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Graviton Compton Scattering and the DHG Sum Rule

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Resumen

La regla de suma DHG relaciona la resta de las secciones eficaces paralela y antiparalela para un proceso del tipo $a\gamma \rightarrow bc$, (donde a, b y c pueden ser cualquier partícula no escalar, por ejemplo: leptones, quarks, fotones, gluones, bosones vectoriales, etc.) con el momento magnético anómalo de la partícula a, entonces esta regla no solo sirve para calcular momentos magnéticos sino que también como una verificación de consistencia entre el resultado clásico y el cuántico de esta regla, como se vio en [1]. Si aplicamos esta regla de suma al proceso $e^-\gamma \rightarrow e^-\gamma$ el resultado clásico sería cero, y con correcciones de loop obtendríamos el momento magnético anómalo del electrón. En esta investigación aplicamos la regla de suma DHG al proceso $e^-g \rightarrow e^-g$ en el contexto de gravedad linealizada, y vemos que el proceso no es unitario aún a nivel de árbol.

Abstract

The DHG sum rule relates the difference between the parallel and antiparallel cross sections of a process of the type $a\gamma \to bc$, (where a, b and c can be any non-scalar particles i.e. leptons, quarks, photons, gluons, vector bosons, etc.), with the anomalous magnetic moment of a, so we can use this rule not only to calculate anomalous magnetic moments but also serves as a consistency check between the classical and quantum results of this rule as seen in [1]. If we apply this rule to the process $e^-\gamma \to e^-\gamma$ the classical result would be zero and with loop corrections we would get the anomalous magnetic moment of the electron. In this investigation we apply the DHG rule to the process $e^-g \to e^-g$ in the context of linearized gravity, without loop corrections, and we see that the process violates unitarity even at tree level.

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Quiero agradecer a todos mis profesores que aportaron en mi educación hasta este momento y un especial agradecimiento a mi profesora de física en el colegio, quién es la que me introdujo a los diversos formalismos de la física.

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Contents

Glossary

Symbol	Meaning
$\eta_{\mu\nu}$	Minkowski metric with signature (+,-,-,-)
$g_{\mu\nu}$	Full metric
$I_{\alpha\beta\mu\nu}$	The identity for symmetric rank 2 tensors i.e. $\frac{1}{2}\left(\eta_{\alpha\mu}\eta_{\beta\nu} + \eta_{\beta\mu}\eta_{\alpha\nu}\right)$
$P_{\alpha\beta\mu\nu}$	$\frac{1}{2}\left(\eta_{\alpha\mu}\eta_{\beta\nu} + \eta_{\beta\mu}\eta_{\alpha\nu} - \eta_{\mu\nu}\eta_{\alpha\beta}\right)$

Chapter 1

Introduction

1.1 Electron's anomalous magnetic moment

The anomalous magnetic moment is the contribution to the magnetic moment of a particle due to loop corrections of the QED vertex. For the case of the electron, it has been measured to incredible precision [3], making it the most accurate prediction in the history of physics. One way to measure it is through the Compton scattering and the DHG sum rule as we will discuss in the following sections.

1.2 Compton Scattering

Compton scattering refers to the process $e^-\gamma \to e^-\gamma$ and has the following diagrams,

(a) (b)

Figure 1.1: Feynman diagrams for Compton scattering

To construct the amplitudes we need the fermion propagator and the QED vertex,

$$D(k) = \frac{i(\not{k} + m)}{k^2 - m^2}$$

$$\tau_\mu = iQ\gamma_\mu$$

From these diagrams we can construct both of the amplitudes involved in this process,

$$
\begin{aligned}
i\mathcal{M}_a &= \bar{u}(p_3)\epsilon_4^{\mu *}\tau_\mu D(k)\tau_\nu \epsilon_2^\nu u(p_1) \\
&= \bar{u}(p_3)\epsilon_4^{\mu *}(-ie\gamma_\mu)\frac{i(\not{k}+m)}{k^2-m^2}(-ie\gamma_\nu)\epsilon_2^\nu u(p_1) \\
&= -i\bar{u}(p_3)\not{\epsilon}_4^*\frac{(\not{p}_1+\not{k}_2+m)}{s^2-m^2}\not{\epsilon}_2 u(p_1)
\end{aligned}
\tag{1.1}
$$

$$
\begin{aligned}
i\mathcal{M}_b &= \bar{u}(p_3)\epsilon_2^{\nu}\tau_\nu D(k)\tau_\mu \epsilon_4^{\mu *} u(p_1) \\
&= \bar{u}(p_3)\epsilon_2^{\nu}(-ie\gamma_\nu)\frac{i(\not{k}+m)}{k^2-m^2}(-ie\gamma_\mu)\epsilon_4^{\mu *} u(p_1) \\
&= -i\bar{u}(p_3)\not{\epsilon}_2\frac{(\not{p}_1-\not{k}_4+m)}{u^2-m^2}\not{\epsilon}_4^* u(p_1)
\end{aligned}
\tag{1.2}
$$

Now we need to know the kinematics of the process to calculate the cross section. We have a collision of a massive particle and a massless one, in the centre-of-mass frame we have

$$
\begin{aligned}
p_1^\mu &= \left(\frac{\sqrt{s}}{2}+\frac{m_e^2}{2\sqrt{s}}, 0, 0, \frac{\sqrt{s}}{2}-\frac{m_e^2}{2\sqrt{s}}\right), \\
k_2^\mu &= \left(\frac{\sqrt{s}}{2}-\frac{m_e^2}{2\sqrt{s}}, 0, 0, \frac{m_e^2}{2\sqrt{s}}-\frac{\sqrt{s}}{2}\right), \\
p_3^\mu &= \left(\frac{\sqrt{s}}{2}+\frac{m_e^2}{2\sqrt{s}}, \left(\frac{\sqrt{s}}{2}-\frac{m_e^2}{2\sqrt{s}}\right)\sin{(\theta_{CM})}, 0, \left(\frac{\sqrt{s}}{2}-\frac{m_e^2}{2\sqrt{s}}\right)\cos{(\theta_{CM})}\right), \\
k_4^\mu &= \left(\frac{\sqrt{s}}{2}+\frac{m_e^2}{2\sqrt{s}}, \left(\frac{m_e^2}{2\sqrt{s}}-\frac{\sqrt{s}}{2}\right)\sin{(\theta_{CM})}, 0, \left(\frac{m_e^2}{2\sqrt{s}}-\frac{\sqrt{s}}{2}\right)\cos{(\theta_{CM})}\right).
\end{aligned}
\tag{1.3}
$$

We can see that all the components of the total tri-momentum are zero and the total energy is $\sqrt{s}$. In the lab frame we have,

$$
\begin{aligned}
p_1^\mu &= (m_e, 0, 0, 0), \\
k_2^\mu &= (\nu, 0, 0, \nu), \\
p_3^\mu &= (E_e, |\vec{p}_3|\sin{(\phi)}, 0, |\vec{p}_3|\cos{(\phi)}), \\
k_4^\mu &= (\nu', \nu'\sin{(\theta_{Lab})}, 0, \nu'\cos{(\theta_{Lab})}).
\end{aligned}
\tag{1.4}
$$

Were ϕ and θ_{Lab} are the scattering angles of the electron and the graviton respectively. With this and the conservation of momentum we can obtain the relation between ν and ν', a relation that we will need for later,

$$\begin{aligned}
p_1^\mu + k_2^\mu &= p_3^\mu + k_4^\mu,\\
(p_1^\mu - p_3^\mu)^2 &= (k_4^\mu - k_2^\mu)^2,\\
2m_e^2 - 2p_1\cdot p_3 &= -2k_2\cdot k_4,\\
2m_e^2 - 2m_eE_e &= -2\nu\nu'(1-\cos(\theta)),\\
m_e(m_e - E_e) &= \nu\nu'(\cos(\theta)-1).
\end{aligned}$$

Using the conservation of energy we have $m_e - E_e = \nu' - \nu$,

$$\begin{aligned}
m_e(\nu' - \nu) &= \nu\nu'(\cos(\theta)-1),\\
\frac{\nu'-\nu}{\nu'\nu} &= \frac{1}{m_e}(\cos(\theta)-1),\\
\frac{1}{\nu} - \frac{1}{\nu'} &= \frac{1}{m_e}(\cos(\theta)-1),\\
\frac{1}{\nu'} &= \frac{1}{m_e}(1-\cos(\theta)) + \frac{1}{\nu},\\
\frac{1}{\nu'} &= \frac{m_e + \nu(1-\cos(\theta))}{m_e\nu},\\
\nu' &= \frac{m_e\nu}{m_e + \nu(1-\cos(\theta))}.
\end{aligned} \tag{1.5}$$

1.3 DHG sum rule

The DHG sum rule [4] [5] is an expression that relates the parallel and antiparallel polarized cross section of processes of the type $a\gamma \to bc$ with the anomalous magnetic moment a,

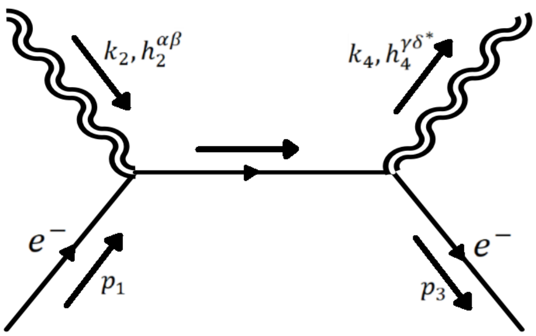
$$\mu_a^2 = \frac{4\pi\alpha S}{m^2}(g-2)^2 = \frac{S}{\pi}\int_{v_{th}}^{\infty}\frac{d\nu}{\nu}(\sigma_P - \sigma_A), \tag{1.6}$$

where α, μ_a^2, S and ν are the fine structure constant, the anomalous magnetic moment of a, the spin of a, and the energy of the photon in the lab frame, respectively. This gives a way of measuring μ_a, and calculate it to an arbitrary level of precision using loop corrections on the amplitudes of the Compton scattering, but also as explained in [1] it gives a way to check the consistency of the theory because at tree level 1.6 has to be zero. This was done for the electroweak theory in [6] and extended to the standard model in [1].

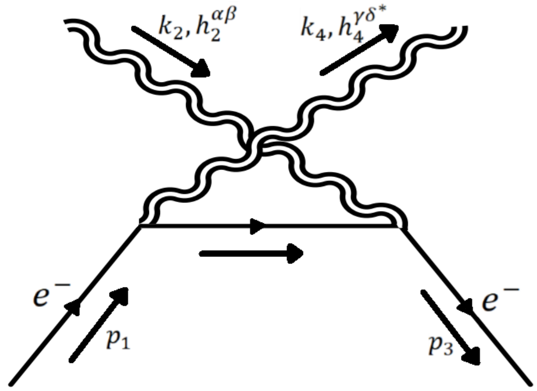
Also, it is worth mentioning what the parallel and antiparallel polarizations mean. The parallel configuration means that the starting helicities of the particles have the same sign (in this case +2 and $+\frac{1}{2}$ or -2 and -$\frac{1}{2}$). In the antiparallel case the starting polarizations have opposite signs (+2 and -$\frac{1}{2}$ or -2 and $+\frac{1}{2}$).

1.4 Graviton Compton scattering

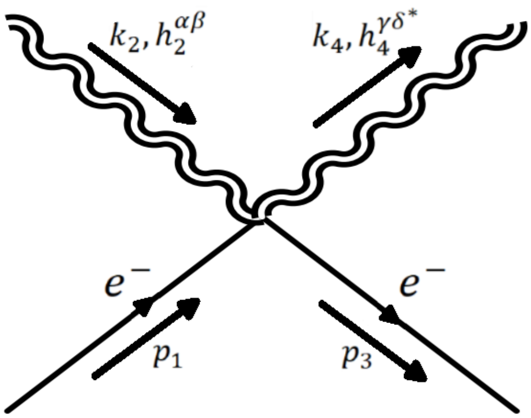
In the graviton Compton scattering is the same as the normal Compton scattering, but instead of photons we use gravitons, so the diagrams for this scattering are [7],



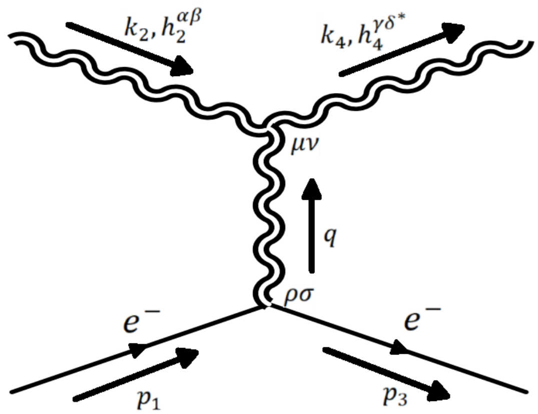
(a)



(b)



(c)



(d)

Figure 1.2: Feynman diagrams for graviton Compton scattering

We see that the process with gravitons shares two diagrams with its photon counterpart plus two new diagrams due to the existence of a vertex with two and three gravitons. The motivation for considering this process is to use it in analogy to the normal Compton scattering in the DHG sum rule to obtain an "anomalous gravitational moment" that could tell us about the interaction of matter and gravity at the subatomic scale.

Chapter 2

Theoretical Framework

2.1 CM and Lab frame cross section

When calculating a differential cross section, it is easier to do so in the centre-of-mass frame, and that is what we did for our calculations, but the DHG sum rule works in the lab frame so we need a relation between a differential cross section in the CM frame and one in the lab frame.

We start from the fact that the number of scattered particles passing through an infinitesimal cross section $d\sigma$ is the same in both frames [8] i.e. $d\sigma(\theta_{CM}, \varphi_{CM}) = d\sigma(\theta_{Lab}, \varphi_{Lab})$ but the solid angle $d\Omega$ does change between frames. From this we can say,

$$\begin{aligned}
\frac{d\sigma}{d\Omega_{Lab}} d\Omega_{Lab} &= \frac{d\sigma}{d\Omega_{CM}} d\Omega_{CM}, \\
\frac{d\sigma}{d\Omega_{Lab}} &= \frac{d\sigma}{d\Omega_{CM}} \frac{d\Omega_{CM}}{d\Omega_{Lab}}, \\
\frac{d\sigma}{d\Omega_{Lab}} &= \frac{d\sigma}{d\Omega_{CM}} \frac{d\cos(\theta_{CM})}{d\cos(\theta_{Lab})} \frac{\varphi_{CM}}{\varphi_{Lab}}.
\end{aligned}$$

Since we have cylindrical symmetry around the beam, $\varphi_{CM} = \varphi_{Lab}$, so we need to find what $\frac{d\cos(\theta_{CM})}{d\cos(\theta_{Lab})}$ is. For that we will use the fact that the Mandelstam variables are invariant under frame transformations. Using (1.3), (1.4) and (1.5) we have,

$$\begin{aligned}
t_{CM} &= t_{Lab}, \\
2\left(\frac{\sqrt{s}}{2} - \frac{m_e^2}{2\sqrt{s}}\right)^2 (\cos(\theta_{CM}) - 1) &= 2\nu\nu'(\cos(\theta_{Lab}) - 1), \\
\frac{(s - m^2)^2}{4s} (\cos(\theta_{CM}) - 1) &= \frac{m_e \nu^2 (\cos(\theta_{Lab}) - 1)}{m_e + \nu(1 - \cos(\theta_{Lab}))}, \\
\cos(\theta_{CM}) &= \frac{4s\nu^2}{(s - m^2)^2} \frac{m_e(\cos(\theta_{Lab}) - 1)}{m_e + \nu(1 - \cos(\theta_{Lab}))} + 1.
\end{aligned}$$

Now we calculate the derivative,

$$\frac{d\cos(\theta_{CM})}{d\cos(\theta_{Lab})} = \frac{4s\nu^2}{(s-m^2)^2}\frac{m_e^2}{(m_e+\nu(1-\cos(\theta_{Lab}))^2},$$

and using $s_{Lab} = m_e^2 + 2m_e\nu$,

$$\begin{aligned}\frac{d\cos(\theta_{CM})}{d\cos(\theta_{Lab})} &= \frac{4(m_e^2+2m_e\nu)\nu^2}{4m_e^2\nu^2}\frac{m_e^2}{(m_e+\nu(1-\cos(\theta_{Lab}))^2},\\ \frac{d\cos(\theta_{CM})}{d\cos(\theta_{Lab})} &= \left(1+\frac{2\nu}{m_e}\right)\frac{m_e^2}{(m_e+\nu(1-\cos(\theta_{Lab}))^2}.\end{aligned} \tag{2.1}$$

2.2 Graviton formalism.

Before we start with the gravity-matter formalism, it would be appropriate to justify why we know that the gravity force carrier has the properties that it has i.e. massless and spin 2.

First the easy one: masslessness, we know that the influence of gravity is infinite, and that can only happen with a massless particle.

Now for spin 2, we have a couple of constraints: first we know that we can't have a consistent and interactive quantum field theory with a massless particle with spin greater that 2 [9], and we also know that gravity is always attractive and that can only happen with a force carrier with even spin [10]. That leaves us with two candidates spin 0 and spin 2.

Let us explore the spin 0 case, because a scalar field doesn't have indices, the only way to couple it with the stress-energy tensor would be through its trace, but the electromagnetic stress-energy tensor is traceless, so photons wouldn't feel the gravitational force, but we know they do, so the only option is for gravitons to have spin 2.

In the linearized gravity formalism, the graviton field $h_{\mu\nu}$ is defined as a first order perturbation over the background metric,

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu},$$

and we can write the inverse metric up to first order as $g^{\mu\nu} = \eta^{\mu\nu} - \kappa h^{\mu\nu}$. Knowing the graviton field we would like to know its gauge transformation structure.

General relativity is invariant under coordinate transformations, from this we will obtain the gauge transformation for $h_{\mu\nu}$.

Consider the following transformation,

$$x^{\mu\prime} = x^\mu + \theta^\mu(x), \tag{2.2}$$

where θ^μ is a small parameter. Under this transformation, we know that the metric transforms as,

$$g'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x^{\mu\prime}}\frac{\partial x^\beta}{\partial x^{\nu\prime}} g_{\alpha\beta},$$

and from (2.2) we can see that,

$$\frac{\partial x^\alpha}{\partial x^{\mu\prime}} = \delta^\alpha_\mu - \frac{\partial \theta^\alpha}{\partial x^{\mu\prime}}.$$

Since θ^μ is small we have,

$$\frac{\partial \theta^\alpha}{\partial x^{\mu\prime}} \approx \frac{\partial \theta^\alpha}{\partial x^{\mu}} = \partial_\mu \theta^\alpha,$$

so the metric transforms (up to first order) as,

$$\begin{aligned} g'_{\mu\nu} &= \frac{\partial x^\alpha}{\partial x^{\mu\prime}} \frac{\partial x^\beta}{\partial x^{\nu\prime}} g_{\alpha\beta}, \\ &= \left(\delta^\alpha_\mu - \frac{\partial \theta^\alpha}{\partial x^{\mu\prime}}\right)\left(\delta^\beta_\nu - \frac{\partial \theta^\beta}{\partial x^{\mu\prime}}\right) g_{\alpha\beta}, \\ &= g_{\mu\nu} - \left(\delta^\alpha_\mu \partial_\nu \theta^\beta + \delta^\beta_\nu \partial_\mu \theta^\alpha\right) g_{\alpha\beta}, \\ &= g_{\mu\nu} - \partial_\nu \theta_\mu - \partial_\mu \theta_\nu. \end{aligned}$$

from this we read

$$h'_{\mu\nu} = h_{\mu\nu} - \partial_\nu \theta_\mu - \partial_\mu \theta_\nu.$$

Now that we have found that $h_{\mu\nu}$ is a gauge field, we introduce a gauge condition, the de Donder gauge [11],

$$g^{\mu\nu}\Gamma^\lambda_{\mu\nu} = 0$$

or, written in terms of $h_{\mu\nu}$,

$$P^{\alpha\beta\gamma\delta}\partial_\alpha h_{\gamma\delta} = \partial_\gamma h^{\gamma\beta} - \frac{1}{2}\partial^\beta h = 0.$$

Before we start with the field equations for $h_{\mu\nu}$, it would be adequate to discuss the degrees of freedom of the field. We start with the 10 degrees of freedom that a symmetric, rank 2 tensor in 4D has, but we have 4 restrictions due to the de Donder gauge. It turns out that the gauge transformation still leaves some freedom, because we can have two fields satisfying the de Donder gauge, and still differ by a vector θ_μ as long as $\Box\theta_\mu = 0$, so that's 4 more restrictions, that leaves us with 2 degrees of freedom for the graviton, and therefore, two helicity configurations ($\pm$ 2).

Now let us find the field equations for $h_{\mu\nu}$ using Einstein's field equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \kappa T_{\mu\nu}. \tag{2.3}$$

We calculate the connection and the Ricci tensor in the first order approximation,

$$\Gamma^\rho_{\lambda\nu} = \frac{1}{2}g^{\mu\rho}\left(\partial_\lambda g_{\mu\nu} + \partial_\nu g_{\lambda\mu} - \partial_\mu g_{\nu\lambda}\right), \tag{2.4}$$

$$R_{\mu\nu} = \partial_\rho \Gamma_\mu{}^\rho{}_\nu - \partial_\mu \Gamma_\rho{}^\rho{}_\nu + \Gamma_\rho{}^\rho{}_\alpha \Gamma_\mu{}^\alpha{}_\nu - \Gamma_\mu{}^\rho{}_\alpha \Gamma_\rho{}^\alpha{}_\nu,$$

For the connection we have

$$\Gamma^{\rho}_{\lambda\nu} = \frac{1}{2}(\eta^{\mu\rho} - \kappa h^{\mu\rho})\kappa\left(\partial_\lambda h_{\mu\nu} + \partial_\nu h_{\lambda\mu} - \partial_\mu h_{\nu\lambda}\right),$$
$$= \frac{\kappa}{2}\left(\partial_\lambda h^{\rho}{}_{\nu} + \partial_\nu h_\lambda{}^{\rho} - \partial^\rho h_{\nu\lambda}\right) + \mathcal{O}(\kappa^2).$$

For the Ricci tensor we immediately see that the last two terms will be of higher order, with this in mind the Ricci tensor, up to first order in κ, is

$$R_{\mu\nu} = \frac{\kappa}{2}\partial_\rho\left(\partial_\nu h^{\rho}{}_{\mu} + \partial_\mu h_\nu{}^{\rho} - \partial^\rho h_{\mu\nu}\right) - \frac{\kappa}{2}\partial_\mu\left(\partial_\rho h^{\rho}{}_{\nu} + \partial_\nu h_\rho{}^{\rho} - \partial^\rho h_{\nu\rho}\right) + \mathcal{O}(\kappa^2),$$
$$= \frac{\kappa}{2}\left(\partial_\rho\partial_\nu h^{\rho}{}_{\mu} + \partial_\rho\partial_\mu h_\nu{}^{\rho} - \Box h_{\mu\nu} - \partial_\mu\partial_\nu h\right) + \mathcal{O}(\kappa^2),$$

and the Ricci scalar is,

$$R = g^{\mu\nu}R_{\mu\nu} = (\eta^{\mu\nu} - \kappa h^{\mu\nu})\frac{\kappa}{2}\left(\partial_\rho\partial_\nu h^{\rho}{}_{\mu} + \partial_\rho\partial_\mu h_\nu{}^{\rho} - \Box h_{\mu\nu} - \partial_\mu\partial_\nu h\right) + \mathcal{O}(\kappa^2),$$
$$= \kappa\left(\partial_\rho\partial_\mu h^{\rho\mu} - \Box h\right) + \mathcal{O}(\kappa^2).$$

Before constructing the field equations for $h_{\mu\nu}$ we will demonstrate that $R_{\mu\nu}$ and R are gauge invariant, starting with the former,

$$R'_{\mu\nu} = \frac{\kappa}{2}\left(\partial_\rho\partial_\nu h'^{\rho}{}_{\mu} + \partial_\rho\partial_\mu h'_\nu{}^{\rho} - \Box h'_{\mu\nu} - \partial_\mu\partial_\nu h'\right),$$
$$= \frac{\kappa}{2}\left(\partial_\rho\partial_\nu h^{\rho}{}_{\mu} - \partial_\rho\partial_\nu\partial^\rho\theta_\mu - \partial_\rho\partial_\nu\partial_\mu\theta^\rho + \partial_\mu\partial_\mu h_\nu{}^{\rho} - \partial_\rho\partial_\mu\partial^\rho\theta_\nu - \partial_\rho\partial_\mu\partial_\nu\theta^\rho\right.$$
$$\left.- \Box h_{\mu\nu} + \partial_\mu\Box\theta_\nu + \partial_\nu\Box\theta_\mu - \partial_\mu\partial_\nu h + 2\partial_\mu\partial_\nu\partial^\rho\theta_\rho\right),$$
$$= \frac{\kappa}{2}\left(\partial_\rho\partial_\nu h^{\rho}{}_{\mu} + \partial_\rho\partial_\mu h_\nu{}^{\rho} - \Box h_{\mu\nu} - \partial_\mu\partial_\nu h\right),$$
$$= R_{\mu\nu}.$$

Now for the Ricci scalar,

$$R' = \kappa\left(\partial_\rho\partial_\mu h'^{\rho\mu} - \Box h'\right),$$
$$= \kappa\left(\partial_\rho\partial_\mu h^{\rho\mu} - \partial_\rho\partial_\mu\partial_\mu\theta^\rho - \partial_\rho\partial_\mu\partial_\rho\theta^\mu - \Box h + 2\Box\partial_\mu\theta^\mu\right),$$
$$= \kappa\left(\partial_\rho\partial_\mu h^{\rho\mu} - \Box h\right),$$
$$= R.$$

With this we ,construct the LHS of (2.3) and apply the harmonic gauge,

$$
\begin{aligned}
R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R &= \frac{\kappa}{2}\left(\partial_\rho\partial_\nu h^\rho{}_\mu + \partial_\rho\partial_\mu h_\nu{}^\rho - \Box h_{\mu\nu} - \partial_\mu\partial_\nu h\right) - \frac{\kappa}{2}(\eta_{\mu\nu} + \kappa h_{\mu\nu})\left(\partial_\rho\partial_\sigma h^{\rho\sigma} - \Box h\right), \\
&= -\frac{\kappa}{2}(\Box h_{\mu\nu} + \eta_{\mu\nu}\left(\partial_\rho\partial_\sigma h^{\rho\sigma} - \Box h\right)) + \frac{\kappa}{2}\left(\partial_\rho\partial_\nu h^\rho{}_\mu + \partial_\rho\partial_\mu h_\nu{}^\rho - \partial_\mu\partial_\nu h\right), \\
&\overset{\text{gauge}}{=} -\frac{\kappa}{2}\Box\left(h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h\right).
\end{aligned}
$$

The field equations for $h_{\mu\nu}$ are,

$$
\Box\left(h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h\right) = -2T_{\mu\nu}. \tag{2.5}
$$

2.2.1 Dirac Lagrangian in curved space time

Before we can write the Dirac Lagrangian we need to talk about the objects called vierbein, they are a set of basis vectors for the tangent space defined locally on a manifold, and serve to translate objects defined in the general coordinates to this local coordinates i.e. $V^a(x) = e^a{}_\mu V^\mu(x)$ and vice versa. The vierbein satisfy the following properties

$$
\begin{aligned}
e^a{}_\mu e^b{}_\nu \eta_{ab} = g_{\mu\nu}, \quad e^a{}_\mu e_{a\nu} = g_{\mu\nu}, \\
e^{a\mu} e_{b\mu} = \delta^a_b, \quad e^{a\mu} e_a{}^\nu = g^{\mu\nu}.
\end{aligned} \tag{2.6}
$$

It is worth mentioning that the Latin indices correspond to the local coordinates and are raised and lowered by η_{ab} and the Greek indices correspond to the general coordinates and are raised and lowered by $g_{\mu\nu}$.

To have a covariant derivative for the objects in the local coordinates we define the following,

$$
D_\mu V^a = \partial_\mu V^a + \omega_\mu{}^a{}_b V^b,
$$

where $\omega_\mu{}^a{}_b$ is the spin connection.

To remain consistent with the covariant derivative of V in the global coordinates we impose that the derivative also transforms in the manner seen above i.e. $D_\mu V^a = e^a{}_\nu D_\mu V^\nu$. For this to be the case, it is easy so see that $D_\mu e^a{}_\nu = 0$ must hold. Expanding this we have,

$$
D_\mu e^a{}_\nu = \partial_\mu e^a{}_\nu - \Gamma^\lambda_{\mu\nu} e^a{}_\lambda + \omega_\mu{}^a{}_b e^b{}_\nu = 0, \tag{2.7}
$$

with this equation and (2.4) we can get an expression for the spin connection in terms of the vierbein.

First we write (2.4) using the properties in (2.6),

$$\begin{aligned}\Gamma^{\lambda}_{\mu\nu} &= \frac{1}{2}g^{\lambda\rho}\left(\partial_\mu(e^c{}_\rho e_{c\nu}) + \partial_\nu(e^d{}_\rho e_{d\mu}) - \partial_\rho(e^f{}_\mu e_{f\nu})\right), \\ &= \frac{1}{2}g^{\lambda\rho}\left(e^c{}_\rho\partial_\mu e_{c\nu} + e_{c\nu}\partial_\mu e^c{}_\rho + e^d{}_\rho\partial_\nu e_{d\mu} + e_{d\mu}\partial_\nu e^d{}_\rho - e^f{}_\mu\partial_\rho e_{f\nu} - e_{f\nu}\partial_\rho e^f{}_\mu\right), \\ &= \frac{1}{2}g^{\lambda\rho}\left(e^c{}_\rho(\partial_\mu e_{c\nu} + \partial_\nu e_{c\mu}) + e^d{}_\nu(\partial_\mu e_{d\rho} - \partial_\rho e_{d\mu}) + e^f{}_\mu(\partial_\nu e_{f\rho} - \partial_\rho e_{f\nu})\right).\end{aligned}$$

Using this in (2.7) we have,

$$\begin{aligned}0 &= \omega_\mu{}^a{}_b e^b{}_\nu + \partial_\mu e^a{}_\nu - e^a{}_\lambda\frac{1}{2}g^{\lambda\rho}\left(e^c{}_\rho(\partial_\mu e_{c\nu} + \partial_\nu e_{c\mu}) + e^d{}_\nu(\partial_\mu e_{d\rho} - \partial_\rho e_{d\mu}) + e^f{}_\mu(\partial_\nu e_{f\rho} - \partial_\rho e_{f\nu})\right), \\ \omega_\mu{}^a{}_b e^b{}_\nu &= \frac{1}{2}e^{a\rho}\left(e^c{}_\rho(\partial_\mu e_{c\nu} + \partial_\nu e_{c\mu}) + e^d{}_\nu(\partial_\mu e_{d\rho} - \partial_\rho e_{d\mu}) + e^f{}_\mu(\partial_\nu e_{f\rho} - \partial_\rho e_{f\nu})\right) - \partial_\mu e^a{}_\nu, \\ \omega_\mu{}^a{}_b e^b{}_\nu e_g{}^\nu &= \frac{1}{2}e^{a\rho}e_g{}^\nu\left(e^c{}_\rho(\partial_\mu e_{c\nu} + \partial_\nu e_{c\mu}) + e^d{}_\nu(\partial_\mu e_{d\rho} - \partial_\rho e_{d\mu}) + e^f{}_\mu(\partial_\nu e_{f\rho} - \partial_\rho e_{f\nu})\right) - e_g{}^\nu\partial_\mu e^a{}_\nu, \\ \omega_\mu{}^a{}_g &= \frac{1}{2}e_g{}^\nu(-\partial_\mu e^a{}_\nu + \partial_\nu e^a{}_\mu) + \frac{1}{2}e^{a\rho}(\partial_\mu e_{g\rho} - \partial_\rho e_{g\mu}) + \frac{1}{2}e^{a\rho}e_g{}^\nu e^f{}_\mu(\partial_\nu e_{f\rho} - \partial_\rho e_{f\nu}).\end{aligned}$$

Rearranging this we have,

$$\omega_{\mu ab} = \frac{1}{2}e_a{}^\nu\left(\partial_\mu e_{b\nu} - \partial_\nu e_{b\mu}\right) - \frac{1}{2}e_b{}^\nu\left(\partial_\mu e_{a\nu} - \partial_\nu e_{a\mu}\right) + \frac{1}{2}e_a{}^\rho e_b{}^\sigma\left(\partial_\sigma e_{c\rho} - \partial_\rho e_{c\sigma}\right)e_\mu{}^c. \tag{2.8}$$

We can couple spinors with our gauge field, following the standard procedure in any gauge theory [12],

$$D_\mu\psi = \partial_\mu\psi + \frac{i}{4}\sigma^{ab}\omega_{\mu ab}\psi,$$

where $\sigma^{ab} = \frac{i}{2}\left[\gamma^a, \gamma^b\right]$.

We now define the gamma matrices in the local coordinates and use the vierbein to have it in the global coordinates, so the Dirac operator is,

$$\not{D} = \gamma^\mu D_\mu = \gamma^a e_a{}^\mu D_\mu.$$

Finally starting with the Dirac Lagrangian in flat space time,

$$\mathcal{L}_m = \frac{i}{2}\bar{\psi}\gamma^\mu\partial_\mu\psi - \frac{i}{2}\left(\partial_\mu\bar{\psi}\right)\gamma^\mu\psi - m\bar{\psi}\psi,$$

we get the Dirac Lagrangian in curved space time,

$$\sqrt{e}\mathcal{L}_m = \sqrt{e}\left(\frac{i}{2}\bar{\psi}\gamma^a e_a{}^\mu D_\mu\psi - \frac{i}{2}\left(D_\mu\bar{\psi}\right)e_a{}^\mu\gamma^a\psi - m\bar{\psi}\psi\right), \tag{2.9}$$

with $e = \det\left(e^a{}_\mu\right)$

2.3 Feynman rules and propagator

In the first part of this section we will calculate the one graviton and two graviton vertices expanding the Dirac Lagrangian to first and second order respectively, and in the second part we will deduce a propagator for the graviton.

2.3.1 Vertices

In the previous section we argued that the Dirac Lagrangian in curved spacetime is,

$$\sqrt{e}\mathcal{L}_m = \sqrt{e}\left(\frac{i}{2}\bar{\psi}\gamma^a {e_a}^\mu D_\mu\psi - \frac{i}{2}\left(D_\mu\bar{\psi}\right){e_a}^\mu\gamma^a\psi - m\bar{\psi}\psi\right).$$

To obtain the vertices we need to expand the vierbein and their related quantities in powers of κ as seen in [13],

$$\begin{aligned} {e^a}_\mu &= {\delta^a}_\mu + \frac{\kappa}{2}{h^a}_\mu - \frac{\kappa^2}{8}h_{\mu\lambda}h^{\lambda a} + \mathcal{O}\left(\kappa^3\right), \\ {e_a}^\mu &= {\delta_a}^\mu - \frac{\kappa}{2}{h_a}^\mu + \frac{3\kappa^2}{8}h^{\mu\lambda}h_{\lambda a} + \mathcal{O}\left(\kappa^3\right). \end{aligned} \tag{2.10}$$

$$e := \det\left({e^a}_\mu\right) = 1 + \frac{\kappa}{2}{h_\mu}^\mu + \frac{\kappa^2}{8}\left({h_\mu}^\mu {h_\nu}^\nu - 2{h_\mu}^\nu {h_\nu}^\mu\right) + \mathcal{O}\left(\kappa^3\right),$$

$$\sqrt{e} = 1 + \frac{\kappa}{4}{h_\mu}^\mu + \frac{\kappa^2}{32}\left({h_\mu}^\mu {h_\nu}^\nu - 4{h_\mu}^\nu {h_\nu}^\mu\right) + \mathcal{O}\left(\kappa^3\right). \tag{2.11}$$

Using this we see that at order κ^0 in the expansion ${e_a}^\mu = \delta_a^\mu$, meaning $\omega_{\mu ab} = 0$ so we have $D_\mu\psi = \partial_\mu\psi$. The lagrangian at this order, $\mathcal{L}_m^{(0)}$ is,

$$\sqrt{e}\mathcal{L}_m^{(0)} = \frac{i}{2}\bar{\psi}\gamma^\mu\partial_\mu\psi - \frac{i}{2}\left(\partial_\mu\bar{\psi}\right)\gamma^\mu\psi - m\bar{\psi}\psi,$$

which is the Dirac Lagrangian in flat spacetime, as expected.

Now, for order κ^1 in the expansion, we will calculate $\omega_{\mu ab}$ as seen in 2.8 term by term,

$$\begin{aligned} A &= \frac{1}{2}{e_a}^\nu\left(\partial_\mu e_{b\nu} - \partial_\nu e_{b\mu}\right), \\ &= \frac{1}{2}(\delta_a^\nu - \frac{\kappa}{2}h_a^\nu)\left(\partial_\mu(\delta_{b\nu} + \frac{\kappa}{2}h_{\nu b}) - \partial_\nu(\delta_{b\mu} + \frac{\kappa}{2}h_{\mu b}\right), \\ &= \frac{1}{2}(\delta_a^\nu - \frac{\kappa}{2}h_a^\nu)\left(\frac{\kappa}{2}\partial_\mu h_{\nu b} - \frac{\kappa}{2}\partial_\nu h_{\mu b}\right), \\ &= \frac{\kappa}{4}\left(\partial_\mu h_{ab} - \partial_a h_{\mu b}\right). \end{aligned} \tag{2.12}$$

$$
\begin{aligned}
B &= \frac{1}{2} e_b{}^{\nu} \left(\partial_\mu e_{a\nu} - \partial_\nu e_{a\mu} \right), \\
&= \frac{1}{2} (\delta_b^\nu - \frac{\kappa}{2} h_b^\nu) \left(\partial_\mu (\delta_{a\nu} + \frac{\kappa}{2} h_{\nu a}) - \partial_\nu (\delta_{a\mu} + \frac{\kappa}{2} h_{\mu a}) \right), \\
&= \frac{1}{2} (\delta_b^\nu - \frac{\kappa}{2} h_b^\nu) \left(\frac{\kappa}{2} \partial_\mu h_{\nu a} - \frac{\kappa}{2} \partial_\nu h_{\mu a} \right), \\
&= \frac{\kappa}{4} \left(\partial_\mu h_{ba} - \partial_b h_{\mu a} \right).
\end{aligned} \tag{2.13}
$$

$$
\begin{aligned}
C &= \frac{1}{2} e_a{}^{\rho} e_b{}^{\sigma} \left(\partial_\sigma e_{c\rho} - \partial_\rho e_{c\sigma} \right) e_\mu{}^{c}, \\
&= \frac{1}{2} (\delta_a^\rho - \frac{\kappa}{2} h_a^\rho)(\delta_b^\sigma - \frac{\kappa}{2} h_b^\sigma) \left(\frac{\kappa}{2} \partial_\sigma h_{\rho c} - \frac{\kappa}{2} \partial_\rho h_{\sigma c} \right) (\delta_\mu^c + \frac{\kappa}{2} h_\mu^c), \\
&= \frac{\kappa}{4} \left(\partial_b h_{a\mu} - \partial_a h_{b\mu} \right).
\end{aligned} \tag{2.14}
$$

using (2.12), (2.13) and (2.14) in (2.8) we have,

$$
\begin{aligned}
\omega_{\mu ab} &= A - B + C, \\
&= \frac{\kappa}{4} \left(\partial_\mu h_{ab} - \partial_a h_{\mu b} \right) - \frac{\kappa}{4} \left(\partial_\mu h_{ba} - \partial_b h_{\mu a} \right) + \frac{\kappa}{4} \left(\partial_b h_{a\mu} - \partial_a h_{b\mu} \right), \\
&= \frac{\kappa}{4} \left(\partial_\mu (h_{ab} - h_{ba}) + \partial_b (h_{\mu a} + h_{a\mu}) - \partial_a (h_{\mu b} + h_{b\mu}) \right), \\
&= \frac{\kappa}{2} \left(\partial_b h_{\mu a} - \partial_a h_{\mu b} \right).
\end{aligned} \tag{2.15}
$$

We will leave this result to the side for a moment, and continue with the covariant derivative,

$$
\begin{aligned}
D_\mu \psi &= \partial_\mu \psi + \frac{i}{4} \sigma^{ab} \omega_{\mu ab} \psi, \\
&= \partial_\mu \psi - \frac{1}{8} \left[\gamma^a, \gamma^b \right] \omega_{\mu ab} \psi.
\end{aligned} \tag{2.16}
$$

Now using 2.16 and remembering that $\bar{\psi} = \psi^\dagger \gamma^0$, we have $D_\mu \bar{\psi}$,

$$
\begin{aligned}
D_\mu \bar{\psi} = (D_\mu \psi)^\dagger \gamma^0 &= \partial_\mu \bar{\psi} - \frac{\kappa}{8} \psi^\dagger \left(\gamma^{b\dagger} \gamma^{a\dagger} - \gamma^{a\dagger} \gamma^{b\dagger} \right) \gamma^0 \omega_{\mu ab}, \\
&= \partial_\mu \bar{\psi} - \frac{\kappa}{8} \psi^\dagger \left(\gamma^0 \gamma^b \gamma^0 \gamma^0 \gamma^a \gamma^0 - \gamma^0 \gamma^a \gamma^0 \gamma^0 \gamma^b \gamma^0 \right) \gamma^0 \omega_{\mu ab}, \\
&= \partial_\mu \bar{\psi} - \frac{\kappa}{8} \bar{\psi} \left(\gamma^b \gamma^a - \gamma^a \gamma^b \right) \omega_{\mu ab}, \\
&= \partial_\mu \bar{\psi} + \frac{\kappa}{8} \bar{\psi} \left[\gamma^a, \gamma^b \right] \omega_{\mu ab}.
\end{aligned} \tag{2.17}
$$

Now we will calculate $\gamma^c e_c{}^\mu D_\mu \psi$ and $D_\mu \bar{\psi} e_c{}^\mu \gamma^c$,

$$
\begin{aligned}
\gamma^c e_c{}^\mu D_\mu \psi &= \gamma^c \left(\delta_c^\mu - \frac{\kappa}{2} h_c^\mu \right) \left(\partial_\mu \psi - \frac{1}{8} \left[\gamma^a, \gamma^b \right] \omega_{\mu ab} \psi \right), \\
&= \gamma^\mu \partial_\mu \psi - \gamma_\nu \frac{\kappa}{2} h^{\nu\mu} \partial_\mu \psi - \frac{1}{8} \gamma^\mu \left[\gamma^a, \gamma^b \right] \omega_{\mu ab} \psi,
\end{aligned}
$$

$$D_\mu\bar{\psi}e_c{}^\mu\gamma^c = \left(\partial_\mu\bar{\psi} + \frac{1}{8}\bar{\psi}\left[\gamma^a,\gamma^b\right]\omega_{\mu ab}\right)\left(\delta^\mu_c - \frac{\kappa}{2}h^\mu_c\right)\gamma^c,$$
$$= \partial_\mu\bar{\psi}\gamma^\mu - \frac{\kappa}{2}h^{\nu\mu}\partial_\mu\bar{\psi}\gamma_\nu + \frac{1}{8}\bar{\psi}\left[\gamma^a,\gamma^b\right]\gamma^\mu\omega_{\mu ab}.$$

With the help of the notation $\bar{\psi}\gamma^\mu\overleftrightarrow{\nabla}_\mu\psi = \bar{\psi}\gamma^\mu\partial_\mu\psi - (\partial_\mu\bar{\psi})\gamma^\mu\psi$, the Lagrangian at first order in κ is,

$$\sqrt{e}\mathcal{L}^{(1)}_m = \sqrt{e}\left(\frac{i}{2}\bar{\psi}\gamma^a e_a{}^\mu D_\mu\psi - \frac{i}{2}\left(D_\mu\bar{\psi}\right)e_a{}^\mu\gamma^a\psi - m\bar{\psi}\psi\right),$$
$$= \left(1+\frac{\kappa}{4}h\right)\left[\frac{i}{2}\bar{\psi}\left(\gamma^\mu\partial_\mu\psi - \gamma_\nu\frac{\kappa}{2}h^{\nu\mu}\partial_\mu\psi - \frac{\kappa}{8}\gamma^\mu\left[\gamma^a,\gamma^b\right]\omega_{\mu ab}\right)\right.$$
$$\left. -\frac{i}{2}\left(\partial_\mu\bar{\psi}\gamma^\mu - \frac{\kappa}{2}\partial_\mu\bar{\psi}h^{\nu\mu}\gamma_\nu + \frac{\kappa}{8}\bar{\psi}\left[\gamma^a,\gamma^b\right]\gamma^\mu\omega_{\mu ab}\right)\psi - m\bar{\psi}\psi\right],$$
$$= \left(1+\frac{\kappa}{4}h\right)\left(\frac{i}{2}\bar{\psi}\gamma^\mu\overleftrightarrow{\nabla}_\mu\psi - m\bar{\psi}\psi - \frac{i\kappa}{4}h^{\mu\nu}\bar{\psi}\gamma_\nu\overleftrightarrow{\nabla}_\mu\psi - \frac{i\kappa}{16}\omega_{\mu ab}\left(\gamma^\mu\left[\gamma^a,\gamma^b\right] + \left[\gamma^a,\gamma^b\right]\gamma^\mu\right)\psi\right).$$

We can rewrite the expression proportional to $\frac{\kappa}{16}$, in the previous expression using the identity $\gamma^\mu\gamma^\nu\gamma^\rho = \eta^{\mu\nu}\gamma^\rho + \eta^{\nu\rho}\gamma^\mu - \eta^{\mu\rho}\gamma^\nu - i\epsilon^{\sigma\mu\nu\rho}\gamma_\sigma\gamma^5$. We get:

$$\gamma^\mu\left[\gamma^a,\gamma^b\right] + \left[\gamma^a,\gamma^b\right]\gamma^\mu = \eta^{\mu a}\gamma^b + \eta^{ab}\gamma^\mu - \eta^{\mu b}\gamma^a - i\epsilon^{\sigma\mu ab}\gamma_\sigma\gamma^5 - \eta^{\mu b}\gamma^a - \eta^{ba}\gamma^\mu + \eta^{\mu a}\gamma^b + i\epsilon^{\sigma\mu ba}\gamma_\sigma\gamma^5,$$
$$+\,\eta^{ab}\gamma^\mu + \eta^{b\mu}\gamma^a - \eta^{a\mu}\gamma^b - i\epsilon^{\sigma ab\mu}\gamma_\sigma\gamma^5 - \eta^{ba}\gamma^\mu - \eta^{a\mu}\gamma^b + \eta^{b\mu}\gamma^a + i\epsilon^{\sigma ba\mu}\gamma_\sigma\gamma^5,$$
$$= -4i\epsilon^{\sigma\mu ab}\gamma_\sigma\gamma^5.$$

This is antisymmetric in all its indices, but in (2.15) we see that its first term is symmetric in μ and a and its second term in μ and b, due to the fact that $h_{\mu\nu}$ is a symmetric tensor. Because of this the last term is zero, so the lagrangian is,

$$\sqrt{e}\mathcal{L}^{(1)}_m = \left(1+\frac{\kappa}{4}h\right)\left(\frac{i}{2}\bar{\psi}\gamma^\mu\overleftrightarrow{\nabla}_\mu\psi - m\bar{\psi}\psi - \frac{i\kappa}{4}h^{\mu\nu}\bar{\psi}\gamma_\nu\overleftrightarrow{\nabla}_\mu\psi\right),$$
$$= \frac{i}{2}\bar{\psi}\gamma^\mu\overleftrightarrow{\nabla}_\mu\psi - m\bar{\psi}\psi - \frac{i\kappa}{4}h^{\mu\nu}\bar{\psi}\gamma_\nu\overleftrightarrow{\nabla}_\mu\psi + \frac{\kappa}{4}h\left(\frac{i}{2}\bar{\psi}\gamma^\mu\overleftrightarrow{\nabla}_\mu\psi - m\bar{\psi}\psi\right),$$
$$= -\frac{i\kappa}{4}h^{\mu\nu}\bar{\psi}\gamma_\nu\overleftrightarrow{\nabla}_\mu\psi + \frac{\kappa}{4}h\left(\frac{i}{2}\bar{\psi}\gamma^\mu\overleftrightarrow{\nabla}_\mu\psi - m\bar{\psi}\psi\right),$$

where in the last step we got rid of the terms of order k^0. With this we can calculate the vertex, replacing the fields with entering plane waves i.e. $\psi = \phi_0 e^{-ip_1x}$, $\bar{\psi} = \bar{\phi}_0 e^{-ip_2x}$ and $h_{\mu\nu} = \epsilon_{\mu\nu}e^{-ikx}$, but before doing that, we will calculate the expression $\bar{\psi}\gamma_\nu\overleftrightarrow{\nabla}_\mu\psi$,

$$
\begin{aligned}
\bar{\psi}\gamma_\nu \overleftrightarrow{\nabla}_\mu \psi &= \bar{\psi}\gamma^\mu \partial_\mu \psi - (\partial_\mu \bar{\psi})\gamma^\mu \psi, \\
&= \left(-ip_{1\mu}\bar{\phi}_0\gamma_\nu\phi_0 + ip_{2\mu}\bar{\phi}_0\gamma_\nu\phi_0\right) e^{-ip_1x}e^{-ip_2x}, \\
&= i\bar{\phi}_0 \left(p_{2\mu}\gamma_\nu - p_{1\mu}\gamma_\nu\right)\phi_0 e^{-ip_1x}e^{-ip_2x}.
\end{aligned} \tag{2.18}
$$

Using this, the Lagrangian is,

$$
\begin{aligned}
\sqrt{e}\mathcal{L}_m^{(1)} &= -\frac{i\kappa}{4}h^{\mu\nu}\bar{\psi}\gamma_\nu \overleftrightarrow{\nabla}_\mu \psi + \frac{\kappa}{4}h\left(\frac{i}{2}\bar{\psi}\gamma^\mu \overleftrightarrow{\nabla}_\mu \psi - m\bar{\psi}\psi\right), \\
&= \frac{\kappa}{4}\epsilon^{\mu\nu}\bar{\phi}_0\left(p_{2\mu}\gamma_\nu - p_{1\mu}\gamma_\nu\right)\phi_0 e^{-ip_1x}e^{-ip_2x}e^{-ikx} + \frac{\kappa}{4}\epsilon^{\mu\nu}\eta_{\mu\nu}\bar{\phi}_0\left(\frac{1}{2}\left(\not{p}_1 - \not{p}_2\right) - m\right)\phi_0 e^{-ip_1x}e^{-ip_2x}e^{-ikx}, \\
&= -\frac{\kappa}{2}\epsilon^{\mu\nu}\bar{\phi}_0\left[\frac{1}{2}\left(p_{1\mu}\gamma_\nu - p_{2\mu}\gamma_\nu\right) - \frac{1}{2}\eta_{\mu\nu}\left(\frac{1}{2}\left(\not{p}_1 - \not{p}_2\right) - m\right)\right]\phi_0 e^{-ip_1x}e^{-ip_2x}e^{-ikx}.
\end{aligned}
$$

For the vertex, we multiply by i and derivate with respect to the amplitudes of the fields,

$$
\begin{aligned}
\frac{\partial^3 i\sqrt{e}\mathcal{L}_m^{(1)}}{\partial\epsilon_{\alpha\beta}\partial\phi_0\partial\bar{\phi}_0} &= -\frac{i\kappa}{4}(\delta_\mu^\alpha\delta_\nu^\beta + \delta_\mu^\beta\delta_\nu^\alpha)\left[\frac{1}{2}\left({p_1}^\mu\gamma^\nu - {p_2}^\mu\gamma^\nu\right) - \frac{1}{2}\eta^{\mu\nu}\left(\frac{1}{2}\left(\not{p}_1 - \not{p}_2\right) - m\right)\right]e^{-ip_1x}e^{-ip_2x}e^{-ikx}, \\
&= -\frac{i\kappa}{2}\left[\frac{1}{4}\left(\gamma^\alpha({p_1}^\beta - {p_2}^\beta) + \gamma^\beta({p_1}^\alpha - {p_2}^\alpha)\right) - \frac{1}{2}\eta^{\alpha\beta}\left(\frac{1}{2}\left(\not{p}_1 - \not{p}_2\right) - m\right)\right]e^{-ip_1x}e^{-ip_2x}e^{-ikx},
\end{aligned}
$$

where we have utilized $\frac{\partial\epsilon_{\mu\nu}}{\partial\epsilon_{\alpha\beta}} = \frac{1}{2}(\delta_\mu^\alpha\delta_\nu^\beta + \delta_\mu^\beta\delta_\nu^\alpha)$. With this, we have the vertex with two fermions and a graviton,

$$
\tau_{\alpha\beta}(p_1,p_2) = -\frac{i\kappa}{2}\left[\frac{1}{4}\left(\gamma_\alpha(p_{1\beta} - p_{2\beta}) + \gamma_\beta(p_{1\alpha} - p_{2\alpha})\right) - \frac{1}{2}\eta_{\alpha\beta}\left(\frac{1}{2}\left(\not{p}_1 - \not{p}_2\right) - m\right)\right]. \tag{2.19}
$$

Now for order κ^2, we will calculate $\omega_{\mu ab}$ term by term,

$$
\begin{aligned}
A &= \frac{1}{2}{e_a}^\nu\left(\partial_\mu e_{b\nu} - \partial_\nu e_{b\mu}\right), \\
&= \frac{1}{2}\left({\delta_a}^\nu - \frac{\kappa}{2}{h_a}^\nu + \frac{3\kappa^2}{8}h^{\nu\lambda}h_{\lambda a}\right)\left(\partial_\mu\left(\delta_{b\nu} + \frac{\kappa}{2}h_{b\nu} - \frac{\kappa^2}{8}h_{\nu\lambda}{h^\lambda}_b\right) - \partial_\nu\left(\delta_{b\mu} + \frac{\kappa}{2}h_{b\mu} - \frac{\kappa^2}{8}h_{\mu\lambda}{h^\lambda}_b\right)\right), \\
&= \frac{1}{4}\left({\delta_a}^\nu - \frac{\kappa}{2}{h_a}^\nu + \frac{3\kappa^2}{8}h^{\nu\lambda}h_{\lambda a}\right)\left(\partial_\mu\left(\kappa h_{b\nu} - \frac{\kappa^2}{4}h_{\nu\lambda}{h^\lambda}_b\right) - \partial_\nu\left(\kappa h_{b\mu} - \frac{\kappa^2}{4}h_{\mu\lambda}{h^\lambda}_b\right)\right), \\
&= \frac{\kappa}{4}\left(\partial_\mu h_{ab} - \partial_a h_{\mu b}\right) - \frac{\kappa^2}{8}h_a^\nu\left(\partial_\mu h_{\nu b} - \partial_\nu h_{\mu b}\right) - \frac{\kappa^2}{16}\left(\partial_\mu(h_{a\lambda}{h^\lambda}_b) - \partial_a(h_{\mu\lambda}{h^\lambda}_b)\right).
\end{aligned} \tag{2.20}
$$

$$
\begin{aligned}
B &= \frac{1}{2}e_b{}^{\nu}\left(\partial_\mu e_{a\nu} - \partial_\nu e_{a\mu}\right), \\
&= \frac{1}{2}\left(\delta_b{}^{\nu} - \frac{\kappa}{2}h_b{}^{\nu} + \frac{3\kappa^2}{8}h^{\nu\lambda}h_{\lambda b}\right)\left(\partial_\mu\left(\delta_{a\nu} + \frac{\kappa}{2}h_{a\nu} - \frac{\kappa^2}{8}h_{\nu\lambda}h^{\lambda}{}_a\right) - \partial_\nu\left(\delta_{a\mu} + \frac{\kappa}{2}h_{a\mu} - \frac{\kappa^2}{8}h_{\mu\lambda}h^{\lambda}{}_a\right)\right), \\
&= \frac{1}{4}\left(\delta_b{}^{\nu} - \frac{\kappa}{2}h_b{}^{\nu} + \frac{3\kappa^2}{8}h^{\nu\lambda}h_{\lambda b}\right)\left(\partial_\mu\left(\kappa h_{a\nu} - \frac{\kappa^2}{4}h_{\nu\lambda}h^{\lambda}{}_a\right) - \partial_\nu\left(\kappa h_{a\mu} - \frac{\kappa^2}{4}h_{\mu\lambda}h^{\lambda}{}_a\right)\right), \\
&= \frac{\kappa}{4}\left(\partial_\mu h_{ab} - \partial_b h_{\mu a}\right) - \frac{\kappa^2}{8}h^{\nu}_b\left(\partial_\mu h_{\nu a} - \partial_\nu h_{\mu a}\right) - \frac{\kappa^2}{16}\left(\partial_\mu(h_{b\lambda}h^{\lambda}{}_a) - \partial_b(h_{\mu\lambda}h^{\lambda}{}_a)\right).
\end{aligned} \tag{2.21}
$$

$$
\begin{aligned}
C &= \frac{1}{2}e_a{}^{\rho}e_b{}^{\sigma}\left(\partial_\sigma e_{c\rho} - \partial_\rho e_{c\sigma}\right)e_\mu{}^{c}, \\
&= \frac{1}{4}\left(\delta_a{}^{\rho} - \frac{\kappa}{2}h_a{}^{\rho} + \frac{3\kappa^2}{8}h^{\rho\lambda}h_{\lambda a}\right)\left(\delta_b{}^{\sigma} - \frac{\kappa}{2}h_b{}^{\sigma} + \frac{3\kappa^2}{8}h^{\sigma\lambda}h_{\lambda b}\right)\left(\partial_\sigma\left(\kappa h_{c\rho} - \frac{\kappa^2}{4}h_{\rho\lambda}h^{\lambda}{}_c\right)\right. \\
&\left.-\partial_\rho\left(\kappa h_{c\sigma} - \frac{\kappa^2}{4}h_{\sigma\lambda}h^{\lambda}{}_c\right)\right)\left(\delta^c{}_\mu + \frac{\kappa}{2}h^c{}_\mu - \frac{\kappa^2}{8}h_{\mu\lambda}h^{\lambda c}\right), \\
&= \frac{\kappa}{4}\left(\partial_b h_{a\mu} - \partial_a h_{b\mu}\right) - \frac{\kappa^2}{8}h^{\rho}_a\left(\partial_b h_{\rho\mu} - \partial_\rho h_{b\mu}\right) - \frac{\kappa^2}{8}h^{\sigma}_b\left(\partial_\sigma h_{a\mu} - \partial_a h_{\sigma\mu}\right) + \frac{\kappa^2}{8}h^{c}_\mu\left(\partial_b h_{ac} - \partial_a h_{bc}\right) \\
&- \frac{\kappa^2}{16}\left(\partial_b(h_{a\lambda}h^{\lambda}{}_\mu) - \partial_a(h_{b\lambda}h^{\lambda}{}_\mu)\right).
\end{aligned} \tag{2.22}
$$

Using (2.20), (2.21) and (2.22) in (2.8) we have,

$$
\begin{aligned}
\omega_{\mu ab} &= A - B + C, \\
&= \frac{\kappa}{4}\left(\partial_\mu h_{ab} - \partial_a h_{\mu b}\right) - \frac{\kappa^2}{8}h^{\nu}_a\left(\partial_\mu h_{\nu b} - \partial_\nu h_{\mu b}\right) - \frac{\kappa^2}{16}\left(\partial_\mu(h_{a\lambda}h^{\lambda}{}_b) - \partial_a(h_{\mu\lambda}h^{\lambda}{}_b)\right) \\
&- \frac{\kappa}{4}\left(\partial_\mu h_{ab} - \partial_b h_{\mu a}\right) + \frac{\kappa^2}{8}h^{\nu}_b\left(\partial_\mu h_{\nu a} - \partial_\nu h_{\mu a}\right) + \frac{\kappa^2}{16}\left(\partial_\mu(h_{b\lambda}h^{\lambda}{}_a) - \partial_b(h_{\mu\lambda}h^{\lambda}{}_a)\right) \\
&+ \frac{\kappa}{4}\left(\partial_b h_{a\mu} - \partial_a h_{b\mu}\right) - \frac{\kappa^2}{8}h^{\rho}_a\left(\partial_b h_{\rho\mu} - \partial_\rho h_{b\mu}\right) - \frac{\kappa^2}{8}h^{\sigma}_b\left(\partial_\sigma h_{a\mu} - \partial_a h_{\sigma\mu}\right) + \frac{\kappa^2}{8}h^{c}_\mu\left(\partial_b h_{ac} - \partial_a h_{bc}\right) \\
&- \frac{\kappa^2}{16}\left(\partial_b(h_{a\lambda}h^{\lambda}{}_\mu) - \partial_a(h_{b\lambda}h^{\lambda}{}_\mu)\right), \\
&= \frac{\kappa}{2}\left(\partial_b h_{a\mu} - \partial_a h_{b\mu}\right) + \frac{\kappa^2}{4}\left(h^{\nu}_a\partial_\nu h_{\mu b} - h^{\nu}_b\partial_\nu h_{\mu a}\right) + \frac{\kappa^2}{8}\left(h^{\nu}_b\left(\partial_\mu h_{\nu a} + \partial_a h_{\nu\mu}\right) - h^{\nu}_a\left(\partial_\mu h_{\nu b} + \partial_b h_{\nu\mu}\right)\right) \\
&+ \frac{\kappa^2}{8}h^{c}_\mu\left(\partial_b h_{ac} - \partial_a h_{bc}\right) - \frac{\kappa^2}{8}\left(\partial_b(h_{a\lambda}h^{\lambda}{}_\mu) - \partial_a(h_{b\lambda}h^{\lambda}{}_\mu)\right), \\
&= \frac{\kappa}{2}\left(\partial_b h_{a\mu} - \partial_a h_{b\mu}\right) + \frac{\kappa^2}{4}\left(h^{\nu}_a\partial_\nu h_{\mu b} - h^{\nu}_b\partial_\nu h_{\mu a}\right) + \frac{\kappa^2}{8}\left(h^{\nu}_b\left(\partial_\mu h_{\nu a} + \partial_a h_{\nu\mu}\right) - h^{\nu}_a\left(\partial_\mu h_{\nu b} + \partial_b h_{\nu\mu}\right)\right) \\
&+ \frac{\kappa^2}{8}h^{c}_\mu\left(\partial_b h_{ac} - \partial_a h_{bc}\right) - \frac{\kappa^2}{8}\left(h_{a\lambda}\partial_b h^{\lambda}{}_\mu + h^{\lambda}{}_\mu\partial_b h_{a\lambda} - h_{b\lambda}\partial_a h^{\lambda}{}_\mu - h^{\lambda}{}_\mu\partial_a h_{b\lambda}\right), \\
&= \frac{\kappa}{2}\left(\partial_b h_{a\mu} - \partial_a h_{b\mu}\right) + \frac{\kappa^2}{4}\left(h^{\nu}_a\partial_\nu h_{\mu b} - h^{\nu}_b\partial_\nu h_{\mu a}\right) + \frac{\kappa^2}{8}\left(h^{\nu}_b\left(\partial_\mu h_{\nu a} + \partial_a h_{\nu\mu}\right) - h^{\nu}_a\left(\partial_\mu h_{\nu b} + \partial_b h_{\nu\mu}\right)\right) \\
&- \frac{\kappa^2}{8}\left(h_{a\lambda}\partial_b h^{\lambda}{}_\mu - h_{b\lambda}\partial_a h^{\lambda}{}_\mu\right).
\end{aligned} \tag{2.23}
$$

We will leave this result to the side for a moment, and continue calculating $\gamma^c e_c{}^\mu D_\mu \psi$ and $D_\mu \bar{\psi} e_c{}^\mu \gamma^c$. Using (2.16) and (2.17),

$$\gamma^c e_c{}^\mu D_\mu \psi = \gamma^c \left(\delta^\mu_c - \frac{\kappa}{2} h^\mu_c + \frac{3\kappa^2}{8} h^\mu_d h^d_c \right) \left(\partial_\mu \psi - \frac{1}{8} \left[\gamma^a, \gamma^b \right] \omega_{\mu ab} \psi \right),$$

$$= \gamma^\mu \partial_\mu \psi - \gamma_\nu \frac{\kappa}{2} h^{\nu\mu} \partial_\mu \psi + \frac{3\kappa^2}{8} \gamma^c h^\mu_d h^d_c \partial_\mu \psi - \frac{1}{8} \gamma^\mu \left[\gamma^a, \gamma^b \right] \omega_{\mu ab} \psi + \frac{\kappa}{16} \gamma_\nu h^{\nu\mu} \left[\gamma^a, \gamma^b \right] \omega_{\mu ab} \psi.$$

$$D_\mu \bar{\psi} e_c{}^\mu \gamma^c = \left(\partial_\mu \bar{\psi} + \frac{1}{8} \bar{\psi} \left[\gamma^a, \gamma^b \right] \omega_{\mu ab} \right) \left(\delta^\mu_c - \frac{\kappa}{2} h^\mu_c + \frac{3\kappa^2}{8} h^\mu_d h^d_c \right) \gamma^c,$$

$$= \partial_\mu \bar{\psi} \gamma^\mu - \frac{\kappa}{2} h^{\nu\mu} \partial_\mu \bar{\psi} \gamma_\nu + \frac{3\kappa^2}{8} h^\mu_d h^d_c \partial_\mu \bar{\psi} \gamma^c + \frac{1}{8} \bar{\psi} \left[\gamma^a, \gamma^b \right] \gamma^\mu \omega_{\mu ab} - \frac{\kappa}{16} h^{\nu\mu} \bar{\psi} \left[\gamma^a, \gamma^b \right] \gamma_\nu \omega_{\mu ab}.$$

Now the lagrangian is,

$$\sqrt{e} \mathcal{L}^{(2)}_m = \sqrt{e} \left(\frac{i}{2} \bar{\psi} \gamma^a e_a{}^\mu D_\mu \psi - \frac{i}{2} \left(D_\mu \bar{\psi} \right) e_a{}^\mu \gamma^a \psi - m \bar{\psi} \psi \right),$$

$$= \sqrt{e} \left[\frac{i}{2} \bar{\psi} \left(\gamma^\mu \partial_\mu \psi - \gamma_\nu \frac{\kappa}{2} h^{\nu\mu} \partial_\mu \psi + \frac{3\kappa^2}{8} \gamma^c h^\mu_d h^d_c \partial_\mu \psi - \frac{1}{8} \gamma^\mu \left[\gamma^a, \gamma^b \right] \omega_{\mu ab} \psi \right.\right.$$

$$\left. + \frac{\kappa}{16} \gamma_\nu h^{\nu\mu} \left[\gamma^a, \gamma^b \right] \omega_{\mu ab} \psi \right) - \frac{i}{2} \left(\partial_\mu \bar{\psi} \gamma^\mu - \frac{\kappa}{2} h^{\nu\mu} \partial_\mu \bar{\psi} \gamma_\nu + \frac{3\kappa^2}{8} h^\mu_d h^d_c \partial_\mu \bar{\psi} \gamma^c + \frac{1}{8} \bar{\psi} \left[\gamma^a, \gamma^b \right] \gamma^\mu \omega_{\mu ab} \right.$$

$$\left.\left. - \frac{\kappa}{16} h^{\nu\mu} \bar{\psi} \left[\gamma^a, \gamma^b \right] \gamma_\nu \omega_{\mu ab} \right) \psi - m \bar{\psi} \psi \right],$$

$$= \sqrt{e} \left(\frac{i}{2} \bar{\psi} \gamma^\mu \overleftrightarrow{\nabla}_\mu \psi - m \bar{\psi} \psi - \frac{i\kappa}{4} h^{\mu\nu} \bar{\psi} \gamma_\nu \overleftrightarrow{\nabla}_\mu \psi + \frac{3i\kappa^2}{16} h^\mu_d h^{\nu d} \bar{\psi} \gamma_\nu \overleftrightarrow{\nabla}_\mu \psi \right.$$

$$\left. - \frac{i}{16} \omega_{\mu ab} \bar{\psi} \left(\gamma^\mu \left[\gamma^a, \gamma^b \right] + \left[\gamma^a, \gamma^b \right] \gamma^\mu \right) \psi + \frac{i\kappa}{32} h^\mu_\rho \omega_{\mu ab} \bar{\psi} \left(\gamma^\rho \left[\gamma^a, \gamma^b \right] + \left[\gamma^a, \gamma^b \right] \gamma^\rho \right) \psi \right).$$

Now we use (2.23) to analyse the last two terms, we will focus on the last term first. We know that only the first term in (2.23) is order κ^1, so if we multiply by κ the rest of the terms will be of order $\mathcal{O}(\kappa^3)$. We are left with the following expression,

$$\frac{i\kappa}{32} h^\mu_\rho \omega_{\mu ab} \bar{\psi} \left(\gamma^\rho \left[\gamma^a, \gamma^b \right] + \left[\gamma^a, \gamma^b \right] \gamma^\rho \right) \psi = \frac{\kappa^2}{16} h^\mu_\rho \left(\partial_b h_{a\mu} - \partial_a h_{b\mu} \right) \bar{\psi} \left(\epsilon^{\sigma\rho ab} \gamma_\sigma \gamma^5 \right) \psi,$$

$$= -\frac{\kappa^2}{16} \epsilon^{ab\rho\sigma} h^\mu_\rho \left(\partial_b h_{a\mu} - \partial_a h_{b\mu} \right) \bar{\psi} \gamma_\sigma \gamma^5 \psi,$$

Now, for the second to last term we have,

$$\frac{i}{16} \omega_{\mu ab} \bar{\psi} \left(\gamma^\mu \left[\gamma^a, \gamma^b \right] + \left[\gamma^a, \gamma^b \right] \gamma^\mu \right) \psi = \frac{1}{4} \bar{\psi} \epsilon^{\sigma\mu ab} \gamma_\sigma \gamma^5 \psi \left[\frac{\kappa}{2} \left(\partial_b h_{a\mu} - \partial_a h_{b\mu} \right) + \frac{\kappa^2}{4} \left(h^\nu_a \partial_\nu h_{\mu b} - h^\nu_b \partial_\nu h_{\mu a} \right) \right.$$

$$+ \frac{\kappa^2}{8} \left(h^\nu_b \left(\partial_\mu h_{\nu a} + \partial_a h_{\nu\mu} \right) - h^\nu_a \left(\partial_\mu h_{\nu b} + \partial_b h_{\nu\mu} \right) \right)$$

$$\left. - \frac{\kappa^2}{8} \left(h_{a\lambda} \partial_b h^\lambda{}_\mu - h_{b\lambda} \partial_a h^\lambda{}_\mu \right) \right].$$

Due to the symmetry of $h_{\mu\nu}$, the first and second term are zero. Also, the expressions $\partial_\mu h_{\nu a}+\partial_a h_{\nu\mu}$ and $\partial_\mu h_{\nu b}+\partial_b h_{\nu\mu}$ in the third term are symmetric in μ and a, and μ and b respectively, so we are left with,

$$\frac{i}{16}\omega_{\mu ab}\bar{\psi}\left(\gamma^\mu\left[\gamma^a,\gamma^b\right]+\left[\gamma^a,\gamma^b\right]\gamma^\mu\right)\psi = -\frac{\kappa^2}{32}\bar{\psi}\epsilon^{\sigma\mu ab}\gamma_\sigma\gamma^5\psi\left(h_{a\lambda}\partial_b h^\lambda{}_\mu - h_{b\lambda}\partial_a h^\lambda{}_\mu\right),$$
$$= \frac{\kappa^2}{32}\bar{\psi}\epsilon^{\sigma\mu ab}\gamma_\sigma\gamma^5\psi h^\lambda_\mu\left(\partial_b h_{a\lambda}-\partial_a h_{b\lambda}\right),$$
$$= -\frac{\kappa^2}{32}\epsilon^{ab\mu\sigma}h^\lambda_\mu\left(\partial_b h_{a\lambda}-\partial_a h_{b\lambda}\right)\bar{\psi}\gamma_\sigma\gamma^5\psi.$$

With this, the lagrangian is,

$$\begin{aligned}\sqrt{e}\mathcal{L}^{(2)}_m =& \left(1+\frac{\kappa}{4}h_\mu{}^\mu+\frac{\kappa^2}{32}\left(h_\mu{}^\mu h_\nu{}^\nu-4h_\mu{}^\nu h_\nu{}^\mu\right)\right)\left(\frac{i}{2}\bar{\psi}\gamma^\mu\overleftrightarrow{\nabla}_\mu\psi - m\bar{\psi}\psi-\frac{i\kappa}{4}h^{\mu\nu}\bar{\psi}\gamma_\nu\overleftrightarrow{\nabla}_\mu\psi\right.\\ &\left.+\frac{3i\kappa^2}{16}h^\mu_d h^{\nu d}\bar{\psi}\gamma_\nu\overleftrightarrow{\nabla}_\mu\psi+\frac{\kappa^2}{32}\epsilon^{ab\mu\sigma}h^\lambda_\mu\left(\partial_b h_{a\lambda}-\partial_a h_{b\lambda}\right)\bar{\psi}\gamma_\sigma\gamma^5\psi-\frac{\kappa^2}{16}\epsilon^{ab\rho\sigma}h^\mu_\rho\left(\partial_b h_{a\mu}-\partial_a h_{b\mu}\right)\bar{\psi}\gamma_\sigma\gamma^5\psi\right),\\ =& \frac{\kappa^2}{32}\left(h_\mu{}^\mu h_\nu{}^\nu-4h_\mu{}^\nu h_\nu{}^\mu\right)\left(\frac{i}{2}\bar{\psi}\gamma^\mu\overleftrightarrow{\nabla}_\mu\psi-m\bar{\psi}\psi\right)-\frac{i\kappa^2}{16}h^\alpha_\alpha h^{\mu\nu}\bar{\psi}\gamma_\nu\overleftrightarrow{\nabla}_\mu\psi\\ &+\frac{3i\kappa^2}{16}h^\mu_d h^{\nu d}\bar{\psi}\gamma_\nu\overleftrightarrow{\nabla}_\mu\psi-\frac{\kappa^2}{32}\epsilon^{ab\mu\sigma}h^\lambda_\mu\left(\partial_b h_{a\lambda}-\partial_a h_{b\lambda}\right)\bar{\psi}\gamma_\sigma\gamma^5\psi,\end{aligned}$$

where in the last step we got rid of the terms of order k^0 and k^1. Before we calculate the vertex we will rewrite the last term,

$$\epsilon^{ab\mu\sigma}h^\lambda_\mu\left(\partial_b h_{a\lambda}-\partial_a h_{b\lambda}\right) = \epsilon^{ab\mu\sigma}h^\lambda_\mu\partial_b h_{a\lambda}-\epsilon^{ab\mu\sigma}h^\lambda_\mu\partial_a h_{b\lambda},$$
$$= \epsilon^{ab\mu\sigma}h^\lambda_\mu\partial_b h_{a\lambda}-\epsilon^{b\mu a\sigma}h^\lambda_a\partial_b h_{\mu\lambda},$$
$$= \epsilon^{ab\mu\sigma}\left(h^\lambda_\mu\partial_b h_{a\lambda}-h^\lambda_a\partial_b h_{\mu\lambda}\right).$$

Now we will calculate the vertex, by replacing the fields with entering plane waves i.e. $\psi=\phi_0 e^{-ip_1x}$, $\bar{\psi}=\bar{\phi}_0 e^{-ip_2x}$, $h_{\mu\nu}=\epsilon^{(1)}_{\mu\nu}e^{-ik_2x}$ and $h_{\mu\nu}=\epsilon^{(2)}_{\mu\nu}e^{-ik_4x}$. Using (2.18) we have,

$$\begin{aligned}\sqrt{e}\mathcal{L}^{(2)}_m =& \bar{\phi}_0\left[\frac{\kappa^2}{32}\left(\epsilon^{(1)}_{\mu\nu}\eta^{\mu\nu}\epsilon^{(2)}_{\rho\sigma}\eta^{\rho\sigma}-4\epsilon^{(1)\nu}_\mu\epsilon^{(2)\mu}_\nu\right)\left(\frac{1}{2}\left(\not{p}_1-\not{p}_2\right)-m\right)+\frac{\kappa^2}{16}\epsilon^{(2)}_{\rho\sigma}\eta^{\rho\sigma}\epsilon^{(1)\mu\nu}\left(p_{2\mu}\gamma_\nu-p_{1\mu}\gamma_\nu\right)\right.\\ &\left.-\frac{3\kappa^2}{16}\epsilon^{(1)\mu}_d\epsilon^{(2)\nu d}\left(p_{2\mu}\gamma_\nu-p_{1\mu}\gamma_\nu\right)+\frac{i\kappa^2}{32}\epsilon^{ab\mu\sigma}\left(\epsilon^{(1)\lambda}_\mu k_{4b}\epsilon^{(2)}_{a\lambda}-\epsilon^{(2)\lambda}_a k_{2b}\epsilon^{(1)}_{\mu\lambda}\right)\gamma_\sigma\gamma^5\right]\phi_0 e^{-ip_1x}e^{-ip_2x}e^{-ik_1x}e^{-ik_2x}.\end{aligned}$$

Before calculating the derivative with respect to the amplitudes we will calculate the following expression,

$$\frac{\partial^2\epsilon_{\mu\nu}\epsilon_{\rho\sigma}}{\partial\epsilon^{\alpha\beta}\partial\epsilon^{\gamma\delta}} = \frac{1}{4}\left[\left(\eta_{\alpha\mu}\eta_{\beta\nu}+\eta_{\beta\mu}\eta_{\alpha\nu}\right)\left(\eta_{\gamma\rho}\eta_{\sigma\delta}+\eta_{\gamma\sigma}\eta_{\delta\rho}\right)+\left(\eta_{\gamma\mu}\eta_{\delta\nu}+\eta_{\delta\mu}\eta_{\gamma\nu}\right)\left(\eta_{\alpha\rho}\eta_{\beta\sigma}+\eta_{\alpha\sigma}\eta_{\beta\rho}\right)\right],$$
$$= I_{\alpha\beta\mu\nu}I_{\gamma\delta\sigma\rho}+I_{\gamma\delta\mu\nu}I_{\alpha\beta\sigma\rho},$$

which is symmetric under the changes $\alpha \leftrightarrow \beta$, $\gamma \leftrightarrow \delta$ and $\alpha\beta \leftrightarrow \gamma\delta$, as it must be. Therefore,

$$\begin{aligned}
\frac{\partial^4 i\sqrt{e}\mathcal{L}_m^{(2)}}{\partial\epsilon^{\alpha\beta}\partial\epsilon^{\gamma\delta}\partial\bar{\phi}_0\partial\phi_0} &= i\kappa^2\left[\frac{1}{32}\left(2I_{\alpha\beta\mu}{}^{\mu}I_{\gamma\delta\sigma}{}^{\sigma} - 8I_{\alpha\beta}{}^{\mu\nu}I_{\gamma\delta\mu\nu}\right)\left(\frac{1}{2}\left(\not{p}_1 - \not{p}_2\right) - m\right)\right.\\
&+ \frac{1}{16}\left(I_{\alpha\beta}{}^{\mu\nu}I_{\gamma\delta\sigma}{}^{\sigma} + I_{\gamma\delta}{}^{\mu\nu}I_{\alpha\beta\sigma}{}^{\sigma}\right)\left(p_{2\mu}\gamma_\nu - p_{1\mu}\gamma_\nu\right) - \frac{3}{16}\left(I_{\alpha\beta d}{}^{\mu}I_{\gamma\delta}{}^{d\nu} + I_{\gamma\delta d}{}^{\mu}I_{\alpha\beta}{}^{d\nu}\right)\gamma_\nu\left(p_{2\mu} - p_{1\mu}\right)\\
&\left.+\frac{i}{32}\epsilon^{ab\mu\rho}(I_{\alpha\beta\mu\lambda}I_{\gamma\delta a}{}^{\lambda} + I_{\gamma\delta\mu\lambda}I_{\alpha\beta\sigma}{}^{\lambda})\left(k_{4b} - k_{2b}\right)\gamma_\rho\gamma^5\right]e^{-ip_1x}e^{-ip_2x}e^{-ik_1x}e^{-ik_2x},\\
&= i\kappa^2\left[\frac{1}{4}\left(\frac{1}{4}\eta_{\alpha\beta}\eta_{\gamma\delta} - I_{\alpha\beta\gamma\delta}\right)\left(\frac{1}{2}\left(\not{p}_1 - \not{p}_2\right) - m\right)\right.\\
&+ \frac{1}{32}\left(\eta_{\gamma\delta}\left(\delta_\alpha{}^{\mu}\delta_\beta{}^{\nu} + \delta_\alpha{}^{\nu}\delta_\beta{}^{\mu}\right) + \eta_{\alpha\beta}\left(\delta_\gamma{}^{\mu}\delta_\delta{}^{\nu} + \delta_\delta{}^{\mu}\delta_\gamma{}^{\nu}\right)\right)\left(p_{2\mu}\gamma_\nu - p_{1\mu}\gamma_\nu\right)\\
&- \frac{3}{16}\left(I_{\alpha\beta d}{}^{\mu}I_{\gamma\delta}{}^{d\nu} + I_{\gamma\delta d}{}^{\mu}I_{\alpha\beta}{}^{d\nu}\right)\gamma_\nu\left(p_{2\mu} - p_{1\mu}\right)\\
&\left.+\frac{i}{32}\epsilon^{ab\mu\rho}(I_{\alpha\beta\mu\lambda}I_{\gamma\delta a}{}^{\lambda} + I_{\gamma\delta\mu\lambda}I_{\alpha\beta\sigma}{}^{\lambda})\left(k_{4b} - k_{2b}\right)\gamma_\rho\gamma^5\right]e^{-ip_1x}e^{-ip_2x}e^{-ik_1x}e^{-ik_2x},\\
&= i\kappa^2\left[-\frac{1}{4}\left(\frac{1}{4}\eta_{\alpha\beta}\eta_{\gamma\delta} + P_{\alpha\beta\gamma\delta}\right)\left(\frac{1}{2}\left(\not{p}_1 - \not{p}_2\right) - m\right)\right.\\
&- \frac{1}{32}\left(\eta_{\gamma\delta}\left(\gamma_\alpha\left(p_1 - p_2\right)_\beta + \gamma_\beta\left(p_1 - p_2\right)_\alpha\right) + \eta_{\alpha\beta}\left(\gamma_\gamma\left(p_1 - p_2\right)_\delta + \gamma_\delta\left(p_1 - p_2\right)_\gamma\right)\right)\\
&\left.+\frac{3}{16}\left(I_{\alpha\beta d}{}^{\mu}I_{\gamma\delta}{}^{d\nu} + I_{\gamma\delta d}{}^{\mu}I_{\alpha\beta}{}^{d\nu}\right)\gamma_\nu\left(p_{1\mu} - p_{2\mu}\right)\right]e^{-ip_1x}e^{-ip_2x}e^{-ik_1x}e^{-ik_2x}.
\end{aligned}$$

In the last step we notice that the term $I_{\alpha\beta\mu\lambda}I_{\gamma\delta a}{}^{\lambda} + I_{\gamma\delta\mu\lambda}I_{\alpha\beta\sigma}{}^{\lambda}$ is symmetric in μ and a and it is contracted with $\epsilon^{ab\mu\rho}$ so the whole term is zero. The vertex with two fermions and two gravitons is,

$$\begin{aligned}
\tau_{\alpha\beta\gamma\delta}(p_1,p_2) &= i\kappa^2\left[-\frac{1}{4}\left(\frac{1}{4}\eta_{\alpha\beta}\eta_{\gamma\delta} + P_{\alpha\beta\gamma\delta}\right)\left(\frac{1}{2}\left(\not{p}_1 - \not{p}_2\right) - m\right)\right.\\
&- \frac{1}{32}\left(\eta_{\gamma\delta}\left(\gamma_\alpha\left(p_1 - p_2\right)_\beta + \gamma_\beta\left(p_1 - p_2\right)_\alpha\right) + \eta_{\alpha\beta}\left(\gamma_\gamma\left(p_1 - p_2\right)_\delta + \gamma_\delta\left(p_1 - p_2\right)_\gamma\right)\right)\\
&\left.+\frac{3}{16}\left(I_{\alpha\beta d}{}^{\mu}I_{\gamma\delta}{}^{d\nu} + I_{\gamma\delta d}{}^{\mu}I_{\alpha\beta}{}^{d\nu}\right)\gamma_\nu\left(p_{1\mu} - p_{2\mu}\right)\right].
\end{aligned} \tag{2.24}$$

2.3.2 Propagator

For the propagator we start with equation (2.5) and rewrite it as,

$$P^{\mu\nu\alpha\beta}\Box h_{\alpha\beta} = -2T^{\mu\nu}$$

The Green function for this equation is $\Delta_{\alpha\beta\gamma\delta}$, the solution of

$$P^{\mu\nu\alpha\beta}\Box\Delta_{\alpha\beta\gamma\delta}\left(x-x'\right) = -i\delta^{(4)}\left(x-x'\right)I^{\mu\nu}{}_{\gamma\delta},$$

Writing this in the momentum space we have,

$$P^{\mu\nu\alpha\beta}k^2\Delta_{\alpha\beta\gamma\delta}\left(k\right) = iI^{\mu\nu}{}_{\gamma\delta}. \tag{2.25}$$

We propose an Ansatz using symmetric structures constructed with $\eta_{\alpha\beta}$, i.e. $\eta_{\alpha\gamma}\eta_{\beta\delta}+\eta_{\alpha\delta}\eta_{\beta\gamma}$ and $\eta_{\alpha\beta}\eta_{\gamma\delta}$. Note that both are symmetric in $\alpha\beta \leftrightarrow \gamma\delta$. The Ansatz is,

$$\Delta_{\alpha\beta\gamma\delta} = \frac{1}{k^2}(A(\eta_{\alpha\gamma}\eta_{\beta\delta}+\eta_{\alpha\delta}\eta_{\beta\gamma})+B\eta_{\alpha\beta}\eta_{\gamma\delta}).$$

Replacing this in (2.25) we have,

$$\begin{aligned}
P^{\mu\nu\alpha\beta}(A(\eta_{\alpha\gamma}\eta_{\beta\delta}+\eta_{\alpha\delta}\eta_{\beta\gamma})+B\eta_{\alpha\beta}\eta_{\gamma\delta}) &= iI^{\mu\nu}{}_{\gamma\delta},\\
\frac{1}{2}(\eta^{\mu\alpha}\eta^{\nu\beta}+\eta^{\nu\alpha}\eta^{\mu\beta}-\eta^{\mu\nu}\eta^{\alpha\beta})(A(\eta_{\alpha\gamma}\eta_{\beta\delta}+\eta_{\alpha\delta}\eta_{\beta\gamma})+B\eta_{\alpha\beta}\eta_{\gamma\delta}) &= iI^{\mu\nu}{}_{\gamma\delta},\\
A(\delta^{\mu}_{\gamma}\delta^{\nu}_{\delta}+\delta^{\mu}_{\delta}\delta^{\nu}_{\gamma}-\eta^{\mu\nu}\eta_{\gamma\delta})-B\eta^{\mu\nu}\eta_{\gamma\delta} &= iI^{\mu\nu}{}_{\gamma\delta},\\
A(\delta^{\mu}_{\gamma}\delta^{\nu}_{\delta}+\delta^{\mu}_{\delta}\delta^{\nu}_{\gamma})-(A+B)\eta^{\mu\nu}\eta_{\gamma\delta} &= \frac{i}{2}(\delta^{\mu}_{\gamma}\delta^{\nu}_{\delta}+\delta^{\mu}_{\delta}\delta^{\nu}_{\gamma}).
\end{aligned}$$

We see that for the equation to be satisfied $A=-B=\frac{i}{2}$, so the propagator is,

$$\begin{aligned}
\Delta_{\alpha\beta\gamma\delta} &= \frac{1}{k^2}\left(\frac{i}{2}(\eta_{\alpha\gamma}\eta_{\beta\delta}+\eta_{\alpha\delta}\eta_{\beta\gamma})-\frac{i}{2}\eta_{\alpha\beta}\eta_{\gamma\delta}\right),\\
\Delta_{\alpha\beta\gamma\delta} &= \frac{i}{k^2}P_{\alpha\beta\gamma\delta}.
\end{aligned}$$

Chapter 3

Results

In this chapter we will calculate the four amplitudes involved in the process $e^- g \to e^- g$. From this, we calculate the parallel and antiparallel cross sections to obtain the classical DHG sum rule for this process.

3.1 Amplitudes

As seen in the previous section, the vertices are,

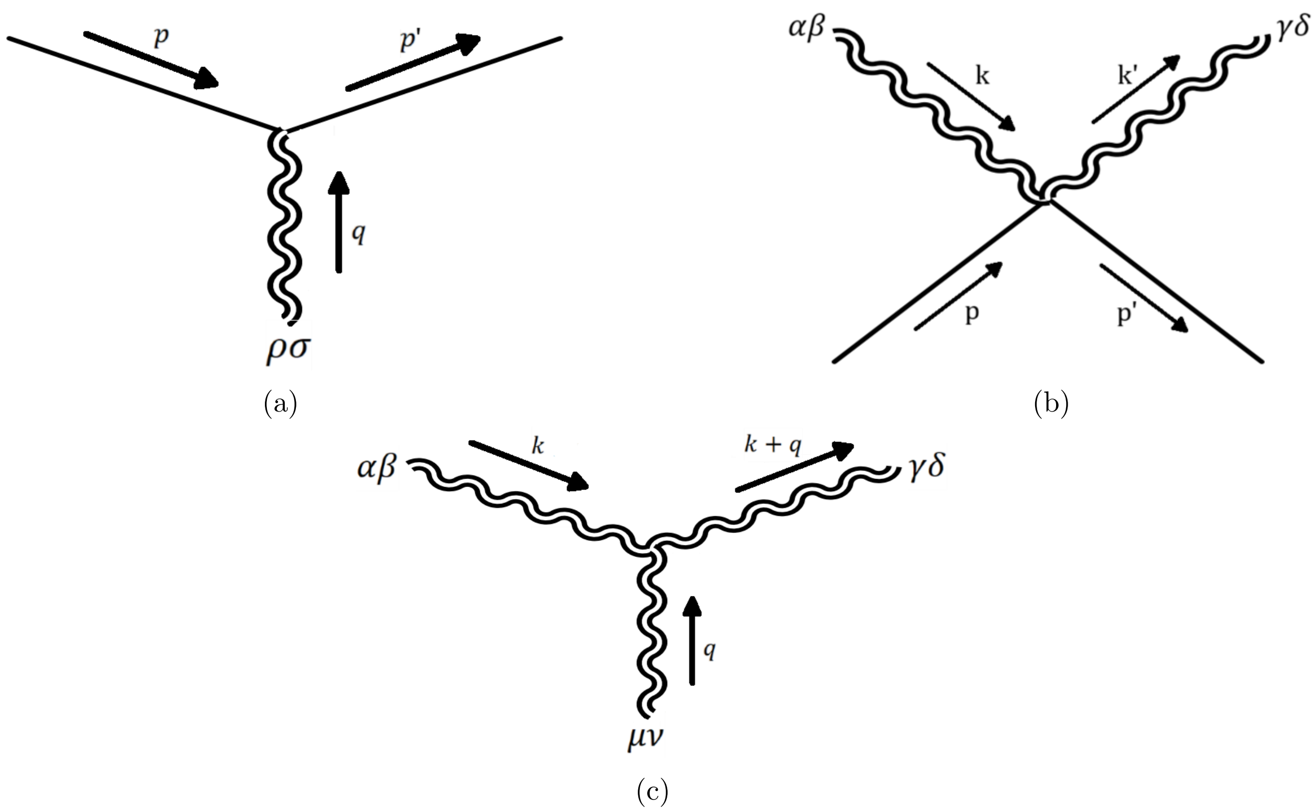


Figure 3.1: vertices for one, two and three gravitons

whose Feynman rules are,

$$\tau_{\rho\sigma}\left(p,p'\right)=\frac{-i\kappa}{2}\left[\frac{1}{4}\left(\gamma_{\rho}\left(p+p'\right)_{\sigma}+\gamma_{\sigma}\left(p+p'\right)_{\rho}\right)-\frac{1}{2}\eta_{\rho\sigma}\left(\frac{1}{2}\left(\not{p}+\not{p}'\right)-m\right)\right], \tag{3.1}$$

$$\begin{aligned}\tau_{\alpha\beta\gamma\delta}(p,p')=i\kappa^{2}\Bigg[&-\frac{1}{4}\left(\frac{1}{4}\eta_{\alpha\beta}\eta_{\gamma\delta}+P_{\alpha\beta\gamma\delta}\right)\left(\frac{1}{2}\left(\not{p}+\not{p}'\right)-m\right)\\&-\frac{1}{32}\left(\eta_{\gamma\delta}\left(\gamma_{\alpha}\left(p+p'\right)_{\beta}+\gamma_{\beta}\left(p+p'\right)_{\alpha}\right)+\eta_{\alpha\beta}\left(\gamma_{\gamma}\left(p+p'\right)_{\delta}+\gamma_{\delta}\left(p+p'\right)_{\gamma}\right)\right)\\&+\frac{3}{16}\left(I_{\alpha\beta d}{}^{\mu}I_{\gamma\delta}{}^{d\nu}+I_{\gamma\delta d}{}^{\mu}I_{\alpha\beta}{}^{d\nu}\right)\gamma_{\nu}\left(p_{\mu}+p'_{\mu}\right)\Bigg],\end{aligned} \tag{3.2}$$

$$\begin{aligned}\tau^{\mu\nu}_{\alpha\beta,\gamma\delta}(k,q)=\frac{i\kappa}{2}\Bigg\{&P_{\alpha\beta,\gamma\delta}\left[k^{\mu}k^{\nu}+(k-q)^{\mu}(k-q)^{\nu}+q^{\mu}q^{\nu}-\frac{3}{2}\eta^{\mu\nu}q^{2}\right]\\&+2q_{\lambda}q_{\sigma}\left[I^{\lambda\sigma,}{}_{\alpha\beta}I^{\mu\nu,}{}_{\gamma\delta}+I^{\lambda\sigma,}{}_{\gamma\delta}I^{\mu\nu,}{}_{\alpha\beta}-I^{\lambda\mu,}{}_{\alpha\beta}I^{\sigma\nu,}{}_{\gamma\delta}-I^{\sigma\nu,}{}_{\alpha\beta}I^{\lambda\mu,}{}_{\gamma\delta}\right]\\&+q_{\lambda}q^{\mu}\left(\eta_{\alpha\beta}I^{\lambda\nu,}{}_{\gamma\delta}+\eta_{\gamma\delta}I^{\lambda\nu,}{}_{\alpha\beta}\right)+q_{\lambda}q^{\nu}\left(\eta_{\alpha\beta}I^{\lambda\mu,}{}_{\gamma\delta}+\eta_{\gamma\delta}I^{\lambda\mu,}{}_{\alpha\beta}\right)\\&-q^{2}\left(\eta_{\alpha\beta}I^{\mu\nu,}{}_{\gamma\delta}+\eta_{\gamma\delta}I^{\mu\nu,}{}_{\alpha\beta}\right)-\eta^{\mu\nu}q^{\lambda}q^{\sigma}\left(\eta_{\alpha\beta}I_{\gamma\delta,\lambda\sigma}+\eta_{\gamma\delta}I_{\alpha\beta,\lambda\sigma}\right)\\&+2q^{\lambda}\left(I^{\sigma\nu,}{}_{\alpha\beta}I_{\gamma\delta,\lambda\sigma}(k-q)^{\mu}+I^{\sigma\mu,}{}_{\alpha\beta}I_{\gamma\delta,\lambda\sigma}(k-q)^{\nu}-I^{\sigma\nu,}{}_{\gamma\delta}I_{\alpha\beta,\lambda\sigma}k^{\mu}-I^{\sigma\mu,}{}_{\gamma\delta}I_{\alpha\beta,\lambda\sigma}k^{\nu}\right)\\&+q^{2}\left(I^{\sigma\mu,}{}_{\alpha\beta}I_{\gamma\delta,\sigma}{}^{\nu}+I_{\alpha\beta,\sigma}{}^{\nu}I^{\sigma\mu,}{}_{\gamma\delta}\right)+\eta^{\mu\nu}q^{\lambda}q_{\sigma}\left(I_{\alpha\beta,\lambda\rho}I^{\rho\sigma,}{}_{\gamma\delta}+I_{\gamma\delta,\lambda\rho}I^{\rho\sigma,}{}_{\alpha\beta}\right)\\&+\left(k^{2}+(k-q)^{2}\right)\left(I^{\sigma\mu,}{}_{\alpha\beta}I_{\gamma\delta,\sigma}{}^{\nu}+I^{\sigma\nu,}{}_{\alpha\beta}I_{\gamma\delta,\sigma}{}^{\mu}-\frac{1}{2}\eta^{\mu\nu}P_{\alpha\beta,\gamma\delta}\right)\\&-\left(k^{2}\eta_{\gamma\delta}I^{\mu\nu,}{}_{\alpha\beta}+(k-q)^{2}\eta_{\alpha\beta}I^{\mu\nu,}{}_{\gamma\delta}\right)\Bigg\},\end{aligned} \tag{3.3}$$

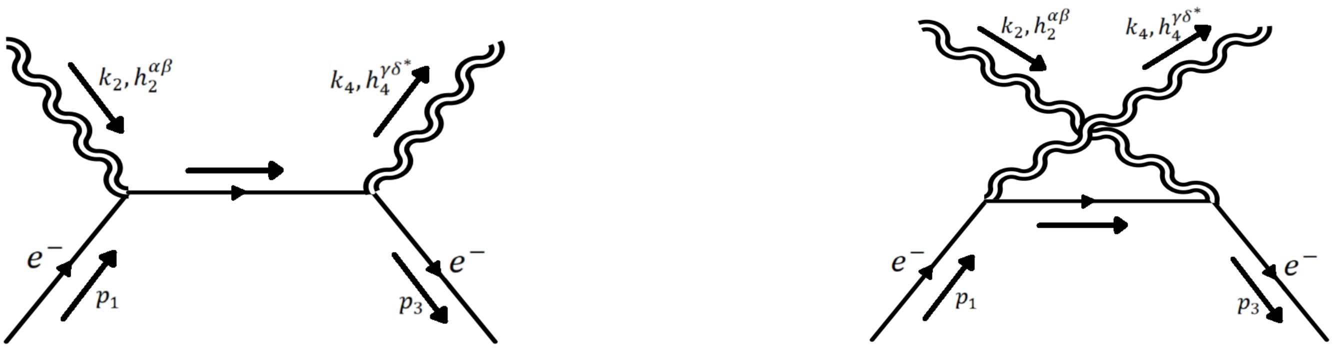
respectively. The triple graviton vertex is given in [7], and the graviton propagator is,

$$D_{\mu\nu,\rho\sigma}(q)=\frac{i}{q^{2}}P_{\mu\nu,\rho\sigma}. \tag{3.4}$$

This is all we need to calculate the amplitudes.

3.1.1 Amplitudes $\mathcal{M}_a$ and $\mathcal{M}_b$

These amplitudes are composed of the vertex 3.1a, but in different arrangements, as seen in the diagrams,



(a) (b)

Figure 3.2: Amplitudes $\mathcal{M}_a$ y $\mathcal{M}_b$

The amplitudes are,

$$i\mathcal{M}_a = \bar{u}(p_3)h_4^{\gamma\delta^*}\tau_{\gamma\delta}(k,p_3)D(k)\tau_{\alpha\beta}(p_1,k)h_2^{\alpha\beta}u(p_1),$$

$$i\mathcal{M}_b = \bar{u}(p_3)h_2^{\alpha\beta}\tau_{\alpha\beta}(k,p_3)D(k)\tau_{\gamma\delta}(p_1,k)h_4^{\gamma\delta^*}u(p_1).$$

We will begin calculating $\mathcal{M}_a$, so we need $\tau_{\alpha\beta}(p_1,-k)h_2^{\alpha\beta}$ and $h_4^{\gamma\delta^*}\tau_{\gamma\delta}(k,-p_3)$,

$$\begin{aligned}
\tau_{\alpha\beta}(p_1,k)h_2^{\alpha\beta} &= \frac{-i\kappa}{2}\left[\frac{1}{4}\left(\gamma_\alpha\left(2p_1+k_2\right)_\beta+\gamma_\beta\left(2p_1+k_2\right)_\alpha\right)-\frac{1}{2}\eta_{\alpha\beta}\left(\frac{1}{2}\left(2\not{p}_1+\not{k}_2\right)-m\right)\right]\epsilon_2^\alpha\epsilon_2^\beta,\\
&= \frac{-i\kappa}{8}\left(\not{\epsilon}_2\left(2p_1+k_2\right)\cdot\epsilon_2+\not{\epsilon}_2\left(2p_1+k_2\right)\cdot\epsilon_2\right),\\
&= \frac{-i\kappa}{2}\not{\epsilon}_2 p_1\cdot\epsilon_2.
\end{aligned}$$

$$\begin{aligned}
\tau_{\gamma\delta}(k,p_3)h_4^{\gamma\delta^*} &= \frac{-i\kappa}{2}\left[\frac{1}{4}\left(\gamma_\gamma\left(2p_3+k_4\right)_\delta+\gamma_\delta\left(2p_3+k_4\right)_\gamma\right)-\frac{1}{2}\eta_{\gamma\delta}\left(\frac{1}{2}\left(2\not{p}_3+\not{k}_4\right)-m\right)\right]\epsilon_4^{\gamma*}\epsilon_4^{\delta*},\\
&= \frac{-i\kappa}{8}\left(\not{\epsilon}_4^*\left(2p_3+k_4\right)\cdot\epsilon_4^*+\not{\epsilon}_4^*\left(2p_3+k_4\right)\cdot\epsilon_4^*\right),\\
&= \frac{-i\kappa}{2}\not{\epsilon}_4^* p_3\cdot\epsilon_4^*.
\end{aligned}$$

With this we have,

$$i\mathcal{M}_a = -\frac{i\kappa^2 p_1\cdot\epsilon_2 p_3\cdot\epsilon_4^*}{4(s-m^2)}\bar{u}_3(p_3)\left(\not{\epsilon}_4^*(\not{p}_1+\not{k}_2+m)\not{\epsilon}_2\right)u(p_1).$$

Now for $\mathcal{M}_b$,

$$\begin{aligned}
\tau_{\alpha\beta}(k,p_3)h_2^{\alpha\beta} &= \frac{-i\kappa}{2}\left[\frac{1}{4}\left(\gamma_\alpha\left(2p_3-k_2\right)_\beta+\gamma_\beta\left(2p_3-k_2\right)_\alpha\right)-\frac{1}{2}\eta_{\alpha\beta}\left(\frac{1}{2}\left(2\not{p}_3-\not{k}_2\right)-m\right)\right]\epsilon_2^\alpha\epsilon_2^\beta,\\
&= \frac{-i\kappa}{8}\left(\not{\epsilon}_2\left(2p_3-k_2\right)\cdot\epsilon_2+\not{\epsilon}_2\left(2p_1-k_2\right)\cdot\epsilon_2\right),\\
&= \frac{-i\kappa}{2}\not{\epsilon}_2 p_3\cdot\epsilon_2.
\end{aligned}$$

$$\begin{aligned}
\tau_{\gamma\delta}(p_1,k)h_4^{\gamma\delta *} &= \frac{-i\kappa}{2}\left[\frac{1}{4}\left(\gamma_\gamma\left(2p_1-k_4\right)_\delta+\gamma_\delta\left(2p_1-k_4\right)_\gamma\right)-\frac{1}{2}\eta_{\gamma\delta}\left(\frac{1}{2}\left(2\not{p}_1-\not{k}_4\right)-m\right)\right]\epsilon_4^{\gamma *}\epsilon_4^{\delta *},\\
&= \frac{-i\kappa}{8}\left(\not{\epsilon}_4^*\left(2p_1-k_4\right)\cdot\epsilon_4^*+\not{\epsilon}_4^*\left(2p_1-k_4\right)\cdot\epsilon_4^*\right),\\
&= \frac{-i\kappa}{2}\not{\epsilon}_4^* p_1\cdot\epsilon_4^*.
\end{aligned}$$

With this we have

$$i\mathcal{M}_b = -\frac{i\kappa^2 p_3\cdot\epsilon_2 p_1\cdot\epsilon_4^*}{4(u-m^2)}\bar{u}_3(p_3)\left(\not{\epsilon}_2(\not{p}_1-\not{k}_4+m)\not{\epsilon}_4^*\right)u(p_1). \tag{3.5}$$

Now we have $\mathcal{M}_a$ and $\mathcal{M}_b$, but the former will always be zero because, as we know, ϵ_2 must be perpendicular to k_2, but k_2 will always be coplanar with p_1(a plane can always be defined with just two vectors), so ϵ_2 also will be perpendicular to p_1 i.e. $\epsilon_2^\pm\cdot p_1=0$. A similar analysis shows that $\epsilon_4^\pm\cdot p_3=0$. Therefore $\mathcal{M}_a$ is always zero, but $\mathcal{M}_b$ isn't.

3.1.2 Amplitude $\mathcal{M}_c$

This amplitude is composed of one vertex, 3.1b, as seen in the diagram,

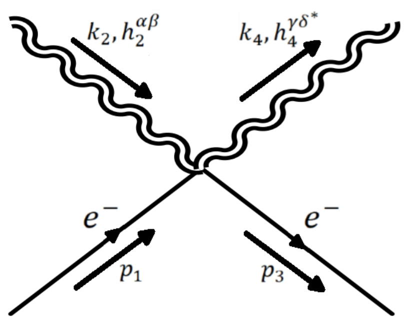


Figure 3.3: Amplitude $\mathcal{M}_c$

From here, the amplitude is,

$$i\mathcal{M}_c = \bar{u}(p_3)h_4^{\gamma\delta *}\tau_{\alpha\beta,\gamma\delta}h_2^{\alpha\beta}u(p_1).$$

First we see that any term in 3.2 that is proportional to $\eta_{\alpha\beta}$ or $\eta_{\gamma\delta}$ will be zero when contracted with the polarizations of the gravitons because $\eta_{\alpha\beta}h_i^{\alpha\beta} = \epsilon_i^\alpha \epsilon_{i\alpha} = 0$. With this in mind we see that the second line is zero.

Now we will do the calculations line by line, starting with the last line because the first one is just,

$$
\begin{aligned}
h_4^{\gamma\delta *}\left(\frac{1}{4}\eta_{\alpha\beta}\eta_{\gamma\delta} + P_{\alpha\beta,\gamma\delta}\right)h_2^{\alpha\beta} &= \frac{1}{2}\epsilon_4^{\gamma *}\epsilon_4^{\delta *}\left(\eta_{\alpha\gamma}\eta_{\beta\delta} + \eta_{\alpha\delta}\eta_{\beta\gamma} - \frac{1}{2}\eta_{\alpha\beta}\eta_{\gamma\delta}\right)\epsilon_2^\alpha\epsilon_2^\beta, \\
&= \frac{1}{2}\left((\epsilon_4^* \cdot \epsilon_2)^2 + (\epsilon_4^* \cdot \epsilon_2)^2\right), \\
&= (\epsilon_4^* \cdot \epsilon_2)^2 .
\end{aligned}
$$

For the last line,

$$
\begin{aligned}
(p_1+p_3)^\epsilon \gamma^\xi h_4^{\gamma\delta *}\left[I_{\xi\phi,\alpha\beta}I^\phi_{\epsilon,\gamma\delta} + I_{\xi\phi,\gamma\delta}I^\phi_{\epsilon,\alpha\beta}\right]h_2^{\alpha\beta} &= (p_1+p_3)^\epsilon\gamma^\xi\frac{\epsilon_4^{\gamma *}\epsilon_4^{\delta *}}{4}\left[(\eta_{\xi\alpha}\eta_{\phi\beta} + \eta_{\xi\beta}\eta_{\phi\alpha})\left(\delta_\gamma^\phi\eta_{\epsilon\delta} + \delta_\delta^\phi\eta_{\epsilon\gamma}\right)\right. \\
&\quad \left. + (\eta_{\xi\gamma}\eta_{\phi\delta} + \eta_{\xi\delta}\eta_{\phi\gamma})\left(\delta_\alpha^\phi\eta_{\epsilon\beta} + \delta_\beta^\phi\eta_{\epsilon\alpha}\right)\right]\epsilon_2^\alpha\epsilon_2^\beta, \\
&= (p_1+p_3)^\epsilon\gamma^\xi\frac{\epsilon_4^{\gamma *}\epsilon_4^{\delta *}}{4}\left[2\left(\eta_{\xi\alpha}\eta_{\gamma\beta}\eta_{\epsilon\delta} + \eta_{\xi\beta}\eta_{\gamma\alpha}\eta_{\epsilon\delta}\right)\right. \\
&\quad \left. + 2\left(\eta_{\xi\gamma}\eta_{\alpha\delta}\eta_{\epsilon\beta} + \eta_{\xi\delta}\eta_{\beta\gamma}\eta_{\epsilon\alpha}\right)\right]\epsilon_2^\alpha\epsilon_2^\beta, \\
&= (p_1+p_3)^\epsilon\gamma^\xi\left[\epsilon_{2\xi}\epsilon_4^* \cdot \epsilon_2\epsilon^*_{4\epsilon} + \epsilon^*_{4\xi}\epsilon_4^* \cdot \epsilon_2\epsilon_{2\epsilon}\right], \\
&= \not{\epsilon}_2\epsilon_4^* \cdot \epsilon_2\epsilon_4^* \cdot p_1 + \not{\epsilon}_4^*\epsilon_4^* \cdot \epsilon_2\epsilon_2 \cdot p_3, \\
&= \epsilon_4^* \cdot \epsilon_2\left[\not{\epsilon}_2\epsilon_4^* \cdot p_1 + \not{\epsilon}_4^*\epsilon_2 \cdot p_3\right],
\end{aligned}
$$

where we used $\epsilon_2^\pm \cdot p_1 = \epsilon_4^\pm \cdot p_3 = 0$. Putting all this together we get

$$
h_4^{\gamma\delta *}\tau_{\alpha\beta,\gamma\delta}h_2^{\alpha\beta} = \frac{i\kappa^2}{2}\left\{\frac{(\epsilon_4^* \cdot \epsilon_2)^2}{2}\left(m - \frac{1}{2}\left(\not{p}_1 + \not{p}_3\right)\right) + \frac{3\epsilon_4^* \cdot \epsilon_2}{8}\left(\not{\epsilon}_2\epsilon_4^* \cdot p_1 + \not{\epsilon}_4^*\epsilon_2 \cdot p_3\right)\right\}.
$$

Finally, using the relations $\bar{u}_3(p_3)(\not{p}_1 + \not{p}_3 - 2m)u_1(p_1) = 0$, allows us to write

$$
\begin{aligned}
i\mathcal{M}_c &= \bar{u}(p_3)h_4^{\gamma\delta *}\tau_{\alpha\beta,\gamma\delta}h_2^{\alpha\beta}u_1(p1), \\
&= \frac{3i\kappa^2}{16}\epsilon_4^* \cdot \epsilon_2\bar{u}(p_3)\left(\not{\epsilon}_2\epsilon_4^* \cdot p_1 + \not{\epsilon}_4^*\epsilon_2 \cdot p_3\right)u(p_1).
\end{aligned} \tag{3.6}
$$

3.1.3 Amplitude $\mathcal{M}_d$

This amplitude is composed of the vertices 3.1a and 3.1c, so the Feynman diagram is,

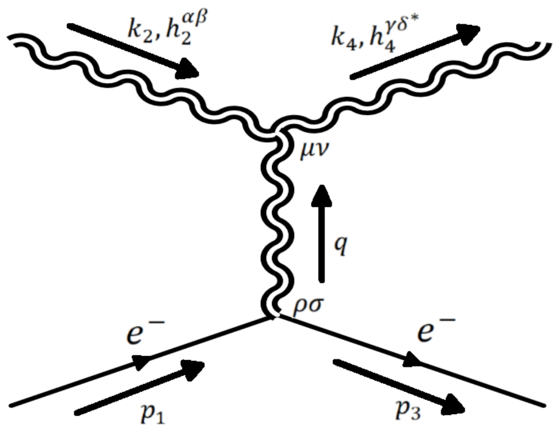


Figure 3.4: Amplitude $\mathcal{M}_d$

then the amplitude is,

$$i\mathcal{M}_d = \bar{u}(p_3)\tau^{\rho\sigma}D_{\rho\sigma,\mu\nu}h_4^{\gamma\delta *}\tau^{\mu\nu}_{\alpha\beta,\gamma\delta}h_2^{\alpha\beta}u(p_1).$$

let us calculate this in parts, starting with $h_4^{\gamma\delta *}\tau^{\mu\nu}_{\alpha\beta,\gamma\delta}h_2^{\alpha\beta}$, but before doing that we notice that any term proportional to $\eta_{\alpha\beta}$ or $\eta_{\gamma\delta}$ in (3.3) will be zero when contracted with the graviton polarizations, as seen in the previous section. Then the third, forth and last line will be zero, and for the first line we will use the previous result $h_4^{\gamma\delta *}P_{\alpha\beta,\gamma\delta}h_2^{\alpha\beta} = (\epsilon_4^* \cdot \epsilon_2)^2$. For the second line we will focus on the first and third term,

$$\begin{aligned} 2q_\lambda q_\sigma h_4^{\gamma\delta *}\left[I^{\lambda\sigma,}{}_{\alpha\beta}I^{\mu\nu,}{}_{\gamma\delta} - I^{\lambda\mu,}{}_{\alpha\beta}I^{\sigma\nu,}{}_{\gamma\delta}\right]h_2^{\alpha\beta} &= q_\lambda q_\sigma \frac{\epsilon_4^{\gamma *}\epsilon_4^{\delta *}}{2}\left[\left(\delta^\lambda_\alpha\delta^\sigma_\beta + \delta^\lambda_\beta\delta^\sigma_\alpha\right)\left(\delta^\mu_\gamma\delta^\nu_\delta + \delta^\mu_\delta\delta^\nu_\gamma\right)\right. \\ &\quad \left. - \left(\delta^\lambda_\alpha\delta^\mu_\beta + \delta^\lambda_\beta\delta^\mu_\alpha\right)\left(\delta^\sigma_\gamma\delta^\nu_\delta + \delta^\sigma_\delta\delta^\nu_\gamma\right)\right]\epsilon_2^\alpha\epsilon_2^\beta, \\ &= \frac{q_\lambda q_\sigma}{2}\left[4\epsilon_2^\lambda\epsilon_2^\sigma\epsilon_4^{\mu *}\epsilon_4^{\nu *} - 4\epsilon_2^\lambda\epsilon_2^\mu\epsilon_4^{\sigma *}\epsilon_4^{\nu *}\right], \\ &= 2\left[(q\cdot\epsilon_2)^2\epsilon_4^{\mu *}\epsilon_4^{\nu *} - q\cdot\epsilon_2 q\cdot\epsilon_4^*\epsilon_2^\mu\epsilon_4^{\nu *}\right]. \end{aligned}$$

For the remaining terms we see that they are similar to the other ones, in the second line but with the exchange $\alpha\beta \leftrightarrow \gamma\delta$, and that results in $\epsilon_2^\mu \leftrightarrow \epsilon_4^{\mu *}$, so the full second line is,

$$\begin{aligned} 2q_\lambda q_\sigma h_4^{\gamma\delta *}\left[I^{\lambda\sigma,}{}_{\alpha\beta}I^{\mu\nu,}{}_{\gamma\delta} - I^{\lambda\mu,}{}_{\alpha\beta}I^{\sigma\nu,}{}_{\gamma\delta} + I^{\lambda\sigma,}{}_{\gamma\delta}I^{\mu\nu,}{}_{\alpha\beta} - I^{\sigma\nu,}{}_{\alpha\beta}I^{\lambda\mu,}{}_{\gamma\delta}\right]h_2^{\alpha\beta} &= 2\left[(q\cdot\epsilon_2)^2\epsilon_4^{\mu *}\epsilon_4^{\nu *} - q\cdot\epsilon_2 q\cdot\epsilon_4^*\epsilon_2^\mu\epsilon_4^{\nu *}\right. \\ &\quad \left. + (q\cdot\epsilon_4^*)^2\epsilon_2^\mu\epsilon_2^\nu - q\cdot\epsilon_2 q\cdot\epsilon_2^*\epsilon_2^\nu\epsilon_4^{\mu *}\right]. \end{aligned}$$

For the fifth line, starting with the first two terms, we have

$$
\begin{aligned}
2q^\lambda h_4^{\gamma\delta *}\left[I^{\sigma\nu,}{}_{\alpha\beta}I_{\gamma\delta,\lambda\sigma}(k-q)^\mu + I^{\sigma\mu,}{}_{\alpha\beta}I_{\gamma\delta,\lambda\sigma}(k-q)^\nu\right]h_2^{\alpha\beta} &= q^\lambda \frac{\epsilon_4^{\gamma *}\epsilon_4^{\delta *}}{2}\left[\left(\delta^\sigma_\alpha\delta^\nu_\beta + \delta^\sigma_\beta\delta^\nu_\alpha\right)\left(\eta_{\gamma\lambda}\eta_{\delta\sigma}+\eta_{\gamma\sigma}\eta_{\delta\lambda}\right)(k_2-q)^\mu\right.\\
&\quad \left. + \left(\delta^\sigma_\alpha\delta^\mu_\beta + \delta^\sigma_\beta\delta^\mu_\alpha\right)\left(\eta_{\gamma\lambda}\eta_{\delta\sigma}+\eta_{\gamma\sigma}\eta_{\delta\lambda}\right)(k_2-q)^\nu\right]\epsilon_2^\alpha\epsilon_2^\beta,\\
&= \frac{q^\lambda}{2}\left[4\epsilon_4^*\cdot\epsilon_2\epsilon_2^\nu\epsilon^*_{4\lambda}(k_2-q)^\mu + 4\epsilon_4^*\cdot\epsilon_2\epsilon_2^\mu\epsilon^*_{4\lambda}(k_2-q)^\nu\right],\\
&= 2\epsilon_4^*\cdot\epsilon_2 q\cdot\epsilon_4^*\left[\epsilon_2^\nu(k_2-q)^\mu + \epsilon_2^\mu(k_2-q)^\nu\right],
\end{aligned}
$$

now the other two,

$$
\begin{aligned}
2q^\lambda h_4^{\gamma\delta *}\left[I^{\sigma\nu,}{}_{\gamma\delta}I_{\alpha\beta,\lambda\sigma}k^\mu + I^{\sigma\mu,}{}_{\gamma\delta}I_{\alpha\beta,\lambda\sigma}k^\nu\right]h_2^{\alpha\beta} &= q^\lambda \frac{\epsilon_4^{\gamma *}\epsilon_4^{\delta *}}{2}\left[\left(\delta^\sigma_\gamma\delta^\nu_\delta + \delta^\sigma_\delta\delta^\nu_\gamma\right)\left(\eta_{\alpha\lambda}\eta_{\beta\sigma}+\eta_{\alpha\sigma}\eta_{\beta\lambda}\right)k_2^\mu\right.\\
&\quad \left. + \left(\delta^\sigma_\gamma\delta^\mu_\delta + \delta^\sigma_\delta\delta^\mu_\gamma\right)\left(\eta_{\alpha\lambda}\eta_{\beta\sigma}+\eta_{\alpha\sigma}\eta_{\beta\lambda}\right)k_2^\nu\right]\epsilon_2^\alpha\epsilon_2^\beta,\\
&= \frac{q^\lambda}{2}\left[4\epsilon_4^*\cdot\epsilon_2\epsilon_{2\lambda}\epsilon_4^{\nu *}k_2^\mu + 4\epsilon_4^*\cdot\epsilon_2\epsilon_{2\lambda}\epsilon_4^{\mu *}k_2^\nu\right],\\
&= 2\epsilon_4^*\cdot\epsilon_2 q\cdot\epsilon_2\left[\epsilon_4^{\nu *}k_2^\mu + \epsilon_4^{\mu *}k_2^\nu\right],
\end{aligned}
$$

With this, the fifth line is,

$$
\begin{aligned}
&2q^\lambda h_4^{\gamma\delta *}\left[I^{\sigma\nu,}{}_{\alpha\beta}I_{\gamma\delta,\lambda\sigma}(k-q)^\mu + I^{\sigma\mu,}{}_{\alpha\beta}I_{\gamma\delta,\lambda\sigma}(k-q)^\nu\right.\\
&\left.-I^{\sigma\nu,}{}_{\gamma\delta}I_{\alpha\beta,\lambda\sigma}k^\mu - I^{\sigma\mu,}{}_{\gamma\delta}I_{\alpha\beta,\lambda\sigma}k^\nu\right]h_2^{\alpha\beta} = 2\epsilon_4^*\cdot\epsilon_2\left(q\cdot\epsilon_4^*\left[\epsilon_2^\nu(k_2-q)^\mu + \epsilon_2^\mu(k_2-q)^\nu\right] - q\cdot\epsilon_2\left[\epsilon_4^{\nu *}k_2^\mu + \epsilon_4^{\mu *}k_2^\nu\right]\right).
\end{aligned}
$$

The term proportional to q^2 in the sixth line,

$$
\begin{aligned}
q^2 h_4^{\gamma\delta *}\left[I^{\sigma\mu,}{}_{\alpha\beta}I_{\gamma\delta,\sigma}{}^\nu + I_{\alpha\beta,\sigma}{}^\nu I^{\sigma\mu,}{}_{\gamma\delta}\right]h_2^{\alpha\beta} &= q^2\frac{\epsilon_4^{\gamma *}\epsilon_4^{\delta *}}{4}\left[\left(\delta^\sigma_\alpha\delta^\mu_\beta + \delta^\sigma_\beta\delta^\mu_\alpha\right)\left(\eta_{\gamma\sigma}\delta^\nu_\delta + \eta_{\delta\sigma}\delta^\nu_\gamma\right)\right.\\
&\quad \left. + \left(\delta^\sigma_\gamma\delta^\mu_\delta + \delta^\sigma_\delta\delta^\mu_\gamma\right)\left(\eta_{\alpha\sigma}\delta^\nu_\beta + \eta_{\beta\sigma}\delta^\nu_\alpha\right)\right]\epsilon_2^\alpha\epsilon_2^\beta,\\
&= \frac{q^2}{4}\left[4\epsilon_4^*\cdot\epsilon_2\epsilon_2^\mu\epsilon_4^{\nu *} + 4\epsilon_4^*\cdot\epsilon_2\epsilon_2^\nu\epsilon_4^{\mu *}\right],\\
&= q^2\epsilon_4^*\cdot\epsilon_2\left[\epsilon_2^\mu\epsilon_4^{\nu *} + \epsilon_2^\nu\epsilon_4^{\mu *}\right],
\end{aligned}
$$

and the remaining term in the sixth line,

$$
\begin{aligned}
\eta^{\mu\nu}q^\lambda q_\sigma h_4^{\gamma\delta *}\left[I_{\alpha\beta\lambda\rho}I^{\rho\sigma,}{}_{\gamma\delta} + I_{\gamma\delta\lambda\rho}I^{\rho\sigma,}{}_{\alpha\beta}\right]h_2^{\alpha\beta} &= \eta^{\mu\nu}q^\lambda q_\sigma\frac{\epsilon_4^{\gamma *}\epsilon_4^{\delta *}}{4}\left[\left(\eta_{\alpha\lambda}\eta_{\beta\rho}+\eta_{\alpha\rho}\eta_{\beta\lambda}\right)\left(\delta^\rho_\gamma\delta^\sigma_\delta + \delta^\rho_\delta\delta^\sigma_\gamma\right)\right.\\
&\quad \left. + \left(\eta_{\gamma\lambda}\eta_{\delta\rho}+\eta_{\gamma\rho}\eta_{\delta\lambda}\right)\left(\delta^\rho_\alpha\delta^\sigma_\beta + \delta^\rho_\beta\delta^\sigma_\alpha\right)\right]\epsilon_2^\alpha\epsilon_2^\beta,\\
&= \frac{\eta^{\mu\nu}q^\lambda q_\sigma}{4}\left[4\epsilon_4^*\cdot\epsilon_2\epsilon_{2\lambda}\epsilon_4^{\sigma *} + 4\epsilon_4^*\cdot\epsilon_2\epsilon^*_{4\lambda}\epsilon_2^\sigma\right],\\
&= 2\eta^{\mu\nu}\epsilon_4^*\cdot\epsilon_2 q\cdot\epsilon_4^* q\cdot\epsilon_2.
\end{aligned}
$$

And lastly for the seventh line we focus on the first two terms, as we already know the result for the third,

$$\left(k^2+(k-q)^2\right) h_4^{\gamma\delta *}\left[I^{\sigma\mu,}{}_{\alpha\beta}I_{\gamma\delta,\sigma}{}^{\nu}+I^{\sigma\nu,}{}_{\alpha\beta}I_{\gamma\delta,\sigma}{}^{\mu}\right] h_2^{\alpha\beta} = (k_2-q)^2\frac{\epsilon_4^{\gamma *}\epsilon_4^{\delta *}}{4}\left[\left(\delta^\sigma_\alpha\delta^\mu_\beta+\delta^\sigma_\beta\delta^\mu_\alpha\right)\left(\eta_{\gamma\sigma}\delta^\nu_\delta+\eta_{\delta\sigma}\delta^\nu_\gamma\right)\right.$$
$$\left.+\left(\delta^\sigma_\alpha\delta^\nu_\beta+\delta^\sigma_\beta\delta^\nu_\alpha\right)\left(\eta_{\gamma\sigma}\delta^\mu_\delta+\eta_{\delta\sigma}\delta^\mu_\gamma\right)\right]\epsilon_2^\alpha\epsilon_2^\beta,$$
$$=(k_2-q)^2\frac{1}{4}\left[4\epsilon_4^*\cdot\epsilon_2\epsilon_2^\mu\epsilon_4^{\nu *}+4\epsilon_4^*\cdot\epsilon_2\epsilon_2^\nu\epsilon_4^{\mu *}\right],$$
$$=(k_2-q)^2\left[\epsilon_4^*\cdot\epsilon_2\epsilon_2^\mu\epsilon_4^{\nu *}+\epsilon_4^*\cdot\epsilon_2\epsilon_2^\nu\epsilon_4^{\mu *}\right].$$

combining all this we have,

$$h_4^{\gamma\delta *}\tau^{\mu\nu}_{\alpha\beta,\gamma\delta}h_2^{\alpha\beta}=\frac{i\kappa}{2}\left\{(\epsilon_2\cdot\epsilon_4^*)^2\left[k^\mu k^\nu+(k-q)^\mu(k-q)^\nu+q^\mu q^\nu-\frac{3}{2}\eta^{\mu\nu}q^2\right]\right.$$
$$+2\left[(q\cdot\epsilon_4^*)^2\epsilon_2^\mu\epsilon_2^\mu+(q\cdot\epsilon_2)^2\epsilon_4^{\mu *}\epsilon_4^{\nu *}-q\cdot\epsilon_2 q\cdot\epsilon_4^*(\epsilon_2^\mu\epsilon_4^{\nu *}+\epsilon_2^\nu\epsilon_4^{\mu *})\right]$$
$$+2\epsilon_4^*\cdot\epsilon_2(q\cdot\epsilon_4^*\left[\epsilon_2^\nu(k_2-q)^\mu+\epsilon_2^\mu(k_2-q)^\nu\right]-q\cdot\epsilon_2\left[\epsilon_4^{\nu *}k_2^\mu+\epsilon_4^{\mu *}k_2^\nu\right])$$
$$+q^2\epsilon_4^*\cdot\epsilon_2\left[\epsilon_2^\mu\epsilon_4^{\nu *}+\epsilon_2^\nu\epsilon_4^{\mu *}\right]+2\eta^{\mu\nu}\epsilon_4^*\cdot\epsilon_2 q\cdot\epsilon_4^* q\cdot\epsilon_2$$
$$\left.+(k_2-q)^2\left[\epsilon_4^*\cdot\epsilon_2\epsilon_2^\mu\epsilon_4^{\nu *}+\epsilon_4^*\cdot\epsilon_2\epsilon_2^\nu\epsilon_4^{\mu *}-\frac{1}{2}\eta^{\mu\nu}(\epsilon_2\cdot\epsilon_4^*)^2\right]\right\}.$$

Due to momentum conservation $q^\mu=k_4^\mu-k_2^\nu$, then

$$h_4^{\gamma\delta *}\tau^{\mu\nu}_{\alpha\beta,\gamma\delta}h_2^{\alpha\beta}=\frac{i\kappa}{2}\left\{(\epsilon_2\cdot\epsilon_4^*)^2\left[k_2^\mu k_2^\nu+(2k_2-k_4)^\mu(2k_2-k_4)^\nu+(k_4-k_2)^\mu(k_4-k_2)^\nu+3\eta^{\mu\nu}k_2\cdot k_4\right]\right.$$
$$+2\left[(k_2\cdot\epsilon_4^*)^2\epsilon_2^\mu\epsilon_2^\nu+(k_4\cdot\epsilon_2)^2\epsilon_4^{\mu *}\epsilon_4^{\nu *}+k_4\cdot\epsilon_2 k_2\cdot\epsilon_4^*(\epsilon_2^\mu\epsilon_4^{\nu *}+\epsilon_2^\nu\epsilon_4^{\mu *})\right]$$
$$-2\epsilon_4^*\cdot\epsilon_2(k_2\cdot\epsilon_4^*\left[\epsilon_2^\nu(2k_2-k_4)^\mu+\epsilon_2^\mu(2k_2-k_4)^\nu\right]+k_4\cdot\epsilon_2\left[\epsilon_4^{\nu *}k_2^\mu+\epsilon_4^{\mu *}k_2^\nu\right])$$
$$-2k_2\cdot k_4\epsilon_4^*\cdot\epsilon_2\left[\epsilon_2^\mu\epsilon_4^{\nu *}+\epsilon_2^\nu\epsilon_4^{\mu *}\right]-2\eta^{\mu\nu}\epsilon_4^*\cdot\epsilon_2 k_2\cdot\epsilon_4^* k_4\cdot\epsilon_2$$
$$\left.-4k_2\cdot k_4\left[\epsilon_4^*\cdot\epsilon_2\epsilon_2^\mu\epsilon_4^{\nu *}+\epsilon_4^*\cdot\epsilon_2\epsilon_2^\nu\epsilon_4^{\mu *}-\frac{1}{2}\eta^{\mu\nu}(\epsilon_2\cdot\epsilon_4^*)^2\right]\right\}.$$

To simplify things we define $\frac{i\kappa}{2}A^{\mu\nu}=h_4^{\gamma\delta *}\tau^{\mu\nu}_{\alpha\beta,\gamma\delta}h_2^{\alpha\beta}$, which is symmetric. Now we calculate $\frac{i\kappa}{2}D_{\rho\sigma,\mu\nu}A^{\mu\nu}$,

$$\frac{i\kappa}{2}D_{\rho\sigma,\mu\nu}A^{\mu\nu}=\frac{\kappa}{8k_2\cdot k_4}\left(\eta_{\mu\rho}\eta_{\nu\sigma}A^{\mu\nu}+\eta_{\mu\sigma}\eta_{\nu\rho}A^{\mu\nu}-\eta_{\mu\nu}\eta_{\rho\sigma}A^{\mu\nu}\right),$$
$$=\frac{\kappa}{8k_2\cdot k_4}\left(A_{\rho\sigma}+A_{\sigma\rho}-\eta_{\rho\sigma}A\right),$$
$$=\frac{\kappa}{4k_2\cdot k_4}\left(A_{\rho\sigma}-\frac{1}{2}\eta_{\rho\sigma}A\right).$$

Lastly, we calculate $\frac{\kappa}{4k_2\cdot k_4}\tau^{\rho\sigma}\left(A_{\rho\sigma}-\frac{1}{2}\eta_{\rho\sigma}A\right)$

$$
\begin{aligned}
\frac{\kappa}{4k_2\cdot k_4}\tau^{\rho\sigma}\left(A_{\rho\sigma}-\frac{1}{2}\eta_{\rho\sigma}A\right) &= \frac{-i\kappa^2}{8k_2\cdot k_4}\left[\frac{1}{4}\left(\gamma^\rho\left(p_1+p_3\right)^\sigma+\gamma^\sigma\left(p_1+p_3\right)^\rho\right)\left(A_{\rho\sigma}-\frac{1}{2}\eta_{\rho\sigma}A\right)\right. \\
&\left.-\frac{1}{2}\eta^{\rho\sigma}\left(A_{\rho\sigma}-\frac{1}{2}\eta_{\rho\sigma}A\right)\left(\frac{1}{2}\left(\not{p}_1+\not{p}_3\right)-m\right)\right], \\
&= \frac{-i\kappa^2}{8k_2\cdot k_4}\left[\frac{1}{2}\gamma^\rho\left(p_1+p_3\right)^\sigma\left(A_{\rho\sigma}-\frac{1}{2}\eta_{\rho\sigma}A\right)+\frac{1}{2}A\left(\frac{1}{2}\left(\not{p}_1+\not{p}_3\right)-m\right)\right], \\
&= \frac{-i\kappa^2}{16k_2\cdot k_4}\left[\gamma^\rho\left(p_1+p_3\right)^\sigma A_{\rho\sigma}-\frac{1}{2}\left(\not{p}_1+\not{p}_3\right)A+A\left(\frac{1}{2}\left(\not{p}_1+\not{p}_3\right)-m\right)\right], \\
&= \frac{-i\kappa^2}{16k_2\cdot k_4}\left[\gamma^\rho\left(p_1+p_3\right)^\sigma A_{\rho\sigma}-mA\right].
\end{aligned}
$$

We also need A,

$$
\begin{aligned}
A &= \left(\epsilon_2\cdot\epsilon_4^*\right)^2\left[-4k_2\cdot k_4-2k_2\cdot k_4+12k_2\cdot k_4\right]+4k_4\cdot\epsilon_2 k_2\cdot\epsilon_4^*\epsilon_2\cdot\epsilon_4^*-0 \\
&-4k_2\cdot k_4(\epsilon_4^*\cdot\epsilon_2)^2-8\epsilon_4^*\cdot\epsilon_2 k_2\cdot\epsilon_4^* k_4\cdot\epsilon_2-0, \\
&= 2k_2\cdot k_4\left(\epsilon_2\cdot\epsilon_4^*\right)^2-4\epsilon_4^*\cdot\epsilon_2 k_2\cdot\epsilon_4^* k_4\cdot\epsilon_2,
\end{aligned}
$$

and $\gamma^\rho\left(p_1+p_3\right)^\sigma A_{\rho\sigma}$, expression that can be simplified using $\epsilon_2^\pm\cdot p_1=\epsilon_4^\pm\cdot p_3=0$, $\epsilon_2^\pm\cdot p_3=-\epsilon_2^\pm\cdot k_4$, $\epsilon_4^\pm\cdot p_1=-\epsilon_4^\pm\cdot k_2$ and $(k_2-k_4)\cdot(p_1+p_3)=0$

$$
\begin{aligned}
\gamma^{\rho}\left(p_1+p_3\right)^{\sigma} A_{\rho\sigma} = {} & \left(\epsilon_2\cdot\epsilon_4^*\right)^2 [\not{k}_2 k_2\cdot(p_1+p_3)+(2\not{k}_2-\not{k}_4)(2k_2-k_4)\cdot(p_1+p_3) \\
& +(\not{k}_4-\not{k}_2)(k_4-k_2)\cdot(p_1+p_3)+3(\not{p}_1+\not{p}_3)k_2\cdot k_4\Big] \\
& +2\left[\left(k_2\cdot\epsilon_4^*\right)^2\not{\epsilon}_2\epsilon_2\cdot p_3+\left(k_4\cdot\epsilon_2\right)^2\not{\epsilon}_4^*\epsilon_4^*\cdot p_1+k_4\cdot\epsilon_2 k_2\cdot\epsilon_4^*(\not{\epsilon}_2\epsilon_4^*\cdot p_1+\not{\epsilon}_4^*\epsilon_2\cdot p_1)\right] \\
& -2\epsilon_4^*\cdot\epsilon_2(k_2\cdot\epsilon_4^*\left[\not{\epsilon}_2(2k_2-k_4)\cdot(p_1+p_3)+(2\not{k}_2-\not{k}_4)\epsilon_2\cdot p_3\right] \\
& +k_4\cdot\epsilon_2\left[\not{\epsilon}_4^* k_2\cdot(p_1+p_3)+\not{k}_2\epsilon_4^*\cdot p_1\right]) \\
& -2k_2\cdot k_4\epsilon_4^*\cdot\epsilon_2\left[\not{\epsilon}_2\epsilon_4^*\cdot p_1+\not{\epsilon}_4^*\epsilon_2\cdot p_3\right]-2(\not{p}_1+\not{p}_3)\epsilon_4^*\cdot\epsilon_2 k_2\cdot\epsilon_4^* k_4\cdot\epsilon_2 \\
& -4k_2\cdot k_4\left[\epsilon_4^*\cdot\epsilon_2(\not{\epsilon}_2\epsilon_4^*\cdot p_1+\not{\epsilon}_4^*\epsilon_2\cdot p_3)-\frac{1}{2}(\not{p}_1+\not{p}_3)(\epsilon_2\cdot\epsilon_4^*)^2\right], \\
= {} & \left(\epsilon_2\cdot\epsilon_4^*\right)^2\left[(3\not{k}_2-\not{k}_4)k_2\cdot(p_1+p_3)+5(\not{p}_1+\not{p}_3)k_2\cdot k_4\right] \\
& -2\left[\left(k_2\cdot\epsilon_4^*\right)^2\not{\epsilon}_2\epsilon_2\cdot k_4+\left(k_4\cdot\epsilon_2\right)^2\not{\epsilon}_4^*\epsilon_4^*\cdot k_2+k_4\cdot\epsilon_2 k_2\cdot\epsilon_4^*(\not{\epsilon}_2\epsilon_4^*\cdot k_2+\not{\epsilon}_4^*\epsilon_2\cdot k_4)\right] \\
& -2\epsilon_4^*\cdot\epsilon_2(k_2\cdot\epsilon_4^*\left[\not{\epsilon}_2 k_2\cdot(p_1+p_3)-(2\not{k}_2-\not{k}_4)\epsilon_2\cdot k_4\right] \\
& +k_4\cdot\epsilon_2\left[\not{\epsilon}_4^* k_2\cdot(p_1+p_3)-\not{k}_2\epsilon_4^*\cdot k_2\right]) \\
& +6k_2\cdot k_4\epsilon_4^*\cdot\epsilon_2\left[\not{\epsilon}_2\epsilon_4^*\cdot k_2+\not{\epsilon}_4^*\epsilon_2\cdot k_4\right]-2(\not{p}_1+\not{p}_3)\epsilon_4^*\cdot\epsilon_2 k_2\cdot\epsilon_4^* k_4\cdot\epsilon_2, \\
= {} & \left(\epsilon_2\cdot\epsilon_4^*\right)^2\left[(3\not{k}_2-\not{k}_4)k_2\cdot(p_1+p_3)+5(\not{p}_1+\not{p}_3)k_2\cdot k_4\right] \\
& -4k_4\cdot\epsilon_2 k_2\cdot\epsilon_4^*(\not{\epsilon}_2\epsilon_4^*\cdot k_2+\not{\epsilon}_4^*\epsilon_2\cdot k_4) \\
& -2\epsilon_4^*\cdot\epsilon_2(k_2\cdot(p_1+p_3)(\not{\epsilon}_2 k_2\cdot\epsilon_4^*+\not{\epsilon}_4^* k_4\cdot\epsilon_2)-(3\not{k}_2-\not{k}_4)k_2\cdot\epsilon_4^*\epsilon_2\cdot k_4) \\
& +6k_2\cdot k_4\epsilon_4^*\cdot\epsilon_2\left[\not{\epsilon}_2\epsilon_4^*\cdot k_2+\not{\epsilon}_4^*\epsilon_2\cdot k_4\right]-2(\not{p}_1+\not{p}_3)\epsilon_4^*\cdot\epsilon_2 k_2\cdot\epsilon_4^* k_4\cdot\epsilon_2, \\
= {} & \left(\epsilon_2\cdot\epsilon_4^*\right)^2\left[(3\not{k}_2-\not{k}_4)k_2\cdot(p_1+p_3)+5(\not{p}_1+\not{p}_3)k_2\cdot k_4\right] \\
& -2(\not{\epsilon}_2\epsilon_4^*\cdot k_2+\not{\epsilon}_4^*\epsilon_2\cdot k_4)(2k_4\cdot\epsilon_2 k_2\cdot\epsilon_4^*-3k_2\cdot k_4\epsilon_4^*\cdot\epsilon_2+\epsilon_4^*\cdot\epsilon_2 k_2\cdot(p_1+p_3)) \\
& +2\epsilon_4^*\cdot\epsilon_2 k_2\cdot\epsilon_4^*\epsilon_2\cdot k_4(3\not{k}_2-\not{k}_4-\not{p}_1-\not{p}_3).
\end{aligned}
$$

Combining the last two results,

$$\begin{aligned}\frac{\kappa}{4k_2\cdot k_4}\tau^{\rho\sigma}\left(A_{\rho\sigma}-\frac{1}{2}\eta_{\rho\sigma}A\right) &= \frac{-i\kappa^2}{16k_2\cdot k_4}\left[\gamma^\rho\left(p_1+p_3\right)^\sigma A_{\rho\sigma}-mA\right],\\
&= \frac{-i\kappa^2}{16k_2\cdot k_4}\Big[(\epsilon_2\cdot\epsilon_4^*)^2\Big[(3\not{k}_2-\not{k}_4)k_2\cdot(p_1+p_3)+5(\not{p}_1+\not{p}_3)k_2\cdot k_4\Big]\\
&\quad -2(\not{\epsilon}_2\epsilon_4^*\cdot k_2+\not{\epsilon}_4^*\epsilon_2\cdot k_4)(2k_4\cdot\epsilon_2k_2\cdot\epsilon_4^*-3k_2\cdot k_4\epsilon_4^*\cdot\epsilon_2+\epsilon_4^*\cdot\epsilon_2k_2\cdot(p_1+p_3))\\
&\quad +2\epsilon_4^*\cdot\epsilon_2k_2\cdot\epsilon_4^*\epsilon_2\cdot k_4(3\not{k}_2-\not{k}_4-\not{p}_1-\not{p}_3)-2mk_2\cdot k_4\left(\epsilon_2\cdot\epsilon_4^*\right)^2\\
&\quad +4m\epsilon_4^*\cdot\epsilon_2k_2\cdot\epsilon_4^*k_4\cdot\epsilon_2\Big],\\
&= \frac{-i\kappa^2}{16k_2\cdot k_4}\Big[(\epsilon_2\cdot\epsilon_4^*)^2\Big[(3\not{k}_2-\not{k}_4)k_2\cdot(p_1+p_3)+(5(\not{p}_1+\not{p}_3)-2m)k_2\cdot k_4\Big]\\
&\quad -2(\not{\epsilon}_2\epsilon_4^*\cdot k_2+\not{\epsilon}_4^*\epsilon_2\cdot k_4)(2k_4\cdot\epsilon_2k_2\cdot\epsilon_4^*-3k_2\cdot k_4\epsilon_4^*\cdot\epsilon_2+\epsilon_4^*\cdot\epsilon_2k_2\cdot(p_1+p_3))\\
&\quad +2\epsilon_4^*\cdot\epsilon_2k_2\cdot\epsilon_4^*\epsilon_2\cdot k_4(3\not{k}_2-\not{k}_4-\not{p}_1-\not{p}_3+2m)\Big].\end{aligned}$$

Finally we have that $\mathcal{M}_d$ is,

$$\begin{aligned}i\mathcal{M}_d &= \frac{-i\kappa^2}{16k_2\cdot k_4}\bar{u}_3(p_3)\Big[(\epsilon_2\cdot\epsilon_4^*)^2\Big[(3\not{k}_2-\not{k}_4)k_2\cdot(p_1+p_3)+(5(\not{p}_1+\not{p}_3)-2m)k_2\cdot k_4\Big]\\
&\quad -2(\not{\epsilon}_2\epsilon_4^*\cdot k_2+\not{\epsilon}_4^*\epsilon_2\cdot k_4)(2k_4\cdot\epsilon_2k_2\cdot\epsilon_4^*-3k_2\cdot k_4\epsilon_4^*\cdot\epsilon_2+\epsilon_4^*\cdot\epsilon_2k_2\cdot(p_1+p_3))\\
&\quad +2\epsilon_4^*\cdot\epsilon_2k_2\cdot\epsilon_4^*\epsilon_2\cdot k_4(3\not{k}_2-\not{k}_4-\not{p}_1-\not{p}_3+2m)\Big]u_1(p_1).\end{aligned}$$

To simplify the expression we use $\bar{u}_3(p_3)(\not{p}_1+\not{p}_3-2m)u_1(p_1)=0$ and $\bar{u}_3(p_3)(\not{k}_2-\not{k}_4)u_1(p_1)=0$. With this we have,

$$\begin{aligned}i\mathcal{M}_d &= \frac{-i\kappa^2}{16k_2\cdot k_4}\bar{u}_3(p_3)\Big[(\epsilon_2\cdot\epsilon_4^*)^2\Big[2\not{k}_2k_2\cdot(p_1+p_3)+4(\not{p}_1+\not{p}_3)k_2\cdot k_4\Big]\\
&\quad -2(\not{\epsilon}_2\epsilon_4^*\cdot k_2+\not{\epsilon}_4^*\epsilon_2\cdot k_4)(2k_4\cdot\epsilon_2k_2\cdot\epsilon_4^*-3k_2\cdot k_4\epsilon_4^*\cdot\epsilon_2+\epsilon_4^*\cdot\epsilon_2k_2\cdot(p_1+p_3))\\
&\quad +4\epsilon_4^*\cdot\epsilon_2k_2\cdot\epsilon_4^*\epsilon_2\cdot k_4\not{k}_2\Big]u_1(p_1),\\
&= \frac{i\kappa^2}{4t}\bar{u}_3(p_3)\Big[(\epsilon_2\cdot\epsilon_4^*)^2\Big[\not{k}_2k_2\cdot(p_1+p_3)+2(\not{p}_1+\not{p}_3)k_2\cdot k_4\Big]\\
&\quad -(\not{\epsilon}_2\epsilon_4^*\cdot k_2+\not{\epsilon}_4^*\epsilon_2\cdot k_4)(2k_4\cdot\epsilon_2k_2\cdot\epsilon_4^*-3k_2\cdot k_4\epsilon_4^*\cdot\epsilon_2+\epsilon_4^*\cdot\epsilon_2k_2\cdot(p_1+p_3))\\
&\quad +2\epsilon_4^*\cdot\epsilon_2k_2\cdot\epsilon_4^*\epsilon_2\cdot k_4\not{k}_2\Big]u_1(p_1).\end{aligned} \tag{3.7}$$

All of this result were corroborated with FeynCalc

3.2 Classical Compton DHG sum rule

To calculate the sum rule, we first calculate each of the sixteen possible helicity configuration in Mathematica using (1.1) and (1.2), and the result are,

$$\mathcal{M}^2(+,+,+,+) = \mathcal{M}^2(-,-,-,-) = \frac{4e^4\left(m^4 - su\right)^3}{\left(m^2-s\right)^4\left(m^2-u\right)^2},$$

$$\mathcal{M}^2(+,+,+,-) = \mathcal{M}^2(-,-,-,+) = \mathcal{M}^2(-,+,-,-) = \mathcal{M}^2(+,-,+,+) = \frac{4e^4m^4t^2\left(m^4 - su\right)}{\left(m^2-s\right)^4\left(m^2-u\right)^2},$$

$$\mathcal{M}^2(+,+,-,+) = \mathcal{M}^2(-,-,+,-) = \mathcal{M}^2(-,+,+,+) = \mathcal{M}^2(+,-,-,-) = -\frac{4e^4m^2t\left(m^4 - su\right)^2}{\left(m^2-s\right)^4\left(m^2-u\right)^2},$$

$$\mathcal{M}^2(+,+,-,-) = \mathcal{M}^2(-,-,+,+) = -\frac{4e^4m^2s^2t^3}{\left(m^2-s\right)^4\left(m^2-u\right)^2},$$

$$\mathcal{M}^2(-,+,+,-) = \mathcal{M}^2(+,-,-,+) = -\frac{4e^4m^6t^3}{\left(m^2-s\right)^4\left(m^2-u\right)^2},$$

$$\mathcal{M}^2(-,+,-,+) = \mathcal{M}^2(+,-,+,-) = \frac{4e^4(m^4 - su)((s-m^2)^2 + tm^2)^2}{\left(m^2-s\right)^4\left(m^2-u\right)^2}.$$

NOw we separate them in parallel and anti-parallel contributions, and calculate $\frac{d\sigma_P}{d\Omega_{CM}}$ and $\frac{d\sigma_A}{d\Omega_{CM}}$. With this, and (2.1) we can calculate what we need for the sum rule,

$$\begin{aligned}
\sigma_P - \sigma_A &= 2\pi\int_0^\pi \left(\frac{d\sigma_P}{d\Omega_{CM}} - \frac{d\sigma_A}{d\Omega_{CM}}\right)\frac{d\cos\left(\theta_{CM}\right)}{d\cos\left(\theta_{Lab}\right)}\sin\left(\theta_{Lab}\right)d\theta_{Lab}, \\
&= 2\pi\int_0^\pi \left(\frac{d\sigma_P}{d\Omega_{CM}} - \frac{d\sigma_A}{d\Omega_{CM}}\right)\left(1+\frac{2\nu}{m_e}\right)\frac{m_e^2\sin\left(\theta_{Lab}\right)}{(m_e+\nu(1-\cos\left(\theta_{Lab}\right))^2}d\theta_{Lab}, \\
&= -\frac{\pi\alpha^2}{m^2\nu}\left[\left(1+\frac{m}{\nu}\right)\ln\left(1+\frac{2m}{\nu}\right) - 2\left(1+\frac{\nu^2}{(m+2\nu)^2}\right)\right].
\end{aligned}$$

Now integrating this we have,

$$\begin{aligned}
DHG &= \int_0^\infty \frac{\sigma_P - \sigma_A}{\nu}d\nu \\
&= 0
\end{aligned}$$

3.3 Classical graviton DHG sum rule

Now, as we did before we calculate each of the sixteen possible helicity configuration in Mathematica using (3.5), (3.6) and (3.7) (for the result see 5.1). Then we separate them in parallel and anti-parallel, and calculate $\frac{d\sigma_P}{d\Omega_{CM}}$ and $\frac{d\sigma_A}{d\Omega_{CM}}$ (for the result also see 5.1). with this and (2.1) we can calculate what we need for the sum rule,

$$\begin{aligned}\sigma_P - \sigma_A &= 2\pi \int_0^\pi \left(\frac{d\sigma_P}{d\Omega_{CM}} - \frac{d\sigma_A}{d\Omega_{CM}} \right) \frac{d\cos(\theta_{CM})}{d\cos(\theta_{Lab})} \sin(\theta_{Lab}) d\theta_{Lab}, \\ &= 2\pi \int_0^\pi \left(\frac{d\sigma_P}{d\Omega_{CM}} - \frac{d\sigma_A}{d\Omega_{CM}} \right) \left(1 + \frac{2\nu}{m_e}\right) \frac{m_e^2 \sin(\theta_{Lab})}{(m_e + \nu(1-\cos(\theta_{Lab}))^2} d\theta_{Lab}, \\ &= \infty.\end{aligned}$$

We see that the difference of polarized cross sections diverges. We can pin point the problem if we calculate the total cross section in the CM frame,

$$\begin{aligned}\sigma_T &= 2\pi \int_0^\pi \left(\frac{d\sigma_P}{d\Omega_{CM}} + \frac{d\sigma_A}{d\Omega_{CM}} \right) d\theta_{CM}, \\ &= \infty.\end{aligned} \tag{3.8}$$

We see that it also diverges, we will discuss the reason behind this result and a possible solution for it in the next section.

3.4 Possible unitarity violation

A quick inspection of (3.8) shows that the divergence happens at $\theta_{CM} = 0$, but even if we regularize the integral with a minimal angle, $\sigma_T = 2\pi \int_{\theta_{min}}^\pi \left(\frac{d\sigma_P}{d\Omega_{CM}} + \frac{d\sigma_A}{d\Omega_{CM}} \right) d\theta_{CM}$ the resulting cross section still grows with the energy ($\sigma_T \sim s$), which is a violation of unitarity.

We can try to save unitarity taking inspiration form [14] and calculate $\sigma_P - \sigma_A$ without the triple graviton vertex (for the amplitudes and cross sections see 5.2) and the result is,

$$\begin{aligned}\sigma_P - \sigma_A &= 2\pi \int_0^\pi \left(\frac{d\sigma_P}{d\Omega_{CM}} - \frac{d\sigma_A}{d\Omega_{CM}} \right) \frac{d\cos(\theta_{CM})}{d\cos(\theta_{Lab})} \sin(\theta_{Lab}) d\theta_{Lab}, \\ &= 2\pi \int_0^\pi \left(\frac{d\sigma_P}{d\Omega_{CM}} - \frac{d\sigma_A}{d\Omega_{CM}} \right) \left(1 + \frac{2\nu}{m_e}\right) \frac{m_e^2 \sin(\theta_{Lab})}{(m_e + \nu(1-\cos(\theta_{Lab}))^2} d\theta_{Lab}, \\ &= \frac{mk^4}{30720\pi v^3 (m+2v)^3} \left(v \left(-15m^6 + 195m^5 v + 1540m^4 v^2 + 3450 m^3 v^3 + 2752 m^2 v^4 + 376 m v^5 - 100 v^6 \right) \right. \\ &\left. + \frac{15}{2} m^2 (m+2v)^3 \left(m^2 - 18mv - 20v^2 \right) \ln\left(1 + \frac{2v}{m}\right) \right).\end{aligned}$$

We see that it is finite, a good sign, but if we try to calculate the sum rule it diverges.

We can see that the process still breaks unitarity if we calculate the total cross section,

$$\begin{aligned}\sigma_T &= 2\pi \int_0^\pi \left(\frac{d\sigma_P}{d\Omega_{CM}} + \frac{d\sigma_A}{d\Omega_{CM}} \right) d\theta_{CM}, \\ &= \frac{\kappa^4}{61440\pi \left(m^2 - s\right)^3 s^3} \left(\left(m^2 - s\right) \left(15m^{12} - 108m^{10}s + 503m^8 s^2 + 1888m^6 s^3 + 493m^4 s^4 - 172m^2 s^5 + 21s^6\right) \right. \\ &\left. +120m^4 s^3 \left(11m^4 + 12m^2 s - s^2\right) \log\left[\frac{s}{m^2}\right] \right),\end{aligned}$$

we see that the cross section still grows to fast $\sigma_T \sim s$, so we find that this process breaks unitarity.

We can more clearly see the behaviour of the sum rule by graphing its integrand $\left(\frac{\sigma_P - \sigma_A}{\nu} \text{ vs. } \nu\right)$ and compare the different results for the graviton Compton scattering and its photonic counterpart.

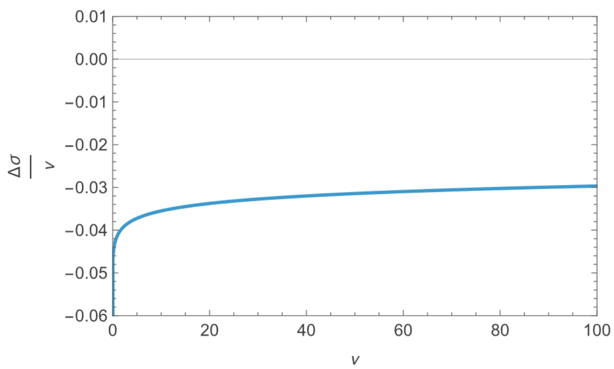


Figure 3.5: Graph $\frac{\sigma_P - \sigma_A}{\nu}$ vs. ν for graviton Compton scattering with triple graviton vertex.

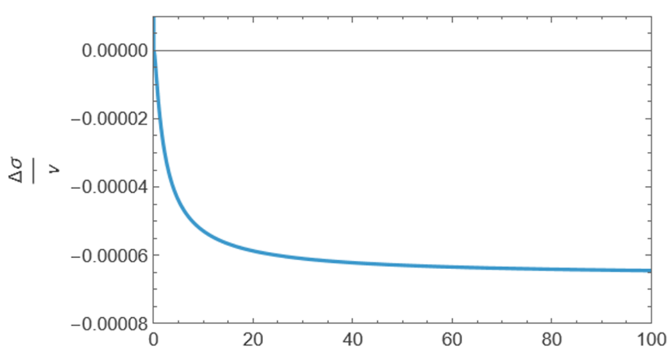


Figure 3.6: Graph $\frac{\sigma_P - \sigma_A}{\nu}$ vs. ν for graviton Compton scattering without triple graviton vertex.

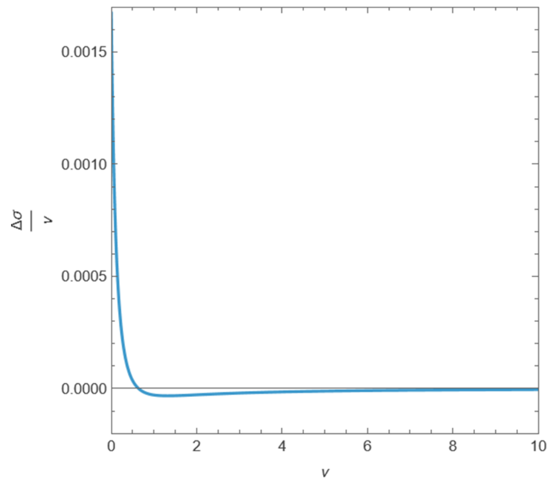
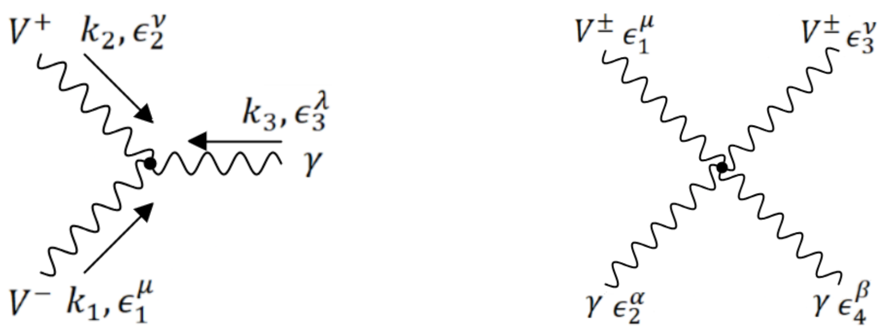


Figure 3.7: Graph $\frac{\sigma_P - \sigma_A}{\nu}$ vs. ν for normal Compton scattering.

We see that one critical difference between 3.5 and 3.6 with 3.7 is that the positive and negative areas can cancel out in the photonic version of the scattering but with gravitons this doesn't happen, meaning that one polarization configuration (parallel or anti-parallel) far dominates the other one for all energies.

3.5 DHG sum rule for spin 1 particle scattering

As a bonus we can wonder about the sum rule for massive spin 1 particles, whose vertices with photons are,



(a) (b)

Figure 3.8: Vertices for photon and a spin 1 particle.

The corresponding Feynman rules and propagator are,

$$\tau_{\mu\nu\lambda}(k_1, k_2, k_3) = ie\left[g_{\mu\nu}(k_1 - k_2)_\lambda + g_{\nu\lambda}(k_2 - k_3)_\mu + g_{\lambda\mu}(k_3 - k_1)_\nu\right] \tag{3.9}$$

$$\tau_{\mu\nu\alpha\beta} = -ie^2\left[2g_{\alpha\beta}g_{\mu\nu} - g_{\alpha\mu}g_{\beta v} - g_{\alpha\nu}g_{\beta\mu}\right] \tag{3.10}$$

$$D_{\mu\nu}(q) = \frac{-i}{q^2 - m^2}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{m^2}\right) \tag{3.11}$$

We can construct the amplitudes $\mathcal{M}_a$, $\mathcal{M}_b$ and $\mathcal{M}_c$ for the diagrams in 3.9.

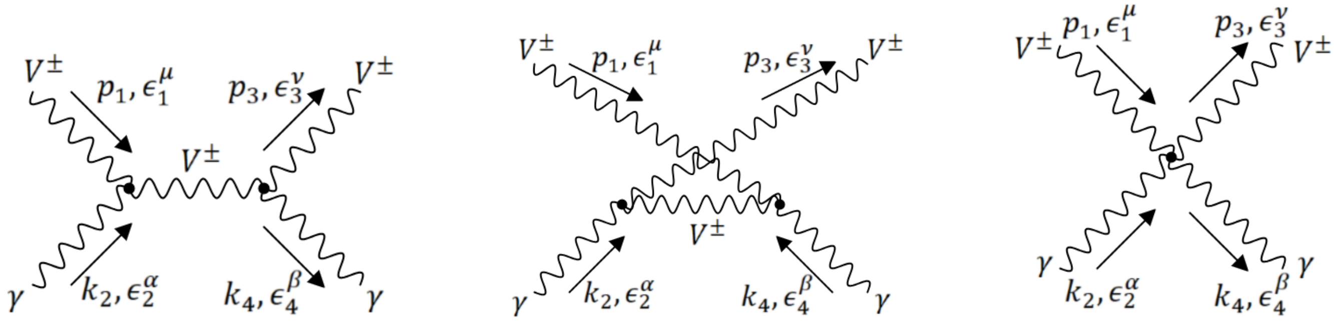


Figure 3.9: Amplitudes for $V^{\pm}\gamma \to V^{\pm}\gamma$

They are

$$i\mathcal{M}_a = \epsilon_3^{\nu *}\tau_{\gamma\sigma\beta}(-p_3, p_1 + k_2, -k_4)\epsilon_4^{\beta *}D_{\sigma\rho}(k)\epsilon_1^{\mu}\tau_{\rho\mu\alpha}(-p_1 - k_2, p_1, k_2)\epsilon_2^{\alpha},$$

$$i\mathcal{M}_b = \epsilon_3^{\nu *}\tau_{\gamma\sigma\alpha}(-p_3, p_1 - k_4, k_2)\epsilon_2^{\alpha}D_{\sigma\rho}(k)\epsilon_1^{\mu}\tau_{\rho\mu\beta}(-p_1 + k_4, p_1, -k_4)\epsilon_4^{\beta *},$$

$$i\mathcal{M}_c = \epsilon_3^{\nu *}\epsilon_4^{\beta *}\tau_{\mu\nu\alpha\beta}\epsilon_1^{\mu}\epsilon_2^{\alpha}.$$

To calculate the amplitudes we use FeynCalc, and the result are,

$$i\mathcal{M}_a = \frac{ie^2}{s - m^2}\epsilon_1 \cdot \epsilon_2\epsilon_3^* \cdot \epsilon_4^*(u - t - m^2),$$

$$i\mathcal{M}_b = \frac{ie^2}{m^2 - u}\left[\epsilon_1 \cdot \epsilon_4^*\epsilon_3^* \cdot \epsilon_2(m^2 + t - s) + 4p_1 \cdot \epsilon_4^*(p_3 \cdot \epsilon_1\epsilon_2 \cdot \epsilon_3^* - p_3 \cdot \epsilon_2\epsilon_1 \cdot \epsilon_3^*)\right.$$
$$\left. + 4p_1 \cdot \epsilon_3^*(p_1 \cdot \epsilon_4^*\epsilon_1 \cdot \epsilon_2 + p_3 \cdot \epsilon_2\epsilon_1 \cdot \epsilon_4) + 4p_3 \cdot \epsilon_1(p_3 \cdot \epsilon_2\epsilon_3^* \cdot \epsilon_4^* - p_1 \cdot \epsilon_3^*\epsilon_2 \cdot \epsilon_4^*)\right],$$

$$i\mathcal{M}_c = ie^2(\epsilon_1 \cdot \epsilon_4^*\epsilon_2 \cdot \epsilon_3^* + \epsilon_1 \cdot \epsilon_2\epsilon_3^* \cdot \epsilon_4^* - 2\epsilon_1 \cdot \epsilon_3^*\epsilon_2 \cdot \epsilon_4^*).$$

As we did before, we calculate each of the possible helicity configurations, which are,

$$\mathcal{M}^2(+,+,+,+) = \mathcal{M}^2(-,-,-,-) = \frac{4e^4s^4t^4}{(m^2 - s)^6\,(m^2 - u)^2},$$

$$\mathcal{M}^2(+,+,+,-) = \mathcal{M}^2(-,-,-,+) = \mathcal{M}^2(-,+,-,-) = \mathcal{M}^2(+,-,+,+) = \frac{4e^4m^4t^2\,(m^4 - su)^2}{(m^2 - s)^6\,(m^2 - u)^2},$$

$$\mathcal{M}^2(+,+,-,+) = \mathcal{M}^2(-,-,+,-) = \mathcal{M}^2(-,+,+,+) = \mathcal{M}^2(+,-,-,-) = \frac{4e^4m^4t^2\,(m^4 - su)^2}{(m^2 - s)^6\,(m^2 - u)^2},$$

$$\mathcal{M}^2(+,+,-,-) = \mathcal{M}^2(-,-,+,+) = \frac{4e^4\,(m^4 - su)^4}{(m^2 - s)^6\,(m^2 - u)^2},$$

$$\mathcal{M}^2(-,+,+,-) = \mathcal{M}^2(+,-,-,+) = \frac{4e^4((s - m^2)^2 + tm^2)^4}{(m^2 - s)^6\,(m^2 - u)^2},$$

$$\mathcal{M}^2(-,+,-,+) = \mathcal{M}^2(+,-,+,-) = \frac{4e^4m^8t^4}{\left(m^2-s\right)^6\left(m^2-u\right)^2},$$

and then, separating them in parallel and anti-parallel, we calculate as before $\sigma_P - \sigma_A$

$$\begin{aligned}
\sigma_P - \sigma_A &= 2\pi\int_0^\pi \left(\frac{d\sigma_P}{d\Omega_{CM}} - \frac{d\sigma_A}{d\Omega_{CM}}\right)\frac{d\cos\left(\theta_{CM}\right)}{d\cos\left(\theta_{Lab}\right)}d\theta_{Lab}, \\
&= 2\pi\int_0^\pi \left(\frac{d\sigma_P}{d\Omega_{CM}} - \frac{d\sigma_A}{d\Omega_{CM}}\right)\left(1+\frac{2\nu}{m_e}\right)\frac{m_e^2}{(m_e+\nu(1-\cos\left(\theta_{Lab}\right))^2}d\theta_{Lab}, \\
&= -\frac{2\pi\alpha^2}{3m^2\nu^4}\left(-\frac{\nu\left(12m^5+69m^4\nu+163m^3\nu^2+252m^2\nu^3+292m\nu^4+152\nu^5\right)}{(m+2\nu)^3}\right. \\
&\left.+\frac{3}{2}\left(4m^3+3m^2\nu+10m\nu^2+4\nu^3\right)\ln\left(1+\frac{2v}{m}\right)\right).
\end{aligned} \tag{3.12}$$

Integrating this we have,

$$\begin{aligned}
DHG &= \int_0^\infty \frac{\sigma_P - \sigma_A}{\nu}d\nu \\
&= 0.
\end{aligned}$$

As expected.

Chapter 4

Conclusions

In the search for information about the gravitational properties of quantum objects we took inspiration from the Compton scattering and the DHG sum rule that can be used for calculating the anomalous magnetic moment of the electron via the difference between the parallel and antiparallel polarized scattering.

We used the graviton Compton scattering in the context of the linearized gravity formalism to see if the sum rule would give us useful information. At first we used all the amplitudes for the process, but we found that the difference of polarized cross sections as well as the total cross section diverges. Then we took out the amplitude with the triple graviton vertex and this time the total cross section was finite but it grew linearly with s breaking unitarity, leading to a divergent result in the sum rule.

This may be problematic because the classical result of the sum rule speaks to the consistency of the theory as seen in [1] so there are two possible solutions for this; either the DHG sum rule is not supposed to work for gravitons (could be that the necessary assumptions that the sum rule needs in order to work are not fulfilled by gravitons in the same way that photons do) or the DHG sum rule does work for gravitons but it does so in other better behaved formalisms for gravity.

Chapter 5

Appendix

5.1 Amplitudes and cross sections with the triple graviton vertex

The polarized amplitudes for the process $e^-g \to e^-g$ including the triple graviton vertex are,

$$\mathcal{M}^2(+,+,+,+) = \mathcal{M}^2(-,-,-,-) = \mathcal{M}^2(+,-,+,-) = \mathcal{M}^2(-,+,-,+) = \frac{\kappa^4 t^2 \left(m^4 - su\right)}{256(m^2-s)^{10}(m^2-u)^2} \left(12m^{10} - 9s^3u(s+u) - 2m^8(17s+4u) + 5m^6 s(3s+5u) + m^2 s^2 \left(9s^2 + 23su - 4u^2\right) - m^4 s \left(18s^2 - su + 3u^2\right)\right)^2,$$

$$\mathcal{M}^2(+,+,+,-) = \mathcal{M}^2(-,-,-,+) = \frac{\kappa^4 \left(m^4 - su\right)^5}{64\left(m^2-s\right)^{10}\left(m^2-u\right)^2 t^2} \left(4m^6 + m^4(15s - 11u) + s(2s+u)(s+3u) + m^2\left(-13s^2 - 12su + 5u^2\right)\right)^2,$$

$$\mathcal{M}^2(+,+,-,+) = \mathcal{M}^2(-,-,+,-) = -\frac{\kappa^4 m^2 s^2 t^3}{64\left(m^2-s\right)^{10}\left(m^2-u\right)^2} \left(-6m^8 + m^4 s(3s - 11u) + 7s^2u(s+u) + 4m^6(4s+u) + m^2 s\left(-5s^2 - 16su + u^2\right)\right)^2,$$

$$\mathcal{M}^2(+,+,-,-) = \mathcal{M}^2(-,-,+,+) = \mathcal{M}^2(+,-,-,+) = \mathcal{M}^2(-,+,+,-) = -\frac{\kappa^4 \left(m^4 - su\right)^4}{64\left(m^2-s\right)^{10}\left(m^2-u\right)^2 t} \left(-4m^7 + m^5(27s+u) + m^3\left(-17s^2 - 28su + u^2\right) + ms\left(2s^2 + 11su + 7u^2\right)\right)^2,$$

$$\mathcal{M}^2(+,-,+,+) = \mathcal{M}^2(-,+,-,-) = \frac{\kappa^4 \left(m^4 - su\right)^3}{64\left(m^2-s\right)^{10}\left(m^2-u\right)^2 t^2} \left(3m^8(13s - 5u) - 2m^6\left(30s^2 + 9su - 7u^2\right) - m^2 s(s-u)\left(15s^2 + 22su + u^2\right) + s^2\left(2s^3 + 3s^2u - 7su^2 - 6u^3\right) + m^4\left(42s^3 + 13s^2u - 4su^2 - 3u^3\right)\right)^2,$$

$$\mathcal{M}^2(+,-,-,-)=\mathcal{M}^2(-,+,+,+)=-\frac{\kappa^4 t^3}{64\left(m^2-s\right)^{10}\left(m^2-u\right)^2}\left(-2m^{11}+m^5s^2(5s-17u)+2m^7s(5s+3u)+ms^3u(5s+7u)\right.$$
$$\left.+m^3s^2\left(-5s^2-10su+u^2\right)\right)^2.$$

Grouping these amplitudes into parallel and antiparallel we have that the differential cross sections are,

$$\frac{d\sigma_P}{d\Omega_{CM}}=$$
$$\frac{\kappa^4}{32768\pi^2\left(m^2-s\right)^{10}s\left(m^2-u\right)^2t^2}\left[4\left(m^4-su\right)^5\left(4m^6+m^4(15s-11u)+s(2s+u)(s+3u)-m^2\left(13s^2+12su-5u^2\right)\right)^2\right.$$
$$-4m^2s^2t^5\left(-6m^8+m^4s(3s-11u)+7s^2u(s+u)+4m^6(4s+u)+m^2s\left(-5s^2-16su+u^2\right)\right)^2$$
$$-4t\left(m^4-su\right)^4\left(-4m^7+m^5(27s+u)-m^3\left(17s^2+28su-u^2\right)+ms\left(2s^2+11su+7u^2\right)\right)^2$$
$$\left.+t^4\left(m^4-su\right)\left(12m^{10}-9s^3u(s+u)-2m^8(17s+4u)+5m^6s(3s+5u)+m^2s^2\left(9s^2+23su-4u^2\right)-m^4s\left(18s^2-su+3u^2\right)\right)^2\right].$$

$$\frac{d\sigma_A}{d\Omega_{CM}}=$$
$$\frac{\kappa^4}{32768\pi^2\left(m^2-s\right)^{10}s\left(m^2-u\right)^2t^2}\left[-4t\left(m^4-su\right)^4\left(m^5(27s+u)-4m^7-m^3\left(17s^2+28su-u^2\right)+ms\left(2s^2+11su+7u^2\right)\right)^2\right.$$
$$-4t^5\left(-2m^{11}+m^5s^2(5s-17u)+2m^7s(5s+3u)+ms^3u(5s+7u)-m^3s^2\left(5s^2+10su-u^2\right)\right)^2$$
$$+t^4\left(m^4-su\right)\left(12m^{10}-9s^3u(s+u)-2m^8(17s+4u)+5m^6s(3s+5u)+m^2s^2\left(9s^2+23su-4u^2\right)-m^4s\left(18s^2-su+3u^2\right)\right)^2$$
$$+4\left(m^4-su\right)^3\left(3m^8(13s-5u)-2m^6\left(30s^2+9su-7u^2\right)-m^2s(s-u)\left(15s^2+22su+u^2\right)+s^2\left(2s^3+3s^2u-7su^2-6u^3\right)\right.$$
$$\left.\left.+m^4\left(42s^3+13s^2u-4su^2-3u^3\right)\right)^2\right].$$

5.2 Amplitudes and cross sections without the triple graviton vertex

The polarized amplitudes for the process $e^-g\to e^-g$, not including the triple graviton vertex are,

$$\mathcal{M}^2(+,+,+,+)=\mathcal{M}^2(-,-,-,-)=\frac{\kappa^4t^4\left(m^4-su\right)\left(4m^6-3m^4s+m^2s(3s-u)-3s^2u\right)^2}{256\left(m^2-s\right)^8\left(m^2-u\right)^2},$$

$$\mathcal{M}^2(+,+,+,-)=\mathcal{M}^2(-,-,-,+)=\frac{\kappa^4t^2\left(m^4-su\right)^5}{16\left(m^2-s\right)^8\left(m^2-u\right)^2},$$

$$\mathcal{M}^2(+,+,-,+)=\mathcal{M}^2(-,-,+,-)=-\frac{\kappa^4m^2s^2t^5\left(m^4-su\right)^2}{16\left(m^2-s\right)^8\left(m^2-u\right)^2},$$

$$\mathcal{M}^2(+,+,-,-) = \mathcal{M}^2(-,-,+,+) = -\frac{\kappa^4 m^2 t^3 \left(m^4 - su\right)^4}{16 \left(m^2 - s\right)^8 \left(m^2 - u\right)^2},$$

$$\mathcal{M}^2(+,-,+,+) = \mathcal{M}^2(-,+,-,-) = \frac{\kappa^4 t^2 \left(m^4 - su\right)^3 \left(5m^4 + s(2s - u) - m^2(5s + u)\right)^2}{64 \left(m^2 - s\right)^8 \left(m^2 - u\right)^2},$$

$$\mathcal{M}^2(+,-,+,-) = \mathcal{M}^2(-,+,-,+) = \frac{\kappa^4 t^4 \left(m^4 - su\right) \left(4m^6 - 3m^4 s + m^2 s(3s - u) - 3s^2 u\right)^2}{256 \left(m^2 - s\right)^8 \left(m^2 - u\right)^2},$$

$$\mathcal{M}^2(+,-,-,+) = \mathcal{M}^2(+,-,-,-) = \mathcal{M}^2(-,+,+,-) = \mathcal{M}^2(-,+,+,+) = -\frac{\kappa^4 m^2 t^3 \left(m^4 - su\right)^4}{16 \left(m^2 - s\right)^8 \left(m^2 - u\right)^2}.$$

Grouping these amplitudes into parallel and antiparallel we have that the differential cross sections are,

$$\frac{d\sigma_P}{d\Omega_{CM}} = \frac{\kappa^4 t^2 \left(m^4 - su\right)}{32768\pi^2 \left(m^2 - s\right)^8 s \left(m^2 - u\right)^2} \Big(-16m^2 s^2 t^3 \left(m^4 - su\right) - 16m^2 t \left(m^4 - su\right)^3 + 16 \left(m^4 - su\right)^4$$
$$+t^2 \left(4m^6 - 3m^4 s + m^2 s(3s - u) - 3s^2 u\right)^2\Big).$$

$$\frac{d\sigma_A}{d\Omega_{CM}} = \frac{\kappa^4 t^2 \left(m^4 - su\right)}{32768\pi^2 \left(m^2 - s\right)^8 s \left(m^2 - u\right)^2} \Big(-32m^2 \left(m^4 - su\right)^3 + t^2 \left(4m^6 - 3m^4 s + m^2 s(3s - u) - 3s^2 u\right)^2$$
$$+4 \left(m^4 - su\right)^2 \left(5m^4 + s(2s - u) - m^2(5s + u)\right)^2\Big).$$

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