}$ only contains the Fourier components with $k < \Lambda/b$ and $\phi_{>}$ only with $\Lambda > k > \Lambda/b$

Then,

$$\mathcal{Z} \equiv \int \mathcal{D}\phi_{<} e^{-S_{\text{eff}}(\phi_{<})} \quad (2.21)$$

where we have defined an effective action S_{eff}

$$e^{-S_{\text{eff}}(\phi_{<})} \equiv \int \mathcal{D}\phi_{>} e^{-S(\phi_{<} + \phi_{>})}. \quad (2.22)$$

This is how we realize the "block-spin" transformation.

However, now we have a new short-distance cutoff $a' \equiv ba$ or, equivalently, a high-energy cutoff $\Lambda' \equiv \Lambda/b < \Lambda$. Therefore, we need to rescale the lengths (and time) in order to restore the units, and we proceed as follows:

$$x' = \frac{x}{b} \quad \text{or} \quad k' = kb \quad (2.23)$$

such that $|k'| < \Lambda$ or $|x'| > a$ back again.

We can define many transformations of this type, and get different effective actions, but these transformations must obey some basic principles. The most important one is that this procedure should be compatible with the underlying symmetries of the system. Thus, if the system has a tendency to become anisotropic (either in space or in spacetime) the rescaling will have to be consistent with this fact. For simplicity, in what follows we will assume that the rescaling is isotropic both in space and in spacetime (as for example, in Lorentz invariant system).

Our objective is to find an action S^* that remains invariant under the RG transformation:

$$S_{\text{eff}}^*(\phi_{<}) = S^*(\phi), \quad (2.24)$$

We will call this action a fixed point of the RG.

This scale invariance at a fixed point makes that there can be no scales left in the problem. Therefore, a fixed-point action describes either a system with a vanishing correlation length, $\xi \rightarrow 0$, and hence a divergent energy gap, $E_G \rightarrow \infty$, or a system with a divergent correlation length, $\xi \rightarrow \infty$, and hence a vanishing energy gap, $E_G \rightarrow 0$, in accordance with (1.1) and (1.3).

Fixed points with vanishing correlation lengths (and divergent energy gaps) describe stable phases of matter, as the system cannot spontaneously change in which state it is, changing its properties. Conversely, fixed points with diverging correlation lengths (and vanishing energy gaps) describe systems at criticality (quantum or thermal), and thus correspond to phase transitions.

Let us consider now a system close to a fixed point, to analyze its stability with respect to perturbations. Quite generally we can write the action of the system in the form

$$S(\phi) = S^*(\phi) + \int dx^D \sum_n \lambda_n \phi_n(x), \quad (2.25)$$

where $\{\phi_n(x)\}$ is a complete set of operators defined in the fixed-point theory. Here $\{\lambda_n\}$ are the associated coupling constants. Under a renormalization-group transformation consisting of integrating out high-energy modes $\Lambda \rightarrow b\Lambda$ ($b < 1$) and then rescaling lengths $x \rightarrow \frac{x}{b}$ the action $S^*(\phi)$ remains invariant since it is a fixed point. The operators $\{\phi_n(x)\}$ transform irreducibly under rescalings as

$$\phi_n\left(\frac{x}{b}\right) = b^{\Delta_n} \phi_n(x) \quad (2.26)$$

where Δ_n is the scaling dimension of this operator. Operators that transform irreducibly (i.e. homogeneously) under rescalings are called primary (scaling) fields. Then, under the action of the RG the perturbation transforms as

$$\int dx^D \sum_n \lambda_n \phi_n(x) \rightarrow \int dx^D \sum_n b^{-D+\Delta_n} \lambda_n \phi_n(x) \quad (2.27)$$

Thus, the RG transformation is equivalent to a rescaling of the coupling constants

$$\lambda'_n = \lambda_n b^{\Delta_n - D} \equiv \lambda_n(b) \quad (2.28)$$

Since $b \rightarrow 1^-$, we write $b = e^{-\ell}$ ($\partial\ell = -\partial\ln b$ for future reference) and turn the transformation into a differential change of the coupling constants of terms of the beta function as defined next

$$\beta(\lambda_n) = \frac{\partial \lambda_n}{\partial \ell} = (D - \Delta_n) \lambda_n + \dots, \quad (2.29)$$

where the ellipsis denotes higher-order contributions. The first-order is usually called the tree-level beta function.

We can ask how the dimensionless coupling constants $\bar{\lambda}_n = \lambda_n a^{D-\Delta_n}$ changes under RG transformations, by scaling $\Lambda \rightarrow b\Lambda$ or $a \rightarrow b^{-1}a$, while keeping the dimensional couplings λ_n fixed. For an infinitesimal change $a \rightarrow a + da$ this can be expressed as the condition

$$a \frac{\partial \lambda_n}{\partial a} = 0 \Rightarrow \beta(\bar{\lambda}_n) = a \frac{\partial \bar{\lambda}_n}{\partial a} \Big|_{\lambda_n} = (D - \Delta_n) \bar{\lambda}_n + O(\bar{\lambda}_n^2) \quad (2.30)$$

Now, remembering that dimensional parameters can depend on the choice of units, we will always work with the dimensionless coupling constants, and analyze its flow under RG.

We now see the following.

1. If $D - \Delta_n > 0$, the dimensionless coupling λ_n grows as the momentum scale decreases. In this case $\phi_n(x)$ is a relevant operator ($\Delta_n < D$) and drives the system away from the fixed-point action S^* , into a phase characterized by a new (stable) fixed point.
2. If $D - \Delta_n < 0$, the dimensionless coupling λ_n shrinks as the momentum scale decreases. In this case $\phi_n(x)$ is an irrelevant operator ($\Delta_n > D$), whose effects become asymptotically negligible at long distances and low energies.
3. If $D - \Delta_n = 0$, to leading order, the dimensionless coupling λ_n does not change as the momentum scale decreases. In this case $\phi_n(x)$ is a marginal operator ($\Delta_n = D$). To assess its effects, a higher-order calculation needs to be done. If this operator remains exactly marginal, then it should be included in the fixed point action.

2.2.2 The correlation length

One quantity of interest is the correlation length which we already talked about it in previous sections, but now we will prove its scaling law under RG transformations.

We will begin by considering first the behavior of the correlation length ξ for a theory close to a fixed point, and consider the distance to the fixed points as a coupling constant λ of an operator of the theory. By dimensional analysis (knowing that in fact, ξ is a length), we have

$$\xi = a f(\lambda) \quad (2.31)$$

where $f(\lambda)$ is an undetermined dimensionless function of the coupling constant, and $a \sim \Lambda^{-1}$ is the ultraviolet (UV) cutoff, which in a condensed matter system corresponds to the lattice scale of the lattice regularized system..

By demanding that physical dimensional quantities, such as the correlation length ξ , remain fixed under the action of the renormalization group, we can derive a flow equation that will allow us to determine the function $f(\lambda)$. Thus, the condition

$$\frac{\partial \xi}{\partial \ln a} = 0 = \frac{\partial}{\partial \ln a}(af(\lambda)) \quad (2.32)$$

leads to the differential flow equation for $f(\lambda)$

$$0 = f(\lambda) + \frac{\partial f}{\partial \lambda} \frac{\partial \lambda}{\partial \ln a} \quad (2.33)$$

where we recognize the presence of the beta function, $\beta(\lambda) = a\partial\lambda/\partial a$ in the last term. We can now write the flow equation as

$$0 = f(\lambda) + \frac{\partial f}{\partial \lambda} \beta(\lambda) \quad (2.34)$$

If the beta function is known, we can solve this flow equation, which in this cases is:

$$\frac{\partial \ln f}{\partial \lambda} = -\frac{1}{\beta(\lambda)} \quad (2.35)$$

so that $f(\lambda)$ must be of the form

$$\ln \left(\frac{f(\lambda)}{f(\lambda_0)} \right) = - \int_{\lambda_0}^{\lambda} \frac{d\lambda'}{\beta(\lambda')} \quad (2.36)$$

We will now consider a theory with a fixed point at a value λ^* . A fixed point is defined as the value λ^* at which the beta function vanishes, $\beta(\lambda^*) = 0$. Let us consider the system at two values of the dimensionless coupling constant λ_1 and λ_2 , where the correlation length takes the values $\xi(\lambda_1) = af(\lambda_1)$ and $\xi(\lambda_2) = af(\lambda_2)$, respectively. Then, $f(\lambda_1)$ and $f(\lambda_2)$ have the form

$$f(\lambda_1) = f(\tilde{\lambda}) \exp \left(\int_{\lambda_1}^{\tilde{\lambda}} \frac{d\lambda}{\beta(\lambda)} \right) \quad (2.37)$$

and

$$f(\lambda_2) = f(\tilde{\lambda}) \exp \left(\int_{\lambda_2}^{\tilde{\lambda}} \frac{d\lambda}{\beta(\lambda)} \right) \quad (2.38)$$

where $\tilde{\lambda}$ is some other value of the coupling constant. Thus, the ratio of the correlation lengths $\xi(\lambda_1)$ and $\xi(\lambda_2)$ is determined by the change of the coupling constant:

$$\frac{\xi(\lambda_1)}{\xi(\lambda_2)} = \exp \left(\int_{\lambda_1}^{\tilde{\lambda}} \frac{d\lambda}{\beta(\lambda)} - \int_{\lambda_2}^{\tilde{\lambda}} \frac{d\lambda}{\beta(\lambda)} \right) = \exp \left(\int_{\lambda_1}^{\lambda_2} \frac{d\lambda}{\beta(\lambda)} \right) \quad (2.39)$$

Therefore we find that

$$\xi(\lambda_1) = \xi(\lambda_2) \exp \left(\int_{\lambda_1}^{\lambda_2} \frac{d\lambda}{\beta(\lambda)} \right) \quad (2.40)$$

Close enough to the fixed point at λ^* , we can approximate the beta function as

$$\beta(\lambda) = \beta'(\lambda^*)(\lambda - \lambda^*) + O((\lambda - \lambda^*)^2) \quad (2.41)$$

where $\beta'(\lambda^*) = (d\beta/d\lambda)(\lambda^*)$ is the slope of the beta function at the fixed point. If we take $|\lambda_1 - \lambda^*| \ll \lambda^*$ to be very close to the fixed point and $|\lambda_2 - \lambda^*| \approx \lambda^*$ at some finite distance from it, we have

$$\begin{aligned} \xi(\lambda_1) &= \xi(\lambda_2) \exp \left(\int_{\lambda_1}^{\lambda_2} \frac{d\lambda}{\beta'(\lambda^*)(\lambda - \lambda^*) + \dots} \right) \\ &= \xi(\lambda_2) \exp \left[\frac{1}{\beta'(\lambda^*)} \ln \left(\frac{\lambda_2 - \lambda^*}{\lambda_1 - \lambda^*} \right) + \dots \right] \\ &= \xi(\lambda_2) \left(\frac{\lambda_2 - \lambda^*}{\lambda_1 - \lambda^*} \right)^{1/\beta'(\lambda^*)} (1 + \dots) \end{aligned} \quad (2.42)$$

If the operator with coupling constant λ is a relevant operator, then the slope of the beta function must be positive, $\beta'(\lambda^*) > 0$. In this case, provided that this relevant operator drives the system to a phase with a finite correlation length, which we can take to be $\xi(\lambda_2) \approx a$, we obtained the power law scaling of the correlation length,

$$\xi(\lambda) \approx a \left| \frac{\lambda_2 - \lambda^*}{\lambda - \lambda^*} \right|^v \quad (2.43)$$

where

$$v = \frac{1}{\beta'(\lambda^*)} \quad (2.44)$$

is the universal critical exponent of the correlation length. In conclusion, the correlation length must diverge $\xi(\lambda) \rightarrow \infty$ as the fixed point is approached $\lambda \rightarrow \lambda^*$ with a universal critical exponent ν .

2.2.3 RG flow of the ϕ^4 theory

Now, to make the link between the Kadanoff, Fisher and Wilson's RG and QFT, we note that from Eqs. (2.12), (2.13), (2.16), (2.17), (2.18), which are defined from renormalization and bare actions, we get the scaling of the quantities, so we can calculate the β functions for the ϕ^4 theory, using μ as the scale parameter, then

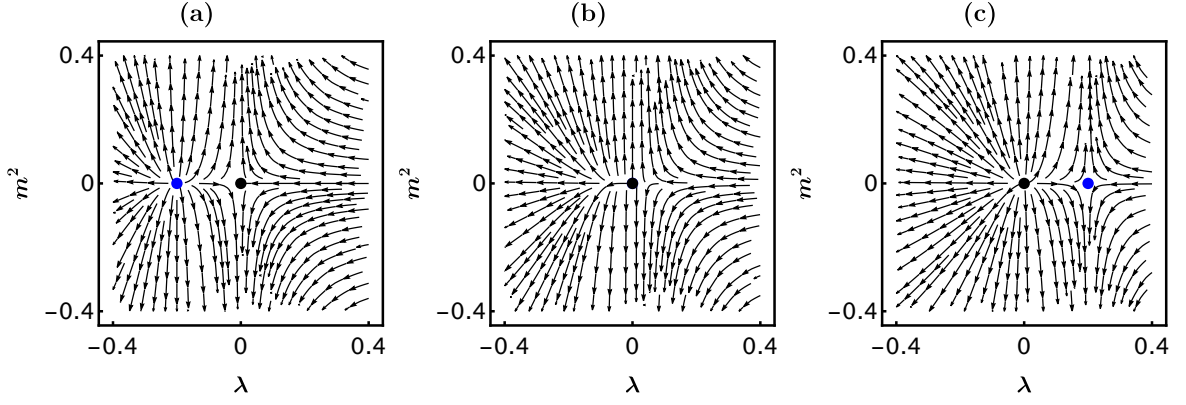


Figure 2.2: The RG flow of m^2 and λ above (panel (a)), at (panel (b)), and below four dimensions (panel (c)). The Wilson–Fisher fixed point (blue) moves from the unphysical region for $\varepsilon < 0$ to the physical region for $\varepsilon > 0$, and exchanges stability along the λ -direction with the Gaussian fixed point (black).

$$\beta_{m^2} \equiv - \left. \frac{dm^2}{d \ln \mu} \right|_{m_0^2, \lambda_0} = 2m^2 - (N_B + 1)m^2\lambda + \mathcal{O}(\text{Two-Loops}), \quad (2.45)$$

$$\beta_\lambda = \varepsilon\lambda - (4N_B + 1)\lambda^2 + \mathcal{O}(\text{Two-Loops}), \quad (2.46)$$

where we rescaled $\lambda^2/8\pi^2 \rightarrow \lambda^2$. With these, we can analyse the flow of these quantities in the $m^2 - \lambda$ plane, see their fixed points (for which the β functions are 0) and analyze their stability, as seen in 2.2.

This theory has two fixed points, called the Gaussian fixed point at $(m^2, \lambda) = (0, 0)$, and the Wilson-Fisher fixed point at

$$(m_\star^2, \lambda_\star) = \left(0, \frac{\varepsilon}{4N_B + 1} \right). \quad (2.47)$$

Above four dimension $\varepsilon < 0$ and the Wilson–Fisher fixed point is at a negative interaction and therefore unphysical. It is also unstable in both λ - and m^2 -directions, i.e. it has two relevant couplings. The Gaussian fixed point, on the other hand, is stable in the λ -direction and unstable only in the μ -direction. Such a fixed point with precisely one unstable direction will be called a critical point.

Below four dimensions $\varepsilon > 0$ and the Wilson–Fisher fixed point moves into the physical region and exchanges stability with the Gaussian fixed point: the Wilson–Fisher fixed point becomes stable in the λ direction while the Gaussian one becomes completely unstable. Critical behavior for $\varepsilon > 0$ is governed by the non-trivial Wilson–Fisher fixed point. In $d = 4$ the Wilson–Fisher and the Gaussian fixed points happen to coincide. A weak positive interaction flows to zero, but since the linear term in β_λ vanishes the flow is slow. This leads to the so-called logarithmic corrections to scaling right at the upper critical

dimension.

Flow near the fixed points follows from the stability matrix M defined by

$$M_{ij} \equiv \left. \frac{\partial \beta_{X_i}}{\partial X_j} \right|_{X_i=X_{*i}} \quad (2.48)$$

The eigenvectors of M with positive eigenvalues determine the relevant (unstable) directions at the given fixed point, and those with negative eigenvalues the irrelevant (stable) directions. In the case of a critical point there is only one such relevant direction, with the eigenvalue y . The correlation length exponent is then given by $\nu = 1/y$.

The stability matrix is

$$M = \begin{pmatrix} -\varepsilon & 0 \\ 0 & 2 - \frac{(N_B+1)\varepsilon}{4N_B+1} \end{pmatrix} \quad (2.49)$$

Then, the correlation length is

$$\nu = \frac{1}{2} + \frac{(N_B+1)\varepsilon}{4N_B+1} + \mathcal{O}(\varepsilon^2) \quad (2.50)$$

For $\varepsilon = 1$ and $N_B = 1$ we obtain $\nu \approx 0.66$, relatively close to the experimental value $\nu = 0.66 \pm 0.02$ in ^4He .

2.3 Gross-Neveu-Yukawa Model

In this section, we discuss the Gross-Neveu-Yukawa (GNY) Model, which has been used in condensed matter physics to explain the interaction-induced transition toward a charge density wave of electrons on the 2D honeycomb lattice that breaks the (Ising) sublattice symmetry, or the disorder-driven transition toward a diffusive metal in a 3D Weyl semi-metal[65]. It is developed from a Gross-Neveu model, after performing a Hubbard-Stratonovich transformation on it. This model has N Dirac fermions and one scalar field (which was introduced by the HS transformation). The model has a continuous $U(N)$ symmetry, and a discrete chiral symmetry that prevents the addition of a fermion mass term to the action. For a specific value of a coefficient of the action, the model undergoes a continuous phase transition.

The (Euclidean) action of the GNY model in d spatial dimensions is as follows

$$\mathcal{S}(\bar{\psi}, \psi, \phi) = \int d\tau \int d^d x \left[\bar{\psi} \cdot (\gamma^\mu \partial_\mu + g\phi) \psi + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}r\phi^2 + \frac{\lambda}{4!}\phi^4 \right], \quad (2.51)$$

where ψ and $\bar{\psi}$ are the Dirac fermion field and its conjugate, ϕ is the scalar field introduced, which will act as the order-parameter of the phase transition studied in this model.

This actions has the following propagators:

$$\begin{aligned}
G_f^{-1}(\omega, \mathbf{k}) &= -i(\gamma_0\omega + \boldsymbol{\gamma} \cdot \mathbf{k}) & G_b^{-1}(\omega, \mathbf{k}) &= \omega^2 + k^2 + m^2 \\
G_f(\omega, \mathbf{k}) &= \frac{i(\gamma_0\omega + \boldsymbol{\gamma} \cdot \mathbf{k})}{\omega^2 + k^2} & G_b(\omega, \mathbf{k}) &= \frac{1}{\omega^2 + k^2 + m^2}
\end{aligned} \tag{2.52}$$

Also, it has a $U(N)$ fermion symmetry corresponding to the transformations,

$$\boldsymbol{\psi}(x) \mapsto \mathbf{U}\boldsymbol{\psi}(x), \quad \bar{\boldsymbol{\psi}}(x) \mapsto \bar{\boldsymbol{\psi}}(x)\mathbf{U}^\dagger, \quad \text{with } \mathbf{U}\mathbf{U}^\dagger = \mathbf{1}$$

and, in addition, in all dimensions, a discrete symmetry which forbids a fermion mass term and odd powers of ϕ in ϕ self-interactions.

In even dimensions, the symmetry can be identified with the discrete chiral $\mathbb{Z}_2$ symmetry (for $d = 4, \gamma_S \equiv \gamma_5$)

$$\boldsymbol{\psi}(x) \mapsto \gamma_S \boldsymbol{\psi}(x), \quad \bar{\boldsymbol{\psi}}(x) \mapsto -\bar{\boldsymbol{\psi}}(x)\gamma_S, \quad \phi(x) \mapsto -\phi(x). \tag{2.53}$$

In odd dimensions, a fermion mass term breaks space parity. In generic dimensions, the discrete symmetry can be expressed as $\phi(x) \mapsto -\phi(x)$ and

$$\mathbf{x} = \{x_1, x_2, \dots, x_d\} \mapsto \tilde{\mathbf{x}} = \{-x_1, x_2, \dots, x_d\}, \quad \begin{cases} \psi(x) \mapsto \gamma_1 \psi(\tilde{x}), \\ \bar{\psi}(x) \mapsto -\bar{\psi}(\tilde{x})\gamma_1 \end{cases}$$

At leading order, the system undergoes a phase transition at $r = r_c = 0$. For $r = m^2 > r_c$, the vacuum expectation value $\langle \phi \rangle$ of ϕ vanishes, the fermions remain massless and the $\mathbb{Z}_2$ symmetry remains unbroken. By contrast, for $r < r_c$, the minimum of the ϕ potential is given by (Considering $V(\phi) = \frac{1}{2}r\phi^2 + \frac{\lambda}{4!}\phi^4$)

$$V'(\phi) = 0 \Rightarrow \langle \phi \rangle = \pm \sqrt{-6r/\lambda} \Rightarrow m_\phi^2 = V''(\phi) = -2r > 0,$$

which shows that the $\mathbb{Z}_2$ symmetry is spontaneously broken. This does not generate Goldstone bosons, because the symmetry is discrete. Then, returning to the action (2.51), replacing the field ϕ by its expectation value, we note that the fermions become massive with a mass (g can always be chosen positive)

$$m_\psi = g|\langle \phi \rangle| = g\sqrt{-6r/\lambda}, \quad \text{and thus} \quad \frac{m_\psi}{m_\phi} = g\sqrt{3/\lambda} \tag{2.54}$$

The physics of the model is characterized by spontaneous $\mathbb{Z}_2$ symmetry breaking and fermion mass generation.

2.3.1 Renormalization

We now perform dimensional analysis of the GNY theory. Since the engineering dimensions of the fermion and boson fields are $\dim[\psi] = d/2$ and $\dim[\phi] = (d+z-2)/2$, the dimension of the Yukawa coupling is $\dim[g] = (2-d+z)/2$. On the other hand, the engineering dimension of the ϕ^4 -coupling is $\dim[\lambda] = 4-d-z$, and the dynamical exponent is the theory is $z = 1$ then the upper critical dimension is $d_{\text{upper}} = 3$, and the GNY model is renormalizable by power counting. To proceed, we will therefore use an ε -expansion about three space dimensions, or equivalently, four space-time dimensions.

Since we calculate only the RG functions, we restrict the calculations to the critical (massless) theory $r = r_c = 0$. The critical renormalized action, in a form suitable for dimensional regularization, can thus be written as,

$$\mathcal{S}_r(\bar{\psi}, \psi, \phi) = \int d\tau \int d^d x \left\{ \bar{\psi}(x) [Z_\psi \gamma^\mu \partial_\mu + g Z_g \mu^{\varepsilon/2} \phi(x)] \psi(x) + \frac{1}{2} Z_\phi (\nabla \phi(x))^2 + \frac{1}{4!} \mu^\varepsilon \lambda Z_\lambda \phi^4(x) \right\} \quad (2.55)$$

where $\varepsilon = 3 - d = 4 - D$, D is space-time dimensions, ψ, ϕ are renormalized fields, μ the physical mass scale, g and λ dimensionless renormalized coupling constants, and Z_ψ, Z_ϕ, Z_g , and Z_λ renormalization constants. The relation between bare (or microscopic) and renormalized coupling constants, denoted g_0, λ_0 and g, λ , respectively, then is

$$g_0 = \mu^{\varepsilon/2} g Z_g / (Z_\psi Z_\phi^{1/2}), \quad \lambda_0 = \mu^\varepsilon \lambda Z_\lambda / Z_\phi^2. \quad (2.56)$$

In the following, we denote by $\tilde{\Gamma}^{(l,n)}$ the vertex functions corresponding to l fermion fields and n scalar fields, in the Fourier representation.

2.3.2 One-loop calculations

An expansion in powers of ϕ, ψ , and $\bar{\psi}$ generates all vertex functions at one-loop order.

Then, the renormalized vertex functions are given by the Feynman diagrams shown in Fig.2.3, and have the form

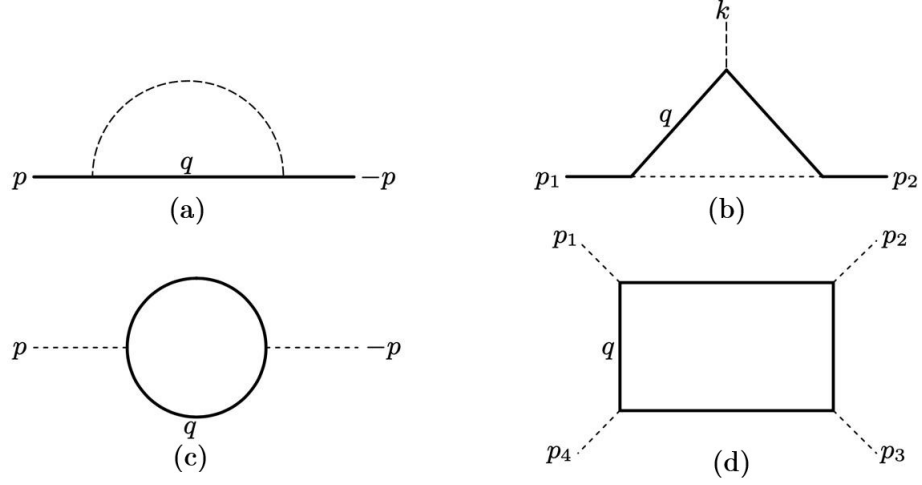


Figure 2.3: Feynman diagrams in the GNY theory at one loop. The dashed lines are the propagator for bosons and the solid lines are the propagator for fermions. (a) Fermionic self-energy diagram. (b) Bosonic self-energy diagram. (c) Vertex correction of Yukawa coupling. (d) Vertex correction of ϕ^4 -coupling

$$\tilde{\Gamma}_r^{(2,0)}(p) = iZ_\psi \not{p} + \frac{g^2 \mu^\varepsilon}{(2\pi)^D} \int \frac{d^D q}{(p-q)^2} \frac{i \not{q}}{q^2}, \quad (2.57)$$

$$\tilde{\Gamma}_r^{(2,1)}(p_1, p_2; k) = gZ_g \mu^{\varepsilon/2} - \frac{g^3 \mu^{3\varepsilon/2}}{(2\pi)^D} \int \frac{d^D q}{(q+p_2)^2} \frac{\not{q}(q \not{q} - \not{k})}{q^2(q-k)^2}, \quad (2.58)$$

$$\tilde{\Gamma}_r^{(0,2)}(p) \Big|_{\lambda=0} = p^2 Z_\phi - \frac{N g^2 \mu^\varepsilon}{(2\pi)^D} \int d^D q \frac{\text{tr } \not{q}(\not{q} + \not{p})}{q^2(p+q)^2}, \quad (2.59)$$

$$\tilde{\Gamma}_r^{(0,4)}(p_1, p_2, p_3, p_4) \Big|_{\lambda=0} = N g^4 \mu^{2\varepsilon} \int \frac{d^D q}{(2\pi)^D} \frac{\text{tr} [q(q + \not{p}_1)(q + \not{p}_1 + \not{p}_2)(q - \not{p}_4)]}{q^2(q+p_1)^2(q+p_1+p_2)^2(q-p_4)^2} \quad (2.60)$$

$$+ 5 \text{ diagrams corresponding to permutations of } \{p_2, p_3, p_4\}. \quad (2.61)$$

with $k \equiv (\omega, \mathbf{k})$ the frequency and momentum. We solve these integrals by dimensional regularization, and in the $\overline{\text{MS}}$ scheme we obtain for the renormalization constants

$$Z_\psi = 1 - \frac{g^2}{(4\pi)^2} \frac{1}{\varepsilon} + O(\text{two loops}) \quad (2.62)$$

$$Z_\phi = 1 - \frac{g^2}{(4\pi)^2} N \frac{1}{\varepsilon} + O(\text{two loops}) \quad (2.63)$$

$$Z_g = 1 + \frac{g^2}{8\pi^2} \frac{1}{\varepsilon}, \quad (2.64)$$

and the renormalization constant for λ is given by

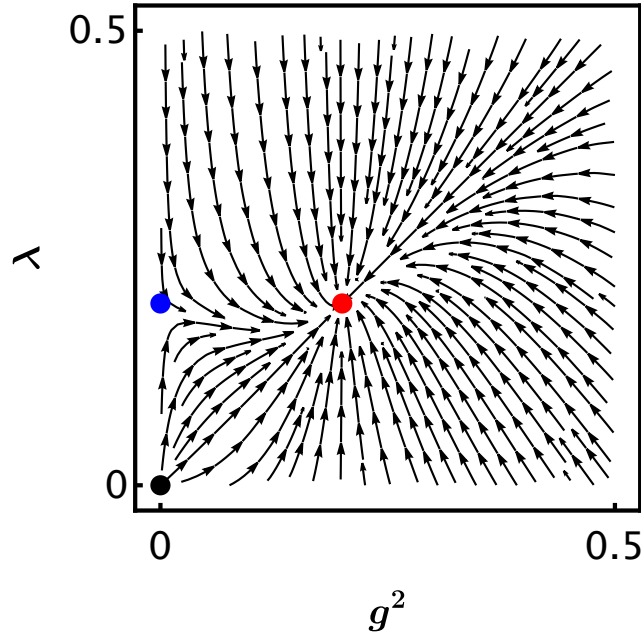


Figure 2.4: Flow diagram of the Yukawa and ϕ^4 coupling in the critical plane $m = 0$, with $N_B = N = 1$ and $\varepsilon = 1$. Wilson-Fisher critical point (blue) is unstable in the Yukawa direction, while the would-be bicritical point lies in the unphysical region of the coupling constant. The Gaussian fixed point (black) is also unstable in both the Yukawa and ϕ^4 coupling constant directions. The only stable GNY fixed point (red) is located at a finite positive value of both Yukawa and ϕ^4 coupling and controls the critical behavior at the transition between the semimetal and the interaction driven insulator.

$$\lambda Z_\lambda = \lambda + \left((N_B + 4) \frac{\lambda^2}{8\pi^2} \frac{1}{\varepsilon} - N \frac{g^2}{2\pi^2} \frac{1}{\varepsilon} \right), \quad (2.65)$$

where N is the number of fermions of the theory.

The coupling constant RG functions $\beta_i(\lambda)$ are then given by

$$\beta_i(\lambda) = - \left. \frac{\partial \lambda_i}{\partial \ln \mu} \right|_\lambda \quad (2.66)$$

A short calculation yields

$$\beta_{g^2} = \varepsilon g^2 - (2N + 3)g^4 + O(\text{two loops}) \quad (2.67)$$

$$\beta_\lambda = \varepsilon \lambda - (4 + N_B)\lambda^2 + 4N g^2(g^2 - \lambda) + O(\text{two loops}) \quad (2.68)$$

here we rescaled $g^2/(8\pi^2) \rightarrow g^2$ and $\lambda/(8\pi^2) \rightarrow \lambda$.

The RG flow is shown in Fig.2.4. For the Gaussian fixed point both directions are relevant, while for the Wilson-Fisher one can see that it becomes unstable in the direction of the Yukawa coupling. The GNY fixed point is

$$(g_\star^2, \lambda_\star) = \left(\frac{\varepsilon}{3 + 2N_F}, \frac{-\sqrt{\left(\varepsilon - \frac{4N_F\varepsilon}{2N_F+3}\right)^2 - \frac{16(-N_B-4)N_F\varepsilon^2}{(2N_F+3)^2}} + \frac{4N_F\varepsilon}{2N_F+3} - \varepsilon}{2(-N_B - 4)} \right). \quad (2.69)$$

Other fixed points are: the Wilson-Fisher point, which is located at $(g_\star^2, \lambda_\star) = (0, \varepsilon/(4 + N_B))$. The trivial Gaussian point $(g_\star^2, \lambda_\star) = (0, 0)$. And another GNY (unphysical) fixed point, at

$$(g_\star^2, \lambda_\star) = \left(\frac{\varepsilon}{3 + 2N_F}, \frac{\sqrt{\left(\varepsilon - \frac{4N_F\varepsilon}{2N_F+3}\right)^2 - \frac{16(-N_B-4)N_F\varepsilon^2}{(2N_F+3)^2}} + \frac{4N_F\varepsilon}{2N_F+3} - \varepsilon}{2(-N_B - 4)} \right) \quad (2.70)$$

Chapter 3

Tilted non-Hermitian Dirac Hamiltonian

Now that we have seen all the physical context, and understood the mathematical tools, we can start to study and analyze the Tilted non-Hermitian Dirac Semimetals. For this, first we construct a minimal Dirac model to include the NH effect and tilt and we calculate some associated quantities such as density of states and optical conductivity.

3.1 Minimal noninteracting model

We start by considering a minimal effective low-energy Hamiltonian-like NH Dirac operator H_{NH} in two spatial dimensions ($d = 2$) which is constructed from the corresponding Hermitian Dirac operator by invoking the mass terms, such that the resulting anti-Hermitian piece of the total (Lorentz-symmetric) NH Dirac operator is linear in momentum [40]. The Hermitian Dirac Hamiltonian reads as $\mathcal{H}_D = \sum_{\mathbf{k}} \Psi_{\mathbf{k}}^\dagger H_D \Psi_{\mathbf{k}}$, where

$$H_D = v_H(\Gamma_x k_x + \Gamma_y k_y) = v_H h_0, \quad (3.1)$$

with v_H as the Hermitian velocity of the Dirac quasiparticles, the Hermitian matrices $\Gamma_x = \sigma_1 \otimes \tau_3$ and $\Gamma_y = \sigma_2 \otimes \tau_0$, chosen here explicitly in this representation for convenience, are mutually anticommuting matrices satisfying the Clifford algebra $\{\Gamma_i, \Gamma_j\} = 2\delta_{ij}$, $i, j = 1, 2$, $\mathbf{k}$ is the momentum, and the form of the Dirac spinor $\Psi_{\mathbf{k}}$ depends on the microscopic details of the system, which is in the following assumed to be four-dimensional, as for instance for the spinless electrons in graphene, with two sublattice and two valley degrees of freedom. The set of Pauli matrices $(\sigma_0, \boldsymbol{\sigma})$ $[(\tau_0, \boldsymbol{\tau})]$ act in the sublattice (valley) space.

To construct the NH counterpart of the Dirac operator in Eq. (3.1), we invoke a Hermitian matrix M , corresponding to a mass term for the Hermitian Dirac operator, and

therefore $\{M, H_D\} = 0$. As such, for any mass matrix, the operator MH_D is anti-Hermitian, $(MH_D)^\dagger = -MH_D$. Using this property, we define the minimal NH Dirac operator as [40]

$$H_{\text{NH}}^0 = (v_H + v_{\text{NH}}M)h_0 = v_H(1 + \beta M)h_0 \quad (3.2)$$

where v_{NH} is the NH velocity parameter, and $\beta = v_{\text{NH}}/v_H$ (always real). Then, the spectrum of the NH operator is $E_{\text{NH}} = \pm v_F|\mathbf{k}|$, and is purely real (imaginary) when $|\beta| < 1$ ($|\beta| > 1$), where $v_F = v_H\sqrt{1 - \beta^2}$ is the effective Fermi velocity of the NH Dirac fermions. For concreteness, we here choose the mass matrix $M = \sigma_1 \otimes \tau_1$, and the velocity parameter $|\beta| < 1$, with the NH Dirac operator thus featuring a purely real spectrum. Our results are, however, independent of the matrix representation.

As the last step, to obtain the minimal tilted NH Dirac Hamiltonian, we also add a Hermitian term which is linear in one of the components of the momentum [95], taken to be k_x for concreteness,

$$H_t^0 = \alpha v_H T k_x, \quad (3.3)$$

with the matrix T such that $[T, \Gamma_i] = 0$ ($i = x, y$) and $[T, \beta] = 0$, corresponding to the tilted Dirac cones. Depending on the spatial symmetries, particularly, the valley exchange symmetry, represented by $(k_x, k_y) \rightarrow (-k_x, k_y)$ and $S_V = \sigma_0 \otimes \tau_1$, we can consider an antisymmetric (symmetric) tilt with $T_{\text{AS}} = \sigma_0 \otimes \tau_0$ ($T_S = \sigma_2 \otimes \tau_2$), which is therefore odd (even) under this symmetry. The Hamiltonian of the minimal model for the NH tilted Dirac semimetal then reads,

$$H_{\text{NH}}^{\text{tilt}}(\mathbf{k}) = (v_H + v_{\text{NH}}M)h_0 + \alpha v_H T k_x. \quad (3.4)$$

Notice that this Hamiltonian preserves (breaks) sublattice exchange symmetry, $(k_x, k_y) \rightarrow (k_x, -k_y)$ and $S_{SL} = \sigma_1 \otimes \tau_0$ for any $\beta \neq 0$, in the case of the antisymmetric (symmetric) tilt term.

For completeness, we also consider the NH Dirac Hamiltonian that includes the term H_t^0 in Eq. (3.3) with the matrix $T = T_{\text{VA}} = \sigma_1 \otimes \tau_0$, which is odd (even) under the valley (sublattice) exchange, and therefore transforms the same as the antisymmetric tilt under these symmetries. However, this term neither commutes nor anticommutes with the Hermitian Dirac Hamiltonian [Eq. (3.1)], $[\Gamma_x, T_{\text{VA}}] = 0$, $\{\Gamma_y, T_{\text{VA}}\} = 0$, and thus introduces a velocity anisotropy in the NH Dirac electronic bands, see Eq. (3.7).

In the case of antisymmetric tilt, the spectrum of the NH tilted Hamiltonian operator features two doubly degenerate bands

$$E_{\text{NH}}^{\text{AS}} = \pm v_F|\mathbf{k}| + \alpha v_H k_x, \quad (3.5)$$

and therefore the tilt acts as an effective (valley-independent) momentum-dependent

chemical potential for the NH Dirac fermions. On the other hand, for the symmetric tilt, the spectrum reads as

$$E_{\text{NH}}^{\text{S}} = \pm v_{\text{F}} |\mathbf{k}| \pm \alpha v_{\text{H}} k_x, \quad (3.6)$$

and acts as an effective momentum-dependent chiral chemical potential, with an opposite sign at the two valleys.

For the non-tilt matrix $T = T_{\text{VA}}$ in Eq. (3.4), the spectrum reads as

$$E_{\text{NH}}^{\text{VA}} = \pm \sqrt{k^2 v_{\text{F}}^2 \pm 2|\alpha| v_{\text{F}} v_{\text{H}} k_x^2 + \alpha^2 v_{\text{H}}^2 k_x^2} = \pm v_{\text{F}} \sqrt{k_y^2 + \left(1 \pm \frac{|\alpha|}{\sqrt{1-\beta^2}}\right) k_x^2}, \quad (3.7)$$

and introduces an effective velocity anisotropy in the Dirac bands since the matrix T_{VA} only partially commutes with the Hermitian Dirac Hamiltonian in Eq. (3.1). Notice that in this case $[M, T_{\text{VA}}] = 0$, the same as in the case of the tilted NH Dirac Hamiltonian.

For further considerations, we first display the fermion propagator for the (symmetrically and antisymmetrically) tilted NH Dirac Hamiltonian that takes the form

$$G_F^{\text{tilt}}(i\omega, \mathbf{k}) = \left\{ - (i\omega + H_0^{\text{t}}) A_{\pm} + B(v_{\text{H}} + v_{\text{NH}} M)(\mathbf{\Gamma} \cdot \mathbf{k}) T_{\text{AS}} k_x + 2\alpha v_{\text{F}} k_x (v_{\text{F}}^2 k^2 T_{\text{AS}} + \alpha v_{\text{H}} k_x (v_{\text{H}} + v_{\text{NH}} M)(\mathbf{\Gamma} \cdot \mathbf{k})) \right\} \frac{1}{A_{\pm}^2 - B^2}, \quad (3.8)$$

where

$$A_{\pm} = A_{\pm}(i\omega, k) = \omega^2 + v_{\text{F}}^2 k^2 \pm \alpha^2 v_{\text{H}}^2 k_x^2, \quad B = B(i\omega, k) = 2i\alpha v_{\text{H}} \omega k_x. \quad (3.9)$$

On the other hand, for the velocity-antisymmetric NH Dirac Hamiltonian, the propagator reads

$$G_F^{\text{VA}}(i\omega, \mathbf{k}) = \left\{ - (i\omega + H_0^{\text{t}}) A + 2i\omega \alpha v_{\text{H}} k_x^2 (v_{\text{H}} + v_{\text{NH}} M) T_{\text{VA}} \Gamma_x \right. \quad (3.10)$$

$$\left. + 2\alpha v_{\text{H}} v_{\text{F}}^2 k_x^2 (k_x + \Gamma_y \Gamma_x k_y) T_{\text{VA}} + 2\alpha^2 v_{\text{H}}^2 k_x (v_{\text{H}} + v_{\text{NH}} M) \Gamma_x k_x^3 \right\} \frac{1}{A^2 - B^2}, \quad (3.11)$$

where

$$A = A(i\omega, \mathbf{k}) = \omega^2 + v_{\text{F}}^2 k^2 + \alpha^2 v_{\text{H}}^2 k_x^2, \quad B = B(i\omega, \mathbf{k}) = 2\alpha v_{\text{H}} v_{\text{F}} k_x^2. \quad (3.12)$$

See appendix A.1.1 for the technical details. As a first step in the analysis of the behavior of the tilted NH Dirac fermions in the presence of the electron-electron interactions, we analyze the DOS.

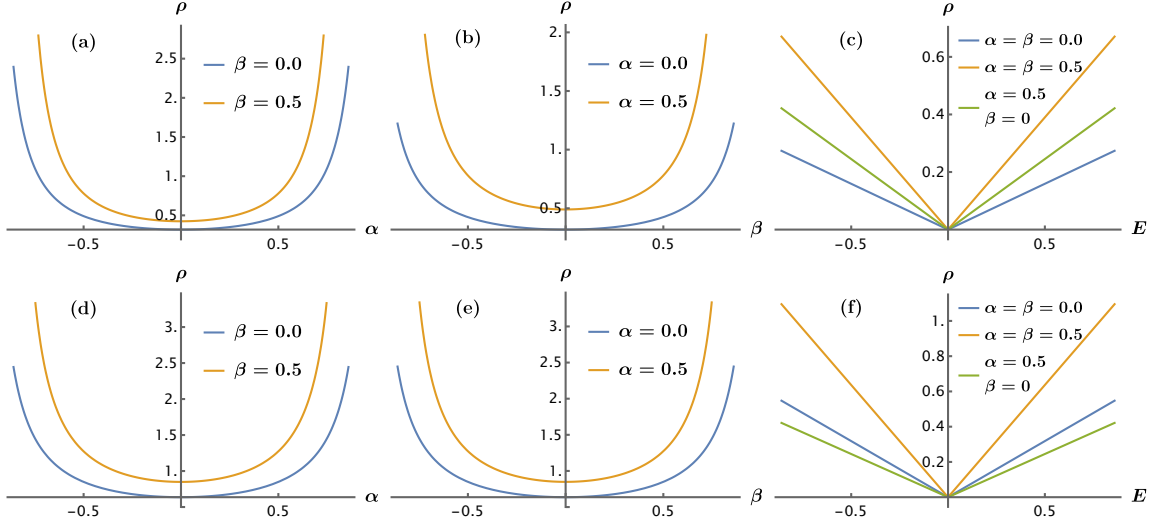


Figure 3.1: The density of states (DOS) for the antisymmetric tilt (velocity anisotropy) case in panels (a), (b) and (c) [(d), (e), and (f)], as given by Eq. (3.14) [Eq. (3.16)]. (a) and (d). The DOS as a function of the tilt parameter for a fixed non-Hermitian (NH) velocity, β . (b) and (e). The DOS as a function of the NH parameter β for a fixed tilt [panel (b)] and velocity anisotropy [panel (e)]. (c) and (f). The DOS as a function of energy for several values of the parameters α and β . In all the plots, we set $v_H = 1$. The plots in panels (a), (b), (d) and (e) are displayed for the value of energy $E = 1$.

3.1.1 Density of states

We calculate the DOS in terms of the (Euclidean) fermion Green's functions given as

$$\rho(E) = -\frac{1}{\pi} \lim_{\eta \rightarrow 0^+} \text{Im} \{ \text{Tr} G_F(i\omega \rightarrow E + i\eta, \mathbf{k}) \}. \quad (3.13)$$

Using the form of the Green's function for the antisymmetric tilt (3.8), we find

$$\rho_{\text{AS}}(E) = \frac{1}{\pi} \frac{v_F |E|}{(v_F^2 - \alpha^2 v_H^2)^{3/2}} = \frac{|E|}{\pi (v_F^{\text{tilt}})^2} \sqrt{\frac{1 - \beta^2}{1 - \alpha^2 - \beta^2}}, \quad (3.14)$$

where the effective tilt-dependent Fermi velocity $v_F^{\text{tilt}} = v_H \sqrt{1 - \alpha^2 - \beta^2}$. In the following, we only consider the subcritically tilted and weak NH case $|\alpha|^2 + |\beta|^2 < 1$, with pointlike Fermi surfaces and the quasiparticles exhibiting a finite lifetime (type-1). Otherwise, NH Dirac Hamiltonian describes type-2 Weyl/Dirac semimetal [95] generalized to the NH case when the tilt parameter $|\alpha| > 1$. For the case of the symmetric tilt, the DOS takes the form

$$\rho_S(E) = \frac{|E|}{\pi v_F^2}, \quad (3.15)$$

with the Fermi velocity is the effective Fermi velocity.

which is the same as for the non-tilted NH Dirac fermions. Finally, for the velocity-

anisotropy [Eq. (3.11)], the DOS reads as

$$\rho_{\text{VA}}(E) = \frac{1}{\pi} \frac{|E|}{v_{\text{F}}^2 - \alpha^2 v_{\text{H}}^2} = \frac{|E|}{\pi (v_{\text{F}}^{\text{tilt}})^2}. \quad (3.16)$$

The plots are shown in Fig. 3.1 with the technical details shown in the appendix A.1.2.

As we can observe, in all the three cases the DOS retains the linear scaling in energy, with the slope depending on the form of the matrix in Eq. (3.2). We notice that for the antisymmetric tilt, the DOS explicitly depends on the NH parameter (β), i.e. it is not solely a function of the effective Fermi velocity, $v_{\text{F}}^{\text{tilt}}$. Second, the symmetric tilt acts as an effective chiral chemical potential and therefore the contributions from the two valleys cancel out, yielding the tilt-free form of the DOS, Eq. (3.15). Finally, in case of the velocity anisotropy, the DOS effectively retains the form as in the tilt-free NH Dirac Hamiltonian with the effective velocity being $v_{\text{F}}^{\text{tilt}}$. When the tilt parameter $\alpha = 0$, both antisymmetric tilt and velocity anisotropy yield the same tilt-free form of the DOS, given by Eq. (3.15).

Such a linear scaling with energy of the DOS implies that the tilted Dirac semimetal remains stable for weak short-range electron-electron interactions. The tendency to a specific ordering at strong coupling then can be deduced from the behavior of the mean-field susceptibility, which we discuss in Sec. 4.1.

3.2 Optical conductivity

We now analyze the NH tilted DSM response to an external electromagnetic field in the Kubo formalism. The polarization bubble (See Figure 3.2) reads:

$$\Pi_{lm}(i\omega) = -\text{Tr} \int_{-\infty}^{\infty} \frac{d\nu}{2\pi} \int \frac{d^2\mathbf{k}}{(2\pi)^2} [J_l G_F(i\omega + i\nu, \mathbf{k}) J_m G_F(i\nu, \mathbf{k})], \quad (3.17)$$

where J_l is the current operator in the l -th direction, which in the x direction is $J_x = (v_{\text{H}} + v_{\text{NH}}M)\Gamma_x + \alpha v_{\text{H}}$, where it shows a tilt contribution, meanwhile in the y direction, it is $J_y = (v_{\text{H}} + v_{\text{NH}}M)\Gamma_y$.

The diagonal element of the polarization tensor $\Pi(i\omega) = (1/2) \sum_{j=1}^2 [\Pi_{jj}(i\omega) - \Pi_{jj}(0)]$ is, for the asymmetric tilt case:

$$\begin{aligned} \Pi_{\text{AS}}(i\omega) = & -N_f \pi^2 \omega \left(3\alpha^4 v_{\text{F}}^2 v_{\text{H}}^4 + 2\alpha^2 v_{\text{F}}^4 v_{\text{H}}^2 - 2\alpha^2 v_{\text{F}}^3 v_{\text{H}}^2 \sqrt{v_{\text{F}}^2 - \alpha^2 v_{\text{H}}^2} + 4v_{\text{F}}^5 \sqrt{v_{\text{F}}^2 - \alpha^2 v_{\text{H}}^2} \right. \\ & \left. - 2\alpha^4 v_{\text{F}} v_{\text{H}}^4 \sqrt{v_{\text{F}}^2 - \alpha^2 v_{\text{H}}^2} - 2\alpha^6 v_{\text{H}}^6 \right) \left\{ \frac{1}{8v_{\text{F}}^3 (v_{\text{F}}^2 - \alpha^2 v_{\text{H}}^2)^{3/2}} \right\}. \end{aligned} \quad (3.18)$$

Then, the zero-temperature optical conductivity is:

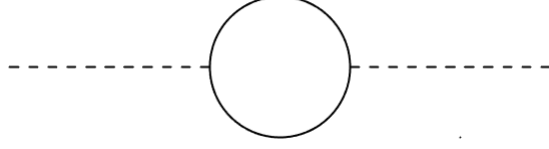


Figure 3.2: Polarization diagram for the Dirac fermion (solid lines) with dashed lines representing an external electromagnetic field.

$$\sigma_{\text{AS}}(\omega) = \frac{\text{Im } \Pi(i\omega \rightarrow \omega + i\eta)}{\omega} \Big|_{\eta \rightarrow 0} \quad (3.19)$$

$$\begin{aligned} \sigma_{\text{AS}}(\omega) = N_f \Big(& 3\alpha^4 v_{\text{F}}^2 v_{\text{H}}^4 + 2\alpha^2 v_{\text{F}}^4 v_{\text{H}}^2 - 2\alpha^2 v_{\text{F}}^3 v_{\text{H}}^2 \sqrt{v_{\text{F}}^2 - \alpha^2 v_{\text{H}}^2} + 4v_{\text{F}}^5 \sqrt{v_{\text{F}}^2 - \alpha^2 v_{\text{H}}^2} \\ & - 2\alpha^4 v_{\text{F}} v_{\text{H}}^4 \sqrt{v_{\text{F}}^2 - \alpha^2 v_{\text{H}}^2} - 2\alpha^6 v_{\text{H}}^6 \Big) \left\{ \frac{1}{32v_{\text{F}}^3 (v_{\text{F}}^2 - \alpha^2 v_{\text{H}}^2)^{3/2}} \right\}, \end{aligned} \quad (3.20)$$

$$\sigma_{\text{AS}}(\omega) = \frac{N_f}{32} \left(\frac{\alpha^2 (-2\alpha^2 + \beta^2 - 1) (\alpha^2 + 2\beta^2 - 2)}{(1 - \beta^2)^{3/2} (-\alpha^2 - \beta^2 + 1)^{3/2}} + \frac{2(\alpha^2 - 2\beta^2 + 2)}{1 - \beta^2} \right). \quad (3.21)$$

For the velocity asymmetric tilt we obtain for the diagonal element of the polarization tensor:

$$\Pi_{\text{VA}}(i\omega) = -N_f \frac{\pi^2 \omega (2v_{\text{F}}^2 - \alpha^2 v_{\text{H}}^2)}{4(v_{\text{F}}^2 - \alpha^2 v_{\text{H}}^2)}, \quad (3.22)$$

while the VA optical conductivity

$$\sigma_{\text{VA}}(\omega) = N_f \frac{2v_{\text{F}}^2 - \alpha^2 v_{\text{H}}^2}{16(v_{\text{F}}^2 - \alpha^2 v_{\text{H}}^2)} = \frac{N_f}{8} \left(1 + \frac{\alpha^2}{2(1 - \alpha^2 - \beta^2)} \right). \quad (3.23)$$

This result is different from the untilted NH case where the $\sigma_{lm}^{(2)} = N_f \frac{e^2}{h} \frac{\pi}{4}$ (Which is shown by the blue line at 3.3 figures (b) and (d)) and does not show any NH parameter dependence, on the contrary, tilt and NH effects boosts the OC near the points $|\alpha| = \sqrt{1 - \beta^2}$. On the contrary, it is implied that NH effects, through the explicit dependence of β , can be explicitly seen in tilted DSMs, and it promotes the study of other observables

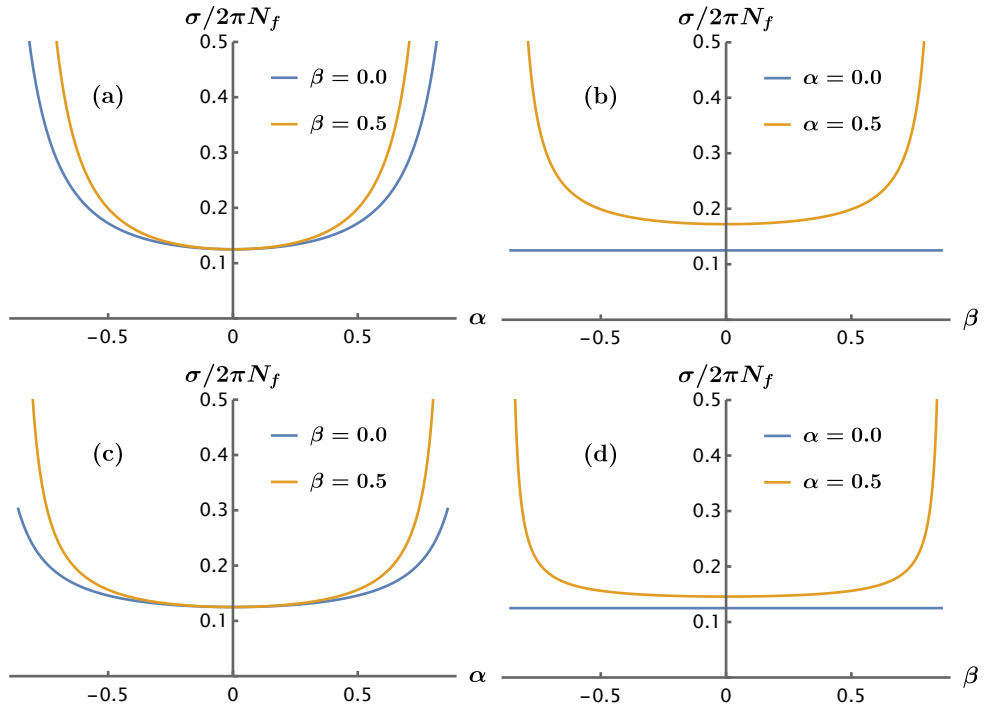


Figure 3.3: Optical conductivity for the asymmetric tilt case in (a), (b) and velocity anisotropy case in (c), (d) with $v_H = 1$ and varying parameters.

in NH Dirac materials.

Chapter 4

Tilted non-Hermitian Dirac Fermions at quantum criticality

After the study of the low energy phenomena, we analyze what occurs at the quantum criticality by a mean-field analysis of the magnetic susceptibility, we carry out the RG analysis of the quantum-critical NH GNY theory with the tilt term to understand what happens in the vicinity the QPT.

4.1 Mean-field susceptibilities for the mass orders

The OP that isotropically gaps out hermitian Dirac fermions can be expressed as an n -component vector $\Phi_j = \langle \Psi^\dagger N_j \Psi \rangle$ such that $\{H_D, N_j\} = 0$, with mutually anticommuting components, $\{N_j, N_k\} = 2\delta_{jk}$. The matrix M in the NH Dirac Hamiltonian induces fragmentation of the possible OPs and we here analyze two cases: (i) A CCM OP for which $[N_j, M] = 0$, for $j = 1, 2, \dots, n$; (ii) An ACM OP for which $\{N_j, M\} = 0$ for $j = 1, 2, \dots, n$. We do not consider a mixed class where the OP matrix partially (anti)commutes with the matrix M .

The ordering tendencies for these two classes of instabilities, showing the influence of the NH parameter (β), can be estimated from their respective (nonuniversal) bare mean-field susceptibilities at zero external frequency and momentum, which for an OP represented by a fermion bilinear $\langle \Psi^\dagger N \Psi \rangle$ reads

$$\chi_N = -\text{Tr} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int \frac{d^d \mathbf{k}}{(2\pi)^d} [N G_F(i\omega, \mathbf{k}) N G_F(i\omega, \mathbf{k})], \quad (4.1)$$

since the (nonuniversal) critical interaction strength for such an ordering $g_{N,*} \sim \chi_N^{-1}$. Here, $G_F(i\omega, \mathbf{k})$ is the fermion propagator for the corresponding noninteracting Hamiltonian and we assume an upper cutoff Λ in the momentum integral, and consider $d = 3$ being the upper critical dimension for the corresponding quantum-critical theory (Sec. 4.2). We furthermore consider the susceptibilities in the two cases with qualitatively different spectrum in the noninteracting NH Dirac Hamiltonian (3.4), namely, the antisymmetric tilt and the velocity anisotropy, while the symmetric tilt is not explicitly considered in the following since it is analogous to its antisymmetric counterpart.

First, in the case of the asymmetric tilt, the susceptibility corresponding to an ACM order has the form

$$\chi_N^{\text{ACM}} = h_{\text{AS}}(v_F, \beta, \alpha) \Lambda^2, \quad (4.2)$$

where

$$\begin{aligned} h_{\text{AS}}(v_F, \beta, \alpha) &= \frac{1}{32\pi^3 v_F} \int [d\phi d\theta] \frac{(1 - \beta^2) (-2\alpha^2 \sin^2 \theta \cos^2 \phi + \alpha^2 + 4) - 2\alpha^2}{(1 - \beta^2)(1 - \beta^2 - \alpha^2 \sin^2 \theta \cos^2 \phi)} \\ &= \frac{(1 + \beta^2) (2 - \alpha^2 - 2\beta^2) \tanh^{-1} \left(\frac{\alpha}{\sqrt{1 - \beta^2}} \right) + 2\alpha (1 - \beta^2)^{3/2}}{8\pi^2 \alpha (1 - \beta^2)^{3/2} v_F}, \end{aligned} \quad (4.3)$$

with $[d\phi d\theta] \equiv \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin \theta$.

For the CCM order, the susceptibility takes the form

$$\chi_N^{\text{CCM}} = f_{\text{AS}}(v_F, \beta, \alpha) \Lambda^2, \quad (4.4)$$

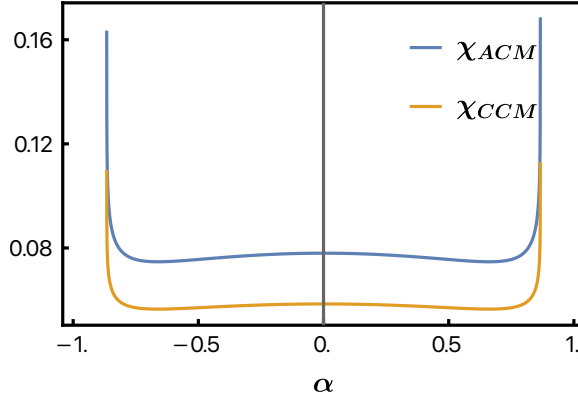


Figure 4.1: The mean-field susceptibility for the anticommuting class mass (ACM) (Eq. (4.2)) and the commuting class mass (CCM) (Eq. (4.4)) orders when noninteracting non-Hermitian (NH) Dirac Hamiltonian features antisymmetric tilt term. We notice that peaks in the susceptibility emerge for the tilt $|\alpha| = \sqrt{1 - \beta^2}$ corresponding to the divergence of the density of states in Eq. (3.14) at the zero effective velocity v_F^{tilt} . The values of Hermitian and NH velocities are $v_H = 1$ and $v_{NH} = 0.5$, respectively, which corresponds to the values of the non-Hermiticity parameter $\beta = 1/2$ and the effective Fermi velocity $v_F = \sqrt{3}/2$.

where

$$\begin{aligned}
 f_{\text{AS}}(v_F, \beta, \alpha) &= \frac{1}{16\pi^3 v_F} \int [d\phi d\theta] \frac{-\alpha^2 (2 \sin^2 \theta \cos^2 \phi + 1) + 4(1 - \beta^2)}{2(1 - \beta^2 - \alpha^2 \sin^2 \theta \cos^2 \phi)} \\
 &= \frac{1}{16\pi^2 v_F \sqrt{1 - \beta^2}} \left[\alpha \ln \left(\frac{\sqrt{1 - \beta^2} - \alpha}{\alpha + \sqrt{1 - \beta^2}} \right) + 4\sqrt{1 - \beta^2} \right. \\
 &\quad \left. + \frac{2}{\alpha}(\beta^2 - 1) \ln \left(\frac{\sqrt{1 - \beta^2} - \alpha}{\alpha + \sqrt{1 - \beta^2}} \right) \right]. \tag{4.5}
 \end{aligned}$$

Both functions also satisfy the scaling as $\sim v_F^{-1}$, with their explicit form for ACM and CCM OPs shown in Fig. 4.1. We notice that the susceptibility is larger for the ACM than for the CCM case, implying the lower critical coupling in the former case. Therefore, the ACM order is favored in the case of antisymmetric tilt, as expected from the fact that the ACM order opens a gap at the band touching points in the tilted Dirac semimetal. Finally, the cusps in the susceptibilities in Fig. 4.1 are a consequence of the vanishing effective Fermi velocities at these specific values of the tilt parameter which makes the system more prone to the instabilities due to the ensuing divergent DOS.

In the velocity anisotropic semimetal, for the CCM case, we obtain

$$\chi_N^{\text{CCM}} = f_{\text{VA}}(v_F, \beta, \alpha) \Lambda^2, \tag{4.6}$$

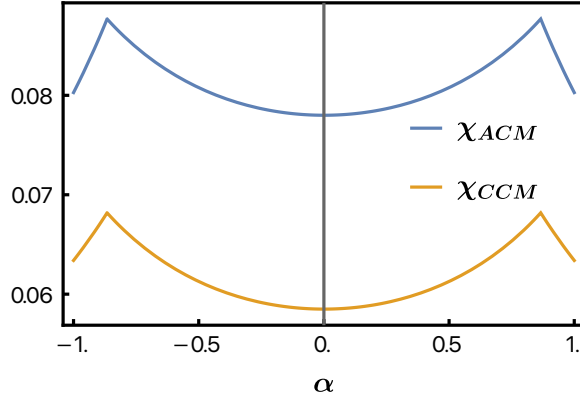


Figure 4.2: The mean-field susceptibility for the ACM [Eq. (4.8)] and CCM [Eq. (4.6)] orders when noninteracting NH Dirac Hamiltonian features velocity-anisotropy term. We notice two cusps corresponding to the absolute value of the velocity-anisotropy parameter $|\alpha| = \sqrt{1 - \beta^2}$, with $\beta = v_{\text{NH}}/v_{\text{H}}$. These cusps coincide with the maximum slope of the density of states, as given by Eq. (3.16), with the effective velocity in Eq. (3.7), and therefore indicate increased propensity toward the instabilities therein. The values of the parameters are the same as in Fig. 4.1.

where

$$f_{\text{VA}}(v_{\text{F}}, \beta, \alpha) = \frac{1}{16\pi^3 v_{\text{F}}} \sum_{\zeta=\pm} \int [d\phi d\theta] \sqrt{\frac{1 - \beta^2}{\alpha \left(2\zeta \sqrt{1 - \beta^2} + \alpha \right) \sin^2(\theta) \cos^2(\phi) + 1 - \beta^2}}. \quad (4.7)$$

Now, for the ACM case, we find the susceptibility of the form

$$\chi_N^{\text{ACM}} = (f_{\text{VA}}(v_{\text{F}}, \beta, \alpha) + h_{\text{VA}}(v_{\text{F}}, \beta, \alpha)) \Lambda^2, \quad (4.8)$$

where

$$h_{\text{VA}}(v_{\text{F}}, \beta, \alpha) = -\frac{1}{16\pi^3} \frac{\beta^2}{v_{\text{F}} \alpha} \sum_{\zeta=\pm} \int [d\phi d\theta] \csc^2(\theta) \sec^2(\phi) \times \zeta \sqrt{\frac{1}{\alpha \left(2\zeta \sqrt{1 - \beta^2} - \alpha \right) \sin^2(\theta) \cos^2(\phi) + 1 - \beta^2}}. \quad (4.9)$$

We notice that the susceptibilities for both CCM and ACM OPs scale as $\sim v_{\text{F}}^{-1}$. Furthermore, the ACM order is also favored in the case of velocity-anisotropic NH Dirac semimetal, as explicitly shown in Fig. 4.2. Notice that susceptibilities for both ACM and CCM orders are increased in comparison to the tilt-free and isotropic semimetal, with a particularly pronounced cusps when the value of the effective velocity reaches zero, and therefore the DOS diverges.

The susceptibilities for both antisymmetrically tilted and velocity-anisotropic NH Dirac

semimetals in $d = 2$ are analyzed in Appendix A.2, and show qualitatively similar behavior as in $d = 3$. Particularly, the ACM order is favored in both cases. However, to address the competition between the different instabilities, together with the emergence of the Yukawa-Lorentz symmetry, we need to consider the RG flows of the velocity parameters and of the tilt in the quantum-critical regime. This is even more pertinent given a rather small difference between the mean-field susceptibilities for the two classes of the OPs, for both the asymmetric tilt (Figs. 4.1) and velocity anisotropy (Fig. 4.2). Motivated by this, we now analyze the quantum-critical behavior of NH Dirac semimetals with the antisymmetric tilt and velocity anisotropy in the vicinity of the quantum phase transition corresponding to the ACM and CCM OPs condensation.

4.2 Quantum-critical theory of the tilted non-Hermitian Dirac fermions

To address the quantum-critical behavior of the tilted NH Dirac fermions, we employ the zero-temperature ($T = 0$) Euclidean action for d -dimensional tilted NH Dirac fermions

$$S_F[\Psi, \Psi^\dagger] = \int d\tau d^d\mathbf{r} \Psi^\dagger [\partial_\tau + H_{\text{NH}}^{\text{tilt}}(\mathbf{k} \rightarrow -i\nabla_{\mathbf{r}})] \Psi, \quad (4.10)$$

where τ is the imaginary time, and $\Psi \equiv \Psi(\tau, \mathbf{r})$.

The bosonic OP fluctuations are described by a Φ^4 theory, with a bosonic action of the form,

$$S_B[\Phi] = \int d\tau d^d\mathbf{r} \left[\sum_{j=1}^n \Phi_j (-\partial_\tau^2 - v_B^2 \nabla^2 + m_B^2) \Phi_j + \frac{\lambda}{2} \left(\sum_{j=1}^n \Phi_j^2 \right)^2 \right], \quad (4.11)$$

where a bosonic propagator $G_B(i\omega, \mathbf{k}) = [\omega^2 + v_B^2 k^2 + m_B^2]^{-1}$. Since we consider the critical plane of the theory, we set $m_B = 0$.

The coupling of the bosonic fluctuations to the NH tilted fermions is described by a Yukawa interaction of the form

$$S_Y = g \int d\tau \int d^d\mathbf{r} \sum_{j=1}^n \Phi_j(\tau, \mathbf{r}) \Psi^\dagger(\tau, \mathbf{r}) N_j \Psi(\tau, \mathbf{r}), \quad (4.12)$$

which encodes the effects of short-range Coulomb interaction in the quantum-critical region, where the fermionic and the bosonic fluctuations are strongly coupled. The Yukawa, and the Φ^4 couplings, g and λ , respectively, are both marginal in $d = 3$, the upper critical dimension of the theory, since their engineering dimensions are $\dim[g^2] = \dim[\lambda] = 3 - d$, with the engineering scaling dimension of momentum set to unity. We thus use the

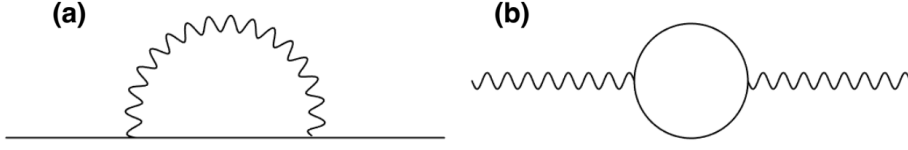


Figure 4.3: Self-energy diagrams for the Dirac fermion (solid lines) in panel (a) and the bosonic order parameter field (wavy lines) in panel (b). The vertex in both diagrams corresponds to the Yukawa coupling in Eq. (4.12).

distance from the upper critical dimension $\epsilon = 3 - d$ as expansion parameter to analyze the quantum-critical theory in $d = 2$ [114, 76]. To this end, we define a Euclidean renormalized action, taking Eqs. (4.10)-(4.12), which reads

$$\begin{aligned}
S_R = \int d\tau d^d \mathbf{r} \Bigg[& Z_\Psi \Psi^\dagger [\partial_\tau - i(Z_{v_H} v_H + Z_{v_{NH}} v_{NH} M)(\Gamma_x \partial_x + \Gamma_y \partial_y) \\
& - iZ_\alpha Z_{v_H} \alpha v_H T \partial_x] \Psi + Z_\Phi \left(\sum_{j=1}^n \Phi_j (-\partial_\tau^2 - Z_{v_B^2} v_B^2 \nabla^2) \Phi_j \right) \\
& + Z_\lambda \frac{\lambda}{2} \left(\sum_{j=1}^n \Phi_j^2 \right)^2 + Z_g g \sum_{j=1}^n \Phi_j(\tau, \mathbf{r}) \Psi^\dagger(\tau, \mathbf{r}) N_j \Psi(\tau, \mathbf{r}) \Bigg], \quad (4.13)
\end{aligned}$$

where Z_Q are renormalization constants. Furthermore, we define renormalization conditions for each parameter relating the renormalized quantities with the bare ones at the cutoff scale,

$$Z_{v_H} v_H = v_H^0 \quad (4.14)$$

$$Z_{v_{NH}} v_{NH} = v_{NH}^0 \quad (4.15)$$

$$Z_\alpha \alpha = \alpha^0 \quad (4.16)$$

$$Z_{v_B} v_B^2 = (v_B^0)^2 \quad (4.17)$$

where the quantities with (without) index 0 in the superscript are bare (renormalized).

To analyze the fate of the Yukawa-Lorentz symmetry at the quantum criticality, we compute the fermionic [Fig. 4.3(a)] and bosonic [Fig. 4.3(b)] self-energy diagrams to the leading order in the ϵ expansion. After computing the self-energy diagrams, as shown in Appendix A.3, we find the renormalization factors Z_Q to the one-loop order in the minimal-subtraction-scheme. Using the obtained factors, we subsequently compute the RG β functions, which are defined as $\beta_Q \equiv dQ/d \ln b|_{Q^0}$, with the bare quantities fixed, and b

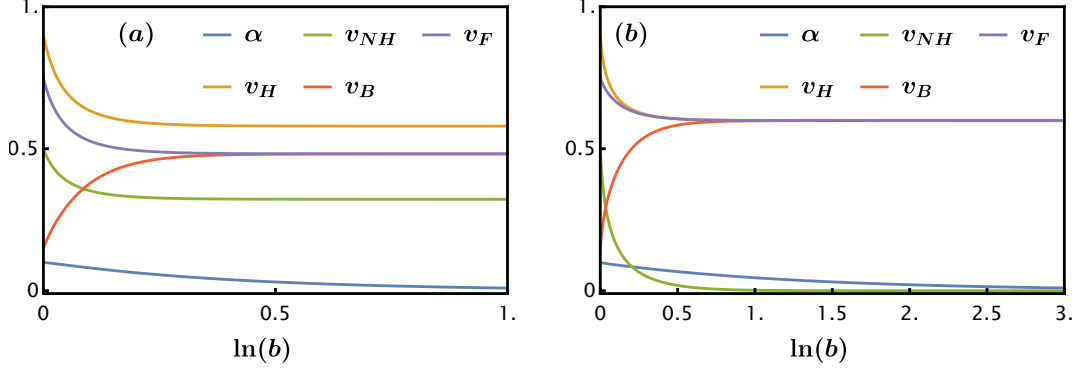


Figure 4.4: Renormalization-group (RG) flow of the velocities in the critical plane ($m_B = 0$) of the Gross-Neveu-Yukawa theory with antisymmetric tilt and the corresponding RG flow equations given by Eqs. (4.18)-(4.21). (a) Commuting class mass order parameter. (b) Anticommuting class mass order parameter. In both cases we set the bare values of the tilt and velocity parameters $\alpha^0 = 0.1$, $v_H^0 = 0.9$, $v_{NH}^0 = 0.5$, $v_B^0 = 0.15$, the Yukawa coupling $g = 1$, and the number of the order-parameter components $n = 1$.

as the RG scale, rescaling the UV cutoff in the theory, $\Lambda \rightarrow \Lambda/b$, in the RG transformation. These RG β functions take the form

$$\beta_{v_H} = -v_H n \frac{g^2}{2} [G(\alpha, v_H, v_{NH}, v_B) - J(\alpha, v_H, v_{NH}, v_B)], \quad (4.18)$$

$$\beta_{v_{NH}} = -v_{NH} n \frac{g^2}{2} [G(\alpha, v_H, v_{NH}, v_B) \mp J(\alpha, v_H, v_{NH}, v_B)], \quad (4.19)$$

$$\beta_\alpha = -\alpha n \frac{g^2}{2} [J(\alpha, v_H, v_{NH}, v_B) - I(\alpha, v_H, v_{NH}, v_B)], \quad (4.20)$$

$$\beta_{v_B} = -v_B n N_f g^2 [F(\alpha, v_H, v_{NH}, v_B) - C(\alpha, v_H, v_{NH}, v_B)], \quad (4.21)$$

where we rescaled the Yukawa coupling, $g^2/(8\pi^2) \rightarrow g^2$, and G, I, J, F, C are functions defined in Appendix A.3. The upper (lower) sign in Eq. (4.19) corresponds to the CCM (ACM) order.

To analyze the behavior of the velocities and tilt under the RG at the criticality, we plot the corresponding RG flows for a fixed value of the Yukawa coupling, g . Ultimately, the form of the RG flows does not depend on the value of this coupling, as also Eqs. (4.18)-(4.21) explicitly show. The corresponding RG flows for the antisymmetric tilt are displayed in the Fig. 4.4, and for the velocity-antisymmetric NH Dirac fermions in Fig. A.1, with CCM or ACM OPs considered in both cases.

We first notice that for both antisymmetric tilt and velocity anisotropy, the qualitative features of the RG flow are similar. Both the tilt and the velocity anisotropy flow to zero under the RG transformation, i.e. they turn irrelevant in the quantum-critical region. The rate of the RG flow, as it can be noticed, is slightly faster for the CCM than for the ACM

OP in the case of the tilt. Furthermore, for the CCM OP in the case of either tilt or velocity anisotropy, the NH velocity flows to a finite value, while the terminal bosonic velocity and the effective Fermi velocity reaches a common value. Therefore the NH Yukawa-Lorentz symmetry is restored in spite of the symmetry-breaking tilt or velocity anisotropy present in the bare (lattice) theory. In the case of ACM order, the NH velocity parameter vanishes, irrespective of the additional (tilt or velocity anisotropy) term, and therefore the system spontaneously restores hermiticity, and at the same time, conventional Lorentz symmetry.

Chapter 5

Conclusions

In summary, our analysis of NH tilted Dirac Fermions shows that, under the influence of this tilt anisotropy, NH effects can be realized on the observables of this system, as shown in Eqs. (3.21)-(3.23) and Fig. 3.3, where originally the optical conductivity did not show any dependence of the NH parameter, making tilt a probe for non-Hermiticity in these types of systems. Additionally, the quantum-critical NH tilted Dirac fermions shows that Yukawa-Lorentz symmetry emerges near the strongly coupled QCP separating the NH Dirac semimetallic and insulating phases. This key feature arises from the irrelevance of the tilt term near the QCP, as our leading-order renormalization group calculations explicitly demonstrate, see Eqs. (4.18)-(4.20). Crucially, the fate of non-Hermiticity hinges on the OP class (ACM or CCM), with ACM leading to complete decoupling from the environment and Hermiticity restoration, while CCM maintains coupling to the environment yet exhibits emergent Yukawa-Lorentz symmetry, see also Fig. 4.4. Furthermore, a velocity anisotropy term turns out to be irrelevant near the QCP (Fig. A.1). These results strongly suggest that Yukawa-Lorentz symmetry is a universal feature of quantum-critical NH Dirac fermions, a prediction that can be tested through numerical lattice simulations.

For the quantum phase transition from NH tilted Dirac semimetal to an ACM OP, even though, the tilt eventually vanishes, its effect on the observables should be non-negligible, due to the finite system size. On the other hand, in the case of the CCM OP, non-Hermiticity close to the QCP should have observable effects on the correlators through a contribution of the finite NH velocity, which we plan to study in near future.

Finally, we would like to point out that our results should motivate further studies of the NH tilted Dirac fermions, as for instance, their instabilities in the case when they feature a finite-size Fermi surface with finite-lifetime quasiparticles (NH type-2 Dirac fermions). The studies of the NH tilted Dirac fermions may also provide some valuable insights regarding the relationship between the condensed matter and low-dimensional gravitational systems, particularly concerning the effects of the Hawking radiation that may be captured through the effective NH Dirac operators discussed here.

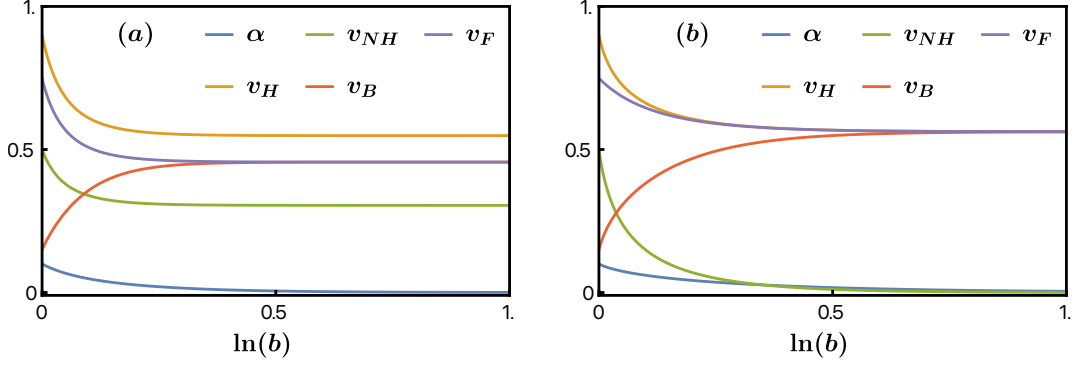


Figure A.1: Renormalization-group (RG) flow of the velocity parameters in the critical plane ($m_B = 0$) of the Gross-Neveu-Yukawa theory for the quantum-critical NH Dirac fermions, given by Eqs. (4.10)-(4.12), with velocity anisotropy. (a) Commuting class mass order parameter. (b) Anticommuting mass order parameter. We use the same values of the parameters as in Fig. 4.4.

Appendix A

A.1 Minimal model

In this appendix, we provide the details of the minimal model, given by the Hamiltonian in Eq. (3.4), for the NH tilted Dirac fermions.

A.1.1 Fermionic propagator

We can obtain our fermion propagator noting that the inverse of the propagator is

$$G_F^{-1}(i\omega, \mathbf{k}) = i\omega - [(v_H + v_{NH}M)(\Gamma_x k_x + \Gamma_y k_y) + \alpha v_H T k_x]. \quad (\text{A.1})$$

In the case of antisymmetric tilt, we obtain

$$\begin{aligned} (G_F^{\text{AS}}(i\omega, \mathbf{k}))^{-1}(i\omega + [(v_H + v_{NH}M)(\mathbf{\Gamma} \cdot \mathbf{k}) + \alpha v_H T_{\text{AS}} k_x]) = & -\omega^2 - v_F^2 k^2 - \alpha^2 v_H^2 k_x^2 \\ & - 2\alpha v_H (v_H + v_{NH}M)(\mathbf{\Gamma} \cdot \mathbf{k}) T_{\text{AS}} k_x. \end{aligned} \quad (\text{A.2})$$

Again, we carry out a multiplication to get a pure scalar function (proportional to the unity matrix) on the right hand side with Γ or T matrices,

$$\begin{aligned} & (G_F^{\text{AS}}(i\omega, \mathbf{k}))^{-1}(i\omega + [(v_H + v_{\text{NH}}M)(\mathbf{\Gamma} \cdot \mathbf{k}) + \alpha v_H T_{\text{AS}} k_x]) \\ & \times (-\omega^2 - v_F^2 k^2 - \alpha^2 v_H^2 k_x^2 + 2\alpha v_H (v_H + v_{\text{NH}}M)(\mathbf{\Gamma} \cdot \mathbf{k}) T_{\text{AS}} k_x) \\ & = (\omega^2 + v_F^2 k^2 + \alpha^2 v_H^2 k_x^2)^2 - 4\alpha^2 v_F^2 v_H^2 k_x^2. \end{aligned} \quad (\text{A.3})$$

Finally, we multiply by the Green's function from the left to find

$$\begin{aligned} G_F^{\text{AS}}(i\omega, \mathbf{k}) = & \left\{ -(i\omega + H_0^{\text{tilt}})A_+ + B(v_H + v_{\text{NH}}M)(\mathbf{\Gamma} \cdot \mathbf{k})T_{\text{AS}}k_x \right. \\ & \left. + 2\alpha v_F k_x (v_F^2 k^2 T_{\text{AS}} + \alpha v_H k_x (v_H + v_{\text{NH}}M)(\mathbf{\Gamma} \cdot \mathbf{k})) \right\} \frac{1}{A_-^2 - B^2}, \end{aligned} \quad (\text{A.4})$$

with

$$A_{\pm} = A_{\pm}(i\omega, \mathbf{k}) = \omega^2 + v_F^2 k^2 \pm \alpha^2 v_H^2 k_x^2, \quad B = B(i\omega, \mathbf{k}) = 2i\alpha v_H \omega k_x. \quad (\text{A.5})$$

In the case of the velocity anisotropy, carrying out the same procedure, and using that $[\Gamma_x, T_{\text{VA}}] = 0$, $\{\Gamma_y, T_{\text{VA}}\} = 0$, we obtain the Green's function in the form

$$\begin{aligned} G_F^{\text{VA}}(i\omega, \mathbf{k}) = & \left\{ -(i\omega + H_0^{\text{tilt}})A_+ + 2i\omega\alpha v_H k_x^2 (v_H + v_{\text{NH}}M)T_{\text{VA}}\Gamma_x \right. \\ & \left. + 2\alpha v_H v_F^2 k_x^2 (k_x + \Gamma_y \Gamma_x k_y)T_{\text{VA}} + 2\alpha^2 v_H^2 k_x (v_H + v_{\text{NH}}M)\Gamma_x k_x^3 \right\} \frac{1}{A^2 - B^2}, \end{aligned} \quad (\text{A.6})$$

where we defined

$$A = A(i\omega, \mathbf{k}) = \omega^2 + v_F^2 k^2 + \alpha^2 v_H^2 k_x^2, \quad B = B(i\omega, \mathbf{k}) = 2\alpha v_H v_F k_x^2. \quad (\text{A.7})$$

A.1.2 Calculation of the density of states

To calculate the DOS in terms of the Green's function, we use

$$\rho(E) = -\frac{1}{\pi} \lim_{\eta \rightarrow 0^+} \text{Im} \{ \text{Tr} G_F(i\omega \rightarrow E + i\eta, \mathbf{k}) \}. \quad (\text{A.8})$$

After taking the trace of the Green's function, we apply the analytical continuation, $i\omega \rightarrow \omega + i\eta$, and use the Sokhotski-Plemelj theorem for the real line,

$$\lim_{\eta \rightarrow 0^+} \frac{1}{x \pm i\eta} = \mp i\pi\delta(x) + \mathcal{P}\left(\frac{1}{x}\right), \quad (\text{A.9})$$

where $\mathcal{P}$ denotes the principal value, while $\delta(x)$ is the delta function.

Now, carrying out this procedure, we calculate the DOS, which for the antisymmetric

tilt yields,

$$\rho_{\text{AS}}(E) = -\frac{1}{\pi} \lim_{\eta \rightarrow 0^+} \text{Im} \left\{ \text{Tr} G_F^{\text{AS}}(i\omega \rightarrow E + i\eta, \mathbf{k}) \right\} \quad (\text{A.10})$$

$$= 2 \sum_{\zeta=\pm} \int \frac{d^2 \mathbf{k}}{(2\pi)^2} \delta[E - (\zeta v_F k + \alpha v_H k_x)], \quad (\text{A.11})$$

$$= \frac{1}{2\pi^2} \sum_{\zeta=\pm} \int_0^{2\pi} d\phi \int_0^\infty dk k \delta[E - (\zeta k v_F + \alpha v_H k \cos \phi)], \quad (\text{A.12})$$

with the argument of each delta function representing the spectrum of a band of the Hamiltonian in Eq. (3.5). The DOS can then be compactly written as

$$\rho_{\text{AS}}(E) = \frac{1}{\pi} \frac{v_F |E|}{(v_F^2 - \alpha^2 v_H^2)^{3/2}} = \frac{\sqrt{1 - \beta^2} |E|}{\pi (1 - \alpha^2 - \beta^2)^{3/2} v_H^2}, \quad (\text{A.13})$$

where $\beta = v_{\text{NH}}/v_H$, and $|\alpha| < 1$, $|\beta| < 1$. This is the form of the DOS for the antisymmetric tilt, given by Eq. (3.14).

The same steps as for the antisymmetric case yield the DOS for the symmetric tilt and velocity anisotropic model as given in Eq. (3.15) and Eq. (3.16), respectively.

A.2 Mean-field susceptibilities in $d = 2$

In Sec. 4.1, we have presented the calculation of the mean-field susceptibilities in the upper critical dimension of the theory, $d = 3$. For completeness, we here carry out the mean-field susceptibility calculations directly in $d = 2$, which for the OP represented by the fermion bilinear $\langle \Psi^\dagger N \Psi \rangle$ reads

$$\chi_N = -\text{Tr} \int_{-\infty}^\infty \frac{d\omega}{2\pi} \int \frac{d^2 \mathbf{k}}{(2\pi)^2} [N G_F(i\omega, \mathbf{k}) N G_F(i\omega, \mathbf{k})], \quad (\text{A.14})$$

with $G_F(i\omega, \mathbf{k})$ as the fermion Green's function.

We first consider the antisymmetric tilt case with the OP matrix N belonging to the CCM class, $[M, N] = 0$,

$$\chi_N^{\text{CCM}} = \left(\frac{\Lambda}{4\pi v_F} \right) \left(\frac{2\sqrt{(1 - \beta^2)(1 - \alpha^2 - \beta^2)} + 2 - \alpha^2 - 2\beta^2}{\sqrt{(1 - \beta^2)(1 - \alpha^2 - \beta^2)}} \right), \quad (\text{A.15})$$

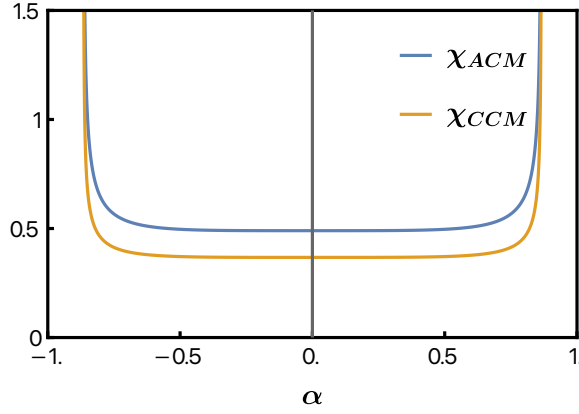


Figure A.2: Mean field susceptibility of the noninteracting non-Hermitian (NH) Dirac Hamiltonian in $d = 2$ with the antisymmetric tilt term, corresponding to the commuting (anticommuting) class mass, given by Eq. (A.15) (Eq. (A.16)). We notice that the susceptibility diverges when $|\alpha| = \sqrt{1 - \beta^2}$, corresponding to the exceptional point of the NH Hamiltonian. We use the values of NH parameter $\beta = 1/2$ and the effective Fermi velocity $v_F = \sqrt{3}/2$.

while in the case of an ACM OP, we find

$$\chi_N^{\text{ACM}} = \left(\frac{\Lambda}{4\pi v_F} \right) \left(\frac{2(1 - \beta^2) \left(\sqrt{(1 - \beta^2)(1 - \alpha^2 - \beta^2)} + \beta^2 + 1 \right) - \alpha^2(\beta^2 + 1)}{(1 - \beta^2)^{3/2} \sqrt{1 - \alpha^2 - \beta^2}} \right). \quad (\text{A.16})$$

Behavior of the susceptibilities for the classes of the mass order in $d = 2$, as shown in Fig. A.2, is qualitatively similar to the situation in $d = 3$, namely, $\chi_N^{\text{ACM}} > \chi_N^{\text{CCM}}$, with the nucleation of the ACM order being more favorable than of the CCM class.

When the system features velocity anisotropy, as given by Eq. (3.4) with the matrix $T = T_{\text{VA}} = \sigma_1 \otimes \tau_0$, the susceptibility for the CCM OP reads

$$\chi_N^{\text{CCM}} = \frac{\Lambda \sqrt{1 - \beta^2}}{(2\pi)^2 v_F} \sum_{\zeta = \pm} \int_0^{2\pi} d\phi \frac{1}{\sqrt{\alpha \left(\alpha + 2\zeta \sqrt{1 - \beta^2} \right) \cos^2(\phi) - \beta^2 + 1}}, \quad (\text{A.17})$$

while for the ACM order we obtain

$$\chi_N^{\text{ACM}} = \chi_N^{\text{CCM}} + \frac{\Lambda \beta^2}{2\pi^2 \alpha v_F} \sum_{\zeta = \pm} \int_0^{2\pi} d\phi \frac{\zeta \sec^2 \phi}{\sqrt{\alpha \left(\alpha - 2\zeta \sqrt{1 - \beta^2} \right) \cos^2(\phi) - \beta^2 + 1}}. \quad (\text{A.18})$$

Since the integrand in the last term in Eq. (A.18) (after taking the sum over $\zeta = \pm$) is positive for any $|\alpha| < 1$ and $|\beta| < 1$, with the spectrum of the corresponding Hamiltonian purely real, we obtain that $\chi_N^{\text{ACM}} > \chi_N^{\text{CCM}}$, as also shown in Fig. A.3. Therefore, the

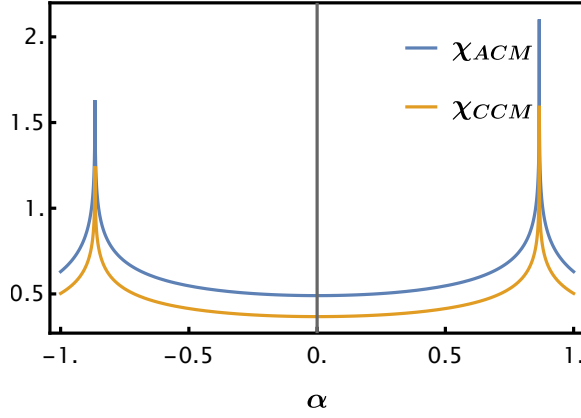


Figure A.3: Mean field susceptibilities of the noninteracting NH Dirac Hamiltonian in $d = 2$ with the velocity-anisotropy term. The susceptibility for the commuting (anticommuting) class mass is given by Eq. (A.15) (Eq. (A.16)). We notice that the susceptibility has cusps when $|\alpha| = \sqrt{1 - \beta^2}$. The value of the anisotropy parameter α corresponding to the cusps coincide with the maximum slope of the density of states, as given by Eq. (3.16), with the effective velocity in Eq. (3.7). The values of the parameters are the same as in Fig. A.2.

velocity-anisotropic tilted NH Dirac semimetal in $d = 2$ shows an enhanced propensity toward the ACM ordering as compared to the CCM class, as it is the case in $d = 3$, see Sec. 4.1.

A.3 Renormalization group analysis

In this section, we present the details of the RG analysis leading to the form of the RG flow equations of the fermionic and bosonic velocities, and the tilt parameter, presented in Sec. 4.2. We employ the leading order ϵ -expansion about $d = 3$ upper critical dimension of the NH Gross-Neveu-Yukawa (GNY) theory with $\epsilon = 3 - d$, and within the minimal subtraction scheme [114, 76]. We compute the self energy diagrams of fermions and bosons as shown in Fig. 4.3(a) and Fig. 4.3(b), respectively, with the form of the Yukawa interaction given in Eq. (4.12).

We here show the technical details for the antisymmetric tilt case, yielding the RG flow equations (4.18)–(4.21) (see also Fig. 4.4). The results for the velocity-anisotropic case are obtained using the same procedure, and with the plot of the RG flows of the velocity parameters displayed in Fig. A.1. However, we omit the details of the calculation for the sake of brevity.

A.3.1 Fermionic self-energy

We first calculate the fermion self-energy, with the corresponding diagram shown in Fig. 4.3(a),

$$\Sigma(i\nu, \mathbf{k}) = g^2 \int [d\omega] \int [d^d \mathbf{q}] [N_i G_F(i\omega, q) N_i G_B(i\nu - i\omega, \mathbf{k} - \mathbf{q})], \quad (\text{A.19})$$

with $\int [d\omega] \equiv \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}$ and $\int [d^d \mathbf{q}] \equiv \int \frac{d^d \mathbf{q}}{(2\pi)^d}$. We then obtain

$$\begin{aligned} \Sigma(i\nu, \mathbf{k}) = & g^2 N_i \int [d\omega] \int [d^d \mathbf{q}] \left[- (i\omega + (v_H + v_{\text{NH}} M)(\mathbf{\Gamma} \cdot \mathbf{q}) + \alpha v_H T_{\text{AS}} q_x) A_+(i\omega, \mathbf{q}) \right. \\ & + B(i\omega, \mathbf{q})(v_H + v_{\text{NH}} M)(\mathbf{\Gamma} \cdot \mathbf{q}) T_{\text{AS}} q_x + 2\alpha v_H q_x (v_F^2 q^2 T_{\text{AS}} + \alpha v_F q_x (v_H + v_{\text{NH}} M)(\mathbf{\Gamma} \cdot \mathbf{q})) \left. \right] \\ & \times N_i \frac{1}{[(A_-(i\omega, \mathbf{q}))^2 - (B(i\omega, \mathbf{q}))^2]((\nu - \omega)^2 + v_B^2(\mathbf{k} - \mathbf{q})^2)}, \end{aligned} \quad (\text{A.20})$$

where $A_{\pm}(i\omega, \mathbf{q})$ is given by Eq. (A.5). For both classes of the mass orders, we find

$$\begin{aligned} \Sigma(i\nu, \mathbf{k}) = & g^2 n \int [d\omega] \int [d^d \mathbf{q}] \left[- i\omega A_+(i\omega, \mathbf{q}) + (v_H \pm v_{\text{NH}} M)(\mathbf{\Gamma} \cdot \mathbf{q})(A_+(i\omega, \mathbf{q}) \right. \\ & - B(i\omega, \mathbf{q}) T_{\text{AS}} - 2\alpha^2 v_H^2 q_x^2) + \alpha v_F T_{\text{AS}} q_x (-A_+(i\omega, \mathbf{q}) + 2v_F^2 q^2) \left. \right] \\ & \times \frac{1}{[(A_-(i\omega, \mathbf{q}))^2 - (B(i\omega, \mathbf{q}))^2][(\nu - \omega)^2 + v_B^2(\mathbf{k} - \mathbf{q})^2]}, \end{aligned} \quad (\text{A.21})$$

where the upper (lower) sign corresponds to CCM (ACM) order.

A.3.1.1 Fermionic field renormalization

To calculate the fermion field renormalization, we extract the term proportional to the identity matrix in the self-energy, and we set external momentum $\mathbf{k} = 0$,

$$\Sigma_{00}(i\nu, \mathbf{k} = 0) = -g^2 n \int [d\omega] \int [d^d \mathbf{q}] \frac{i\omega A_+(i\omega, \mathbf{q})}{[A_-(i\omega, \mathbf{q}))^2 - (B(i\omega, \mathbf{q}))^2][(\nu - \omega)^2 + v_B^2 q^2]}. \quad (\text{A.22})$$

The renormalization condition reads

$$Z_\psi = 1 + \left. \frac{\partial \Sigma_{00}(i\nu, k)}{\partial(i\nu)} \right|_{i\nu=0, \mathbf{k}=0}, \quad (\text{A.23})$$

and after integrating over the Matsubara frequency, we find

$$\begin{aligned} \left. \frac{\partial \Sigma_{00}(i\nu, \mathbf{k})}{\partial(i\nu)} \right|_{i\nu=0, \mathbf{k}=0} &= -g^2 n \int [d^d \mathbf{q}] \frac{\alpha v_{\text{H}}^2 q_x^2 + q^2 (v_{\text{B}} + v_{\text{F}})^2}{2q v_{\text{B}} (\alpha^2 v_{\text{H}}^2 q_x^2 - q^2 (v_{\text{B}} + v_{\text{F}})^2)^2} \\ &= -\frac{ng^2}{(4\pi)^2} G(\alpha, v_{\text{B}}, v_{\text{F}}) \int \frac{dq}{q}, \end{aligned} \quad (\text{A.24})$$

where

$$\begin{aligned} G(\alpha, v_{\text{B}}, v_{\text{F}}) &= \int_0^{2\pi} \frac{d\phi}{\pi} \int_0^\pi d\theta \frac{\sin \theta (\alpha^2 v_{\text{H}}^2 \cos^2 \phi \sin^2 \theta + (v_{\text{B}} + v_{\text{F}})^2)}{v_{\text{B}} (-\alpha^2 v_{\text{H}}^2 \cos^2 \phi \sin^2 \theta + (v_{\text{B}} + v_{\text{F}})^2)^2} \\ &= \frac{4}{v_{\text{B}} ((v_{\text{B}} + v_{\text{F}})^2 - \alpha^2 v_{\text{H}}^2)}. \end{aligned} \quad (\text{A.25})$$

We then invoke the correspondence between hard cutoff and dimensional regularization,

$$\int \frac{dq}{q} \rightarrow \frac{1}{\epsilon} \quad (\text{A.26})$$

to find for the fermion field renormalization

$$Z_\psi = 1 - \frac{ng^2}{(4\pi)^2 \epsilon} G(\alpha, v_{\text{B}}, v_{\text{F}}). \quad (\text{A.27})$$

A.3.1.2 Renormalization of the fermionic velocities

To obtain renormalization factors for fermionic velocities, we need to extract the terms proportional to the matrices Γ in the self-energy, Eq. (A.21), and we choose the term proportional to Γ_x to find the renormalization factor for the velocity v_{H} ,

$$\Sigma_{\Gamma_x}(i\nu, k) = ng^2 \int [d\omega] \int [d^d \mathbf{q}] \left[\frac{v_{\text{H}} q_x (A_+(i\omega, \mathbf{q}) - 2\alpha^2 v_{\text{H}}^2 q_x^2)}{[A_-(i\omega, \mathbf{q})^2 - B(i\omega, \mathbf{q})^2][(\nu - \omega)^2 + v_{\text{B}}^2 (\mathbf{k} - \mathbf{q})^2]} \right]. \quad (\text{A.28})$$

We then obtain

$$\left. \frac{\partial \Sigma_{\Gamma_x}(i\nu, \mathbf{k})}{\partial(v_{\text{H}} k_x)} \right|_{i\nu=0, \mathbf{k}=0} = \frac{ng^2}{(4\pi)^2 \epsilon} J(\alpha, v_{\text{B}}, v_{\text{F}}), \quad (\text{A.29})$$

with

$$\begin{aligned}
J(\alpha, v_B, v_F) &= \frac{1}{\pi} \int [d\phi d\theta] \frac{\cos^2 \phi \sin^2 \theta (-\alpha^2 v_H^2 \cos^2 \phi \sin^2 \theta + (v_B + v_F)^2 (v_F + 2v_B))}{v_B v_F (-\alpha^2 v_H^2 \cos^2 \phi \sin^2 \theta + (v_B + v_F)^2)^2} \\
&= \left[\alpha^2 v_H (-v_B v_F^2 + v_B^2 v_F + v_B^3 + \alpha^2 v_F v_H^2 - v_F^3) \right. \\
&\quad \left. - |\alpha| (\alpha^2 v_H^2 (v_F^2 - v_B^2) + (v_B - v_F) (v_B + v_F)^3) \cosh^{-1} \left(\frac{v_B + v_F}{\sqrt{(v_B + v_F)^2 - \alpha^2 v_H^2}} \right) \right] \\
&\quad \times \frac{4}{\alpha^4 v_B v_F v_H^3 (v_B + v_F - \alpha v_H) (v_B + v_F + \alpha v_H)}. \tag{A.30}
\end{aligned}$$

The renormalization condition for the Hermitian velocity (v_H) reads

$$Z_\psi Z_{v_H} + \frac{\partial \Sigma_{\Gamma_x}(i\nu, \mathbf{k})}{\partial (v_H k_x)} \Big|_{i\nu, \mathbf{k}=0} = 1, \tag{A.31}$$

implying that

$$Z_\psi Z_{v_H} = 1 - \frac{ng^2}{(4\pi)^2 \epsilon} J(\alpha, v_B, v_F), \tag{A.32}$$

which finally yields the Z -factor for the Hermitian velocity,

$$Z_{v_H} = 1 + \frac{g^2 n}{(4\pi)^2 \epsilon} [G(\alpha, v_B, v_F) - J(\alpha, v_B, v_F)]. \tag{A.33}$$

Notice that the factors Z_ψ and Z_{v_H} are independent of the class of the mass orders.

To find the renormalization factor for the NH velocity, v_{NH} , we need to consider the terms proportional to the product ΓM in the self-energy [Eq. (A.21)],

$$\begin{aligned}
\Sigma_{\Gamma_x M}(i\nu, \mathbf{k}) &= \pm ng^2 v_{NH} \int [d\omega] \int [d^d \mathbf{q}] \left[\frac{q_x (A_+(i\omega, \mathbf{q}) - 2\alpha^2 v_H^2 q_x^2)}{[A_-(i\omega, \mathbf{q})^2 - B(i\omega, \mathbf{q})^2]} \right. \\
&\quad \left. \times \frac{1}{[(\nu - \omega)^2 + v_B^2 (\mathbf{k} - \mathbf{q})^2]} \right] \tag{A.34}
\end{aligned}$$

yielding

$$\frac{\partial \Sigma_{\Gamma_x M}(i\nu, \mathbf{k})}{\partial (v_{NH} k_x)} \Big|_{i\nu=0, \mathbf{k}=0} = \pm \frac{ng^2}{(4\pi)^2 \epsilon} J(\alpha, v_B, v_F), \tag{A.35}$$

and therefore the Z -factor for the NH velocity reads

$$Z_{v_{NH}} = 1 + \frac{g^2 n}{(4\pi)^2 \epsilon} [G(\alpha, v_B, v_F) \mp J(\alpha, v_B, v_F)]. \tag{A.36}$$

Notice that the factor $Z_{v_{\text{NH}}}$ does depend on the class of the mass order, with the upper (lower) sign corresponding to the CCM (ACM) OP.

A.3.1.3 Renormalization of the tilt parameter

To find the tilt renormalization factor, we extract the terms proportional to T_{AS} , and we set $\mathbf{k} = k_x \hat{\mathbf{e}}_x$ in the fermion self-energy in Eq. (A.21),

$$\Sigma_{T_{\text{AS}}}(i\nu, \mathbf{k}) = g^2 n \int [d\omega] \int [d^d \mathbf{q}] \frac{\alpha v_{\text{H}} q_x (-A_+(i\omega, q) + 2v_{\text{F}}^2 q^2)}{[A_-(i\omega, \mathbf{q})]^2 - (B(i\omega, \mathbf{q}))^2][(\nu - \omega)^2 + v_{\text{B}}^2 (\mathbf{k} - \mathbf{q})^2]}. \quad (\text{A.37})$$

After integrating over Matsubara frequencies, we obtain

$$\left. \frac{\partial \Sigma_{T_{\text{AS}}}(i\nu, \mathbf{k})}{\partial (v_{\text{F}} k_x)} \right|_{i\nu=0, \mathbf{k}=0} = \frac{g^2 n \alpha}{(4\pi)^2 \epsilon} I(\alpha, v_{\text{B}}, v_{\text{F}}), \quad (\text{A.38})$$

where, after the angular integrations, we obtain

$$I(\alpha, v_{\text{B}}, v_{\text{F}}) = \frac{4}{\alpha^2 v_{\text{B}} v_{\text{H}}^3} \left(\frac{\alpha^2 v_{\text{H}}^3 - v_{\text{F}} v_{\text{H}} (v_{\text{B}} + v_{\text{F}})}{(v_{\text{B}} + v_{\text{F}})^2 - \alpha^2 v_{\text{H}}^2} + \frac{v_{\text{F}}}{|\alpha|} \cosh^{-1} \left(\frac{v_{\text{B}} + v_{\text{F}}}{\sqrt{(v_{\text{B}} + v_{\text{F}})^2 - \alpha^2 v_{\text{H}}^2}} \right) \right). \quad (\text{A.39})$$

The renormalization condition for the tilt parameter reads

$$Z_{\psi} Z_{\alpha} Z_{v_{\text{H}}} \alpha v_{\text{H}} k_x + v_{\text{H}} k_x \left. \frac{\partial \Sigma_{T_{\text{AS}}}(i\nu, \mathbf{k})}{\partial (v_{\text{H}} k_x)} \right|_{i\nu, \mathbf{k}=0} = \alpha v_{\text{H}} k_x, \quad (\text{A.40})$$

and we find

$$Z_{\psi} Z_{\alpha} Z_{v_{\text{H}}} = 1 - \frac{1}{\alpha} \left. \frac{\partial \Sigma_{T_{\text{AS}}}(i\nu, \mathbf{k})}{\partial (v_{\text{H}} k_x)} \right|_{i\nu=0, k=0} = 1 - \frac{g^2 n}{(4\pi)^2 \epsilon} I(\alpha, v_{\text{B}}, v_{\text{F}}), \quad (\text{A.41})$$

which implies the form of the renormalization parameter for the tilt

$$Z_{\alpha} = 1 + \frac{g^2 n}{(4\pi)^2 \epsilon} [J(\alpha, v_{\text{B}}, v_{\text{F}}) - I(\alpha, v_{\text{B}}, v_{\text{F}})]. \quad (\text{A.42})$$

A.3.2 Bosonic self-energy

We here calculate the bosonic self-energy, with the Feynman diagram shown in Fig. 4.3(b). To facilitate the calculation, we furthermore take $\mathbf{k} = k_x \hat{\mathbf{e}}_x$, so that the self-energy reads

$$\Pi(i\nu, \mathbf{k}) = -\frac{g^2}{2} \int [d\omega] \int [d^d \mathbf{q}] \text{Tr} [N_i G_F(i\omega, \mathbf{q}) N_i G_F(i\nu + i\omega, \mathbf{k} + \mathbf{q})]. \quad (\text{A.43})$$

Using this expression for the bosonic self-energy, we subsequently calculate the bosonic field renormalization and the bosonic velocity renormalization factors for both the CCM and ACM OPs.

A.3.2.1 Bosonic field renormalization

The boson field renormalization is calculated using:

$$\left. \frac{\partial \Pi(i\nu, \mathbf{k})}{\partial \nu^2} \right|_{i\nu, \mathbf{k}=0} = -\frac{g^2 n}{4\pi^2 \epsilon} F(\alpha, v_B, v_F). \quad (\text{A.44})$$

For the CCM order, we find

$$\begin{aligned} F_{\text{CCM}}(\alpha, v_B, v_F) &= \frac{1}{4} \int [d\phi d\theta] \frac{\csc^2(\theta) \sec^2(\phi)}{16} \left(\frac{4 \sin^2(\theta) \cos^2(\phi) + 2}{v_F^3} \right. \\ &\quad \left. + \left(\frac{2 \sin^2(\theta) \cos^2(\phi) - 1}{(v_F - \alpha \sin(\theta) v_H \cos(\phi))^3} + \frac{2 \sin^2(\theta) \cos^2(\phi) - 1}{(v_F + \alpha \sin(\theta) v_H \cos(\phi))^3} \right) \right), \end{aligned} \quad (\text{A.45})$$

which further simplifies to

$$\begin{aligned} F_{\text{CCM}}(\alpha, v_B, v_F) &= \frac{1}{16v_F^3 \left(\frac{\alpha^2 v_H^2}{v_F^2} - 1 \right)^2} \left(\frac{3|\alpha|v_H (v_F^2 - \alpha^2 v_H^2)^2}{v_F^5} \log \left(\frac{2v_F}{|\alpha|v_H + v_F} - 1 \right) \right. \\ &\quad \left. - \frac{v_H^4}{v_F^4} - \frac{14\alpha^2 v_H^2}{v_F^2} + 8 \right). \end{aligned} \quad (\text{A.46})$$

For the ACM case, following the analogous steps, we obtain

$$\begin{aligned} F_{\text{ACM}}(\alpha, v_B, v_F) &= \frac{v_H}{8v_F^5 \left(\frac{\alpha^2 v_H^2}{v_F^2} - 1 \right)^2} \left(\alpha v_F \left(3 \left(\frac{\alpha^2 v_H^2}{v_F^2} - 1 \right)^2 \tanh^{-1} \left(\frac{\alpha v_H}{v_F} \right) - \frac{\alpha v_H}{v_F} \right) \right. \\ &\quad \left. + 2v_H \left(\frac{2\alpha^4 v_H^4}{v_F^4} - \frac{3\alpha^2 v_H^2}{v_F^2} - \frac{3\alpha v_H (v_F^2 - \alpha^2 v_H^2)^2}{v_F^5} \tanh^{-1} \left(\frac{\alpha v_H}{v_F} \right) + 2 \right) \right). \end{aligned} \quad (\text{A.47})$$

The renormalization condition has the form

$$Z_\phi - \left. \frac{\partial \Pi(i\nu, \mathbf{k})}{\partial \nu^2} \right|_{i\nu, \mathbf{k}=0} = 1, \quad (\text{A.48})$$

which yields

$$Z_\phi = 1 - \frac{g^2 n}{4\pi^2 \epsilon} F(\alpha, v_B, v_F). \quad (\text{A.49})$$

Notice that the function $F(\alpha, v_B, v_F)$, and therefore also the renormalization factor Z_ϕ depends on the class of the mass order. We, however, do not include this dependence in the last equation explicitly to simplify the notation.

A.3.2.2 Bosonic velocity renormalization

To find the renormalization factor for the bosonic velocity, we need the term of the order k^2 in the bosonic self-energy

$$\Pi_{v_B^2 k^2}(i\nu, \mathbf{k}) \Big|_{i\nu, \mathbf{k}=0} = -\frac{ng^2 N_F}{4\pi^2 \epsilon} C(\alpha, v_B, v_F), \quad (\text{A.50})$$

with the function depending explicitly on the class of the mass order, as we show in the following.

We first consider the CCM case, which yields

$$\begin{aligned} C_{\text{CCM}}(\alpha, v_B, v_F) = & -\frac{1}{4} \int [d\phi d\theta] \frac{1}{8v_B^2 (v_F^3 - \alpha^2 v_F \sin^2(\theta) v_H^2 \cos^2(\phi))^3} \left(v_F^8 (8 - 12 \sin^2(\theta) \cos^2(\phi)) \right. \\ & + \alpha^2 v_F^6 v_H^2 (14 \sin^4(\theta) \cos^4(\phi) + 3 \sin^2(\theta) \cos^2(\phi) - 9) \\ & + \alpha^4 v_F^4 v_H^4 (-14 \sin^6(\theta) \cos^6(\phi) + 9 \sin^4(\theta) \cos^4(\phi) + 2 \sin^2(\theta) \cos^2(\phi) + 2) \\ & + \alpha^6 v_F^2 \sin^2(\theta) v_H^6 \cos^2(\phi) (4 \sin^6(\theta) \cos^6(\phi) - 11 \sin^2(\theta) \cos^2(\phi) + 1) \\ & \left. + \alpha^8 \sin^4(\theta) v_H^8 \cos^4(\phi) (2 \sin^2(\theta) \cos^2(\phi) + 1) \right), \end{aligned} \quad (\text{A.51})$$

and after the angular integrations, we find

$$\begin{aligned} C_{\text{CCM}} = & -\frac{1}{48|\alpha|^3 v_B^2 v_F^4 v_H^3} \left(4|\alpha| v_F v_H (2\alpha^2 v_F^2 v_H^2 - 3v_F^4 + 3\alpha^4 v_H^4) \right. \\ & \left. + 3(-4\alpha^4 v_F^2 v_H^4 + 6\alpha^2 v_F^4 v_H^2 - 2v_F^6 + \alpha^6 v_H^6) \log \left(\frac{2v_F}{|\alpha| v_H + v_F} - 1 \right) \right). \end{aligned} \quad (\text{A.52})$$

Similarly, for the ACM case, we obtain

$$\begin{aligned} C_{\text{ACM}} = & \frac{1}{24\alpha^3 v_B^2 v_F^6 v_H^3} \left(6v_F^8 \tanh^{-1} \left(\frac{\alpha v_H}{v_F} \right) - 6\alpha v_F^7 v_H - 6(3\alpha^2 + 2) v_F^6 v_H^2 \tanh^{-1} \left(\frac{\alpha v_H}{v_F} \right) \right. \\ & + 4\alpha(2\alpha^2 + 3) v_F^5 v_H^3 + 12\alpha^2(\alpha^2 + 3) v_F^4 v_H^4 \tanh^{-1} \left(\frac{\alpha v_H}{v_F} \right) - 6\alpha^3(\alpha^2 + 2) v_F^3 v_H^5 \\ & \left. - 6\alpha^6 v_H^8 \tanh^{-1} \left(\frac{\alpha v_H}{v_F} \right) - 3\alpha^4(\alpha^2 + 8) v_F^2 v_H^6 \tanh^{-1} \left(\frac{\alpha v_H}{v_F} \right) \right). \end{aligned} \quad (\text{A.53})$$

The renormalization condition for the bosonic reads

$$Z_\phi Z_{v_B^2} - \Pi_{v_B^2 k^2}(i\nu, \mathbf{k}) \Big|_{i\nu, \mathbf{k}=0} = 1, \quad (\text{A.54})$$

which yields

$$Z_\phi Z_{v_B^2} = 1 - \frac{ng^2 N_F}{4\pi^2 \epsilon} C(\alpha, v_B, v_F). \quad (\text{A.55})$$

Finally, we find the renormalization factor for the bosonic velocity

$$Z_{v_B^2} = 1 + \frac{ng^2 N_F}{4\pi^2 \epsilon} [F(\alpha, v_B, v_F) - C(\alpha, v_B, v_F)]. \quad (\text{A.56})$$

Again, both functions $F(\alpha, v_B, v_F)$ and $C(\alpha, v_B, v_F)$ depend on the class of the mass order, implying the same feature for the renormalization factor $Z_{v_B^2}$, but we do not include this dependence here explicitly to simplify the notation.

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