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Superconductivity in Tilted Dirac Materials

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“Si el río suena, se ahogó una orquesta.”

Camilo Castro Arriaza

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Chapter 1

Introduction

The Dirac crystals featuring Dirac quasiparticles as their low-energy excitations exhibiting a linear dispersion opened up a new paradigm in the modern condensed matter physics [1–3]. They also bridge the gap between the seemingly unrelated phenomena in the realms of high-energy and condensed-matter physics through the emergence of the Dirac and Weyl quasiparticles, being the prime examples in this respect. Such Dirac materials feature an emergent relativistic-like Lorentz symmetry because of the Dirac quasiparticles’ linear energy-momentum dispersion, $E(\mathbf{k}) = v_F k$, with the Fermi velocity, v_F , typically a few hundred times smaller than the speed of light, playing the role of an effective velocity of light in their crystalline universe. Such a relativistic-like Lorentz symmetry turns out to be quite a robust feature of the Dirac materials even when the electron-electron interactions that explicitly break it are taken into account. For instance, the Lorentz symmetry emerges in the deep infrared regime for the two-dimensional (2D) Dirac fermions coupled via the long-range Coulomb interaction, where the speed of light takes over as the common velocity for both the Dirac fermions and the photons [4]. Furthermore, for Yukawa (short-range) interacting Dirac fermions, the quantum-critical point (QCP) separating a semimetallic and a strongly-coupled gapped (insulating or superconducting) phase is Lorentz symmetric [5–9]. This issue has been also recently addressed beyond the realm of the usual Hermitian Dirac materials, by taking into account the effects of the non-Hermiticity, that may arise, e.g. from the coupling with an environment, resulting in the Lorentz symmetry emerging at low energies [10, 11]. These results point towards a ubiquitous emergence of the Lorentz symmetry at low energy in the Dirac materials, even though such a symmetry is absent at the lattice (UV) energy scale.

To obtain further insights into the ubiquity of such an emergent Lorentz symmetry, in this thesis we will consider tilted Dirac materials, featuring low-energy quasiparticles with tilted Dirac dispersion, explicitly breaking this symmetry at the lattice scale already at the noninteracting level. Such an explicit symmetry breaking is realized by the direction of the tilt term in the dispersion of the tilted Dirac fermions (TDFs), which is dictated by the underlying crystalline lattice, and is represented by a single term $\sim \alpha$, with α as the tilt parameter, in a Dirac Hamiltonian, as shown in Eqs. (3.3) and (3.4). Introduction of such a tilt term may be considered as possibly the simplest Lorentz-symmetry-breaking deformation of an otherwise Lorentz-symmetric noninteracting Dirac Hamiltonian. Its effects have been studied in various contexts, for instance, it was argued that TDFs may represent a platform for the simulation of the curved space-time in condensed-matter systems [12–21], while the quantum transport [22–28], the effects of the long- and short-range components of the Coulomb interaction [29–32], charged impurities [33–36], and the magnetic field [37–39] have also been addressed. We emphasize that the tilted Dirac dispersion has been theoretically investigated in the materials using first-principle techniques [40–42]. In fact, the TDFs have been realized in a plethora of quantum materials [43–50], both with the subcritical tilt (type-I), featuring the point-like Fermi surfaces, and in overtilted materials (type-II), with the electron- and hole-like Fermi surfaces, putting the tilted Dirac crystals within the landscape of the quantum materials.

Through this thesis, we explore the effects of the s -wave superconducting instability in the type-I tilted Dirac materials. Our main result is that at the strong-coupling QCP separating the tilted Dirac semimetal and the s -wave SC phase, the tilt operator becomes *irrelevant* and therefore the Lorentz symmetry gets restored. To address this problem, we employ the Gross-Neveu-Yukawa (GNY) theory [51–58], since this instability necessarily occurs at strong coupling, as can be seen already at the mean-field level [Eq. (3.9)], due to a vanishing density of states (DOS) at zero energy [Eq. (3.10)]. To account for the effects of both fermionic and order-parameter (OP) fluctuations in the vicinity of such a QCP, since both the Yukawa and Φ^4 -couplings [Eq. (3.19)] are marginal in three spatial dimensions ($D = 3$), we use the renormalization group (RG) analysis within the $\epsilon = (3 - D)$ -expansion. We demonstrate that the tilt parameter in the mirror-symmetric [Eq. (3.3)] and mirror-symmetry-breaking [Eq. (3.4)] tilted Dirac semimetals are irrelevant at the superconducting QCP, with the corresponding RG β -function given by Eq. (3.56) [see also Fig. 3.4.1], and the flow shown in Fig. 3.4.2, therefore implying the restoration

of the Lorentz symmetry at such a QCP. To further corroborate the ubiquity of this scenario, we show that the transition into a CDW phase also features the emergent Lorentz symmetry at low energy, albeit with a slower rate of the restoration, as shown in Eq. (3.65), see also Fig. 3.4.3 and Fig. 3.4.1. We argue that such an emergence of the Lorentz symmetry is generic for the TDFs interacting via a short-range Coulomb interaction, with the rate that depends on whether the OP commutes or anticommutes with the tilt term. Finally, the anticommutation makes the rate of the restoration faster, as can be seen explicitly in the case of the SC and CDW instabilities.

This thesis is organized as follows. In Chapter 1 we introduce some of the key concepts that will allow the reader to understand the context of the work we have developed. In Chapter 2, we present the theoretical tools that we use in order to solve the problem. In Chapter 3, which contains new results posted on the preprint server arXiv [59], we address the superconducting phase transition for TDFs, presenting our results and arguing the irrelevance of the tilt parameter. Finally, our conclusions and the outlook are presented in Chapter 4.

1.1 Quantum Phase Transitions

The main focus of this thesis will be the semimetal-superconductor phase transition at zero temperature. This transition falls into the category of continuous quantum phase transitions, which will be the principal subject of this section. We will adhere to the simpler modern classification in which phase transitions may be either continuous (second order) or discontinuous (first order). A continuous phase transition is change of phase of a macroscopic system in equilibrium that does not involve latent heat. Phase transitions that do involve latent heat, such as the freezing of water, are called first order or discontinuous [60].

There are several ways to approach the concept of quantum phase transitions. Lets first consider a Hamiltonian $H(g)$, whose degrees of freedom reside on the sites of a lattice, and which varies as a function of a dimensionless coupling, g . Following the evolution of the ground state energy of $H(g)$ as a function of g , we can start the discussion on the idea of phase transitions. For the finite volume case, the ground state energy will be a smooth, analytic function of g . There is a main possibility for an exception, if g couples only to a conserved quantity (i.e. $H(g) = H_0 + gH_1$, where H_0 and H_1 commute) [61]. Thus, H_0 and H_1 can be simultaneously diagonalized

implying that the eigenfunctions are independent of g even when the eigenvalues change with g . Here comes the idea of nonanalyticity, in the last case, there can be a level-crossing where an excited level becomes the ground state at the critical value of the coupling $g = g_c$, creating a point of nonanalyticity in the ground state energy.

Another way to approach quantum phase transitions is by developing an order parameter (OP) which lowers or break the symmetry. In general one speaks of a spontaneous symmetry breakdown if the symmetry group in field space reduces to a subgroup when transitioning from the disordered high-temperature phase to the ordered low-temperature phase. This concept plays a central role in the following work and will be explicitly used in Section 2.1.

Lev Landau introduced the concept of order parameter in 1937 [62] in order to quantify the transformation of a phase of matter at a transition where the change of phase takes place - the phase transition. In Landau's theory, each phase transition is associated with an order parameter ψ that, once the temperature drops below the transition temperature T_c , acquires nonzero value

$$|\psi| = \begin{cases} 0, & (T > T_c) \\ |\psi_0| > 0, & (T < T_c). \end{cases} \quad (1.1)$$

The order parameter can be either a real or complex number (scalar), a vector or a tensor and microscopically is directly related to the expectation value of a certain quantum operator. For example, in a superconductor or superfluid, the order parameter is a complex number related to the expectation value of a bosonic field characterizing the condensate, whose moduo is proportional to the density of the superfluid density.

In general, the order parameter description is good on length scales larger than the a characteristic length. Second order phase transitions invariably have a diverging coherence length ξ . This could be the length scale determining the exponential decay of equal-time correlations in the ground state or the length scale at which some characteristic crossover occurs to the correlations at the longest distances.

The idea that phase transitions are defined as points in the parameter space where some thermodynamic quantity (usually some thermodynamic potential) becomes non-analytic, carries with it an important philosophical discussion. The aforementioned nonanalyticity can arise only in the thermodynamic limit, when the size of the system is assumed to be infinite. This paradoxical statement has been addressed by many

people over the years [63].

1.1.1 Universality Classes and Critical Exponents

Although most of phase transitions in nature are in fact discontinuous, we are interested in continuous phase transitions because of a very curious phenomena. Near the critical point, some properties of completely different physical systems appear to be the same, this phenomenon is called the universality. Thus, some macroscopic properties of a system near a continuous phase transition appear to be rather independent of the microscopic interactions between particles. We will see that some general features of the system, like dimensionality, symmetry and presence or absence of sufficiently long-ranged interactions. This phenomenon of completely different systems exhibiting the same critical behavior is called the universality.

Returning to the idea of divergent correlation length. ξ diverges as

$$\xi^{-1} \sim \Lambda |g - g_c|^\nu, \quad (1.2)$$

where ν is a critical exponent, and Λ is a momentum cutoff (an inverse length scale of the order the inverse lattice spacing). Another characteristic of second order phase transitions is that there exists the characteristic energy scale of fluctuations above the ground state which vanishes as g approach to g_c . We call this energy Δ and could be the energy of the lowest excitation above the ground state for example. In the majority of cases, we will find that, as g approaches g_c , Δ vanishes as

$$\Delta \sim J |g - g_c|^{z\nu} \quad (1.3)$$

The characteristic energy scale vanishes as the z th power of the inverse correlation length

$$\Delta \sim \xi^{-z}. \quad (1.4)$$

The value of z and ν are usually universal or independent of the microscopic properties of the system. Critical exponents are not the only universal quantities. There are others, like scaling functions and universal amplitude ratios. However, the exponents are often the most directly measurable quantities, and their non-trivial and nearly universal experimental values have provided the motivation for the theory of critical phenomena.

Universality in recent numerical work. To better illustrate the concept

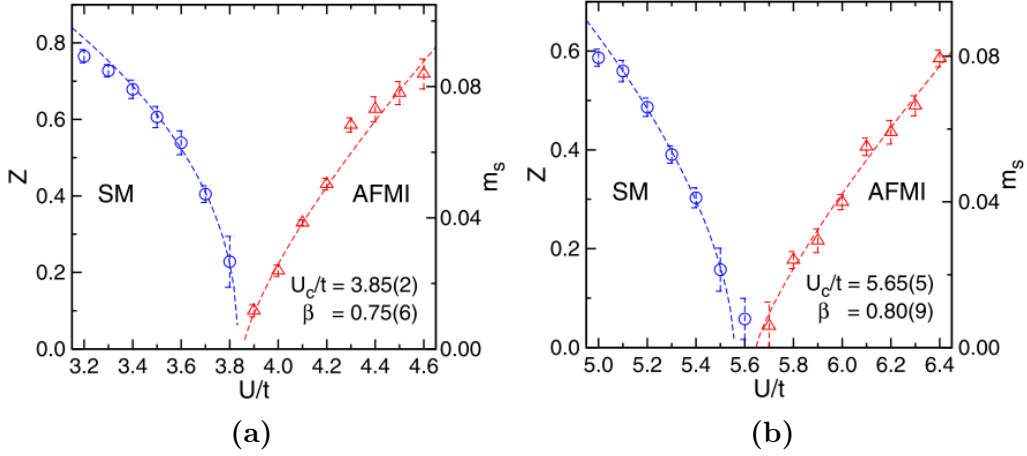


Figure 1.1.1: The ground-state phase diagrams for (a) the honeycomb lattice model and (b) the π -flux model. Open triangles and open circles represent the AF order parameter m_s and the quasiparticle weight Z , respectively. The critical U_c estimated is indicated in the figures along with the critical exponent β . from m_s , assuming a form of $m_s \sim (U - U_c)^\beta$ (red dashed lines) [64].

of universality, take for example the metal-insulator transition of interacting Dirac electrons in the honeycomb lattice model and in the π -flux model, studied in 2016 by Otsuka *et al.* [64]. Focusing in the semimetal (SM) and the antiferromagnetic (AF) insulator continuous quantum phase transition, two quantities can be calculated using Quantum Monte Carlo techniques, the staggered magnetization and the quasiparticle weight. The first quantity reveals the magnetic transition to the AF state, and the latter one directly captures the metal-insulator transition. These transitions are expected to occur at the different critical values of the Hubbard interaction, U_c . This is so because the critical value of tuning parameter for the transition is *non-universal*, i.e. it does explicitly depend on the model. In the discussed example, the Hubbard model is implemented on two different lattices thereby yielding different values for U_c .

The results of the calculation show that the antiferromagnetic order parameter m_s , for both the honeycomb lattice and the π -flux model, is finite above the critical point at U_c , while the value of U_c differs in the cases of the honeycomb and the square π flux lattices. Near the critical point, m_s behaves as the power-law

$$m_s \sim (U - U_c)^\beta \quad (1.5)$$

Then, for the quasiparticle weight Z in the thermodynamic limit U/t , where t is

the hopping parameter, the following form is proposed

$$Z \sim (U_c^{\text{MI}} - U)^{\eta_Z} \quad (1.6)$$

where U_c^{MI} is the metal-insulator transition. In this research, the results show that the critical values for U_c^{MI} are thus consistent with those U_c determined from the AF order parameter m_s within the statistical errors. Fig. 1.1.1 shows the results for Z in both models. It is easy to see that even though they are completely different physical systems, the critical behavior is the same, e.g., the critical exponents take the same values (within the numerical accuracy).

1.2 Dirac/Weyl Materials

The main system that we are going to be working on is Dirac/Weyl materials. The birth of these concepts came from the development of theories, that have their origin in High Energy Physics, applied to condensed matter systems. In 1928, P.A.M Dirac published his famous paper "The Quantum Theory for the Electron" in which he proposed his milestone equation. As the name of the article suggests, the Dirac equation was first conceived as a relativistic quantum mechanical description of the electron that involved a 4-component wave function. In the equation are encoded the concepts of particle/antiparticle, is the start of quantum field theory and, to this day, we know that it describes any fermion with mass and spin-1/2.

Of course, a lot of variations of the Dirac equation came after. In 1929, the mathematician Hermann Weyl proposed a simplified version. He showed that the use of a 2-component field was sufficient to describe massless fermions with definite chirality. In 1937, Ettore Majorana found a variation with real solutions that described a neutral particle that was its own antiparticle. All these variations have found vast applications in modern particle physics and, to this day, they continue to be of great importance.

But given that this is a thesis within the area of condensed matter, we are interested in the low-energy regime. The propagation of slow electrons modes through the periodic potential of a crystal leads to electronic states with peculiar characteristics. In some instances, we can describe these phenomena by means of effective low-energy theories, these theories have the form of the Dirac equation. There are many examples of this type of materials, the most famous being graphene,

whose low energy description is given by a two-dimensional Dirac equation [65]. This low energy description is performed near the so-called Weyl/Dirac points. In this context, when two (four) electronic bands cross, Weyl (Dirac) cones are realized. In the following, we will call such band touching points Dirac points while the number of the crossing bands will be clear from the context, i.e. whether there are two, four or even more bands crossing at such a point in the Brillouin zone.

1.2.1 Dirac and Weyl Fermions

In condensed matter, the lattice breaks the Lorentz symmetry of the vacuum, but in certain systems, such as the honeycomb lattice of graphene, the Lorentz symmetry emerges in a lower energy scale and with a velocity scale v_F , which is much smaller than the speed of light. These systems are described by effective low energy Hamiltonian of the form

$$H = v_F \gamma^0 \gamma^\mu p_\mu + m v_F^2 \gamma^0, \quad (1.7)$$

where $\mu = 0, 1, \dots, d$ label the time ($\mu = 0$ denotes “time”) and space dimensions, and γ^μ are the Dirac matrices. We hereafter employ the Einstein notation. The $d + 1$ gamma matrices are chosen Hermitian, $2^{k+1} \times 2^{k+1}$ -dimensional in both spatial dimensions $d = 2k + 1$ and $d = 2k + 2$, and satisfy the anticommutation relation $\{\gamma^\mu, \gamma^\nu\} = 0$ for $\mu \neq \nu$ (see App. A1 for details of the Clifford algebra). Equivalently, the description of spin- $\frac{1}{2}$ particles is given by the Dirac equation in d spatial dimensions

$$(i\gamma^\mu \partial_\mu - m)\psi = 0. \quad (1.8)$$

In this thesis we consider $\hbar = c = k_B = 1$, unless explicitly stated otherwise. Then, the analysis made by Weyl consists of a simplification in odd spatial dimensions [66]. The description of massless fermions allows to circumvent the problem that helicity depends on the observer. Thus, let’s consider $d = 3$ and define chiral components $\gamma_5 \psi_\pm = \pm \psi_\pm$, where $\gamma_5 = i^k \gamma^0 \dots \gamma^3$, then, we have for the massless Dirac equation

$$i\partial_t \psi_\pm = H_\pm \psi_\pm, \quad (1.9)$$

$$H_\pm = \mp \mathbf{p} \cdot \boldsymbol{\sigma}. \quad (1.10)$$

The Weyl representation implies that fermions propagate parallel (or antiparallel)

to their spin, which defines chirality. This chiral fermions must appear in opposite pairs, as dictated by the Nielsen-Ninomiya theorem [67, 68].

Field theoretical approach. The core calculations of this thesis are performed on the field-theoretical methods, which are based on the formalism of functional integrals. In this context, the fermion fields $\bar{\psi}, \psi$ ($\bar{\psi} = \psi^\dagger \gamma^0$ as usual) are generators of a Grassmann algebra (see App. A2). We will take a look at the action for free fermions in real time (Minkowski metric), although the analytical continuation to imaginary time is what we will use in the rest of the work. In real time, the free action for massive Dirac fields reads

$$S[\bar{\psi}, \psi] = \int dt \int d^d \mathbf{x} \bar{\psi}(t, \mathbf{x}) \{i\gamma^\mu \partial_\mu - m\} \psi(t, \mathbf{x}). \quad (1.11)$$

Such action is of course Lorentz invariant but it will help to study the Lorentz symmetry in this case. Let S be an element of the fundamental representation of the spin group $\text{Spin}(1, d-1)$ and the $d \times d$ matrix $\Lambda(S)$ be its image in the $SO(1, d-1)$ space-time pseudo-orthogonal transformations. Lorentz transformations acting on 4-vectors satisfy

$$\Lambda g \Lambda^T = g \quad (1.12)$$

where $g_{\mu\nu}$ the Minkowski metric. The change of a vector in different frames is defined by $x^\mu \rightarrow \Lambda^\mu_\nu x^\nu$. Since our intention is not to go into detail with the Lorentz group, we will limit ourselves to simply use some properties of the Lorentz group to show the Lorentz invariance of the Dirac Lagrangian:

$$\begin{aligned} \mathcal{L} &= \bar{\psi}(x)(i\gamma^\mu \partial_\mu - m)\psi(x) \\ &= \bar{\psi}(\Lambda^{-1}x')(i\gamma^\mu \Lambda^\nu_\mu \partial'_\nu - m)\psi(\Lambda^{-1}x') \\ &= \bar{\psi}(\Lambda^{-1}x')(iS^{-1}\gamma^\nu S\partial'_\nu - m)\psi(\Lambda^{-1}x') \\ &= \bar{\psi}(\Lambda^{-1}x')S^{-1}(i\gamma^\nu \partial'_\nu - SmS^{-1})S\psi(\Lambda^{-1}x') \\ &= \bar{\psi}'(x')(i\gamma^\nu \partial'_\nu - m)\psi'(x'). \end{aligned} \quad (1.13)$$

Free Euclidian Dirac fermions. In order to make possible the renormalization group and perturbative calculations we will carry out analytic continuation to imaginary time τ [69]. The free action for Euclidian Dirac fermions can be written as

$$S[\bar{\psi}, \psi] = \int_0^\beta d\tau \int d^d \mathbf{x} \bar{\psi}(\partial_\tau - H(\mathbf{p} \rightarrow -i\nabla_{\mathbf{x}}))\psi. \quad (1.14)$$

with β being the inverse of temperature. As we will see later, all the calculations involving the renormalization group came from quantities derived from the partition function in the functional integral representation. Thus partition function for the Dirac fermions is given as the functional integral over the Grassmann fields

$$Z = \int [d\psi, d\bar{\psi}] \exp \left[-S[\bar{\psi}, \psi] + \mu \int_0^\beta d\tau \int d^d \mathbf{x} \bar{\psi}(\tau, \mathbf{x}) \gamma_0 \psi(\tau, \mathbf{x}) \right], \quad (1.15)$$

where $[d\psi d\bar{\psi}] = \prod^n d\psi d\bar{\psi}$ is standard notation, and we introduce the chemical potential μ for illustration. Fermions must satisfy anti-periodic boundary conditions in the time direction:

$$\psi(\tau = 0, \mathbf{x}) = -\psi(\beta, \mathbf{x}), \quad \bar{\psi}(0, \mathbf{x}) = -\bar{\psi}(\beta, \mathbf{x}). \quad (1.16)$$

Since we will deal with quantum phase transitions we consider only zero temperature. Thus, in this case, the boundary condition plays no role. The time direction closes in on itself and we can no longer distinguish between space and time.

1.2.2 Tilted Dirac/Weyl Materials

As we discussed earlier, the relativistic nature of low-energy quasiparticle (electron and hole) excitations in graphene like systems is directly connected to Lorentz symmetry, this is not the case for many condensed matter systems. A considerable amount of condensed matter systems that have linear dispersion and resemble the Dirac/Weyl equation break the Lorentz symmetry. This is also the case for high-energy physics systems [70–73]. One way, among several interesting ways, to break Lorentz symmetry is to tilt the Dirac/Weyl cone. This can be done by adding a term $\hbar v_F \alpha^i p_i$ to the Hamiltonian. Thus, the generalized low-energy Hamiltonian for the tilted Dirac/Weyl fermions is given by

$$H = v_F \gamma^0 \gamma^\mu p_\mu + m v_F^2 \gamma^0 + v_F \alpha^i p_i, \quad (1.17)$$

where α^i are the “tilt parameters” and $i = 1, \dots, d$. Setting the Fermi velocity to 1

we can quickly see the implication of the $\alpha^i p_i$ term in Lorentz symmetry:

$$\begin{aligned}
\mathcal{L} &= \bar{\psi}(i\gamma^\mu \partial_\mu - m + \gamma^0 \alpha^i \partial_i)\psi \\
&= \bar{\psi}(\Lambda^{-1}x')(i\gamma^\mu \Lambda_\mu^\nu \partial'_\nu - m + \gamma^0 \alpha^i \Lambda_i^j \partial'_j)\psi(\Lambda^{-1}x') \\
&= \bar{\psi}(\Lambda^{-1}x')(iS^{-1}\gamma^\nu S \partial'_\nu - m + \gamma^0 \alpha^i \Lambda_i^j \partial'_j)\psi(\Lambda^{-1}x') \\
&= \bar{\psi}(\Lambda^{-1}x')S^{-1}(i\gamma^\nu \partial'_\nu - SmS^{-1} + S\gamma^0 \alpha^i \Lambda_i^j \partial'_j S^{-1})S\psi(\Lambda^{-1}x') \\
&= \bar{\psi}'(x')(i\gamma^\nu \partial'_\nu - m + S\gamma^0 \alpha^i \Lambda_i^j \partial'_j S^{-1})\psi'(x').
\end{aligned} \tag{1.18}$$

This way, the term $\alpha^i p_i$ explicitly breaks the Lorentz symmetry. There have been many theoretical efforts to understand tilted Dirac fermion (TDF). Kawarabayashi *et al.* find a generalized chiral symmetry that protects the tilted Dirac node [74]. For the honeycomb lattice, a generalized hopping model supporting TDC was developed by Kishigi *et al.* [75]. Volovik proposed tilted Dirac/Weyl materials as a new spacetime structure in solid state physics[12, 13].

Materials hosting tilted Dirac/Weyl cones. These tilted Dirac fermions and corresponding tilted Dirac cone materials (TDCMs) can exist in three (3D), two (2D) and one (1D) space dimensions. The first realization of a tilted Dirac cone was the layered organic compound α -(BEDT-TTF)₂I₃ [76], which was discovered in Japan [77]. Nuclear Magnetic Resonance spectroscopy measurements suggest that this material is a strongly correlated system [78] and the change in the Dirac cone is caused by strong Coulomb interactions [79].

A breakthrough came by substituting the iodine with halogens (I→F) in the last organic compound. This way it was discovered that the Dirac cone can be overtilted [80]. This is important because it corresponds to the transition from type-I to type-II Dirac fermions. Strictly speaking, for $|\alpha| < 1$ the Dirac cone is denoted as type-I. For large tilts $|\alpha| > 1$, the Dirac cone is overtilted and called type-II [40], see Fig. 1.2.1 for an illustration, and feature electron- and hole-like Fermi surfaces. For other layered systems, like LaAlO₃/LaNiO₃/LaNiO₃, it is possible to shift the Dirac node by varying the number of LaNiO₃ layers. TDFs are also predicted in the surface of topological insulators [81] and in the antiferromagnetic phase of Iron-based superconductors as excitations emerging from a spin density wave [82]

2D TDFs are also expected in purely 2D materials, i.e., one atom-thick materials. First principle calculations suggest that partially hydrogenated graphene compound, C₆H₂ has a tilted Dirac cone (TDC) in its band structure [84]. Boron in $8Pmmn$ lattice structure is also predicted to host TDFs [85, 86]. In 3D systems, experiments

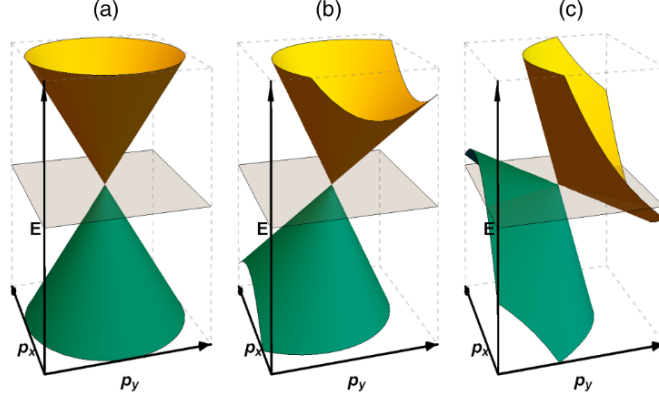


Figure 1.2.1: Illustration of a generic Dirac cone for an isotropic Dirac semimetal (a), type-I Weyl semimetal (b), and type-II Weyl semimetal (c) [83].

confirmed the existence type-II Weyl fermions on WTe_2 [87–89]. Latter, MoTe_2 was predicted [90] and confirmed [46] to be a tilted Weyl material. Last but not least, it has been proposed that 1D TDFs can rise in the sodium termination of zigzag edges in graphene nano-ribbons [91].

1.2.3 Mean-Field Analysis for Dirac fermions

A first approach to study the critical behaviour of Dirac fermions is a mean-field analysis of the action for the massless Dirac fermions. This simple analysis is useful because it gives us clues to the behavior of the interacting Dirac fermions, as we will see in the following. When Fermi energy is pinned at these Dirac points in the Brillouin zone, the density of states vanishes as the system approached the Fermi level and the ground state of the system is a semimetal.

The action for noninteracting massless Dirac fermions in d -dimensional Euclidean space (imaginary time) has the form

$$\begin{aligned} S[\bar{\psi}, \psi] &= \int d\tau \int d^d \mathbf{x} \bar{\psi}(\tau, \mathbf{x}) \not{\partial} \psi(\tau, \mathbf{x}) \\ &\equiv \int d\tau \int d^d \mathbf{x} \bar{\psi}(\tau, \mathbf{x}) (\gamma^0 \partial_0 + v_F \gamma^\alpha \partial_\alpha) \psi(\tau, \mathbf{x}), \end{aligned} \quad (1.19)$$

where $\not{\partial} = \gamma^\mu \partial_\mu$ and $\alpha = 1, \dots, d$. The representation of the Dirac field, and the matrices acting on them, depends on the physical system and its symmetries. Setting $v_F = 1$, the action in the momentum space reads

$$S[\bar{\psi}, \psi] = \int \frac{d\omega}{2\pi} \int \frac{d^d \mathbf{p}}{(2\pi)^d} \bar{\psi}(\omega, \mathbf{p}) (-i\gamma^\mu p_\mu) \psi(\omega, \mathbf{p}). \quad (1.20)$$

where $p_\mu = (\omega, \mathbf{p})$. We consider now a local density-density interaction in 2 spatial dimensions of the form

$$S_{\text{int}}[\bar{\psi}, \psi] = g \int d\tau \int d^2\mathbf{x} (\bar{\psi}(\tau, \mathbf{x}) \psi(\tau, \mathbf{x}))^2 \quad (1.21)$$

The general form of the local density-density fermion interaction is given by $S_{\text{int}}[\bar{\psi}, \psi] = g \int d\tau \int d^2\mathbf{x} (\bar{\psi}(\tau, \mathbf{x}) \hat{M} \psi(\tau, \mathbf{x}))^2$, with $\hat{M}$ as an 4×4 hermitian matrix. Now, in order to formulate the mean-field theory for the transition into an insulating state induced by short-range interaction in two spatial dimensions, $d = 2$, we define the action $S = S + S_{\text{int}}$ (Eqs. (1.19) and (1.22)). We then perform a Hubbard-Stratonovich transformation in order to decouple the interaction by introducing a new field $m = \langle \bar{\psi} \psi \rangle$. We do this in the channel where, in the case of graphene, it corresponds with the charge-density wave instability with the sublattice symmetry broken by the difference of the chemical potentials on the two sublattices. Thus, the action reads

$$S[\bar{\psi}, \psi] = \int d\tau \int d^2\mathbf{x} \left\{ \bar{\psi}(\tau, \mathbf{x}) (\not{\partial} + m) \psi(\tau, \mathbf{x}) - \frac{1}{2g} m^2 \right\}. \quad (1.22)$$

We then obtain the corresponding energy of the ground state at half filling ($\mu = 0$) in the form

$$\mathcal{E}[m] = -\frac{1}{2g} m^2 + \int \frac{d^2\mathbf{q}}{(2\pi)^2} \sqrt{q^2 + m^2}. \quad (1.23)$$

Minimizing above free energy with respect to the order parameter m , we obtain

$$-\frac{1}{g} + \int \frac{d^2\mathbf{q}}{(2\pi)^2} \frac{1}{\sqrt{q^2 + m^2}} = 0, \quad (1.24)$$

and therefore the order parameter m acquires a non-zero expectation value for $g > g_c = 2\pi/\Lambda$, with Λ as the momentum cutoff. The value of the critical interaction involves the momentum cutoff and is therefore non-universal, as expected on general grounds. Thus

$$\frac{1}{g} - \frac{1}{g_c} = \int \frac{d^2\mathbf{q}}{(2\pi)^2} \left(\frac{1}{\sqrt{q^2 + m^2}} - \frac{1}{q} \right). \quad (1.25)$$

Performing the integral and keeping the leading term in m/Λ yields

$$\frac{1}{g_c} - \frac{1}{g} = \frac{m}{2\pi}. \quad (1.26)$$

From the last equation we can obtain an important conclusion

$$m \sim (g - g_c), \quad (1.27)$$

implying that the correlation-length exponent $\nu = 1$ for the phase transition of the Dirac fermions in the mean-field analysis. $m \sim \xi^{-z} \sim (g - g_c)^{z\nu}$, with ξ as the correlation-length, and $z = 1$ for the dynamical exponent. The transition from the semimetallic ground state to the insulating one takes place at the quantum critical point at a finite interaction strength. Since the system is two-dimensional and the Dirac fermions are already gapless in the semimetallic phase, we expect the fluctuations at the transition to play an important role for the ensuing quantum phase transition, which is, indeed, the case, as will be discussed later in this thesis.

1.3 Superconductivity and the BCS Theory

Superconductivity is one of the most important and challenging problems in quantum statistics and condensed matter physics. In 1911, Heike Kamerlingh-Onnes found that the electric resistance of mercury suddenly vanished at temperatures near 4 **K**, the same is true to other materials close to absolute zero. Many brilliant physicists worked to develop a theory of the mechanism underlying superconductivity, however, it took 46 years to give a microscopic description of the phenomena.

In 1957, John Bardeen, Leon Cooper, and Robert Schrieffer published their complete theory of superconductivity. The description is based on the fact that the interaction between electrons, resulting from virtual exchange of phonons, is attractive when the energy difference between the electrons states involved is less than the phonon energy [92]. The introduction of the Cooper pairs as charge carriers and the accurate prediction of the superconducting energy gap Δ , are two of the key features of this microscopic theory

Although many works followed the line of BCS theory, there was another direction that produced key breakthroughs. The Ginzburg-Landau theory of superconductivity, proposed 7 years before BCS theory, establishes a complex order parameter ψ in the

context of Landau's theory for second order phase transitions.

Since then, both theories have opened a door to a plethora of results and understanding of this phenomena. In the rest of this section we will take a brief look at some key concepts in superconductivity that will help the reader to understand the main subject of this work.

1.3.1 Phenomenology of Superconductivity

A key concept to start the understanding of superconductivity is that it is a definite macroscopic state of matter, it is a macroscopic phenomenon. In this section we explore the most important feature of a superconductor, namely, how it responds to external magnetic fields. For many practical uses a phenomenological description is enough thus, an electrodynamic treatment is worth to be presented.

Perfect Diamagnets. When an external field $\mathbf{H}$ is applied to a superconductor, all the flux lines are expelled from its interior, so $\mathbf{B}$ is zero inside the material. This is what is known as perfect diamagnetism. This property can be explained by supposing superconductors to be solids in which electrons accelerate in the presence of electric fields, therefore, we start from the standard Drude model for electrical conductivity

$$m\dot{\mathbf{v}} = e\mathbf{E} - \frac{m\mathbf{v}}{\tau}, \quad (1.28)$$

here $\mathbf{v}$ is the drift velocity and τ is a phenomenological parameter describing the scattering time. Since the conductivity is $\sigma = e^2\tau n/m$ and in a superconductor is expected that $\sigma \rightarrow \infty$, we are left with $m\mathbf{v} = e\mathbf{E}$ and using $\mathbf{J} = -en\mathbf{v}$, we write

$$\frac{\partial \mathbf{J}}{\partial t} = \frac{ne^2}{m}\mathbf{E}. \quad (1.29)$$

Taking the curl on both sides and making use of the Ampère law in Gaussian units ($\nabla \times \mathbf{H} = \frac{1}{c}(4\pi\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t})$)

$$\begin{aligned} 0 &= \frac{\partial}{\partial t} \left(\nabla \times \mathbf{J} + \frac{ne^2}{m}\mathbf{B} \right) \\ &= \frac{\partial}{\partial t} \left(\frac{4\pi}{c} \nabla \times \nabla \times \frac{\mathbf{B}}{\mu} - \nabla \times \frac{\partial \mathbf{D}}{\partial t} + \frac{ne^2}{m}\mathbf{B} \right). \end{aligned} \quad (1.30)$$

We take into account only slowly varying currents so that $\frac{\partial \mathbf{D}}{\partial t} = 0$. Integrating in

time we obtain

$$-\lambda_L^2 \nabla^2 (\mathbf{B} - \mathbf{B}_0) + (\mathbf{B} - \mathbf{B}_0) = 0. \quad (1.31)$$

This equation tell us that the magnetic field inside the material decays exponentially from $\mathbf{B}$ to $\mathbf{B}_0$. In the Meissner and Ochsenfelds observation [93] it was concluded that, in the superconductor state, the magnetic field is totally expelled from the material, this is know as *Meissner effect*. The above equation and the experimental observation allows us to set $\mathbf{B}_0$ to zero. This way we obtain the London equation

$$\mathbf{B} - \lambda_L^2 \nabla^2 \mathbf{B} = 0, \quad (1.32)$$

where the London penetration depth is

$$\lambda_L = \sqrt{\frac{mc^2}{4\pi\mu ne^2}}. \quad (1.33)$$

Zero resistance. Under the critical temperature T_c of a material, the DC electrical resistivity ρ drops to zero. This possibly leads to a effect know as *persistent currents*. If a current is set up in a superconductor, it will flow forever without any driving voltage. It is important to mention that zero resistivity and perfect diamagnetism are two independent properties of a superconductor.

Critical magnetic field. The application of a sufficiently strong magnetic field destroys the superconducting state and returns the material to its normal state. We denote the critical value of the magnetic field as $H_c(T)$, which is a function of temperature T . Usually, the temperature dependence of H_c is approximated by

$$H_c(T) = H_c(0) \left(1 - \frac{T^2}{T_c^2}\right). \quad (1.34)$$

In the presence of an applied magnetic field, superconductors can behave in two ways. Type-I superconductors have only one critical magnetic field $H_c(T)$; fields below are excluded from the bulk material due to Meissner effect. On the other hand, type-II superconductor has two critical magnetic fields. For fields between the two critical fields $H_{c1}(T) < H < H_{c2}(T)$, there is a partial penetration by the magnetic flux, therefore, the material contains both normal and superconducting regions. Outside of that regime, the material behaves as a Type-I superconductor.

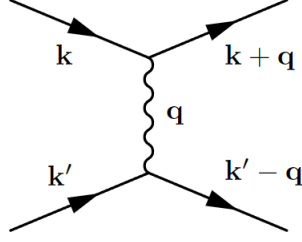


Figure 1.3.1: Electron-electron interaction by exchange of a virtual phonon.

1.3.2 The Cooper Pair and the Energy Gap

The physical mechanism responsible for the BCS interaction is the effective electron-phonon interaction, realized through the exchange of phonons by electrons. The corresponding physical process is schematized in Fig. 1.3.1. On the other hand, the process can be thought of as an effective electron-electron interaction. Namely, after we integrate out the phonon degrees of freedom, we obtain the Fourier component of the effective interaction of the form [94]

$$U_{\mathbf{k}\mathbf{q}} = \frac{2|M_{\mathbf{q}}|^2\hbar\omega_{\mathbf{q}}}{[E(\mathbf{k}+\mathbf{q}) - E(\mathbf{k})]^2 - (\hbar\omega_{\mathbf{q}})^2}, \quad (1.35)$$

where $M_{\mathbf{q}}$ is the matrix element for phonon-electron interaction, $\omega_{\mathbf{q}}$ is the phonon frequency and, $E(\mathbf{k})$ is the electron's energy measured relative to the Fermi energy. The corresponding effective Hamiltonian is

$$H = \sum_{\mathbf{k},\sigma} E(\mathbf{k})c_{\mathbf{k},\sigma}^{\dagger}c_{\mathbf{k},\sigma} + \sum_{\mathbf{k},\mathbf{k}',\mathbf{q},\sigma,\sigma'} \frac{|M_{\mathbf{q}}|^2\hbar\omega_{\mathbf{q}}}{[E(\mathbf{k}+\mathbf{q}) - E(\mathbf{k})]^2 - (\hbar\omega_{\mathbf{q}})^2} \times c_{\mathbf{k}+\mathbf{q},\sigma}^{\dagger}c_{-\mathbf{k}'-\mathbf{q},-\sigma}^{\dagger}c_{\mathbf{k}',-\sigma'}c_{\mathbf{k},\sigma}. \quad (1.36)$$

Here $c_{\mathbf{k},\sigma}^{\dagger}$ ($c_{\mathbf{k},\sigma}$) are the fermionic creation (annihilation) operator in the momentum space and σ is a spin index. We construct the wave function for the Cooper pair by creating two electrons above the Fermi surface $|FS\rangle$ with no net momentum. This is because we are interested in calculating the interaction energy of the pair ΔE , which is maximized when $\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{K} = 0$. We also assume that both electron spins are antiparallel. Thus, the wave function reads

$$\psi = \sum_{\mathbf{k}} a(\mathbf{k})c_{\mathbf{k}\uparrow}^{\dagger}c_{-\mathbf{k}\downarrow}^{\dagger}|FS\rangle, \quad (1.37)$$

where $a(\mathbf{k})$ is the Fourier transform of the spatial pair wave function and the pair

creation operator $\sum_{\mathbf{k}} a(\mathbf{k}) c_{\mathbf{k}\uparrow}^\dagger c_{-\mathbf{k}\downarrow}^\dagger$ is written as a sum over pairs in the momentum space and $|FS\rangle = \prod_{k < k_F} c_{\mathbf{k}\uparrow}^\dagger c_{-\mathbf{k}\downarrow}^\dagger |0\rangle$ defines the Fermi sea. As seen in Eq. (1.35), if $U_{\mathbf{k}\mathbf{q}}$ is positive the interaction is repulsive, while it is attractive when negative. Cooper noticed that the interaction is not strongly momentum-dependent, being attractive only within a finite energy range, defined by the energy $\hbar\omega_{\mathbf{q}}$, the latter referred to as the Debye energy. Therefore, we take $U_{\mathbf{k}\mathbf{q}}$ to be

$$U_{\mathbf{k}\mathbf{q}} = \begin{cases} -U & |E(\mathbf{k} + \mathbf{q}) - E(\mathbf{k})| \leq \hbar\omega_{\mathbf{q}} \\ 0 & \text{otherwise,} \end{cases} \quad (1.38)$$

hence the Hamiltonian is

$$H = \sum_{\mathbf{k}, \sigma} E(\mathbf{k}) c_{\mathbf{k}, \sigma}^\dagger c_{\mathbf{k}, \sigma} - \frac{U}{2} \sum_{\mathbf{k}, \mathbf{q}, \sigma} c_{\mathbf{k}+\mathbf{q}, \sigma}^\dagger c_{-\mathbf{k}-\mathbf{q}, -\sigma}^\dagger c_{-\mathbf{k}, -\sigma} c_{\mathbf{k}, \sigma}. \quad (1.39)$$

From here, we can obtain one of the main predictions of the BCS theory. We then find for the energy

$$E = \langle \Psi | H | \Psi \rangle = 2 \sum_{\mathbf{k}} E(\mathbf{k}) |a(\mathbf{k})|^2 - U \sum_{\mathbf{k}\mathbf{q}} a^*(\mathbf{k} + \mathbf{q}) a(\mathbf{k}). \quad (1.40)$$

To determine $a(\mathbf{k})$ we vary E subject to the condition $\sum_{\mathbf{k}} |a(\mathbf{k})|^2 = 1$. Thus we obtain the following equation

$$(2E(\mathbf{k}) - \lambda) a(\mathbf{k}) = U \sum_{\mathbf{k}'} a(\mathbf{k}'). \quad (1.41)$$

To satisfy the restriction we remember that V takes only nonzero value for energies in the range E_F to $E_F + \hbar\omega_{\mathbf{q}}$. The same is also true for $a(\mathbf{k})$, and the sum in the last equation run for infinite values of $\mathbf{k}'$, therefore

$$a(\mathbf{k}) = \frac{U \sum_{\mathbf{k}} a(\mathbf{k})}{2E(\mathbf{k}) - \lambda}, \quad \sum_{\mathbf{k}} a(\mathbf{k}) = \sum_{E_F < E(\mathbf{k}) < E_F + \hbar\omega_{\mathbf{q}}} \frac{U \sum_{\mathbf{k}} a(\mathbf{k})}{2E(\mathbf{k}) - \lambda} \quad (1.42)$$

Finally, we take the complex conjugate of the last equation, multiply by $a(\mathbf{k})$, and sum over $\mathbf{k}$. Now we can fix the Lagrange parameter to $\lambda = E$ in order to agree with equation (1.40). Thus,

$$1 = U \sum_{E(\mathbf{k})} \frac{1}{2E(\mathbf{k}) - E} = U \int_{E_F}^{E_F + \hbar\omega_{\mathbf{q}}} d\epsilon \frac{g(\epsilon)}{2\epsilon - E} \quad (1.43)$$

where we replace the summation by an integral over energy, $\sum_{E(\mathbf{k})} \rightarrow \int_{E_F}^{E_F + \hbar\omega_{\mathbf{q}}} g(\epsilon)$ and introduce the density of states $g(\epsilon)$. We can approximate $g(\epsilon) \approx g(E_F)$ because of the narrow range of interaction. Evaluating the integral we get

$$E = 2E_F - \frac{2\hbar\omega_{\mathbf{q}}e^{-2/g(E_F)U}}{1 - e^{-2/g(E_F)U}} \quad (1.44)$$

In the weak coupling limit ($Ug(E_F) \ll 1$)

$$E \approx 2E_F - 2\hbar\omega_{\mathbf{q}}e^{-2/g(E_F)U} \quad (1.45)$$

From the last equation we conclude that not matter how weak the attractive interaction is, the bound state formed by two electrons known as Cooper pair, is always lower in energy than $2E_F$. One can show that all other solutions of (1.40) lead to energies $> 2E_F$. The energy E of the bound state is not analytical function of U as $U \rightarrow 0$, therefore, E cannot be expanded in powers of U implying that the result for E cannot be obtained by a perturbation theory approach in powers of U .

Furthermore, the energy in Eq. (1.45) coincides with the energy gap of the superconducting ground state [94]

$$\Delta = 2\hbar\omega_{\mathbf{q}}e^{-2/g(E_F)U}. \quad (1.46)$$

Mean-field theory for superconductivity: It is useful to take a brief look at the mean field analysis of the BCS theory. The mean-field theory introduces the gap parameter as

$$\Delta = |\Delta|e^{i\phi} = - \sum_{\mathbf{k}'} U_{\mathbf{k}'\mathbf{k}} \langle c_{-\mathbf{k}'\downarrow} c_{\mathbf{k}'\uparrow} \rangle, \quad (1.47)$$

which is an intensive quantity and the order parameter of the theory. Here we consider $\sigma = \{\uparrow, \downarrow\}$, this way we have an s -wave or zero total spin state. It is also a macroscopic property of superconductors that has both an amplitude and a phase ϕ . An additional advantage of mean-field analysis is that it allows us to understand the nature of the excited states. The order parameter sets the size of the gap in the excitation spectrum.

Decoupling the interaction with the order parameter we obtain the mean-field Hamiltonian for the BCS theory

$$H_{\text{MF}} = \sum_{\mathbf{k}, \sigma} E(\mathbf{k}) c_{\mathbf{k}, \sigma}^\dagger c_{\mathbf{k}, \sigma} - \sum_{\mathbf{k}} \Delta^* c_{-\mathbf{k} \downarrow} c_{\mathbf{k} \uparrow} - \sum_{\mathbf{k}} \Delta c_{\mathbf{k} \uparrow}^\dagger c_{-\mathbf{k} \downarrow}^\dagger + \sum_{\mathbf{k}} \Delta \langle c_{\mathbf{k} \uparrow}^\dagger c_{-\mathbf{k} \downarrow}^\dagger \rangle. \quad (1.48)$$

The order parameter Δ is an emergent many-body property whose fluctuations become negligible in the thermodynamic limit. This statement can be made more rigorous by a path integral approach, in this way the quantum variable is fully taken into account.

1.3.3 Field Theoretical Approach to BCS Theory

Lets first consider the Hamiltonian (1.39) in a simpler form,

$$H = \sum_{\mathbf{k}, \sigma} E(\mathbf{k}) c_{\mathbf{k}, \sigma}^\dagger c_{\mathbf{k}, \sigma} - U A^\dagger A, \quad (1.49)$$

where $A^\dagger = \sum_{\mathbf{k}, \mathbf{k}'} c_{\mathbf{k} \uparrow}^\dagger c_{-\mathbf{k} \downarrow}^\dagger$ and $A = \sum_{\mathbf{k}, \mathbf{k}'} c_{-\mathbf{k}' \downarrow} c_{\mathbf{k}' \uparrow}$ are pairing operators. The BCS theory can be physically interpreted as the Bose-Einstein condensation of Cooper pairs, this is due to the fact that the order parameter for the BCS transition is proportional to the expectation value of the pairing operators. Next, we proceed by writing the problem as a path integral

$$Z = \int [dc^\dagger, dc] \exp \left\{ - \int d\tau \left[\sum_{\mathbf{k}, \sigma} c_{\mathbf{k}, \sigma}^\dagger (\partial_\tau + E(\mathbf{k})) c_{\mathbf{k}, \sigma} - U A^\dagger A \right] \right\}. \quad (1.50)$$

To deal with the interaction term, the condensate of Cooper pairs may be conveniently treated by performing Hubbard-Stratonovich transformation

$$e^{U \int d\tau A^\dagger A} = \int [d\Delta^*, d\Delta] e^{\int d\tau [\Delta^* A + A^\dagger \Delta + \frac{1}{U} \Delta^* \Delta]}, \quad (1.51)$$

where $\Delta^*(\tau)$ and $\Delta(\tau)$ are complex fluctuating fields. Then, the action takes the form

$$S = \int d\tau \left[\sum_{\mathbf{k}, \sigma} c_{\mathbf{k}, \sigma}^\dagger (\partial_\tau + E(\mathbf{k})) c_{\mathbf{k}, \sigma} + \Delta^* A + A^\dagger \Delta - \frac{1}{U} \Delta^* \Delta \right]. \quad (1.52)$$

Nambu spinor representation. Now it is convenient to introduce the Nambu's spinor representation proposed by Yoichiro Nambu [95]. Nambu introduced an *isospin* construct that describes the (electron-hole) space. Therefore, the spinors

describe electrons and holes and not up and down electron spins.

$$\psi_{\mathbf{k}} = \begin{pmatrix} c_{\mathbf{k}\uparrow} \\ c_{-\mathbf{k}\downarrow}^\dagger \end{pmatrix}, \quad \psi_{\mathbf{k}}^\dagger = \begin{pmatrix} c_{\mathbf{k}\uparrow}^\dagger, c_{-\mathbf{k}\downarrow} \end{pmatrix}. \quad (1.53)$$

Nambu spinors behave like convectional fermionic fields, with the following anticommutaion relation

$$\{\psi_{\mathbf{k},\alpha}, \psi_{\mathbf{k},\beta}^\dagger\} = \delta_{\alpha\beta} \delta_{\mathbf{k}\mathbf{k}'}. \quad (1.54)$$

The Nambu's formalism allow to express the action in a very convenient way.

$$h_{\mathbf{k}} = \begin{pmatrix} E(\mathbf{k}) & \Delta \\ \Delta^* & -E(\mathbf{k}) \end{pmatrix} \quad (1.55)$$

It is worth noticing that we can express the $\psi_{\mathbf{k}}^\dagger h_{\mathbf{k}} \psi_{\mathbf{k}} = \psi_{\mathbf{k}}^\dagger \mathbf{h}_{\mathbf{k}} \cdot \boldsymbol{\tau} \psi_{\mathbf{k}}$ by taking $\Delta = \Delta_1 + i\Delta_2$ and $\mathbf{h}_{\mathbf{k}} = (\Delta_1, \Delta_2, E(\mathbf{k}))$. The vector $\mathbf{h}_{\mathbf{k}}$ plays the role of a field acting in isospin space . Here we used the Pauli isospin matrices

$$\boldsymbol{\tau} = (\tau_1, \tau_2, \tau_3) = \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right). \quad (1.56)$$

Therefore, the partition function in the Nambu's representation takes the form

$$Z = \int [d\psi^\dagger, d\psi, d\Delta^*, d\Delta] e^{-\int d\tau [\sum_{\mathbf{k}} \psi_{\mathbf{k}}^\dagger (\delta_\tau + h_{\mathbf{k}}) \psi_{\mathbf{k}} + \frac{1}{U} \Delta^\dagger \Delta]}. \quad (1.57)$$

In the next chapters we will explore the physics that emerges from all of the phenomena that were presented in this introduction. The main goal is to develop the theory of the superconducting quantum phase transition in a tilted Dirac fermion system, an aim that we hopefully made well motivated after revisiting some fundamental concepts of Dirac materials, superconductivity and quantum criticality.

Chapter 2

Theoretical Framework

In the development of a quantum field theory for tilted Dirac superconductor transition we will make use of several well established mathematical tools and field theories. In this chapter we introduce the Ginzburg-Landau theory, the Gross-Neveu-Yukawa theory, and the renormalization group (RG). These tools are fundamental for our analysis and are useful to the physical interpretation of the phenomena that we will investigate.

2.1 Ginzburg-Landau-Wilson Theory

The basis of the Ginzburg-Landau-Wilson (GLW) theory is its application to the description of the behavior of complex systems undergoing continuous phase transitions, such as the transition between different states of matter, for instance, liquid and gas or ferromagnetic and paramagnetic phases. In our case, we will use the theory to describe the fluctuations of the order parameter, which behaves as a boson, and its role in the superconducting transition.

We construct the Ginzburg-Landau-Wilson theory using an N -component fluctuating field $\phi \equiv (\phi_1(\mathbf{r}), \dots, \phi_N(\mathbf{r}))$ in D euclidean dimensions. For this section, we define the GLW action as follows

$$S[\phi] = \int d\tau d^D \mathbf{r} \left[\phi(-\partial_\tau^2 - \nabla^2 + m^2)\phi + \frac{\lambda}{2}\phi^4 \right], \quad (2.1)$$

In quantum field theory, a mathematical structure like the above Ginzburg-Landau functional is called a ϕ^4 theory. Equipped with different symmetries, these ϕ^4 theories have become a very useful tool to study the critical behaviour in many physical

systems. Their relevance for understanding these phenomena was emphasized by Wilson, and Fisher [96–98].

The GLW partition function with a general number of real components N serves as a sort of meta-model. For $N = 1$, such action would be symmetric only with respect to an change of sign of the fields, $\phi(\mathbf{r}) \rightarrow -\phi(\mathbf{r})$. This is the same Z_2 symmetry as the Ising model. The $N = 2$ theory describes the superfluid transition and the XY model. For $N = 3$ real components the action is invariant under an arbitrary global rotation of the fields $\phi_i(\mathbf{r}) \rightarrow M_{ij}\phi_j(\mathbf{r})$, with the real matrix $M^T M = 1$, implying that the action is invariant under the group $SO(3)$. GLW theory with $N = 3$ thus has the same symmetry as the Heisenberg model and therefore is expected to describe the magnetic system of Heisenberg spins near the critical point.

All calculations to obtain observables involve the theory propagator

$$G(i\omega, \mathbf{p}) = \frac{1}{\omega^2 + p^2 + m_B^2}. \quad (2.2)$$

Feynman diagrams and correlation functions. The n -point correlation function in the ϕ^4 -theory is given by the integral of the form

$$\langle \phi(r_1) \dots \phi(r_n) \rangle = Z^{-1} \int [d\phi] \phi(r_1) \dots \phi(r_n) e^{-S[\phi]} \quad (2.3)$$

where $Z = \int [d\phi] e^{-S[\phi]}$ is the partition function. In order to calculate the field integral we can do a perturbative approach, keeping the free action and expanding the partition function in powers of $V_I(\phi)$ ($V_I(\phi) = \frac{\lambda}{2}\phi^4$ in our case). The n -correlation function can be evaluated using Wick's theorem in terms of the Gaussian ($V_I(\phi) = 0$) two-point correlation function. Such contributions turn out to be proportional to the 2-point correlation function calculated at the same (space-time) point, and are thus of the form

$$\langle \phi^2(r) \rangle_0 = \int [d\omega] \int [d^D \mathbf{p}] \frac{1}{\omega^2 + p^2 + m_B^2} \quad (2.4)$$

which are infinite for $D \geq 2$. This is called UV divergence, because they follow from the large momentum behaviour of the perturbative contributions. Actually, expectation values of product of fields at coinciding points (local monomials) are always divergent. In conclusion, we need a method to address these divergences.

2.1.1 Spontaneous Symmetry Breaking

Landau's mean-field theory. To study the critical phenomena, it is very useful to employ a saddle point approximation of the GWL theory. First, only for this case, let us consider $T \neq 0$ and that the fields are independent of imaginary time. This way we can integrate over the time and redefine the action as follows

$$S = \frac{1}{k_B T} \int d^D \mathbf{r} \left[\phi(-\nabla^2 + \mu)\phi + \frac{\lambda}{2}\phi^4 \right], \quad (2.5)$$

where μ is a chemical potential. In this approach, we approximate the partition function with the value of the integral at its maximum. In this context, the free energy of the system is simply $F = k_B T S[\phi_0]$, where ϕ_0 minimizes the action, determined by

$$\frac{\delta S}{\delta \phi} \Big|_{\phi_0} = 0. \quad (2.6)$$

The solution ϕ_0 is assumed to be static i.e. it is independent of the coordinates, and therefore takes either trivial value ($\phi_0 = 0$) or it is equal to

$$|\phi_0|^2 = \frac{m^2}{\lambda}. \quad (2.7)$$

Taking the m^2 to be the tuning parameter, the system is in the normal phase for $m^2 < 0$, $\phi_0 = 0$. When $m^2 > 0$, we obtain that ϕ_0 is finite

$$S = -\frac{(m^2)^2}{2\lambda} \frac{\mu}{k_B T} < 0. \quad (2.8)$$

In the saddle point approximation, the transition occurs at the critical temperature T_c where the function $\mu(T)$ change sign. For $\mu > 0$, ϕ_0 becomes a maximum of the action, as shown in Fig. 2.1.1. For $N = 2$, since the value of ϕ_0 determines the ordered phase, it can be taken as the order parameter of the superfluid phase transition.

It is important to notice that the absolute value of ϕ_0 is fixed by the condition Eq. (2.7), while leaving its phase arbitrary. Any choice of phase breaks the $U(1)$ invariance of the action (2.5) under the transformation

$$\phi(\mathbf{r}) \rightarrow e^{i\theta} \phi(\mathbf{r}). \quad (2.9)$$

This mechanism is known as *spontaneous symmetry breaking*.

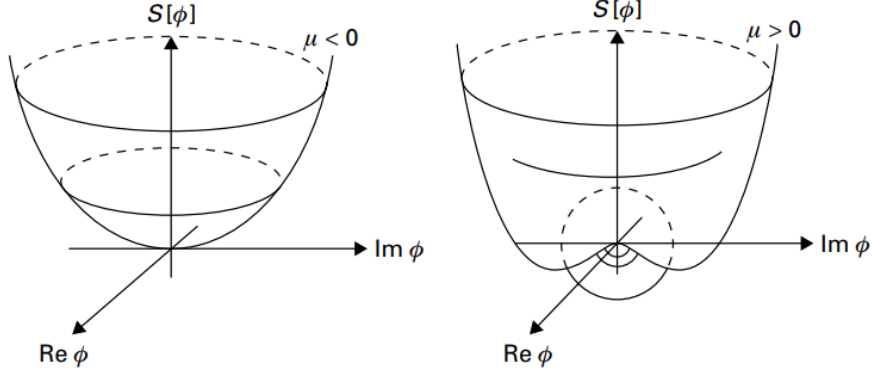


Figure 2.1.1: The action S as a function of the constant complex field ϕ . For $\mu < 0$ there is a unique minimum at $\phi_0 = 0$. On the other hand, at $\mu > 0$, the action resembles a “Mexican hat” and have a minima at fine ϕ_0 [60].

2.1.2 Upper Critical Dimension

Landau’s mean-field theory, at least in the normal phase, may be understood as the zeroth order approximation in the coupling λ . It is expected that better approximations would be obtained by performing a perturbation theory in coupling λ but, in order for the perturbation approach to succeed, the coupling parameter needs to be small. Near the critical point, the critical behaviour depends of the dimensionality of the system, so we get two types of behaviours: for $d > 4$, as $m^2 \rightarrow 0$, $\lambda \rightarrow 0$, and the critical behaviour is well described by the mean-field theory; for $d < 4$, on the other hand, $m^2 \rightarrow 0$, $\lambda \rightarrow \infty$, and perturbation theory does not apply in the critical region [60]. Moreover, due to the locality of quantum field theories, perturbative calculations are full of short distance singularities, or in the Fourier space, full of divergences at large momenta (UV divergences), with the form shown in Eq. (2.4).

At the critical point, a system of fluctuating fields with a quartic self-interaction (as the ϕ^4 -theory) therefore has the important scaling properties described correctly by the mean-field theory when $d > 4$, while otherwise this is not the case. Therefore, there is a dimension separating the regimes where the critical behavior is determined exactly by the mean-field theory from where the critical behavior is non-trivial. Such a special dimensionality is called the *upper critical dimension*. For a theory with an arbitrary power r of the field in the interaction term ϕ^r , there exists the upper critical dimension

$$D_c = \frac{r}{\frac{r}{2} - 1}. \quad (2.10)$$

A way to identify the upper critical dimension is by means of a dimensional analysis. Dimensionless coupling constants are the technical seal for the upper critical dimension. Thus, in the ϕ^4 -theory, the upper critical dimension is $D_c = 4$.

We can now proceed with the dimensional analysis of the following action, which is dimensionless,

$$S = \int d^d \mathbf{r} \left[\phi(-\partial_\tau^2 - \nabla^2 + m^2)\phi + V_I(\phi) \right]. \quad (2.11)$$

We first define the dimension of the length to be $\dim[x] = -1$, and thus $\dim[\nabla] = 1$, implying that $\dim[\phi] = \frac{1}{2}(d - 2)$. In general, we assume that the perturbation V_I is a linear combination of local monomials of degree r in the ϕ field of the form (for our case, we do not consider monomials with derivative in the fields)

$$V(\phi) = \int d^d \mathbf{r} V_r(\phi, \mathbf{r}). \quad (2.12)$$

Then, we can define the dimension of V by

$$\dim[V] = \dim[\phi]r - d, \quad (2.13)$$

as developed in [99].

2.2 The Renormalization Group

We already know that a direct perturbative approach breaks down under the upper critical dimension. In four dimensions, the integrals of two- and four-point functions diverge. With the help of so-called regularization procedures they can be made finite by modifying its short-distance or large-energy-momentum structure. A regularization parameter is introduced, so that all divergences of the integrals appear as singularities in this parameter. Changing the properties of the action generates an effective field theory, which only valid at an energy scale during the renormalization.

In dimensional regularization, the measure of momentum integration is changed by allowing the continuous dimension D . The idea is to calculate a Feynman integral for a continuous dimension $D = 4 - \epsilon$ and expand them in powers of ϵ , which is also known as ϵ -*expansion*, introduced by 't Hooft and Veltman [100]. This regularization has the following advantages: first, it preserves all symmetries of the theory. Second, it allows an easy identification of the divergences. Third, it regularizes at the same

time IR-divergences in massless theories.

Our renormalization procedure is mainly formulated in terms of vertex (one-particle irreducible or 1PI) functions,

$$\Gamma^n(x_1, \dots, x_n) = \langle \phi(x_1) \dots \phi(x_n) \rangle_{1\text{PI}}. \quad (2.14)$$

which have simpler properties than correlation functions.

Renormalizability. We check renormalizability classifying the vertices by means of dimensional analysis.

If at least one vertex has positive dimension, $\dim[V] > 0$, the theory is called **non-renormalizable**.

When all vertices of the theory have strictly negative dimensions, only a finite number of Feynman diagrams are superficially divergent. The corresponding field theory is called **super-renormalizable**.

Renormalizable theories are characterized by the property that at least one vertex has dimension 0, and no vertex has a positive dimension. For the case of ϕ^4 -theory we have

$$\dim[\phi^4] = 4 - D \quad (2.15)$$

Implying that the vertex is renormalizable in $D = 4$ [99].

2.2.1 Renormalization of the ϕ^4 Theory

In this section we present our algorithm of renormalization using the ϕ^4 -theory as an example. We employ the *minimal subtraction* (MS) scheme, which consists in performing simultaneous expansions of all Feynman integrals in powers of ϵ , followed by a subtraction of terms containing ϵ -poles. Especially for a study of the critical region in $D < 4$ dimensions, this procedure has an important advantage. It can be used to remove divergences in $D = 4$, but also at small finite values of ϵ at zero mass, so that we can access the physics under $D = 4$.

Within the MS scheme, observables can be obtained by multiplying in each correlation function, the fields, the mass, and the coupling constant by compensating factors, the renormalization constants Z_ϕ , Z_{m^2} and Z_λ , which are to be called field or wave function, mass, and coupling renormalization constant, respectively.

We introduce the renormalization constants Z_ϕ , Z_{m^2} and Z_λ to obtain the

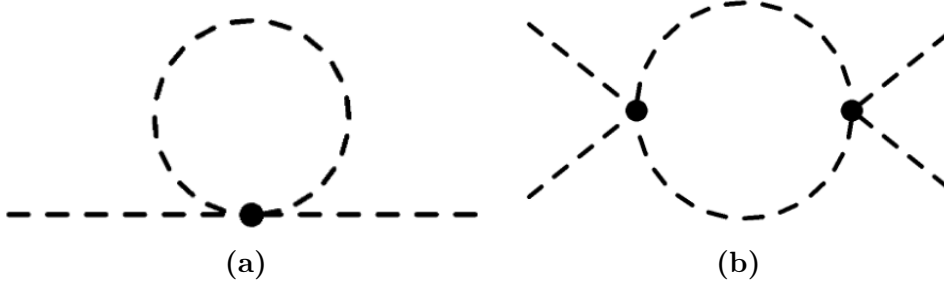


Figure 2.2.1: Diagrams contributing to the ϕ^4 renormalization at one-loop order. The Bosonic self-energy (a) only renormalizes the mass m and diagram (b) contributes to the renormalization of the coupling λ .

renormalized action, from which we calculate the correlation functions

$$S_R = \int d\tau \int d^D \mathbf{r} \left[Z_\phi \phi (-\partial_\tau^2 - \nabla^2) \phi + Z_{m^2} m^2 \phi^2 + Z_\lambda \frac{\lambda \mu^\epsilon}{2} \phi^4 \right]. \quad (2.16)$$

Here, we introduce μ as the RG (momentum) scale, so that the couplings are dimensionless. Together with condition $\phi_0 = Z_\phi^{1/2} \phi$, the following renormalization conditions allow us to recover the bare action (2.1)

$$m_0^2 \equiv \frac{Z_{m^2}}{Z_\phi} m^2 \mu^2 \quad (2.17)$$

$$\lambda_0 \equiv \frac{Z_\lambda}{Z_\phi^2} \mu^\epsilon \lambda. \quad (2.18)$$

From now on, the objects with (without) the subscript are the bare (renormalized) ones. All of the physics that describes the critical phenomena is contained in the renormalization constants.

The Feynman diagrams contributing to renormalization are shown in Fig. 2.2.1. We are going to be making use of the vertex functional $\Gamma^{(n)}$, which is the Legendre transform of the generator of connected correlation functions. This functional contains only one-particle irreducible (1PI) Feynman diagrams. At the one-loop order, the counterterms which are necessary to make the two-point vertex function $\Gamma_R^{(2)}(\mathbf{k}, \Omega)$ finite is given by the contribution of the bosonic self-energy (see Fig. 2.2.1a)

$$\Gamma_R^{(2)}(0, 0) = Z_{m^2} m^2 + \frac{\lambda}{2} (2N_B + 2) \int [d\omega] \int [d^D \mathbf{q}] \frac{1}{\omega^2 + q^2 + m^2}. \quad (2.19)$$

The correction to the four-point vertex function (depicted in Fig. 2.2.1b) yields

the renormalization for the coupling λ

$$\Gamma_R^{(4)}(\mathbf{k}, \Omega) = Z_\lambda \frac{\lambda \mu^\epsilon}{2} - \left(\frac{\lambda}{2}\right)^2 \frac{1}{2} (4N_B + 1) \int \int \frac{[d\omega][d^D \mathbf{q}]}{\omega^2 + q^2 + m^2} \frac{1}{(\omega + \Omega)^2 + (\mathbf{q} + \mathbf{k})^2 + m^2}, \quad (2.20)$$

where $\mathbf{k}$ and Ω are external momentum and frequency, respectively. We solved the integrals using dimensional regularization with $D = 4 - \epsilon$ dimensions in the critical plane $m = 0$, and setting the external momentum and frequency to zero. This allows us to isolate the divergences in order to obtain the renormalization constants

$$Z_{m^2} = 1 + (N_B + 1) \frac{\lambda}{8\pi^2} \frac{1}{\epsilon}, \quad (2.21)$$

$$Z_\lambda = 1 + (4N_B + 1) \frac{\lambda}{8\pi^2} \frac{1}{\epsilon}. \quad (2.22)$$

Since there is no counterterm proportional to $\mathbf{k}$ the renormalization constant for the field is $Z_\phi = 1$ at one-loop order. Using the above renormalization conditions we can obtain the RG β functions to the leading order in the ϵ -expansion. We define the β function for $X = \{m^2, \lambda\}$ as $\beta_X \equiv -dX/d\ln(\mu)|_{\{X_0\}}$, with $(\{X_0 = \{m_0^2, \lambda_0\}\})$ fixed. Thus, the β functions for the ϕ^4 -theory are

$$\beta_{m^2} = 2m^2 - (N_B + 1)m^2\lambda + \mathcal{O}(\text{Two-loops}), \quad (2.23)$$

$$\beta_\lambda = \epsilon\lambda - (4N_B + 1)\lambda^2 + \mathcal{O}(\text{Two-loops}), \quad (2.24)$$

where we rescaled $\lambda^2/8\pi^2 \rightarrow \lambda^2$. These functions reflect how the couplings change as a function of the scale of the theory. The RG procedure may be understood as a flow in the space of coupling constants generated by the above differential equations. Three different scenarios are presented in Fig. 2.2.2.

The easiest way to understand qualitatively the flow in the m^2 - λ plane is to look for the fixed points at which $\beta_{m^2} = \beta_\lambda = 0$ and analyze their stability. The differential recursion relations in Eqs. (2.23) and (2.24) have two fixed points: the Gaussian in $m^2 = 0$, and $\lambda = 0$, and the Wilson-Fisher at

$$(m_\star^2, \lambda_\star) = \left(0, \frac{\epsilon}{(4N_B + 1)}\right). \quad (2.25)$$

Classification of fixed points. The behaviour of the couplings in the RG flow allows us to classify them in at least in three different types. First, if a coupling

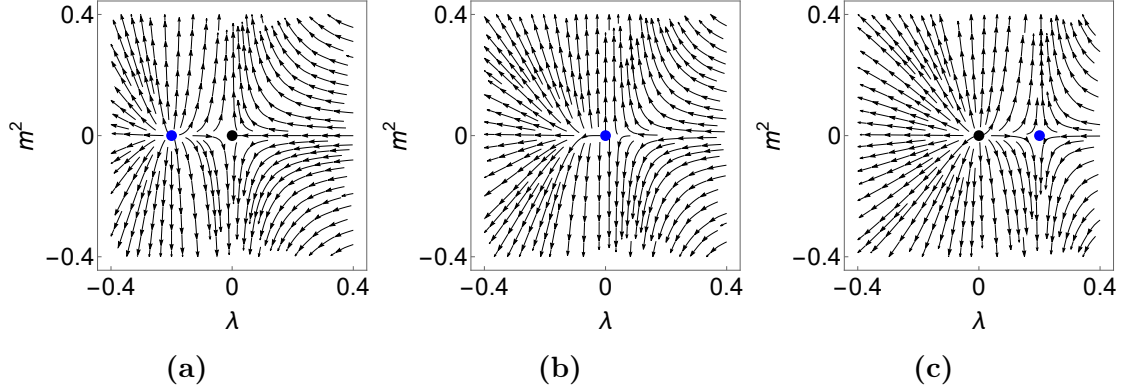


Figure 2.2.2: RG flow of the renormalized interaction λ and the mass m^2 with $N_B = 1$. (a) above 4 dimensions ($\epsilon = -1 < 0$). (b) at 4 dimensions ($\epsilon = 0$). (c) Below four dimensions ($\epsilon = 1 > 0$). The blue dot correspond to the Wilson-Fisher fixed point and the black one to the Gaussian fixed point. The Wilson-Fisher fixed point moves from the unphysical region for $\epsilon < 0$ to the physical region for $\epsilon > 0$. We work with the vertex functional, so we don't consider the external lines of the diagrams

grows, i.e. $\beta_X > 0$, then its is called **relevant**. Second, if a coupling shrinks, i.e. $\beta_X < 0$, then its is called **irrelevant**. Finally, if a coupling neither grows nor shrinks, i.e. $\beta_X = 0$, it is called a **marginal coupling**.

Below four dimensions, the Wilson-Fisher point moves to the physical region and it is stable in the λ direction and unstable in the m^2 direction. A fixed point with exactly one unstable direction is called **critical point**. Thus, the critical behaviour for $\epsilon > 0$ is governed by the Wilson-Fisher critical point. As we approach $D = 4$ from below, the Wilson-Fisher point moves to Gaussian point until they coincide at $D = 4$.

In dimensions higher than four, the Wilson-Fisher fixed point is at a negative interaction and therefore unphysical. It is also unstable in both directions of the $m^2 - \lambda$ plane, therefore, it has two relevant couplings. An important commentary needs to be made here: above four dimensions, i.e. for $\epsilon < 0$, the mean-field approximation, which neglects interactions, is exact in the critical region because the critical point is the non-interacting Gaussian fixed point, with a stable in the λ direction, and an unstable direction in the m^2 direction. This implies that small interactions are irrelevant and flow to zero.

The flow near the fixed points can be described by means of the stability matrix M defined by

$$M_{ij} = \frac{\partial \beta_{X_i}}{\partial X_j} \Big|_{X_i = X_{*i}}. \quad (2.26)$$

The eigenvectors of M with positive values determines the relevant (unstable) direction at the given fixed point, and those with negative eigenvalues the irrelevant (stable) directions. There is only one such relevant direction in the case of a critical point, with the eigenvalue y . This allow us to calculate the correlation length exponent given by $\nu = 1/y$. The stability matrix take the form

$$M = \begin{pmatrix} -\epsilon & 0 \\ 0 & 2 - \frac{(N_B+1)\epsilon}{4N_B+1} \end{pmatrix}. \quad (2.27)$$

The correlation length exponent is therefore

$$\nu = \frac{1}{2} + \frac{(N_B + 1)\epsilon}{4(4N_B + 1)} + \mathcal{O}(\epsilon^2). \quad (2.28)$$

For $\epsilon = 1$ and $N_B = 1$ we obtain $\nu \approx 0.6$, relatively close to the experimental value $\nu = 0.66 \pm 0.02$ in ${}^4\text{He}$ [101].

2.3 Gross-Neveu-Yukawa Model

The Gross-Neveu-Yukawa (GNY) model is obtained by performing a Hubbard-Stratonovich transformation in the Gross-Neveu model [102]. This procedure introduces a scalar field and a Yukawa interaction. Therefore, our model that can be defined in continuous dimensions, involves N Dirac fermions and one scalar field. Furthermore, it possesses a continuous $U(N)$ symmetry, and a discrete chiral symmetry that prevents the addition of a fermion mass term to the action. Similar to the ϕ^4 -theory, the model undergoes spontaneous symmetry breaking of the $\mathbb{Z}_2$ symmetry, which leads to spontaneous fermion mass generation.

In order to address the quantum criticality of Dirac fermions in the RG framework, we will promote the order parameter field to a dynamical variable described by the action (2.1) of the ϕ^4 -theory. In addition, massless Dirac fermions are coupled to the bosonic order-parameter fluctuations through the Yukawa term

$$S_Y[\bar{\psi}, \psi, \phi] = \int d\tau \int d^d \mathbf{r} g \phi \bar{\psi} \psi. \quad (2.29)$$

Therefore, the total action takes the form

$$S[\bar{\psi}, \psi, \phi] = \int d\tau \int d^d \mathbf{r} \left[\bar{\psi} \gamma^\mu \partial_\mu \psi + g \phi \bar{\psi} \psi - (\partial_\mu \phi)^2 + m^2 \phi^2 + \frac{\lambda}{2} \phi^4 \right] \quad (2.30)$$

We work in a d -dimensional euclidean spacetime. The field $\psi = \psi_i$ with $i \in \{1, \dots, N_f\}$ denotes a $N = d_\gamma N_f$ component spinor. N_f is the number of fermion flavor and $d_\gamma = 4$ is the dimension of the Clifford algebra of a single flavor of four-component Dirac fermions. Standard notation is used $\not{\partial} = \gamma^\mu \partial_\mu$ with $\mu \in \{1, \dots, d-1\}$ and, because of the properties of the action under the spin group, the conjugate of the Dirac field is given by $\bar{\psi} = \psi^\dagger \gamma_0$.

As mention before, the action has a $U(N)$ fermion symmetry corresponding to the transformations,

$$\psi(r) \mapsto U \psi(r), \quad \bar{\psi}(r) \mapsto \bar{\psi}(r) U^\dagger, \quad \text{with} \quad U U^\dagger = 1, \quad (2.31)$$

and a discrete symmetry which forbids a fermion mass term and odd powers of ϕ . In even dimensions the symmetry correspond to a chiral $\mathbb{Z}_2$ symmetry (fermion mass breaks space reflection symmetry in odd dimensions)

$$\psi(r) \mapsto \gamma_S \psi(r), \quad \bar{\psi}(r) \mapsto -\bar{\psi}(r) \gamma_S, \quad \phi(r) \mapsto -\phi(r), \quad (2.32)$$

where γ_S is defined by (A.2) (for $d=4$, $\gamma_S \equiv \gamma_5$). The set of matrices γ together with matrix γ_S form a representation of the generators of the Clifford algebra.

2.3.1 Renormalization Group Near Four Dimensions

First we perform dimensional analysis of the GNY theory. Since the engineering dimensions of the fermion and boson fields are $\dim[\psi] = d/2$ and $\dim[\phi] = (d+z-2)/2$. The dimension of the Yukawa coupling is $\dim[g] = (2-d+z)/2$ and the dimension of the ϕ^4 -coupling is $\dim[\lambda] = -d-z+4$. Previously, from Eq. (3.9) we find that the dynamical exponent $z = 1$, this implies that $\dim[g] = \dim[\lambda] = 0$ for $d = 3$, and then $d_c = 3$ is the upper critical dimension. The theory is renormalizable and to proceed, we will therefore use an ϵ -expansion about three spatial or equivalently four space-time dimensions D .

We introduce Grassmann sources $\bar{\eta}(\mathbf{r}) \equiv \bar{\eta}$, $\eta(\mathbf{r}) \equiv \eta$ and a real scalar source

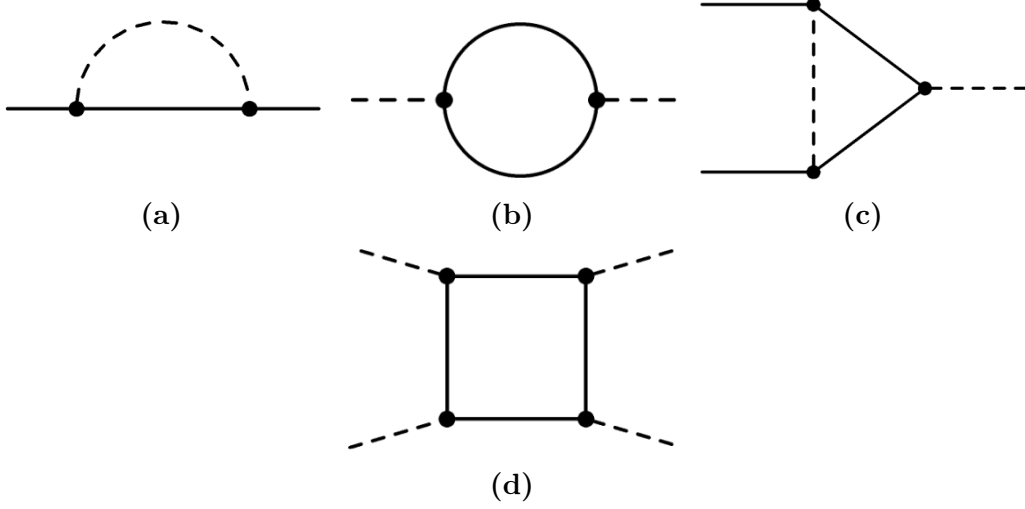


Figure 2.3.1: Feynman diagrams in the GNY theory. The dashed lines are the propagator for bosons and the solid lines are the propagator for fermions. (a) One-loop fermionic self-energy diagram. (b) One-loop bosonic self-energy diagram. (c) One-loop vertex correction of Yukawa coupling. (d) One-loop vertex correction of ϕ^4 -coupling.

$J(\mathbf{r}) \equiv J$ to make the generator for both ψ and ϕ correlation functions of the theory

$$Z[\bar{\eta}, \eta, J] = \int [d\psi d\bar{\psi} d\phi] \exp \left\{ -S[\bar{\psi}, \psi, \phi] + \int d^D \mathbf{r} [\bar{\eta}\psi + \bar{\psi}\eta + J\phi] \right\}. \quad (2.33)$$

Since we calculate only the RG functions, we restrict the calculations to the critical (massless) theory $m^2 = m_c^2 = 0$. The renormalized action reads ($\epsilon = 4 - D$)

$$S_R = \int d\tau \int d^d \mathbf{r} \left[Z_\psi \bar{\psi} \gamma^\mu \partial_\mu \psi + Z_g g \mu^{\epsilon/2} \phi \bar{\psi} \psi - Z_\phi (\partial_\mu \phi)^2 + Z_\lambda \frac{\lambda \mu^\epsilon}{2} \phi^4 \right] \quad (2.34)$$

where μ defines the energy scale which parameterizes the RG flow of the couplings. The explicit dependence of the action on the renormalization energy scale μ reflects that the coupling constants g and λ are dimensionless. The renormalization constants Z_ψ, Z_ϕ, Z_g and Z_λ yield the relations between bare and renormalized coupling constants, which read

$$g_0 = g \mu^{\epsilon/2} Z_g / (Z_\psi Z_\phi^{1/2}), \quad \lambda_0 = \lambda \mu^\epsilon Z_\lambda / Z_\phi^2. \quad (2.35)$$

Together with the diagrams in Fig. 2.2.1, the Feynman diagrams at one-loop

order contributing to renormalization are depicted in Fig. 2.3.1. The correction to the correlation functions $\Gamma_R^{(f,b)}$, where f and b correspond to a number of fermion and scalar fields in the correlator, respectively, are given by

$$\Gamma_R^{2,0}(p) = iZ_\psi \not{p} + g^2 \mu^\epsilon \int [d^D \mathbf{q}] \frac{1}{(p-q)^2} \frac{i \not{q}}{q^2}, \quad (2.36)$$

$$\Gamma_R^{2,1}(p_1, p_2, k) = gZ_g \mu^{\epsilon/2} - g^3 \mu^{3\epsilon/2} \int [d^D \mathbf{q}] \frac{1}{(q+p_2)^2} \frac{\not{q}(\not{q}-\not{k})}{q^2(q-k)^2}, \quad (2.37)$$

$$\Gamma_R^{0,2}(p)|_{\lambda=0} = p^2 Z_\phi - g^2 \mu^\epsilon \int [d^D \mathbf{q}] \text{Tr} \frac{\not{q}(\not{q}-\not{p})}{q^2(p+q)^2}, \quad (2.38)$$

$$\Gamma_R^{0,4}(p_1, p_2, p_3, p_4)|_{\lambda=0} = g^4 \mu^{2\epsilon} \int [d^D \mathbf{q}] \frac{\text{Tr} [\not{q}(\not{q}+\not{p}_1)(\not{q}+\not{p}_1+\not{p}_2)(\not{q}-\not{p}_4)]}{q^2(q+p_1)^2(q+p_1+p_2)^2(q-p_4)^2}, \quad (2.39)$$

with $k \equiv (\omega, \mathbf{k})$ the frequency and the momentum, whether internal or external, and we integrate in $D = 4 - \epsilon$ space-time dimensions. Again, we solve the integrals using dimensional regularization with $D = 4 - \epsilon$ dimensions in the critical hyperplane $m = 0$, and setting the external momentum and frequency to zero. Thus, from the above equations we obtain the renormalization constant for the g and λ coupling

$$Z_\Psi = 1 - \frac{g^2}{(4\pi)^2} \frac{1}{\epsilon}, \quad (2.40)$$

$$Z_\Phi = 1 - \frac{g^2}{4\pi^2} N_F \frac{1}{\epsilon}, \quad (2.41)$$

$$Z_g = 1 + \frac{g^2}{8\pi^2} \frac{1}{\epsilon}. \quad (2.42)$$

For the Z_λ renormalization constant, we use the following renormalization condition

$$\lambda Z_\lambda = \lambda + \left((N_B + 4) \frac{\lambda^2}{8\pi^2} \frac{1}{\epsilon} - N_f \frac{g^4}{2\pi^2} \frac{1}{\epsilon} \right). \quad (2.43)$$

Having found all the renormalization constants, a short calculation yields

$$\beta_{g^2} = \epsilon g^2 - (2N_F + 3)g^4, \quad (2.44)$$

$$\beta_\lambda = \epsilon \lambda - (4 + N_B)\lambda^2 + 4N_F g^2 (g^2 - \lambda), \quad (2.45)$$

where we rescaled $g^2/8\pi^2 \rightarrow g^2$ and $\lambda/8\pi^2 \rightarrow \lambda$. The RG flow is presented in Fig. 2.3.2. For the Gaussian fixed point both directions are relevant, while for the Wilson-Fisher one can see that it becomes unstable in the direction of the Yukawa

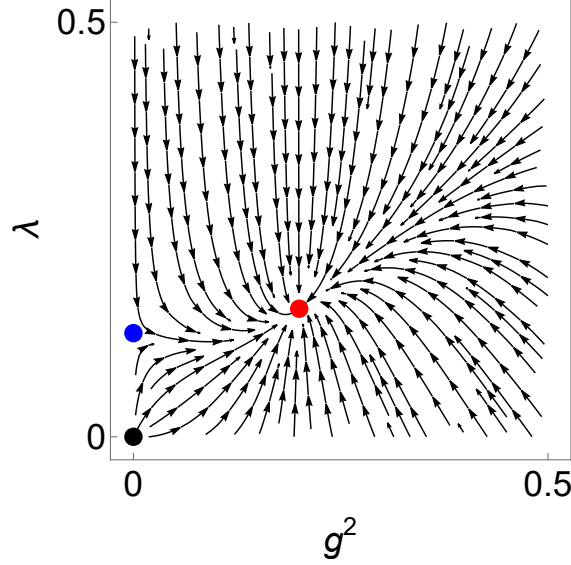


Figure 2.3.2: Flow diagram of the Yukawa and ϕ^4 coupling in the critical plane $m^2 = 0$. The blue dot is the WF fixed point, and the black one is the Gaussian trivial point. The only stable fixed point is the GNY fixed point (red dot), located at a finite positive value of both Yukawa and ϕ^4 coupling and controls the critical behavior at the transition between the semimetal and the interaction driven insulator. The plot is for $N_B = 4$ and $N_F = 1$, this is done just to make clearer the difference between the WF point and the GNY point in the λ direction.

coupling. The GNY fixed point is

$$(g_\star^2, \lambda_\star) = \left(\frac{\epsilon}{3 + 2N_F}, \frac{-\sqrt{\left(\epsilon - \frac{4N_F\epsilon}{2N_F+3}\right)^2 - \frac{16(-N_B-4)N_F\epsilon^2}{(2N_F+3)^2}} + \frac{4N_F\epsilon}{2N_F+3} - \epsilon}{2(-N_B - 4)} \right). \quad (2.46)$$

The above β functions yield other fixed points: The Wilson-Fisher point is located at $(g_\star^2, \lambda_\star) = (0, \epsilon/(4 + N_B))$; Obviously, the trivial Gaussian point $(g_\star^2, \lambda_\star) = (0, 0)$; The non-physical GNY fixed point, which is unphysical because $\lambda_\star < 0$, located at

$$(g_\star^2, \lambda_\star) = \left(\frac{\epsilon}{3 + 2N_F}, \frac{\sqrt{\left(\epsilon - \frac{4N_F\epsilon}{2N_F+3}\right)^2 - \frac{16(-N_B-4)N_F\epsilon^2}{(2N_F+3)^2}} + \frac{4N_F\epsilon}{2N_F+3} - \epsilon}{2(-N_B - 4)} \right). \quad (2.47)$$

Anomalous dimensions for the fermionic and the bosonic fields are obtained from the field-strength renormalization constants after restoring the external momentum k and setting it to be equal to the renormalization scale μ . This yields an extra factor of $\mu^{-\epsilon}$ multiplying g^2 in the field strength renormalization for both fermion and boson

fields, which vanishes in their final expressions. Nevertheless, this piece is needed for obtaining the anomalous dimension of the fields. The anomalous dimensions η_ϕ and η_ψ for the bosonic ϕ and fermionic field ψ are obtained by evaluating the logarithmic derivatives of the renormalization constants $\eta_i = -\partial Z_i / \partial \ln \mu$, with $i = \phi, \psi$, at a given fixed point

$$\eta_\phi = 2N_F g_\star^2, \tag{2.48}$$

$$\eta_\psi = \frac{1}{2} g_\star^2. \tag{2.49}$$

The anomalous dimension simply measures how much the scaling dimension deviates from its free-field value.

Chapter 3

Superconductivity in tilted Dirac materials

Now that we have seen the context and the mathematical tools, we are in a position to address the quantum-critical behavior of the tilted Dirac superconductor. First, we will build a low-energy effective theory that takes into account all the symmetries of graphene as a starting point. Then, as in the previous chapter, we make use of the renormalization group to access the physics beyond the mean field theory. Throughout this chapter we explore the behavior of the Lorentz symmetry as we renormalize the corresponding quantum-critical theory, and, in particular we study the effects of the Dirac cone's tilt. This chapter is based on Ref. [59].

3.1 The low-energy theory

We consider a low-energy effective theory for the Dirac fermion quasiparticles around the two inequivalent valleys, as for instance, two Dirac points at the two inequivalent corners of the Brillouin zone in graphene. To construct the Nambu-doubled basis for the Dirac superconductor, we first write an eight-component Dirac spinor given by [103]

$$\Psi_{\omega,\mathbf{k}} = \begin{pmatrix} c_{\uparrow}^{(+)} & c_{\uparrow}^{(-)} & c_{\downarrow}^{(+)} & c_{\downarrow}^{(-)} \end{pmatrix}^T, \quad c_s^v = \begin{pmatrix} a_s^v & b_s^v \end{pmatrix}, \quad (3.1)$$

where a_s^v and b_s^v are the fermionic fields corresponding to two different (A and B) sublattices with valley index $v = (\pm)$, and spin projection $s = \uparrow, \downarrow$. Then, we

introduce a particle-hole degree of freedom and write a doubled, sixteen-component Dirac-Nambu spinor as

$$\Psi_{\omega, \mathbf{k}} \rightarrow \Psi_{\text{Nam}} = \begin{pmatrix} \Psi_{\omega, \mathbf{k}} \\ \Gamma_{210} \Psi_{-\omega, -\mathbf{k}}^* \end{pmatrix}. \quad (3.2)$$

We conveniently write matrices acting on the sixteen-dimensional Dirac-Nambu spinors as $\Gamma_{\mu\nu\lambda\rho} = \eta_\mu \otimes \sigma_\nu \otimes \tau_\lambda \otimes \alpha_\rho$. Here, $\{\alpha\}, \{\tau\}, \{\sigma\}$ and $\{\eta\}$ are sets of Pauli matrices operating on the sublattice, valley, spin and particle-hole degrees of freedom, respectively, with $\mu, \nu, \lambda, \rho = 0, 1, 2, 3$, and the index 0 corresponding to the 2×2 unity matrix. Notice that the lower component of the Dirac-Nambu spinor in Eq. (3.2) is a time-reversal partner of the upper one, since the matrix $\Gamma_{210} = \sigma_2 \otimes \tau_1 \otimes \alpha_0$ is the unitary part of the time reversal operator in the undoubled basis. Furthermore, the form of the Γ -matrices dictates that any two of them either commute or anticommute. To simplify notation, we suppress subscript ‘‘Nam’’ hereafter, $\Psi \equiv \Psi_{\text{Nam}}$ and $\Psi^\dagger \equiv \Psi_{\text{Nam}}^\dagger$.

Given this Dirac-Nambu representation, we can write the noninteracting low energy effective Hamiltonian for the tilted Dirac fermions in the form $\hat{H} = \sum_{\mathbf{k}} \Psi^\dagger (\hat{\mathcal{H}}(\mathbf{k}) - \mu_{\text{tot}} \Gamma_{3000}) \Psi$, with μ_{tot} as the chemical potential, and

$$\hat{\mathcal{H}}(\mathbf{k}) = v \Gamma_{3031} k_x + v \Gamma_{3002} k_y + \alpha v \Gamma_{3030} k_x, \quad (3.3)$$

where v is the Fermi velocity and α represents the (dimensionless) tilt parameter. We here consider only case with $|\alpha| < 1$ (type-1) featuring the point-like Fermi surfaces with the TDF quasiparticle excitations, while the overtilted (type-2) system with $|\alpha| > 1$ hosts a particle- and a hole-like Fermi surfaces. Notice that the tilt term can be thought of as a momentum-dependent chiral chemical potential, and it commutes with the (total) chemical potential represented by the matrix Γ_{3000} . We hereafter set $v = 1$ and consider the system at half filling, $\mu_{\text{tot}} = 0$, unless otherwise stated.

The form of the single-particle Hamiltonian implies its invariance under time reversal, generated by $\mathcal{T} = \Gamma_{0210} K$, with K as the complex conjugation, $\text{SU}(2)$ spin rotations generated by Γ_{0s00} , with $s = \{1, 2, 3\}$, as well as the valley (sublattice) exchange, represented by $V = \Gamma_{0010}$, $k_x \rightarrow -k_x$ ($S = \Gamma_{0001}$, $k_y \rightarrow -k_y$) [104]. Finally, the unitary particle-hole symmetry is represented by $\mathcal{C} = \Gamma_{1000}$, so that $\{\mathcal{C}, \hat{\mathcal{H}}(\mathbf{k})\} = 0$, implying that its anti-unitary counterpart is represented by the operator $\mathcal{C}\mathcal{T}$. Notice that $\tilde{\mathcal{C}} = \Gamma_{2000}$ can also be taken as a unitary particle-hole operator, which is a

consequence of the U(1) symmetry of the noninteracting Hamiltonian generated by the matrix Γ_{3000} representing the electric charge operator in the Dirac-Nambu representation employed here. Finally, we emphasize that the choice of the tilt term ($\sim \alpha$) in Eq. (3.3) ensures its invariance under all the mentioned symmetries.

We point out that the mirror-symmetry breaking tilted Dirac Hamiltonian, with the sign of the tilt equal for all the valleys (or other fermion flavors) is obtained by breaking the valley exchange or mirror symmetry discussed above, and we refer to it as *asymmetric* tilt. The corresponding Hamiltonian then takes the form

$$\hat{\mathcal{H}}_{\text{AS}}(\mathbf{k}) = v\Gamma_{3031}k_x + v\Gamma_{3002}k_y + \alpha v\Gamma_{3000}k_x, \quad (3.4)$$

with the tilt term proportional to the matrix representing the chemical potential. Therefore, the asymmetric tilt term effectively represents a momentum-dependent chemical potential. As we show in the following, our conclusions regarding the restoration of the Lorentz symmetry in the case of the interacting fully symmetric tilted Dirac Hamiltonian in Eq. (3.3) equally apply to the asymmetrically tilted Hamiltonian in Eq. (3.4).

3.2 Mean-field analysis

As in section 1.2.3, to motivate the analysis of the strong-coupling superconducting instability of the tilted Dirac semimetal, we first discuss mean-field (MF) theory for a transition from the semimetallic to a fully gapped ordered state, which is independent of the nature of the state, i.e. whether the state is an insulator or a superconductor. For concreteness, we here subscribe to the *s*-wave superconducting order parameter, which reads

$$\phi(\mathbf{r}, \tau) = \langle \Psi^\dagger(\mathbf{r}, \tau)(\Gamma_{1000} \cos \varphi + i\Gamma_{2000} \sin \varphi)\Psi(\mathbf{r}, \tau) \rangle, \quad (3.5)$$

where φ is the U(1) phase of the superconducting order parameter (OP). This OP anticommutes with the particle number operator $N = \Gamma_{3000}$, and commutes with all the three generators of spin rotations, Γ_{0s00} , as dictated by its *s*-wave nature. Notice that both matrices in the *s*-wave OP *anticommute* with the tilted Dirac Hamiltonian in Eq. (3.3) which ensures that the underlying ordered state is fully gapped. In the MF analysis, we set the OP to be a constant, $\phi(\mathbf{r}, \tau) \equiv m$, and fix

the SC phase $\varphi = 0$. We assume that such a condensation is propelled by an effective local density-density interaction of the form

$$\mathcal{H}_{\text{int}} = \tilde{g} \int d^2\mathbf{r} [\Psi^\dagger(\mathbf{r}, \tau) \Gamma_{1000} \Psi(\mathbf{r}, \tau)]^2. \quad (3.6)$$

Then, we formulate the MF theory for the transition into such a state by taking the Hamiltonian

$$H_{\text{MF}} = \int d^2\mathbf{r} \Psi^\dagger(\mathbf{r}) \left\{ \hat{\mathcal{H}}(\mathbf{k} \rightarrow -i\nabla_{\mathbf{r}}) + \mathcal{H}_{\text{int}} \right\} \Psi(\mathbf{r}), \quad (3.7)$$

with the first term representing the free tilted Dirac Hamiltonian in Eq. (3.3), together with the short-range interaction part, given by Eq. (3.6). We then perform a Hubbard-Stratonovich transformation in the s -wave channel to obtain the corresponding energy of the ground state at half filling ($\mu = 0$) in the form,

$$\begin{aligned} \mathcal{E}[m] &= -\frac{1}{2\tilde{g}}m^2 + 2 \sum_{w=\pm} \int \frac{d^2\mathbf{k}}{(2\pi)^2} \left\{ w\alpha k_x + \sqrt{k^2 + m^2} \right\} \\ &= -\frac{1}{2\tilde{g}}m^2 + 4 \int \frac{d^2\mathbf{k}}{(2\pi)^2} \sqrt{k^2 + m^2}. \end{aligned} \quad (3.8)$$

Therefore, the MF ground state energy is tilt-independent, yielding the same form of the gap equation as in the untilted case, after minimization with respect to m , and implies that

$$m \sim (\tilde{g} - \tilde{g}_c), \quad (3.9)$$

with the tilt-independent critical coupling $\tilde{g}_c \sim \Lambda$ scaling with the ultraviolet cutoff, defining the energy scale up to which the Dirac dispersion holds. Therefore, the transition into the s -wave (and, more generally, any gapped) phase takes place at a strong coupling, which is a consequence of the vanishing density of states (DOS) for the tilted Dirac fermions

$$\rho(E, \alpha) = -\frac{1}{\pi} \lim_{\delta \rightarrow 0} \text{Im} \{ \text{Tr} G_F(i\omega \rightarrow \omega + i\delta, \mathbf{k}) \} = \frac{4}{\pi} \frac{|E|}{(1 - \alpha^2)^{3/2}}, \quad (3.10)$$

for a single 16-component Dirac flavor. It is therefore expected that at any finite-tilt fixed point of the RG, the critical value of the interaction decreases with respect to the untilted ($\alpha = 0$) value. Latter we will explicitly show this phenomena by means of the β functions for the Yukawa coupling although for a point that turns out to be

unstable.

3.3 Gross-Neveu-Yukawa quantum-critical field theory

We now address the behavior of the $(2 + 1)$ –dimensional tilted s –wave Dirac superconductor at the quantum-critical point within the framework of the Gross-Neveu-Yukawa (GNY) field theory. Unlike in the section 2.3, this time we introduce a new base to represent the symmetries of the new system we are dealing with.

3.3.1 S –wave superconducting instability

First, since both noninteracting Hamiltonian and the OP are spin singlets, we discard the spin subspace hereafter. We therefore consider eight-component massless Dirac fermions endowed with N_F copies, which interact with a complex bosonic order parameter field via Yukawa interaction (g). The s –wave superconducting OP is in the form given by Eq. (3.9), which after discarding the spin subspace, reads

$$\phi(\mathbf{r}, \tau) = \langle \Psi^\dagger(\mathbf{r}, \tau)(\Gamma_{100} \cos \varphi + i\Gamma_{200} \sin \varphi)\Psi(\mathbf{r}, \tau) \rangle, \quad (3.11)$$

Hereafter, $\Psi \equiv \Psi(\mathbf{r}, \tau)$, $\Psi^\dagger \equiv \Psi^\dagger(\mathbf{r}, \tau)$ and $\phi \equiv \phi(\mathbf{r}, \tau)$. The dynamics of the system is then described by an imaginary time (Euclidean) action

$$S = S_F + S_Y + S_B, \quad (3.12)$$

with the noninteracting fermionic part of the form

$$S_F = \int d\tau d^D \mathbf{r} \Psi^\dagger \left[\partial_\tau + \hat{\mathcal{H}}(\mathbf{k} \rightarrow -i\nabla_{\mathbf{r}}) \right] \Psi, \quad (3.13)$$

where τ is the imaginary time and $\hat{\mathcal{H}}(\mathbf{k})$ represents the Hamiltonian for noninteracting massless TDFs in Eq. (3.3). This time, we rewrite the coupling of the massless Dirac fermions to the OP fluctuations making use of the imaginary and real parts of the OP. The Yukawa interaction thus take the form

$$S_Y = g \int d\tau d^D \mathbf{r} [(\text{Re}\phi)\Psi^\dagger \Gamma_{100} \Psi + (\text{Im}\phi)\Psi^\dagger i\Gamma_{200} \Psi]. \quad (3.14)$$

Lastly, the dynamics of the OP is described again by the standard Ginzburg-Landau action

$$S_B = \int d\tau d^D \mathbf{r} \left[\phi(-\partial_\tau^2 - \nabla^2 + m_B^2)\phi + \frac{\lambda}{2}\phi^4 \right], \quad (3.15)$$

where m_B^2 is the tuning parameter for the continuous phase transition with $m_B^2 > 0$ ($m_B^2 < 0$) in the $U(1)$ symmetric (symmetry broken) phase. We furthermore set the bosonic and fermionic velocities to be equal to unity in the critical region [7], see also the consequences of Eq. (3.38).

The fermionic and the bosonic propagators read, respectively,

$$G_F(i\omega, \mathbf{k}) = \left\{ i\omega A_- + B(k_x \Gamma_{001} + k_y \Gamma_{032}) + A_-(\alpha k_x \Gamma_{330} + \Gamma_{331} k_x + \Gamma_{302} k_y) \right. \\ \left. + 2\alpha k_x [\Gamma_{330} k^2 + \alpha k_x (k_x \Gamma_{331} + k_y \Gamma_{302})] \right\} \frac{1}{A_+^2 - B^2}, \quad (3.16)$$

and

$$G_B(i\omega, \mathbf{k}) = \frac{1}{\omega^2 + k^2 + m_B^2}. \quad (3.17)$$

where $A_\pm \equiv A_\pm(i\omega, \mathbf{k}) = -\omega^2 \pm \alpha^2 k_x^2 - k^2$ and $B \equiv B(i\omega, \mathbf{k}) = 2\alpha i\omega k_x$. For a detail derivation of the Fermionic propagator see App. B.

We point out that the form of the propagator corresponding to the asymmetrically tilted system, described by the Hamiltonian in Eq (3.4), preserves the form in Eq. (3.16), since the asymmetric tilt commutes with the untilted piece of the Dirac Hamiltonian. Explicitly, the propagator of the asymmetric TDFs reads

$$G_{F,AS}(i\omega, \mathbf{k}) = \left\{ i\omega A_- + B(k_x \Gamma_{031} + k_y \Gamma_{002}) + A_-(\alpha k_x \Gamma_{300} + \Gamma_{331} k_x + \Gamma_{302} k_y) \right. \\ \left. + 2\alpha k_x [\Gamma_{300} k^2 + \alpha k_x (k_x \Gamma_{331} + k_y \Gamma_{302})] \right\} \frac{1}{A_+^2 - B^2}, \quad (3.18)$$

and such a structure of the propagator, as we show in the following, implies that all the conclusions derived for the symmetric tilt equally apply to the asymmetric case. We therefore consider only the fully symmetric TDFs.

3.4 Renormalization group analysis

Next, we proceed with the analysis of the $U(1)$ GNY field theory. The engineering

(bare) dimensions of the couplings are $\dim[g^2] = \dim[\lambda] = 3 - D$ and for the tilt parameter $\dim[\alpha] = 0$, implying that $D = 3$ is the upper critical dimension of the theory. We therefore employ an ϵ -expansion about the upper critical dimension, $\epsilon = 3 - D$, to access the quantum-critical behavior in $D = 2$. To this end, we define the action $S_R = \int d\tau d\mathbf{r} L_R$, where the renormalized Lagrangian is

$$L_R = \Psi^\dagger [Z_\Psi (\partial_\tau - iZ_v v \Gamma_{331} \partial_x - iZ_v v \Gamma_{302} \partial_y - iZ_v Z_\alpha \alpha v \Gamma_{330} \partial_x)] \Psi + Z_g L_Y + Z_\phi \phi (-\partial_\tau^2 - \nabla^2) \phi + Z_{m_B} m_B^2 \phi^2 + Z_\lambda \frac{\lambda}{2} \phi^4, \quad (3.19)$$

where Z_j 's are renormalization constants, and we restore the Fermi velocity for completeness.

To obtain RG flow of the couplings, we integrate out the fermionic and bosonic modes with frequency $-\infty < \omega < \infty$ and use dimensional regularization in $D = 3 - \epsilon$ spatial dimensions within the minimal subtraction scheme to handle the ensuing divergences. The one-loop diagrams for that contribute to the renormalization are given in Fig. 2.2.1 and Fig. 2.3.1. The computation of the fermionic self energy is given by Fig. 2.3.1a and the bosonic self energy contribution comes from the diagram 2.3.1b.

The short-range interaction is encoded through the Yukawa coupling between the Dirac fermion quasiparticles and the bosonic degrees of freedom representing the fluctuations of the s -wave superconducting phase

$$S_Y = g \int d\tau d^D \mathbf{r} [(\text{Re}\phi) \Psi^\dagger \Gamma_{100} \Psi + (\text{Im}\phi) \Psi^\dagger i \Gamma_{200} \Psi] \equiv \int d\tau d^D \mathbf{r} L_Y. \quad (3.20)$$

It is convenient to rewrite the coupling of the bosonic order parameter to the fermionic field explicitly in terms of the real and imaginary parts of the OP,

$$L_Y = g [(\text{Re}\phi) \Psi^\dagger \Gamma_{100} \Psi + (\text{Im}\phi) \Psi^\dagger i \Gamma_{200} \Psi] = g [\phi \Psi^\dagger P_+ \Psi + \phi^* \Psi^\dagger P_- \Psi], \quad (3.21)$$

where the matrices $P_\pm$ are defined as

$$P_\pm = \frac{1}{2} (\Gamma_{100} \mp i \Gamma_{200}), \quad (3.22)$$

and we employ eight-dimensional representation of the Γ -matrices.

3.4.1 Fermionic Self Energy

We compute to the leading-order the zero-temperature ($T = 0$) self-energy for tilted Dirac fermions (Fig. 2.3.1a) in the critical hyperplane (all masses set to zero)

$$\Sigma_F(i\Omega, \mathbf{k}) = g^2 \int [d\omega] \int [d^D \mathbf{q}] \{P_+ G_F(i\omega, \mathbf{q}) P_- + P_- G_F(i\omega, \mathbf{q}) P_+\} G_b(i\Omega - i\omega, \mathbf{q} - \mathbf{k}) \quad (3.23)$$

where $[d\omega] = d\omega/2\pi$ and $[d^D \mathbf{q}] = d^D \mathbf{q}/(2\pi)^D$. The bosonic and fermionic propagators are given in Eqs. (3.16) and (3.17), respectively. The explicit form of the fermionic self-energy is

$$\begin{aligned} \Sigma_F(i\Omega, \mathbf{k}) &= g^2 \int [d\omega] \int [d^D \mathbf{q}] \{P_+ G_F(i\omega, \mathbf{q}) P_- + P_- G_F(i\omega, \mathbf{q}) P_+\} G_B(i\Omega - i\omega, \mathbf{q}) \\ &= g^2 \int [d\omega] \int [d^D \mathbf{q}] \{P_+, P_-\} [i\omega A_- + B(q_x \Gamma_{001} + q_y \Gamma_{032}) - \alpha q_x (A_- + 2q^2) \Gamma_{330} \\ &\quad - (A_- + 2\alpha^2 q_x^2)(q_x \Gamma_{331} + q_y \Gamma_{302})] \frac{1}{(\omega^2 - \alpha^2 q_x^2 + q^2)^2 + (2\alpha\omega q_x)^2} \\ &\quad \times \frac{1}{(\omega - \Omega)^2 - (\mathbf{q} - \mathbf{k})^2}, \end{aligned} \quad (3.24)$$

with the anticommutator $\{P_+, P_-\} = 1$, as can be directly obtained from Eq. (3.22); $A_\pm \equiv A_\pm(i\omega, \mathbf{k}) = -\omega^2 \pm \alpha^2 k_x^2 - k^2$ and $B \equiv B(i\omega, \mathbf{k}) = 2\alpha i\omega k_x$. We here also set Fermi velocity $v = 1$, since it does not get renormalized due to the relativistic invariance of the Yukawa interaction Eq. (3.14). The algebraic structure of the integrand allows us to systematically calculate the corresponding renormalization constants (Z 's), which we explicitly show in the following. Notice that the negative signs multiplying the last two terms in the above self-energy result from the anticommutation of the OP matrices ($P_\pm$) with the tilt matrix Γ_{330} .

Fermionic field renormalization: For the renormalization constant Z_Ψ , only the terms proportional to the unity matrix contribute. Thus after setting the external momentum to zero, $\mathbf{k} = 0$, and performing the integral over ω , we obtain

$$\Sigma_{F,000}(i\Omega, 0) = -g^2 \int [d^D \mathbf{q}] \frac{i\Omega (4q^2 + \alpha^2 q_x^2 + \Omega^2)}{2q (16q^4 + 8q^2 (\Omega^2 - \alpha^2 q_x^2) + (\alpha^2 q_x^2 + \Omega^2)^2)}. \quad (3.25)$$

Now, to find the fermion field renormalization Z_Ψ , we take

$$\lim_{i\Omega \rightarrow 0} \frac{\partial \Sigma_{F,000}(i\Omega, 0)}{\partial(i\Omega)} = -\frac{g^2}{(4\pi)^2} G(\alpha) \int_\lambda^\Lambda dq \frac{1}{q}, \quad (3.26)$$

with $\lambda = \Lambda/b$, where the Wilsonian RG parameter $b > 1$, and $b - 1 \ll 1$. Then, we use the correspondence between the hard cutoff and dimensional regularization

$$\int_{\lambda}^{\Lambda} \frac{dq}{q} \rightarrow \frac{1}{\epsilon}, \quad (3.27)$$

and the function $G(\alpha)$ reads

$$G(\alpha) = \int_0^{2\pi} \frac{d\phi}{\pi} \int_0^{\pi} d\theta \sin(\theta) \frac{(4 + \alpha^2 \sin^2(\theta) \cos^2(\phi))}{(4 - \alpha^2 \sin^2(\theta) \cos^2(\phi))^2} = \frac{4}{4 - \alpha^2}, \quad (3.28)$$

Notice that the function $G(\alpha)$ is strictly positive for $|\alpha| < 1$. We then obtain the renormalization constant for the fermion field through the following renormalization condition

$$Z_{\Psi} = 1 + \lim_{i\Omega \rightarrow 0} \frac{\partial \Sigma_{f,000}(i\Omega, 0)}{\partial i\Omega} = 1 - \frac{g^2}{(4\pi)^2} G(\alpha) \frac{1}{\epsilon}. \quad (3.29)$$

Renormalization factor for the tilt parameter: The renormalization constant for the tilt parameter, Z_{α} , is obtained from the term in the fermionic self-energy proportional to the matrix Γ_{330} , as dictated by the form of the tilt term in the bare theory, see Eq. (3.3). This, together with the form of the self-energy, in turn, dictates the following renormalization condition for the tilt,

$$-Z_{\Psi} Z_{\alpha} \alpha k_x - k_x \frac{\partial \Sigma_{F,330}(i\Omega, \mathbf{k})}{\partial k_x} \Big|_{i\Omega \rightarrow 0, \mathbf{k} \rightarrow 0} = -\alpha k_x, \quad (3.30)$$

where $\Sigma_{F,330}(i\Omega, \mathbf{k})$ is the term proportional to the matrix Γ_{330} in the fermionic self-energy in Eq. (3.24),

$$\Sigma_{F,330}(i\Omega, \mathbf{k}) = g^2 \int [d\omega] \int [d^D \mathbf{q}] \frac{\alpha q_x (\omega^2 + \alpha^2 q_x^2 - q^2)}{[(\omega^2 - \alpha^2 q_x^2 + q^2)^2 + (2\alpha\omega q_x)^2][(\omega - \Omega)^2 - (\mathbf{q} - \mathbf{k})^2]}. \quad (3.31)$$

We notice that we here explicitly used marginality of the Fermi velocity v in Eq. (3.3), due to the relativistic invariance of the Yukawa term, see Eq. (3.14).

We then take a finite external momentum $\mathbf{k} = k_x \mathbf{e}_x$ in Eq. (3.31), and keep only the terms proportional to Γ_{330} , which, by virtue of the renormalization condition in Eq. (3.30), yields,

$$Z_{\alpha} Z_{\Psi} = 1 - \frac{g^2}{(4\pi)^2 \epsilon} I(\alpha), \quad (3.32)$$

where

$$I(\alpha) = \int_0^{2\pi} \frac{d\phi}{\pi} \int_0^\pi d\theta \frac{\cos^2 \phi \sin^3 \theta [-8 + \alpha^2 \cos^2 \phi \sin^2 \theta]}{(4 - \alpha^2 \cos^2 \phi \sin^2 \theta)^2}. \quad (3.33)$$

Finally, using the form of the fermion-field renormalization Z_Ψ in Eq. (3.29), we find

$$Z_\alpha = 1 + \frac{g^2}{(4\pi)^2 \epsilon} [G(\alpha) - I(\alpha)], \quad (3.34)$$

with the function $G(\alpha)$ given by Eq. (3.28).

Renormalization factor for the Fermi velocity: We here show that the renormalization factor of the Fermi velocity, Z_v , is equal to unity, as a consequence of the Lorentz invariance of the Yukawa vertex in Eq. (3.14). Therefore, the Fermi velocity remains marginal, as is the case in the tilt-free case. To show this feature, we consider only one component of the velocity, taken to be along the x -axis for concreteness. The corresponding renormalization condition reads

$$Z_\Psi Z_v + \frac{\partial \Sigma_{F,331}(i\Omega, \mathbf{k})}{\partial k_x} \Big|_{i\Omega \rightarrow 0, \mathbf{k} \rightarrow 0} = 1, \quad (3.35)$$

where the term $\Sigma_{f,331}$ is the part of the self-energy in Eq. (3.24) multiplied by the matrix Γ_{331} , which explicitly reads

$$\Sigma_{F,331}(i\Omega, \mathbf{k}) = g^2 \int [d\omega] \int [d^D \mathbf{k}] \frac{q_x (\omega^2 + q^2 - \alpha^2 q_x^2)}{[(\omega^2 + q^2 - \alpha^2 q_x^2)^2 + (2\alpha\omega q_x)^2][(\omega - \Omega)^2 + (\mathbf{q} - \mathbf{k})^2]}. \quad (3.36)$$

An explicit calculation then yields

$$\frac{\partial \Sigma_{F,331}(i\Omega, \mathbf{k})}{\partial k_x} \Big|_{i\Omega \rightarrow 0, \mathbf{k} \rightarrow 0} = \frac{g^2}{2(2\pi)^3 \epsilon} \int_0^{2\pi} d\phi \int_0^\pi d\theta \frac{\cos^2 \phi \sin^3 \theta (12 - \alpha^2 \cos^2 \phi \sin^2 \theta)}{(4 - \alpha^2 \cos^2 \phi \sin^2 \theta)^2}, \quad (3.37)$$

which, after performing the angular integrals, yields

$$\frac{\partial \Sigma_{F,331}(i\Omega, \mathbf{k})}{\partial k_x} \Big|_{i\Omega \rightarrow 0, \mathbf{k} \rightarrow 0} = \frac{g^2}{(4\pi)^2} G(\alpha) \frac{1}{\epsilon}, \quad (3.38)$$

with the function $G(\alpha)$ given by Eq. (3.28). Using Eq. (3.29) and the renormalization condition (3.38), we obtain that $Z_v = 1$, implying that the Fermi velocity remains marginal (unrenormalized) for the TDFs to the leading order in the ϵ -expansion, once the velocity isotropy is reached. Notice that we here considered only the x -component of the velocity, but the same conclusion holds for other components

after invoking the irrelevance of the tilt parameter.

3.4.2 Bosonic Self Energy

In order to obtain the RG equations for the Yukawa coupling, we compute the self energy for the bosonic field and the correction to the Yukawa vertex. First, we compute the bosonic self energy (Fig. 2.3.1b), which reads

$$\Sigma_B(i\Omega, \mathbf{k}) = -g^2 \int [d\omega] \int [d^D \mathbf{q}] \text{Tr} [P_+ G_F(i\omega + i\Omega, \mathbf{q} + \mathbf{k}) P_- G_F(i\omega, \mathbf{q})], \quad (3.39)$$

with the negative sign arising from the trace over the fermionic fields, and we set external momentum to zero ($\mathbf{k} = 0$). After taking the trace, the integral is of the form

$$\begin{aligned} \Sigma_B(i\Omega, 0) = & -4g^2 N_F \int [d\omega] \int [d^D \mathbf{q}] \left(4k_x^4 k^2 \alpha^4 - B(\omega, \mathbf{q}) B(\omega + \Omega, \mathbf{q}) q^2 \right. \\ & + 4k_x^2 k^2 \alpha^2 (A_-(\omega + \Omega, \mathbf{q}) + k^2) + 4k_x^2 k^2 \alpha^2 A_-(\omega, \mathbf{q}) \\ & \left. + A_-(\omega, \mathbf{q}) A_-(\omega + \Omega, \mathbf{q}) (k^2 + \alpha^2 k_x^2 + \omega(\omega + \Omega)) \right) \\ & \times \left(\frac{1}{[A_+(\omega, \mathbf{q})]^2 - B^2(\omega, \mathbf{q})} \right) \left(\frac{1}{[A_+(\omega + \Omega, \mathbf{q})]^2 - B^2(\omega + \Omega, \mathbf{q})} \right), \end{aligned} \quad (3.40)$$

with N_F as the number of eight-component Dirac fermion flavors. Taking the derivative, after performing the integral over the frequency and the momentum, we obtain

$$\lim_{\Omega \rightarrow 0} \frac{\partial \Sigma_B(i\Omega, 0)}{\partial \Omega^2} = -\frac{g^2}{4\pi^2} N_F F(\alpha) \frac{1}{\epsilon}, \quad (3.41)$$

where function $F(\alpha)$ is defined as

$$F(\alpha) = \frac{1}{(1 - \alpha^2)^2}. \quad (3.42)$$

With the above definition of the bosonic self-energy, the bosonic wave-function renormalization is determined by the renormalization condition

$$Z_\Phi - \frac{\partial \Sigma_B(i\Omega, 0)}{\partial \Omega^2} \Big|_{\Omega \rightarrow 0} = 1, \quad (3.43)$$

implying that the renormalization constant for the bosonic fields is

$$Z_\Phi = 1 + \left. \frac{\partial \Sigma_B(i\Omega, 0)}{\partial \Omega^2} \right|_{\Omega \rightarrow 0} = 1 - \frac{g^2}{4\pi^2} N_F F(\alpha) \frac{1}{\epsilon}. \quad (3.44)$$

3.4.3 Correction to Yukawa Vertex

The correction to the Yukawa vertex (Fig. 2.3.1c) reads

$$\begin{aligned} \delta g = & g^2 \int [d\omega] \int [d^D \mathbf{q}] \{ (P_+ G_F(i\Omega_1 + i\Omega_2 - i\omega, \mathbf{k} + \mathbf{p} - \mathbf{q}) P_+ G_F(i\Omega_1 - i\omega, \mathbf{k} - \mathbf{q}) P_-) \\ & + (P_- G_F(i\omega + i\Omega_1 + i\omega_2, \mathbf{q} + \mathbf{k} + \mathbf{q}) P_+ G_F(i\omega + i\Omega_1, \mathbf{q} + \mathbf{k}) P_+) \\ & + (P_+ G_F(i\Omega_1 - \Omega_2 - i\omega, \mathbf{k} - \mathbf{p} - \mathbf{q}) P_- G_F(i\Omega_1 - i\omega, \mathbf{k} - \mathbf{q}) P_-) \\ & + (P_- G_F(i\Omega_1 - i\Omega_2 + i\omega, \mathbf{k} - \mathbf{p} - \mathbf{q}) P_- G_F(i\omega + i\Omega_1, \mathbf{q} + \mathbf{k}) P_+) \} \frac{1}{\omega^2 + q^2}. \end{aligned} \quad (3.45)$$

Using the relation $P_\pm P_\pm = 0$, the form of the fermionic propagator in Eq. (3.16), and that all the matrices either commute or anticommute with the matrices $P_\pm$, it can be readily shown that the correction of the Yukawa vertex identically vanishes, $\delta g = 0$, after taking all the external momenta and frequencies to be equal to zero. Therefore, the renormalization for the Yukawa coupling has the form

$$Z_\phi^{1/2} Z_\Psi g_0 \mu^{-\epsilon/2} = g, \quad (3.46)$$

which, together with the form of the renormalization constants Z_Ψ and Z_ϕ , in Eqs. (3.29) and (3.44), respectively, allow us to obtain the β function for the Yukawa coupling.

3.4.4 Correction to the ϕ^4 Vertex

The first contribution to the λ coupling comes from the already know result form the ϕ^4 vertex (Fig. 2.2.1b), which reads

$$\delta \lambda_\lambda = (N_B + 4) \frac{\lambda^2}{8\pi^2} \frac{1}{\epsilon}, \quad (3.47)$$

where, N_B is the number of complex components of the bosonic order parameter field.

The correction to the λ coupling due to the Yukawa interaction (Fig. 2.3.1d),

with all external momenta and frequencies set to zero, reads

$$\delta\lambda_Y = -g^4 \int [d\omega] \int [d^D \mathbf{q}] \text{Tr} [P_+ G_F(i\omega, \mathbf{q}) P_- G_F(i\omega, \mathbf{q}) P_+ G_F(i\omega, \mathbf{q}) P_- G_F(i\omega, \mathbf{q})]. \quad (3.48)$$

After taking the trace and integrate over the frequency and momentum, we obtain the divergent part of this correction

$$\delta\lambda_Y = -\frac{g^4}{2\pi^2} N_F F(\alpha) \frac{1}{\epsilon}, \quad (3.49)$$

where the function $F(\alpha)$ is defined in Eq. (3.42). Finally, the renormalization condition of the λ reads

$$Z_\phi^2 \lambda_0 \mu^{-\epsilon} - (\delta\lambda_Y + \delta\lambda_\lambda) = \lambda, \quad (3.50)$$

which ultimately yields the β function for the λ coupling.

3.4.5 β -functions for a tilted Dirac s -wave superconductor

Let's complete our results for the renormalization constants

$$Z_\Psi = 1 - \frac{g^2}{(4\pi)^2} G(\alpha) \frac{1}{\epsilon}, \quad (3.51)$$

$$Z_\Phi = 1 - \frac{g^2}{4\pi^2} N_F F(\alpha) \frac{1}{\epsilon} \quad (3.52)$$

$$Z_\alpha = 1 + \frac{g^2}{(4\pi)^2} [G(\alpha) - I(\alpha)] \frac{1}{\epsilon}, \quad (3.53)$$

while the renormalization constant for the velocity $Z_v = 1$, as shown in Sec. 3.4.1. Here, the functions are defined as $F(\alpha) = (1 - \alpha^2)^{-2}$, $G(\alpha) = (1 - (\alpha/2)^2)^{-1}$, while $I(\alpha)$ is given by Eq. (3.33).

The computation of the vertex diagrams to the same order gives the renormalization conditions for the Yukawa coupling constant (Sec. 3.4.3)

$$Z_\phi^{1/2} Z_\Psi g_0 \mu^{-\epsilon/2} = g, \quad (3.54)$$

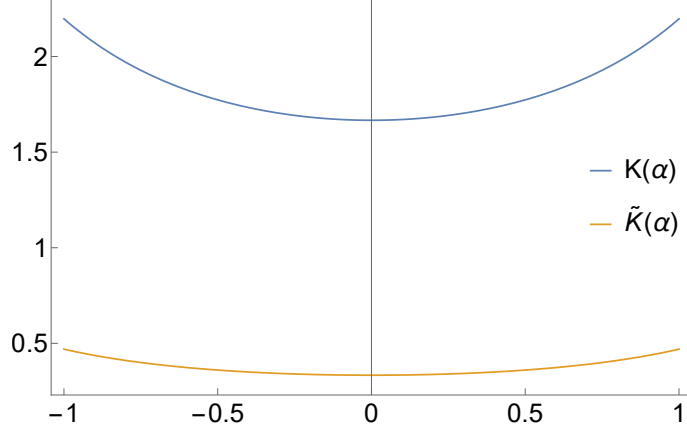


Figure 3.4.1: The form of the functions $K(\alpha)$ and $\tilde{K}(\alpha)$ governing the renormalization-group flow of the tilt parameter at the transition from the tilted Dirac semimetal to the s -wave superconducting (SC) and the charge-density wave (CDW) phases, respectively. The corresponding β function is given by Eq. (3.56) [Eq. (3.65)] for the s -wave SC [CDW] instability.

and for the Φ^4 coupling (Sec. 3.4.4)

$$Z_\phi^2 \lambda_0 \mu^{-\epsilon} - \left((N_B + 4) \frac{\lambda^2}{8\pi^2} - \frac{g^4}{2\pi^2} N_F F(\alpha) \right) \frac{1}{\epsilon} = \lambda. \quad (3.55)$$

Here, the couplings with (without) the subscript are the bare (renormalized) ones. Furthermore, μ is the RG (momentum) scale, and we consider N_B complex components for the bosonic OP.

Using the above renormalization conditions, we obtain the RG β functions to the leading order in the ϵ -expansion, in the critical hyperplane $m_B^2 = 0$, see Eq. (3.19). We define the β function for a coupling constant X as $\beta_X \equiv -dX/d \ln(\mu)|_{\{X_0\}}$, with all the bare couplings ($\{X_0\}$) fixed. Since the renormalization constant for the velocity $Z_v = 1$, the renormalization condition $Z_v v = v_0$ implies that the Fermi velocity is marginal, $\beta_v = 0$. Furthermore, the renormalization condition for the tilt term, $Z_\alpha \alpha = \alpha_0$ implies that

$$\beta_\alpha = -\alpha \frac{g^2}{2} [G(\alpha) - I(\alpha)] \equiv -\alpha g^2 K(\alpha), \quad (3.56)$$

with the function $K(\alpha)$ as strictly positive for $|\alpha| < 1$, as shown in Fig. 3.4.1. Therefore, the tilt parameter is *irrelevant*, and flows to zero as the system approaches the critical point governing the transition into the s -wave superconducting state. Ultimately, the Lorentz symmetry is restored at the QCP and the quantum-criticality

is controlled by the GNY QCP for $\alpha = 0$. To confirm this scenario, using the renormalization condition in Eq. (3.54), we furthermore find the β -function for the Yukawa coupling

$$\beta_{g^2} = \epsilon g^2 - [2N_F F(\alpha) + G(\alpha)]g^4, \quad (3.57)$$

where we rescaled $g^2/8\pi^2 \rightarrow g^2$. The RG flow in the (g^2, α) plane is shown in Fig. 3.4.2, which indeed corroborates the irrelevance of the tilt parameter, and the ensuing restoration of the Lorentz symmetry at the nontrivial QCP with the value given by Eq. (3.57),

$$g_\star^2 = \frac{\epsilon}{[2N_F F(0) + G(0)]}. \quad (3.58)$$

For completeness, we also show the RG flow equation for the Φ^4 -coupling for the tilted Dirac superconductor,

$$\beta_\lambda = \epsilon\lambda - (4 + N_B)\lambda^2 + 4N_F g^2 F(\alpha)(g^2 - \lambda), \quad (3.59)$$

which is obtained from the renormalization condition in Eq. (3.55), after rescaling $X/8\pi^2 \rightarrow X$, with $X = \{g^2, \lambda\}$. Notice that the β functions in Eqs. (3.57) and (3.56) for the untilted Dirac fermion ($\alpha = 0$) coincide with the ones obtained in Ref. [54]. The corresponding fixed points of the RG flow are obtained from the β -functions for the Yukawa and the λ couplings after taking $\alpha = 0$, as implied by the irrelevance of the tilt term (Eq. (3.56)). Only one of them is stable in the critical plane ($m_B^2 = 0$), and therefore represents the critical point, which is located at [54]

$$(g_\star^2, \lambda_\star) = \frac{\epsilon}{X + 4N_F} \left(1, X + \sqrt{X^2 + 8(1 - X)(4 + N_B)} \right), \quad (3.60)$$

where $X = 1 - 2N_F$. Notice that $\lambda_\star$ is positive, and therefore corresponds to a second-order phase transition, only for $N_F > 1/2$, or equivalently, in the Dirac system featuring at least one eight-component Dirac fermion flavor.

We emphasize here that identical renormalization factors and vertex corrections are obtained for the asymmetric TDFs, since the tilt matrix in this case anticommutes with the SC OP and commutes with the untilted part of the Dirac Hamiltonian, which is the same as for the symmetric TDFs, while all the form factors in the propagators for the asymmetric [Eq. (3.18)] and symmetric [Eq. (3.16)] TDFs are identical.

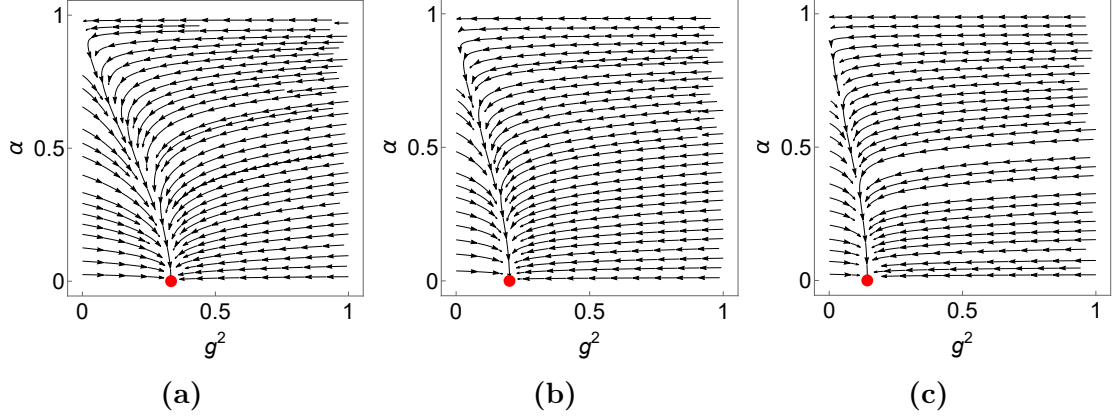


Figure 3.4.2: One-loop RG flow of the tilt parameter α and the Yukawa coupling g^2 at fixed $\epsilon = 1$ in the case of the s -wave instability. The red dot corresponds to the Gross-Neveu-Yukawa superconducting quantum-critical point (QCP), for different number of eight-component Dirac flavors (N_F). RG flow is shown for (a) $N_F = 1$; (b) $N_F = 2$; (c) $N_F = 3$. The analytical form of the Lorentz-symmetric QCP ($\alpha = 0$) is given by Eq. (3.60).

3.4.6 Comparison with Charge-Density-Wave Instability

We here revise the problem of a charge-density wave (CDW) instability breaking the sublattice symmetry of a tilted Dirac semimetal [30], in light of the restoration of the Lorentz symmetry at the superconducting QCP. The CDW OP takes the form,

$$\Phi(\mathbf{r}, \tau) = \langle \Psi^\dagger(\mathbf{r}, \tau) \Gamma_{303} \Psi(\mathbf{r}, \tau) \rangle, \quad (3.61)$$

implying that the Yukawa action reads

$$S_Y = g \int d\tau d^D \mathbf{r} \, \Phi \, \Psi^\dagger \Gamma_{303} \Psi, \quad (3.62)$$

with the OP matrix anticommuting with the untilted part of the Dirac Hamiltonian in Eq. (3.3), and therefore gaps it out. We are interested here only in the RG flow of the tilt parameter and the Yukawa coupling, and therefore consider the fermionic

self-energy, which reads

$$\begin{aligned}
\Sigma_F(i\Omega, \mathbf{k}) &= g^2 \int [d\omega] \int [d^D \mathbf{q}] \Gamma_{303} G_F(i\omega, \mathbf{q}) \Gamma_{303} G_B(i\Omega - i\omega, \mathbf{q}) \\
&= g^2 \int [d\omega] \int [d^D \mathbf{q}] \left[i\omega A_- - B(q_x \Gamma_{001} + q_y \Gamma_{032}) + \alpha q_x (A_- + 2q^2) \Gamma_{330} \right. \\
&\quad \left. - (A_- + 2\alpha^2 q_x^2)(q_x \Gamma_{331} + q_y \Gamma_{302}) \right] \frac{1}{[A_+^2 + B^2][(\omega - \Omega)^2 - (\mathbf{q} - \mathbf{k})^2]}, \tag{3.63}
\end{aligned}$$

which is obtained directly from the self-energy for the SC OP, Eq. (3.23), taking into account that the CDW OP commutes with the tilt matrix, Γ_{330} . This form of the fermionic self-energy, in turn, yields the renormalization constant for the field (Z_Ψ) in the form given by Eq. (3.51), while the factor Z_α reads

$$Z_\alpha = 1 + \frac{g^2}{(4\pi)^2} [G(\alpha) + I(\alpha)] \frac{1}{\epsilon}, \tag{3.64}$$

with the function $G(\alpha)$ and $I(\alpha)$ defined in Eq. (3.53) and Eq. (3.33), respectively. The corresponding RG flow equation for the tilt is then straightforwardly obtained from the renormalization condition $Z_\alpha \alpha = \alpha_0$, yielding

$$\beta_\alpha = -\alpha \frac{g^2}{2} [G(\alpha) + I(\alpha)] \equiv -\alpha g^2 \tilde{K}(\alpha), \tag{3.65}$$

with the strictly positive function $\tilde{K}(\alpha)$ shown in Fig. 3.4.1. Therefore, the Lorentz symmetry gets restored for the CDW instability, but much slower than in the SC case, since $\tilde{K}(\alpha) < K(\alpha)$, for $|\alpha| < 1$, as can be seen in Fig. 3.4.1. Furthermore, the flow of the Yukawa coupling in this case is given by

$$\beta_{g^2} = \epsilon g^2 - (N_F + 3)G(\alpha)g^4, \tag{3.66}$$

with the stable QCP realized at

$$g_\star^2 = \frac{\epsilon}{(3 + N_F)G(0)}. \tag{3.67}$$

The corresponding RG flow in the (g^2, α) plane is displayed in Fig. 3.4.3, which corroborates the restoration of the Lorentz symmetry at the QCP governing the transition into the CDW phase, and making the line of critical points found in Ref. [30] eventually unstable.

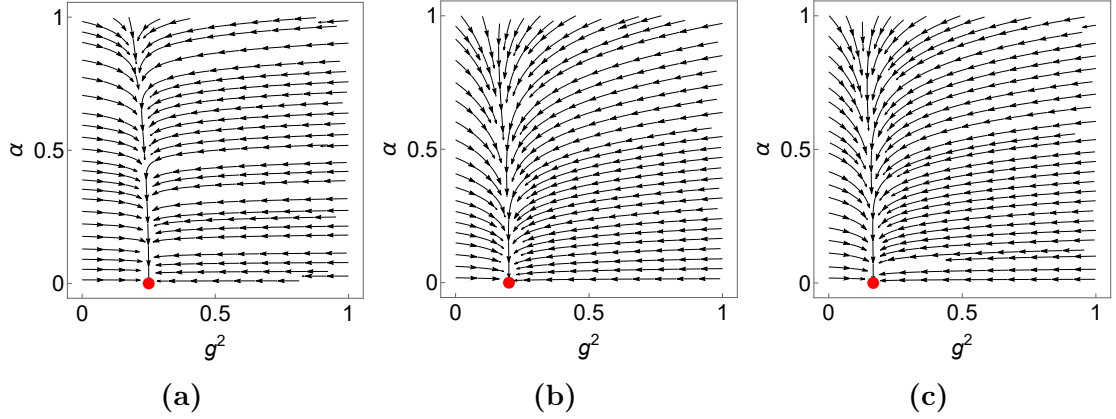


Figure 3.4.3: One-loop RG flow of the tilt parameter α and the Yukawa coupling g^2 at fixed $\epsilon = 1$ in the case of the charge-density-wave (CDW) instability. The red dot corresponds to the Gross-Neveu-Yukawa superconducting quantum-critical point (QCP), for different number of eight-component Dirac flavors (N_F). RG flow is shown for (a) $N_F = 1$; (b) $N_F = 2$; (c) $N_F = 3$. The analytical form of the Lorentz-symmetric QCP ($\alpha = 0$) is given by Eq. (3.67).

We point out that the sign in front of the strictly negative function $I(\alpha)$ (for $|\alpha| < 1$) in Eq. (3.65), is a direct consequence of the *commutation* of the tilt matrix (Γ_{330}) and the CDW OP Γ_{303} , which is diagonal in the Nambu space. This *positive* sign is responsible for slowing down the RG flow of the tilt with respect to the SC case, implying that the restoration of the Lorentz symmetry takes longer RG time.

In general, we can observe that the rate of the RG flow for the tilt parameter, and therefore the rate of the Lorentz symmetry restoration is directly related to the (anti)commutation relation of the OP with the tilt matrix. Namely, since all the Γ -matrices square to unity and either commute or anticommute with each other, an OP that commutes with the tilt matrix is expected to make the tilt parameter *less irrelevant* than an anticommuting OP. In particular, an insulating (particle-hole) instability diagonal in other flavor subspaces (valley, sublattice, etc.) should render the tilt less irrelevant than a superconducting (particle-particle) instability, as the examples of the CDW and the s -wave instabilities explicitly demonstrate.

Finally, the form of the propagator of the asymmetric TDFs implies that the β -functions at CDW QCP for the asymmetric TDFs are of the same form as for the symmetric TDFs.

Chapter 4

Conclusions and Future Work

In this thesis we have gone through the ideas of quantum phase transitions, we put special emphasis in the concept of order parameter to access the broken symmetry phase. We introduce the basic concepts of superconductivity and establish the Nambu representation to build a quantum field theory of it. We saw how normalization group approach allows us to access the critical behaviour under the upper critical dimension. Finally, we address the tilted Dirac fermion system. To this day, much work remains to be done in the field of Dirac materials. This thesis attempt is to contribute with a little but very interesting outcome.

Our findings are presented in Sec 3 of this thesis, where we have shown that the Lorentz symmetry is restored at the QCP separating the tilt Dirac semimetal from a gapped s -wave SC instability through the irrelevance of the tilt operator and its ultimate vanishing at the QCP. As we further argued such a symmetry restoration is a quite generic feature of the strongly interacting TDFs, irrespective of whether they break a discrete mirror lattice symmetry at the noninteracting level, i.e. whether they feature symmetric [Eq. (3.3)] or asymmetric [Eq. (3.4)] tilt.

Our results can be routinely tested in the lattice quantum Monte Carlo simulations [105] of the Hubbard interacting TDFs. In light of this, we emphasize that that even though the tilt parameter is irrelevant, and ultimately vanishes at the QCP, it may take long RG time to reach this value. Therefore in any finite-size system the effects of a finite tilt may still be appreciable, as given by the solution of the flow equations (3.56), (3.57) and (3.59). See also Fig. 3.4.2.

Directly pertinent to the SC instability of the TDFs is whether the coupling to a dynamical $U(1)$ gauge field can lead to the breaking of the Lorentz symmetry, which is an issue that we plan to investigate in future. In particular, it would be interesting

to see whether such a coupling could help stabilization of a charged QCP at a finite coupling.

Our work should motivate further studies of the interaction effects in the non-Hermitian tilted Dirac materials with the coupling to the environment explicitly taken into account through non-Hermitian terms in the Hamiltonian-like operator. In particular, it will be important to address whether the restoration of the Lorentz symmetry is also featured in the non-Hermitian Dirac semimetals, similarly to the case of untilted Dirac fermions [10].

Finally, we would like to point out that these results may be relevant for future studies of the strong coupling instabilities of type-II Dirac semimetals with electron- and hole Fermi surfaces, and should spur further research efforts in this direction.

Appendix A

Mathematical Tools

A1 Clifford algebra and Orthogonal groups

Let γ^μ , $\mu = 1, \dots, d$, be the generators of an associative algebra on $\mathbb{R}$, satisfying the following anticommutation relations

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\delta^{\mu\nu} \mathbf{I}, \quad (\text{A.1})$$

where $\mathbf{I}$ is the unit matrix. These matrices generate a Clifford algebra $\mathcal{C}(d)$. From the relation (A.1), it follows that the elements of $\mathcal{C}(d)$ form a real vector space of dimension 2^d , spanned by $\mathbf{I}$ and the products $\gamma^{\mu_1} \gamma^{\mu_2} \dots \gamma^{\mu_p}$, with $\mu_1 < \mu_2 < \dots < \mu_p$.

$$\gamma_S \equiv \gamma^{d+1} = i^{-d/2} \prod_{i=1}^d \gamma^i \quad (\text{A.2})$$

We now perform a linear transformation $\gamma \rightarrow \gamma'$ acting on the generators

$$\gamma'^\nu = R^\nu_\mu \gamma^\mu \quad (\text{A.3})$$

where R of elements R^ν_μ is a real matrix and $\det\{R\} \neq 0$. The elements γ' form an equivalent set of generators if they satisfy the relations

$$\gamma'^\mu \gamma'^\nu + \gamma'^\nu \gamma'^\mu = R^\mu_\rho R^\nu_\sigma (\gamma^\rho \gamma^\sigma + \gamma^\sigma \gamma^\rho) = 2R^\mu_\rho R^{\nu\rho} = 2\delta^{\mu\nu} \mathbf{I} \quad (\text{A.4})$$

Thus, the matrix R must be orthogonal and the relation (A1) are invariant under the orthogonal group $O(d)$ (rotation and reflections in the Euclidian d-dimensional

space) [99].

A2 Grassmann Algebra

A Grassmann (or exterior) algebra $\mathfrak{A}$ on $\mathbb{R}$ or $\mathbb{C}$ is an associative algebra constructed from a unit 1 and a set of generators $\{\theta_i\}$ that satisfy the anticommutation relations

$$\theta_i \theta_j + \theta_j \theta_i = 0, \quad \forall i, j. \quad (\text{A.5})$$

1. if the number n of generators is finite, the elements of the algebra form a vector space of finite dimension 2^n over $\mathbb{R}$ or $\mathbb{C}$. All elements can be written as linear combinations of the element A_ν , $\nu = 1, \dots, 2^n$ of the form

$$A_\nu \in \{1 \text{ and } \{\theta_{i_1} \theta_{i_2} \dots \theta_{i_p}, \text{ with } i_1 < i_2 < \dots < i_p, 1 \leq p \leq n\} \quad (\text{A.6})$$

2. $\mathfrak{A}$ is a graded algebra: for each monomial $\theta_{i_1} \dots \theta_{i_p}$, one can associate an integer p that counts the number of generators in a product. In particular, A_p and A_q are two monomials of degree p and q respectively, then

$$A_q A_p = (-1)^{pq} A_p A_q \quad (\text{A.7})$$

3. Elements of $\mathfrak{A}$ are invertible if and only if in the expansion in the basis (A.6) the term of degree zero do not vanish.
4. All elements in a Grassmann algebra, which are a function of generator θ_i , are first degree polynomials, that is, affine functions.

This Grassmann algebra brings with it very interesting properties related to differentiation and integration with respect to Grassmann numbers that actually makes calculation easier and are used for the quantization of the Dirac fields considered in this Thesis.

Appendix B

Fermionic Propagator

In this appendix we present the derivation of the fermionic propagator corresponding to the Hamiltonian of the tilted Dirac fermions, Eq. 3.3. We start from the following definition of the propagator

$$G_F(i\omega, \mathbf{k}) = [i\omega - \mathcal{H}(\mathbf{k})]^{-1}. \quad (\text{B.1})$$

We now write the propagator as

$$G_F(i\omega, \mathbf{k}) = \frac{1}{i\omega - \mathcal{H}(\mathbf{k})} \frac{i\omega + \mathcal{H}(\mathbf{k})}{i\omega + \mathcal{H}(\mathbf{k})}. \quad (\text{B.2})$$

Solving the product

$$\begin{aligned} (i\omega - \mathcal{H}(\mathbf{k}))(i\omega + \mathcal{H}(\mathbf{k})) &= -\omega^2 - [k_x^2 + k_y^2 + \alpha^2 k_x^2 + 2\alpha k_x k_y \Gamma_{330} \Gamma_{302} + 2\alpha k_x^2 \Gamma_{330} \Gamma_{331}] \\ &= -\omega^2 - [k^2 + \alpha^2 k_x^2 + 2\alpha k_x k_y \Gamma_{032} + 2\alpha k_x^2 \Gamma_{001}]. \end{aligned} \quad (\text{B.3})$$

Multiplying once again by the unity matrix written in the convenient form as in the last term on the right hand side of Eq. (B.2), we obtain

$$\begin{aligned} G_F(i\omega, \mathbf{k}) &= \frac{i\omega + \mathcal{H}(\mathbf{k})}{-\omega^2 - k^2 - \alpha^2 k_x^2 - [2\alpha k_x k_y \Gamma_{032} + 2\alpha k_x^2 \Gamma_{001}]} \\ &= \frac{i\omega + \mathcal{H}(\mathbf{k})}{A_- - [2\alpha k_x k_y \Gamma_{032} + 2\alpha k_x^2 \Gamma_{001}]} \frac{A_- + [2\alpha k_x k_y \Gamma_{032} + 2\alpha k_x^2 \Gamma_{001}]}{A_- + [2\alpha k_x k_y \Gamma_{032} + 2\alpha k_x^2 \Gamma_{001}]} \\ &= \frac{(i\omega + \mathcal{H}(\mathbf{k}))(A_- + [2\alpha k_x k_y \Gamma_{032} + 2\alpha k_x^2 \Gamma_{001}])}{A_-^2 - [(2\alpha k_x k_y)^2 + (2\alpha k_x^2)]}, \end{aligned} \quad (\text{B.4})$$

where we have defined $A_- = -\omega^2 - k^2 - \alpha^2 k_x^2$. Now we simplify the product

$$\begin{aligned}
(i\omega + \mathcal{H}(\mathbf{k}))(A_- + [2\alpha k_x k_y \Gamma_{032} + 2\alpha k_x^2 \Gamma_{001}]) &= i\omega A_- + B(k_x \Gamma_{001} + k_y \Gamma_{032}) \\
&+ A_-(\alpha k_x \Gamma_{330} + \Gamma_{331} k_x + \Gamma_{302} k_y) \\
&+ 2\alpha k_x [\Gamma_{330} k^2 + \alpha k_x (k_x \Gamma_{331} + k_y \Gamma_{302})],
\end{aligned} \tag{B.5}$$

where we defined $B \equiv B(i\omega, \mathbf{k}) = 2\alpha i\omega k_x$. Finally, defining $A_+ = -\omega^2 - k^2 + \alpha^2 k_x^2$, the propagator takes the form

$$\begin{aligned}
G_F(i\omega, \mathbf{k}) &= \left\{ i\omega A_- + B(k_x \Gamma_{001} + k_y \Gamma_{032}) + A_-(\alpha k_x \Gamma_{330} + \Gamma_{331} k_x + \Gamma_{302} k_y) \right. \\
&\quad \left. + 2\alpha k_x [\Gamma_{330} k^2 + \alpha k_x (k_x \Gamma_{331} + k_y \Gamma_{302})] \right\} \frac{1}{A_+^2 - B^2}.
\end{aligned} \tag{B.6}$$

For the case of asymmetric tilt, the procedure is exactly the same, and we only need to consider the matrix Γ_{300} in the term with the tilt parameter. Thus, the fermionic propagator in this case reads

$$\begin{aligned}
G_{F,AS}(i\omega, \mathbf{k}) &= \left\{ i\omega A_- + B(k_x \Gamma_{031} + k_y \Gamma_{002}) + A_-(\alpha k_x \Gamma_{300} + \Gamma_{331} k_x + \Gamma_{302} k_y) \right. \\
&\quad \left. + 2\alpha k_x [\Gamma_{300} k^2 + \alpha k_x (k_x \Gamma_{331} + k_y \Gamma_{302})] \right\} \frac{1}{A_+^2 - B^2}.
\end{aligned} \tag{B.7}$$

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