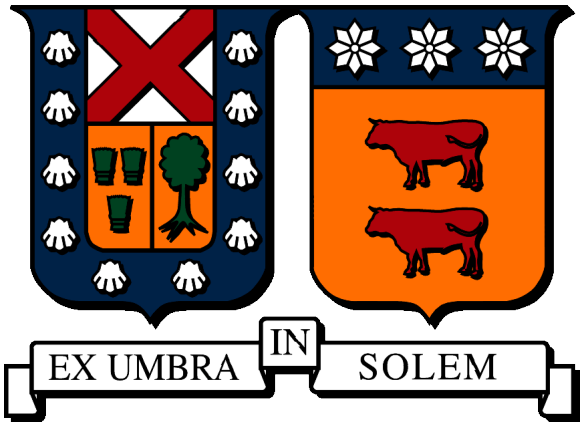
UNIVERSIDAD TÉCNICA
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Departamento de Matemática
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Stabilization of a wave equation with boundary disturbance and dynamic boundary condition

Tesis presentada por:

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Como requisito para optar al título profesional de Ingeniera Civil Matemática y al grado de Magíster en Ciencias Mención Matemática

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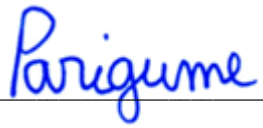
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Resumen

En esta tesis se estudia la estabilización y el buen planteamiento de un sistema híbrido que incorpora una ecuación tipo onda y está sujeto a una perturbación desconocida en la frontera que acompaña al control. El objetivo principal es diseñar un control que permita obtener un decaimiento exponencial de la energía sin requerir el conocimiento de la perturbación. Inspirados en las técnicas de Lyapunov utilizadas por Rao [28] para el sistema no perturbado, introducimos una ley de realimentación que incorpora un operador signo multivaluado. Suponiendo que se conoce una cota para la perturbación, se demuestra que, bajo ciertas condiciones sobre los parámetros del sistema, la energía decae exponencialmente.

Para estudiar el buen planteamiento del sistema en lazo cerrado, se introduce un cambio de variable que permite trasladar la perturbación desde la condición de frontera hacia la ecuación diferencial y reformular el sistema transformado como una inclusión diferencial abstracta en un espacio de Hilbert. El problema de buen planteamiento se reduce entonces a demostrar que el operador asociado es monótono maximal. Para la monotonía, se introduce un producto interno equivalente inspirado en el funcional de Lyapunov. La maximalidad se obtiene mediante un argumento basado en la reducción de la condición de frontera a un problema lineal auxiliar.

Finalmente, se consideran dos niveles de regularidad. Para datos iniciales y perturbaciones suficientemente regulares, se obtiene una única solución fuerte. Para datos menos regulares, se obtiene una única solución generalizada, mediante un argumento de aproximación. En ambos casos, se demuestra que el decaimiento exponencial de la energía se preserva.

Abstract

This thesis studies the stabilization and well-posedness of a hybrid system that incorporates a wave-type equation and is subject to an unknown boundary disturbance acting together with the control. The main objective is to design a control that provides exponential decay of the energy without requiring knowledge of the disturbance. Inspired by the Lyapunov techniques used by Rao [28] for the undisturbed system, we introduce a feedback law incorporating a multivalued sign operator. Assuming that a bound for the disturbance is known, we prove that, under suitable conditions on the system parameters, the energy decays exponentially.

To study the well-posedness of the closed-loop system, we introduce a change of variables that moves the disturbance from the boundary condition to the differential equation and reformulates the transformed system as an abstract differential inclusion in a Hilbert space. The well-posedness problem is then reduced to proving that the associated operator is maximal monotone. For monotonicity, we introduce an equivalent inner product inspired by the Lyapunov functional. Maximality is achieved through an argument based on reducing the boundary condition to an auxiliary linear problem.

Finally, two levels of regularity are considered. For sufficiently regular initial data and disturbances, we obtain a unique strong solution. For less regular data, we obtain a unique generalized solution using an approximation argument. In both cases, we prove that the exponential decay of the energy is preserved.

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Para todas las mujeres que hacen matemática,
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Contents

1 Introduction **7**

2 Preliminaries **11**

2.1 Multivalued operators 11

2.2 Monotone operators 12

2.3 Maximal monotone operators 13

2.4 Maximal monotone operators and evolution equations 16

2.5 Multivalued Sign Operator 17

3 Feedback Control Design **20**

3.1 Energy Functional 20

3.2 Construction of the Auxiliary Function 21

3.3 Lyapunov Functional 22

3.4 Energy Decay Estimates 24

4 Well-posedness **32**

4.1 Change of Variables 33

4.2 Abstract Formulation and Maximal Monotonicity 36

4.2.1 Hilbert Space and Operator Definition 37

4.2.2 Modified Inner Product 38

4.2.3 Monotonicity of the Operator 40

4.2.4 Maximality of the Operator 45

4.2.5 Density of the Domain 50

4.3 Well-posedness of the closed-loop system 53

5 Conclusions **60**

Bibliography **63**

Chapter 1

Introduction

This thesis lies within the theory of control of partial differential equations (PDEs), specifically stabilization problems. Control theory studies how inputs can be used to influence the behavior of a system and, in particular, to drive its state towards a desired equilibrium. The mathematical theory of control and stabilization is developed, for example, in [7, 25, 21, 24].

We study the stabilization of a hyperbolic PDE with an unknown boundary disturbance. More precisely, we consider a simplified model of an overhead crane consisting of a cable of length ℓ, a motor-driven platform, and a rigid load of mass M.

The behavior of the system is described by the following hybrid system:

$$\begin{cases} y_{tt} - (ay_x)_x = 0, & 0 < x < \ell, \quad t > 0, & \text{(1.0.1a)} \\ (ay_x)(0,t) = u(t) + d(t), & t > 0, & \text{(1.0.1b)} \\ (ay_x)(\ell,t) + My_{tt}(\ell,t) = 0, & t > 0, & \text{(1.0.1c)} \\ y(x,0) = y_0(x), \quad y_t(x,0) = y_1(x), & 0 < x < \ell. & \text{(1.0.1d)} \end{cases}$$

where $u = u(t)$ is the control input and $d = d(t)$ is an unknown boundary disturbance. The coefficient $a = a(x)$ represents the stiffness coefficient of the cable and satisfies

$$a \in H^1(0,\ell), \qquad a(x) \geq a_0 > 0, \quad \forall\, x \in [0,\ell], \tag{1.0.2}$$

where a_0 is a positive constant. The condition at $x = \ell$ is a dynamic boundary condition associated with the load.

The stabilization problem for the undisturbed system, that is, for $d = 0$, was studied by Rao in [28]. The system considered there is the same as (1.0.1), with the spatial interval

normalized to $(0,1)$. The presence of an unknown boundary disturbance changes the control problem in a fundamental way. Indeed, let $u_*(t)$ denote the feedback law proposed by Rao which is given by

$$u_*(t) = \alpha y(0,t) + f(y_t(0,t)), \qquad \alpha > 0, \tag{1.0.3}$$

where $f \in C(\mathbb{R})$ is a strictly increasing function satisfying $f(0) = 0$. In the absence of a disturbance, this feedback is designed to stabilize the system. However, it does not in general preserve this property in the presence of a persistent boundary disturbance.

To see this, consider the closed-loop system obtained by applying (1.0.3) in the presence of a disturbance:

$$\begin{cases} y_{tt} - (ay_x)_x = 0, & 0 < x < 1, \quad t > 0, & \text{(1.0.4a)} \\ (ay_x)(0,t) - \alpha y(0,t) - f(y_t(0,t)) = d(t), & t > 0, & \text{(1.0.4b)} \\ (ay_x)(1,t) + My_{tt}(1,t) = 0, & t > 0, & \text{(1.0.4c)} \\ y(x,0) = y_0(x), \quad y_t(x,0) = y_1(x), & 0 < x < 1. & \text{(1.0.4d)} \end{cases}$$

Choose

$$d(t) = -\alpha, \qquad y_0(x) = 1, \qquad y_1(x) = 0. \tag{1.0.5}$$

Then $y(x,t) \equiv 1$ is a solution of (1.0.4). Since the closed-loop system considered by Rao is well-posed, for the initial data in (1.0.5), the solution is unique. Since this solution is a nonzero constant, it cannot converge to zero as $t \to \infty$.

On the other hand, if the disturbance were known, one could formally compensate for it by taking

$$u(t) = u_*(t) - d(t).$$

However, since $d(t)$ is unknown, this compensation cannot be implemented. Consequently, a feedback law must be constructed that does not require knowledge of the disturbance and is robust with respect to its presence.

In this thesis, we consider two levels of regularity for the disturbance, corresponding to the two types of solutions obtained in the well-posedness analysis. We assume that

$$d \in W^{3,1}_{\text{loc}}(0,\infty), \qquad d(0) = d_t(0) = 0, \tag{A1.1}$$

$$d \in W^{2,1}_{\text{loc}}(0,\infty), \qquad d(0) = d_t(0) = 0, \tag{A1.2}$$

and there exists a known constant $D > 0$ such that

$$|d(t)| \le D, \qquad \forall\, t \ge 0. \tag{A2}$$

The stabilization of PDEs with unknown disturbances has been studied for different types of equations. For heat equations, distributed disturbances and boundary disturbances have been considered in [14, 1, 19, 15, 16, 23]. For wave equations, results for distributed disturbances can be found in [9], while boundary disturbances have been studied in [13, 10, 26, 17]. For beam equations, distributed and boundary disturbances have been considered in [1, 11, 18, 22]. Finally, boundary disturbances for Schrödinger equations have been studied in [12, 20]. The assumptions on the disturbance used in this thesis are consistent with assumptions considered in this literature.

The feedback design analysis, developed in Chapter 3, combines the Lyapunov techniques considered by Rao in [28] with the use of multivalued operators to address the unknown boundary perturbation, specifically the multivalued sign operator. Section 3.1 introduces the energy functional, while Section 3.2 presents the auxiliary function used in the stability analysis. Section 3.3 defines the Lyapunov functional and establishes the properties that will be used in Section 3.4 to construct the feedback law and obtain the exponential energy decay.

The use of the multivalued sign operator transforms the closed-loop system into a differential inclusion. Therefore, the well-posedness analysis presents difficulties due to the nonlinear and multivalued nature of the closed-loop system and the presence of the disturbance in the boundary condition. To address these difficulties, Section 4.1 introduces a change of variables that moves the perturbation from the boundary condition to the partial differential equation. This transformation allows the closed-loop system to be expressed in an abstract form involving a time-independent operator. More precisely, the transformed system is equivalent to a differential inclusion of the form:

$$\frac{d}{dt}X(t) + \mathcal{A}X(t) \ni F(t),$$

on a suitable Hilbert space H. This formulation reduces the well-posedness problem to the study of the operator $\mathcal{A}$, in particular, to the proof of its monotonicity and maximality.

Section 4.2 develops this abstract formulation. The Hilbert space and a modified inner product are introduced to handle the terms arising from the boundary condition at $x = 0$, which would otherwise be difficult to deal with in the monotonicity analysis. The subsequent sections establish the monotonicity and maximality of $\mathcal{A}$. In particular, Section 4.2.4 presents a novel and different argument to prove the maximality of the operator. Finally, Section 4.3 applies the theory of maximal monotone operators and nonlinear semigroups developed in [3, 29, 2, 5, 6, 8, 27] to obtain the well-posedness of the closed-loop system.

The mathematical preliminaries required for these arguments are presented in Chapter 2.

This chapter introduces the functional analysis, maximal monotone operator theory, and semigroup theory used throughout the thesis. Chapter 5 concludes the thesis with a discussion of the main results.

Chapter 2

Preliminaries

This chapter introduces the main concepts and results from the theory of maximal monotone operators that will be used in the well-posedness analysis of the closed-loop system.

Throughout this chapter, H denotes a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and associated norm $\| \cdot \|$. We denote by $\mathcal{P}(H)$ the set of all subsets of H.

2.1 Multivalued operators

We begin with the notion of a multivalued operator. This concept is required because the feedback law considered in this thesis contains the multivalued sign operator.

Definition 2.1.1 (Multivalued operator)**.** A multivalued operator on H is a mapping from H into $P(H)$.

The domain of $\mathcal{A}$ is defined by

$$D(\mathcal{A}) = \{X \in H : \mathcal{A}X \neq \emptyset\},$$

and its range is

$$R(\mathcal{A}) = \bigcup_{X \in H} \mathcal{A}X.$$

If $\mathcal{A}X$ contains at most one element for every $X \in H$, then $\mathcal{A}$ is called single-valued. Otherwise, $\mathcal{A}$ is called multivalued.

The graph of $\mathcal{A}$ is the set

$$G(\mathcal{A}) = \{(X, Y) \in H \times H : X \in D(\mathcal{A}),\ Y \in \mathcal{A}X\}.$$

2.2 Monotone operators

We now introduce monotone operators. The definitions and results in this section can be found in [3], [2] and [29].

Definition 2.2.1 (Monotone operator)**.** A multivalued operator $\mathcal{A}$ on H is said to be monotone if

$$\langle Y - \widetilde{Y}, X - \widetilde{X} \rangle \geq 0 \tag{2.2.1}$$

for all

$$X, \widetilde{X} \in D(\mathcal{A}), \qquad Y \in \mathcal{A}X, \qquad \widetilde{Y} \in \mathcal{A}\widetilde{X}.$$

When $\mathcal{A}$ is single-valued, condition (2.2.1) becomes

$$\langle \mathcal{A}X - \mathcal{A}\widetilde{X}, X - \widetilde{X} \rangle \geq 0, \qquad \forall\, X, \widetilde{X} \in D(\mathcal{A}).$$

Remark 2.2.2. If $\mathcal{A}$ is a linear operator, its monotonicity can be checked by the simpler condition:

$$\langle \mathcal{A}X, X \rangle \geq 0, \qquad \forall\, X \in D(\mathcal{A}).$$

The following characterization will be useful in the study of maximal monotone operators.

Proposition 2.2.3. *Let $\mathcal{A}$ be an operator on H. Then $\mathcal{A}$ is monotone if and only if*

$$\|X - \widetilde{X}\| \leq \|X - \widetilde{X} + \lambda(Y - \widetilde{Y})\| \tag{2.2.2}$$

for all

$$X, \widetilde{X} \in D(\mathcal{A}), \qquad Y \in \mathcal{A}X, \qquad \widetilde{Y} \in \mathcal{A}\widetilde{X}, \qquad \lambda > 0.$$

An operator satisfying (2.2.2) is also called accretive.

Proof. Assume first that $\mathcal{A}$ is monotone. For $X, \widetilde{X} \in D(\mathcal{A})$, $Y \in \mathcal{A}X$, $\widetilde{Y} \in \mathcal{A}\widetilde{X}$, and $\lambda > 0$, we have

$$\|X - \widetilde{X} + \lambda(Y - \widetilde{Y})\|^2 = \|X - \widetilde{X}\|^2 + \lambda^2 \|Y - \widetilde{Y}\|^2 + 2\lambda \langle Y - \widetilde{Y}, X - \widetilde{X} \rangle.$$

By monotonicity,

$$\langle Y - \widetilde{Y}, X - \widetilde{X} \rangle \geq 0,$$

and therefore

$$\|X - \widetilde{X} + \lambda(Y - \widetilde{Y})\|^2 \geq \|X - \widetilde{X}\|^2.$$

Taking square roots gives (2.2.2).

Conversely, assume that (2.2.2) holds. Squaring both sides gives:

$$0 \leq \lambda^2 \|Y - \widetilde{Y}\|^2 + 2\lambda \langle Y - \widetilde{Y}, X - \widetilde{X} \rangle.$$

Since $\lambda > 0$, division by λ yields:

$$0 \leq \lambda \|Y - \widetilde{Y}\|^2 + 2\langle Y - \widetilde{Y}, X - \widetilde{X} \rangle.$$

Letting $\lambda \to 0^+$, we obtain:

$$\langle Y - \widetilde{Y}, X - \widetilde{X} \rangle \geq 0.$$

Hence, $\mathcal{A}$ is monotone. □

2.3 Maximal monotone operators

The definitions and results in this section can be found in [3].

Definition 2.3.1 (Maximal monotone operator)**.** A monotone operator $\mathcal{A}$ on H is called maximal monotone if there is no monotone extension of $\mathcal{A}$. Equivalently,

$$G(\mathcal{A}) \subseteq G(\mathcal{B}), \quad \mathcal{B} \text{ monotone} \quad \Longrightarrow \quad \mathcal{A} = \mathcal{B}.$$

In other words, $\mathcal{A}$ is maximal monotone if and only if $\mathcal{A}$ is monotone and for all $(X, Y) \in H \times H$ such that

$$\langle Y - \widetilde{Y}, X - \widetilde{X} \rangle \geq 0 \quad \forall (\widetilde{X}, \widetilde{Y}) \in G(\mathcal{A}),$$

then $(X, Y) \in G(\mathcal{A})$.

Remark 2.3.2. Maximal monotone operators are also referred to as maximal accretive or m-accretive operators.

There is an equivalent characterization that is useful when proving maximality. But first, we state an auxiliary result.

Theorem 2.3.3. *Let C be a closed convex subset of H and let $\mathcal{A}$ be a monotone operator on C. Then, for every $Z \in H$, there exists $W \in C$ such that*

$$\langle Y + W, X - W \rangle \geq \langle Z, X - W \rangle, \quad \forall (X, Y) \in G(\mathcal{A}) \tag{2.3.1}$$

For a proof of this theorem, see [3, Theorem 2.1]. It will be used in the proof of the next proposition.

Proposition 2.3.4. *Let $\mathcal{A}$ be an operator on H. The following properties are equivalent:*

(i) $\mathcal{A}$ *is maximal monotone.*

(ii) $\mathcal{A}$ *is monotone and* $R(I+\mathcal{A}) = H$.

(iii) For every $\lambda > 0$, $(I+\lambda\mathcal{A})^{-1}$ *is a contraction defined on all of* H.

Remark 2.3.5. Proving the range condition is equivalent to proving the existence of a solution to a stationary problem associated with $\mathcal{A}$. Note that the range condition means that, for every $Y \in H$, there exists $X \in D(\mathcal{A})$ such that

$$Y \in X + \mathcal{A}X.$$

In the single-valued case, this is equivalent to $X + \mathcal{A}X = Y$.

Proof. We first prove $(iii) \Rightarrow (ii)$. Since $(I+\mathcal{A})^{-1}$ is defined on all of H, its domain is H. By the definition of the inverse operator,

$$D\big((I+\mathcal{A})^{-1}\big) = R(I+\mathcal{A}),$$

and hence

$$R(I+\mathcal{A}) = H.$$

Moreover, the contraction property implies the monotonicity of $\mathcal{A}$. More explicitly, for $X, \widetilde{X} \in D(\mathcal{A})$, $Y \in \mathcal{A}X$, and $\widetilde{Y} \in \mathcal{A}\widetilde{X}$,

$$\|X - \widetilde{X}\| \le \|X - \widetilde{X} + \lambda(Y - \widetilde{Y})\|,$$

and Proposition 2.2.3 gives monotonicity.

We next prove $(ii) \Rightarrow (i)$. Assume that $\mathcal{A}$ is monotone and

$$R(I+\mathcal{A}) = H.$$

Let $\mathcal{B}$ be a monotone operator such that

$$G(\mathcal{A}) \subseteq G(\mathcal{B}).$$

Take $X \in D(\mathcal{B})$ and $Y \in \mathcal{B}X$. Since $R(I+\mathcal{A}) = H$, there exists $\widetilde{X} \in D(\mathcal{A})$ such that

$$X + Y \in \widetilde{X} + \mathcal{A}\widetilde{X}.$$

Since $G(\mathcal{A}) \subseteq G(\mathcal{B})$, we also have $X - \widetilde{X} + Y \in \mathcal{B}\widetilde{X}$. Then, by monotonicity of $\mathcal{B}$,

$$0 \le \langle (X - \widetilde{X} + Y) - Y, \widetilde{X} - X\rangle = -\langle \widetilde{X} - X, \widetilde{X} - X\rangle \le 0$$

Therefore, $X = \widetilde{X}$ and $Y \in AX$. Thus,

$$G(\mathcal{B}) \subseteq G(\mathcal{A}),$$

and consequently $\mathcal{A} = \mathcal{B}$. Hence $\mathcal{A}$ is maximal monotone.

Finally, assume that $\mathcal{A}$ is maximal monotone. By Theorem 2.3.3, applied with $C = H$, for every $Y \in H$ there exists $X \in H$ such that

$$\langle \widetilde{Y} + X, \widetilde{X} - X \rangle \geq \langle Y, \widetilde{X} - X \rangle$$

for all $(\widetilde{X}, \widetilde{Y}) \in G(\mathcal{A})$. Equivalently,

$$\langle \widetilde{Y} - (Y - X), \widetilde{X} - X \rangle \geq 0$$

for all $(\widetilde{X}, \widetilde{Y}) \in G(\mathcal{A})$. The maximal monotonicity of $\mathcal{A}$ then implies

$$Y - X \in \mathcal{A}X.$$

Consequently,

$$Y \in X + \mathcal{A}X,$$

which proves

$$R(I + \mathcal{A}) = H.$$

The same argument applies to $\lambda\mathcal{A}$ for every $\lambda > 0$, since multiplication by a positive scalar preserves maximal monotonicity. Therefore,

$$R(I + \lambda\mathcal{A}) = H, \qquad \lambda > 0,$$

so $(I + \lambda\mathcal{A})^{-1}$ is defined on all of H. Moreover, Proposition 2.2.3 gives

$$\|X - \widetilde{X}\| \leq \|X - \widetilde{X} + \lambda(Y - \widetilde{Y})\|.$$

Hence the inverse is a contraction. □

2.4 Maximal monotone operators and evolution equations

The theory of maximal monotone operators provides a powerful framework for studying nonlinear evolution equations. By introducing an appropriate operator $\mathcal{A}$ acting on a suitable Hilbert space H, a given evolution problem can be reformulated in the abstract form:

$$\frac{d}{dt}X(t) + \mathcal{A}X(t) \ni F(t), \tag{2.4.1}$$

where $\mathcal{A}$ encapsulates the spatial structure of the problem and is independent of time, while the temporal dynamics are described by the differential equation on the Hilbert space H. Furthermore, this framework allows us to study nonlinear problems and differential inclusions, since $\mathcal{A}$ is a multivalued operator.

Once the evolution problem has been written in the form (2.4.1), the well-posedness analysis can be reduced to the study of the operator $\mathcal{A}$. In particular, it is enough to verify that $\mathcal{A}$ is maximal monotone in order to apply Theorem 2.4.1 and obtain well-posedness of the evolution problem.

Theorem 2.4.1 (Kato)**.** *Let $\mathcal{A}$ be a maximal monotone operator on the Hilbert space H. For every $X_0 \in D(\mathcal{A})$ and every absolutely continuous function $F : [0,T] \longrightarrow H$, there exists a unique absolutely continuous function $X : [0,T] \longrightarrow H$ such that*

$$\begin{cases} \dfrac{d}{dt}X(t) + \mathcal{A}X(t) \ni F(t), \quad \text{a.e. } t \in (0,T), & \text{(2.4.2a)} \\ X(0) = X_0. & \text{(2.4.2b)} \end{cases}$$

Moreover, X is Lipschitz continuous on $[0,T]$ and

$$X(t) \in D(\mathcal{A}), \qquad t \in [0,T].$$

The proof of Theorem 2.4.1 requires several results from the theory of maximal monotone operators and nonlinear semigroups and is not included here. A proof can be found in [29, Theorem 4.1].

Remark 2.4.2. Here, absolutely continuous means $F \in W^{1,1}(0,T;H)$.

Definition 2.4.3 (Generalized solution)**.** Let $\mathcal{A}$ be a maximal monotone operator on H, let $F \in L^1(0,T;H)$, and let $X_0 \in H$. A function $X \in C([0,T];H)$ is called a generalized solution of (2.4.1) with initial condition $X(0) = X_0$, if there exist sequences $\{F^n\}_{n\in\mathbb{N}} \subset W^{1,1}(0,T;H)$ and $\{X_0^n\}_{n\in\mathbb{N}} \subset D(\mathcal{A})$ such that

$$F^n \longrightarrow F \quad \text{in } L^1(0,T;H), \qquad X_0^n \longrightarrow X_0 \quad \text{in } H,$$

and the corresponding solutions X^n given by Theorem 2.4.1 satisfy

$$X^n \longrightarrow X \quad \text{in } C([0,T];H).$$

The following result extends Theorem 2.4.1 to initial data in H and source terms in $L^1(0,T;H)$.

Theorem 2.4.4. *Let $\mathcal{A}$ be a maximal monotone operator on H. For every $X_0 \in \overline{D(\mathcal{A})}$ and every $F \in L^1(0,T;H)$, there exists a unique generalized solution $X \in C([0,T];H)$ of (2.4.2).*

The result follows from the approximation argument associated with Theorem 2.4.1; see [29, Theorem 4.1A].

2.5 Multivalued Sign Operator

Definition 2.5.1 (Multivalued sign operator)**.** We define the multivalued sign operator as the mapping $\operatorname{sign} : \mathbb{R} \to P(\mathbb{R})$ given by:

$$\operatorname{sign}(x) = \begin{cases} \dfrac{x}{|x|}, & \text{if } x \neq 0, \\ [-1,1], & \text{if } x = 0. \end{cases} \tag{2.5.1}$$

In systems subject to unknown disturbances, the available information about the disturbance is often a uniform-in-time upper bound for its norm, such as (A2). This information can be used in the design of feedback controls. In particular, the multivalued sign operator allows us to construct a control term that can be used to compensate for the effect of the unknown disturbance. For this purpose, we use the property:

$$x\delta = |x|, \qquad \forall \delta \in \operatorname{sign}(x), \tag{2.5.2}$$

which is also written as $x \operatorname{sign}(x) = |x|$. This property will be used later in the energy estimates.

The next proposition shows that the multivalued sign operator belongs to the class of maximal monotone operators.

Proposition 2.5.2. *The multivalued sign operator is maximal monotone on* $\mathbb{R}$.

Proof. Let $\sigma \in \operatorname{sign}(x)$ and $\widetilde{\sigma} \in \operatorname{sign}(\widetilde{x})$. We first show that the operator is monotone.

If $x, \widetilde{x} > 0$, then

$$\sigma = \widetilde{\sigma} = 1,$$

and therefore

$$(\sigma - \widetilde{\sigma})(x - \widetilde{x}) = 0.$$

The same conclusion holds if $x, \widetilde{x} < 0$.

If $x > 0$ and $\widetilde{x} < 0$, then

$$\sigma = 1, \qquad \widetilde{\sigma} = -1,$$

and hence

$$(\sigma - \widetilde{\sigma})(x - \widetilde{x}) = 2(x - \widetilde{x}) > 0.$$

The case $x < 0$ and $\widetilde{x} > 0$ is analogous.

It remains to consider the case where one of the two points is zero. Suppose, for example, that $x > 0$ and $\widetilde{x} = 0$. Then

$$\sigma = 1, \qquad \widetilde{\sigma} \in [-1, 1],$$

and therefore

$$(\sigma - \widetilde{\sigma})(x - \widetilde{x}) = (1 - \widetilde{\sigma})x \geq 0.$$

The remaining cases are analogous. Thus, $(\sigma - \widetilde{\sigma})(x - \widetilde{x}) \geq 0$ and sign is monotone.

Now, we prove maximality. It is enough to show that $R(I + \operatorname{sign}) = \mathbb{R}$.

Let $x \in \mathbb{R}$. We consider two cases. If $|x| \leq 1$, we take $\widetilde{x} = 0$. Since $x \in [-1, 1] = \operatorname{sign}(0)$, we have:

$$x \in \widetilde{x} + \operatorname{sign}(\widetilde{x}).$$

Next, if $|x| > 1$, we take:

$$\widetilde{x} = x - \frac{x}{|x|}.$$

Then $\widetilde{x} \neq 0$ and

$$\operatorname{sign}(\widetilde{x}) = \frac{\widetilde{x}}{|\widetilde{x}|} = \frac{x}{|x|}.$$

Therefore,

$$x = \widetilde{x} + \frac{x}{|x|} \in \widetilde{x} + \operatorname{sign}(\widetilde{x}).$$

Thus, for every $x \in \mathbb{R}$, there exists $\widetilde{x} \in \mathbb{R}$ such that

$$x \in \widetilde{x} + \operatorname{sign}(\widetilde{x}).$$

Hence, sign is maximal monotone on $\mathbb{R}$. □

Remark 2.5.3. While we have provided a direct proof of the maximal monotonicity of the multivalued sign operator on $\mathbb{R}$, alternative proofs can be found in the literature. For example, see Example 2.3.4 in [3].

Chapter 3

Feedback Control Design

In this chapter, we establish the uniform exponential decay of the energy $E(t)$ for the hybrid system (1.0.1) by designing a robust boundary control law $u(t)$ capable of mitigating disturbances. Throughout this chapter, we assume that $y(x,t)$ is a solution to the hybrid system (1.0.1) possessing sufficient regularity for all performed mathematical manipulations to be well-defined, and that the unknown disturbance $d(t)$ is globally bounded (A2).

Our stability analysis is mathematically inspired by the work of Rao [28], where a perturbed Lyapunov functional approach was successfully employed to deduce the exponential decay of a classic overhead crane model in the absence of disturbances.

The main difference between our work and that of [28] lies in the robustness of the control law against perturbations. To eliminate the adverse effects of the unknown disturbance $d(t)$, we design a non-linear control law that contains a multivalued sign operator. This strategy allows us to successfully cancel the disturbance and guarantees the stability of the system.

3.1 Energy Functional

To analyze the stability of the hybrid system, we begin by defining the energy functional associated with the open-loop system (1.0.1) as follows:

$$E(t) = \frac{1}{2}\left\{\int_0^\ell \left(y_t^2(x,t) + a(x)y_x^2(x,t)\right)dx + \alpha y^2(0,t) + My_t^2(\ell,t)\right\}, \tag{3.1.1}$$

where $\alpha > 0$ is a positive constant to be fixed later. Assuming that the solution y has

sufficient regularity, we compute the time derivative of the energy:

$$E_t(t) = \int_0^\ell (y_t y_{tt} + a y_x y_{xt})\, dx + \alpha y(0,t) y_t(0,t) + M y_t(\ell,t) y_{tt}(\ell,t).$$

By simplifying and integrating by parts the second term of the integral, we obtain:

$$\begin{aligned} E_t(t) &= \int_0^\ell y_t y_{tt}\, dx - \int_0^\ell (a y_x)_x y_t\, dx + a y_x y_t|_0^\ell + \alpha (y y_t)(0,t) + M(y_t y_{tt})(\ell,t) \\ &= \int_0^\ell (y_{tt} - (a y_x)_x)\, y_t\, dx + \{(a y_x)(\ell,t) + M y_{tt}(\ell,t)\}\, y_t(\ell,t) - \{(a y_x)(0,t) - \alpha y(0,t)\}\, y_t(0,t) \end{aligned}$$

Using the dynamics of the system (1.0.1), we observe that:

1. From (1.0.1a), the term $\int_0^\ell (y_{tt} - (a y_x)_x) y_t\, dx = 0$.
2. From (1.0.1c), the term $(a y_x)(\ell,t) + M y_{tt}(\ell,t) = 0$.

Therefore, the derivative of the energy simplifies to:

$$E_t(t) = -\{(a y_x)(0,t) - \alpha y(0,t)\}\, y_t(0,t). \tag{3.1.2}$$

Finally, substituting the control condition at the boundary $x = 0$ from (1.0.1b), we obtain:

$$E_t(t) = y_t(0,t)\{-u(t) - d(t) + \alpha y(0,t)\}. \tag{3.1.3}$$

3.2 Construction of the Auxiliary Function

To establish the exponential decay of the energy, we introduce an auxiliary function denoted by $\phi(x)$. Let $R > 0$ be a positive scaling parameter to be determined later. We define ϕ as the solution to the following differential equation:

$$\begin{cases} \phi_x(x) + \phi(x) \left| \dfrac{a_x(x)}{a(x)} \right| = R, \quad x \in (0,\ell), \\ \phi(\ell) = -MR. \end{cases} \tag{3.2.1}$$

To calculate the explicit solution of (3.2.1), we multiply the differential equation by the integrating factor $\mu(x)$ defined by:

$$\mu(x) = \exp\left(\int_x^\ell \left| \frac{a_x(s)}{a(s)} \right| ds \right).$$

Then, we integrate the resulting expression over the interval (x, ℓ) and utilize the boundary condition $\phi(\ell) = -MR$. This procedure shows that the solution is given by:

$$\phi(x) = -R\left\{M\mu(x) + \int_x^\ell \exp\left(\int_x^s \left|\frac{a_x(r)}{a(r)}\right| dr\right) ds\right\}. \tag{3.2.2}$$

Since $\mu(x) > 0$ and the integral term is nonnegative for all $x \in [0, \ell]$, it is clear from (3.2.2) that $\phi(x) < 0$ for all $x \in [0, \ell]$. From (3.2.1) and the fact that ϕ is negative, we derive the following essential properties:

$$\begin{cases} \phi_x = R - \phi\left|\dfrac{a_x}{a}\right| \geq R, \\ a\left(\dfrac{\phi}{a}\right)_x = \phi_x - \phi\dfrac{a_x}{a} \geq R, \\ \phi(0) \leq \phi(\ell) = -MR < 0. \end{cases} \tag{3.2.3}$$

These properties indicate that ϕ is a negative and strictly increasing function, and they will be used in the next section to construct a Lyapunov functional.

3.3 Lyapunov Functional

To prove the main stability result, we introduce the perturbation functional $\rho(t)$ defined by:

$$\rho(t) = 2\int_0^\ell \phi y_t y_x \, dx + Ry(0,t)\left(\int_0^\ell y_t \, dx + My_t(\ell, t)\right), \tag{3.3.1}$$

Furthermore, for a given parameter $\varepsilon > 0$, we define the Lyapunov functional $V(t)$ as:

$$V(t) = E(t) + \varepsilon\rho(t). \tag{3.3.2}$$

Lemma 3.3.1. *There exists a positive constant $C_1 > 0$ such that:*

$$|\rho(t)| \leq C_1 E(t), \quad \forall t \geq 0, \tag{3.3.3}$$

where C_1 is given explicitly by:

$$C_1 := 2\left\|\frac{\phi}{\sqrt{a}}\right\|_{L^\infty(0,\ell)} + \left(\sqrt{\frac{M}{\alpha}} + \sqrt{\frac{\ell}{\alpha}}\right) R. \tag{3.3.4}$$

Proof. By applying the absolute value operator to the definition of $\rho(t)$ given in (3.3.1) and employing the triangle inequality, we obtain:

$$|\rho(t)| \leq 2\left|\int_0^\ell \phi y_t y_x\, dx\right| + R|y(0,t)|\left|\int_0^\ell y_t\, dx\right| + RM|y(0,t)||y_t(\ell,t)|.$$

Multiplying and dividing appropriately by $\sqrt{a(x)}$ and α, we rewrite the inequality as follows:

$$|\rho(t)| \leq 2\int_0^\ell \left|\frac{\phi}{\sqrt{a}}(\sqrt{a}y_x)y_t\right| dx + \frac{R}{\sqrt{\alpha}}\sqrt{\alpha}|y(0,t)|\int_0^\ell |y_t\cdot 1|\, dx + \frac{RM}{\sqrt{\alpha}}\sqrt{\alpha}|y(0,t)||y_t(\ell,t)|.$$

Next, we apply the Cauchy-Schwarz inequality to the integral terms:

$$\begin{aligned}|\rho(t)| \leq & 2\left(\int_0^\ell y_t^2 dx\right)^{1/2}\left(\int_0^\ell \frac{\phi^2}{a} a y_x^2 dx\right)^{1/2} \\ & + \frac{R}{\sqrt{\alpha}}\sqrt{\alpha}|y(0,t)|\left(\int_0^\ell 1^2 dx\right)^{1/2}\left(\int_0^\ell y_t^2 dx\right)^{1/2} \\ & + \frac{R\sqrt{M}}{\sqrt{\alpha}}\sqrt{\alpha}|y(0,t)|\sqrt{M}|y_t(\ell,t)|.\end{aligned}$$

Then, we deduce:

$$\begin{aligned}|\rho(t)| \leq & 2\left\|\frac{\phi}{\sqrt{a}}\right\|_{L^\infty(0,\ell)}\left(\int_0^\ell y_t^2 dx\right)^{1/2}\left(\int_0^\ell a y_x^2 dx\right)^{1/2} \\ & + \frac{\sqrt{\ell}R}{\sqrt{\alpha}}\sqrt{\alpha}|y(0,t)|\left(\int_0^\ell y_t^2 dx\right)^{1/2} \\ & + \frac{R\sqrt{M}}{\sqrt{\alpha}}\sqrt{\alpha}|y(0,t)|\sqrt{M}|y_t(\ell,t)|\end{aligned}$$

By applying Young's inequality $ab \leq \frac{1}{2}(a^2+b^2)$, we get:

$$\begin{aligned}|\rho(t)| \leq & 2\left\|\frac{\phi}{\sqrt{a}}\right\|_{L^\infty(0,\ell)}\cdot\frac{1}{2}\left(\int_0^\ell (y_t^2 + a y_x^2) dx\right) \\ & + \frac{\sqrt{\ell}R}{\sqrt{\alpha}}\cdot\frac{1}{2}\left(\alpha y(0,t)^2 + \int_0^\ell y_t^2 dx\right) \\ & + \frac{R\sqrt{M}}{\sqrt{\alpha}}\cdot\frac{1}{2}\left(\alpha y(0,t)^2 + M y_t(\ell,t)^2\right).\end{aligned}$$

Since all terms are non-negative, we can bound each separate term by adding the remaining missing positive elements required to fully reconstruct $E(t)$ as defined in (3.1.1). Consequently, we arrive at:

$$|\rho(t)| \leq \left\{2\left\|\frac{\phi}{\sqrt{a}}\right\|_{L^\infty(0,\ell)} + \left(\sqrt{\frac{M}{\alpha}} + \sqrt{\frac{\ell}{\alpha}}\right)R\right\}E(t).$$

Therefore, by defining the constant $C_1 > 0$ by (3.3.4), we conclude (3.3.3). This completes the proof of the lemma. □

Lemma 3.3.2. *Given any $K > 1$, there exists a parameter $\varepsilon_0 > 0$ such that for all $0 < \varepsilon \leq \varepsilon_0$, the Lyapunov functional $V(t)$ satisfies*

$$\frac{1}{\sqrt{K}} V(t) \leq E(t) \leq \sqrt{K} V(t), \quad \forall t \geq 0. \tag{3.3.5}$$

Proof. Let $\varepsilon > 0$. From definition (3.3.2), it is straightforward to see that

$$|V(t) - E(t)| = \varepsilon |\rho(t)|.$$

By applying (3.3.3), we obtain:

$$(1 - \varepsilon C_1) E(t) \leq V(t) \leq (1 + \varepsilon C_1) E(t).$$

If $0 < \varepsilon \leq \varepsilon_0$, with $\varepsilon_0 := C_1^{-1}(1 - 1/\sqrt{K})$, then it follows that

$$\frac{1}{\sqrt{K}} E(t) \leq V(t) \leq \left(2 - \frac{1}{\sqrt{K}}\right) E(t). \tag{3.3.6}$$

as $0 \leq \left(\sqrt{K} - 1\right)^2$ is equivalent to $2 - \frac{1}{\sqrt{K}} \leq \sqrt{K}$, from (3.3.6) we conclude (3.3.5), which finishes the proof of the lemma. □

3.4 Energy Decay Estimates

In this section, we combine the energy derivative, the perturbation functional estimates, and the robust boundary control design to prove the main stability result of this chapter. By properly tuning the system parameters, we establish that the Lyapunov functional satisfies a differential inequality that guarantees the exponential decay of the system's energy.

Theorem 3.4.1 (Exponential Decay)**.** *Given $K > 1$, there exists $\varepsilon_* > 0$ such that, for all $\varepsilon \in (0, \varepsilon_*]$, if α is given by*

$$\alpha := 2\left(1 + \frac{\varepsilon \phi(0)}{a(0)}\right) \varepsilon R, \tag{3.4.1}$$

then the energy functional satisfies

$$E(t) \leq K E(0) e^{-\omega t}, \quad \forall t \geq 0, \tag{3.4.2}$$

where $\omega = \frac{\varepsilon R}{\sqrt{K}}$.

Proof. To begin the proof of the exponential stability, we decompose the total perturbation functional $\rho(t) = \widetilde{\rho}(t) + \hat{\rho}(t)$ into its components, defined respectively by

$$\widetilde{\rho}(t) = 2\int_0^\ell \phi y_t y_x \, dx, \tag{3.4.3}$$

$$\hat{\rho}(t) = Ry(0,t)\left(\int_0^\ell y_t \, dx + My_t(\ell,t)\right). \tag{3.4.4}$$

Step 1: Estimates for the derivative of $\widetilde{\rho}(t)$.
From (3.4.3) we have:

$$\widetilde{\rho}'(t) = 2\int_0^\ell \phi\left(y_{tt}y_x + y_t y_{xt}\right) dx.$$

By substituting the equation (1.0.1a) from the open-loop system into the term y_{tt} and doing an integration by parts on the second integral, we obtain:

$$\begin{aligned}
\widetilde{\rho}'(t) &= 2\int_0^\ell \phi(ay_x)_x y_x \, dx + 2\int_0^\ell \phi y_t y_{tx} \, dx \\
&= 2\int_0^\ell \phi(ay_x)_x y_x \, dx + 2\int_0^\ell \phi\, \partial_x\left(\frac{1}{2}y_t^2\right) dx \\
&= 2\int_0^\ell \phi(ay_x)_x y_x \, dx + 2\left[\phi\cdot\frac{1}{2}y_t^2\right]_0^\ell - 2\int_0^\ell \phi_x\cdot\frac{1}{2}y_t^2 \, dx \\
&= 2\int_0^\ell \phi(ay_x)_x y_x \, dx + \left[\phi y_t^2\right]_0^\ell - \int_0^\ell \phi_x y_t^2 \, dx.
\end{aligned}$$

Next, we expand the integrand of the first term on the right-hand side as follows:

$$\phi(ay_x)_x y_x = \phi\left(a_x y_x + a y_{xx}\right) y_x = \phi a_x y_x^2 + \phi a y_x y_{xx} = \phi a_x y_x^2 + \phi a\, \partial_x\left(\frac{1}{2}y_x^2\right).$$

Substituting this back to the previous equation and integrating by parts once again leads to:

$$\begin{aligned}
\widetilde{\rho}'(t) &= 2\left\{\int_0^\ell \phi a_x y_x^2 \, dx + \int_0^\ell \phi a\, \partial_x\left(\frac{1}{2}y_x^2\right) dx\right\} + \left[\phi y_t^2\right]_0^\ell - \int_0^\ell \phi_x y_t^2 \, dx \\
&= 2\left\{\int_0^\ell \phi a_x y_x^2 \, dx + \frac{1}{2}\left[\phi a y_x^2\right]_0^\ell - \int_0^\ell (\phi a)_x\left(\frac{1}{2}y_x^2\right) dx\right\} + \left[\phi y_t^2\right]_0^\ell - \int_0^\ell \phi_x y_t^2 \, dx \\
&= 2\int_0^\ell \left(\phi a_x - \frac{1}{2}(\phi a)_x\right) y_x^2 \, dx + \left[\phi a y_x^2\right]_0^\ell + \left[\phi y_t^2\right]_0^\ell - \int_0^\ell \phi_x y_t^2 \, dx.
\end{aligned} \tag{3.4.5}$$

We now exploit the second property stated in (3.2.3). Multiplying that expression by the coefficient $a(x)$ yields:

$$a\left(\frac{\phi}{a}\right)_x a = \phi_x a - \phi a_x,$$

which allows us to simplify the integrand coefficient in (3.4.5) via

$$\phi a_x - \frac{1}{2}(\phi a)_x = \phi a_x - \frac{1}{2}\phi_x a - \frac{1}{2}\phi a_x = -\frac{1}{2}(\phi_x a - \phi a_x) = -\frac{1}{2} a \left(\frac{\phi}{a}\right)_x a.$$

By placing this identity back into (3.4.5), the derivative of $\widetilde{\rho}(t)$ reduces to

$$\widetilde{\rho}'(t) = -\int_0^\ell a \left(\frac{\phi}{a}\right)_x a y_x^2 \, dx + \left[\phi a y_x^2\right]_0^\ell + \left[\phi y_t^2\right]_0^\ell - \int_0^\ell \phi_x y_t^2 \, dx. \tag{3.4.6}$$

Thanks to (3.2.3), we can upper bound several terms in (3.4.6) as follows:

$$\begin{aligned} \phi(\ell)(a y_x^2)(\ell, t) = -MR(a y_x^2)(\ell, t) \leq 0, \quad & -\int_0^\ell a \left(\frac{\phi}{a}\right)_x a y_x^2 \, dx \leq -R \int_0^\ell a y_x^2 \, dx, \\ \phi(\ell) y_t^2(\ell, t) = -MR y_t^2(\ell, t), \quad & -\int_0^\ell \phi_x y_t^2 \, dx \leq -R \int_0^\ell y_t^2 \, dx. \end{aligned}$$

Consequently, from (3.4.6) we arrive at:

$$\widetilde{\rho}'(t) \leq -R \int_0^\ell a y_x^2 \, dx - \phi(0)(a y_x^2)(0, t) - MR y_t^2(\ell, t) - \phi(0) y_t^2(0, t) - R \int_0^\ell y_t^2 \, dx. \tag{3.4.7}$$

Step 2: Estimates for the derivative of $\hat{\rho}(t)$.
By utilizing the PDE (1.0.1a) and the dynamic boundary condition (1.0.1c) defined in the open-loop system, the derivative of the perturbation functional $\hat{\rho}(t)$ satisfies

$$\begin{aligned} \hat{\rho}'(t) &= R y_t(0, t) \left(\int_0^\ell y_t \, dx + M y_t(\ell, t) \right) + R y(0, t) \left(\int_0^\ell y_{tt} \, dx + M y_{tt}(\ell, t) \right) \\ &= R y_t(0, t) \left(\int_0^\ell y_t \, dx + M y_t(\ell, t) \right) + R y(0, t) \left(\int_0^\ell (a y_x)_x \, dx + M y_{tt}(\ell, t) \right) \\ &= R y_t(0, t) \left(\int_0^\ell y_t \, dx + M y_t(\ell, t) \right) + R y(0, t) \left((a y_x)(\ell, t) - (a y_x)(0, t) + M y_{tt}(\ell, t) \right) \\ &= R y_t(0, t) \left(\int_0^\ell y_t \, dx + M y_t(\ell, t) \right) + R y(0, t) \left(-M y_{tt}(\ell, t) - (a y_x)(0, t) + M y_{tt}(\ell, t) \right) \\ &= R y_t(0, t) \left(\int_0^\ell y_t \, dx + M y_t(\ell, t) \right) - R y(0, t)(a y_x)(0, t). \end{aligned} \tag{3.4.8}$$

We now apply Young's inequality with $p = q = 2$ to get:

$$R \left(y_t(0, t) \cdot \int_0^\ell y_t \, dx \right) \leq \frac{R}{2} \int_0^\ell y_t^2 \, dx + \frac{R}{2} \ell y_t^2(0, t),$$

$$MR \left(y_t(0, t) \cdot y_t(\ell, t) \right) \leq \frac{MR}{2} y_t^2(\ell, t) + \frac{MR}{2} y_t^2(0, t).$$

Substituting these upper bounds back into (3.4.8), we can establish

$$\hat{\rho}'(t) \leq \frac{R}{2}\int_0^\ell y_t^2\,dx + \frac{MR}{2}y_t^2(\ell,t) + \frac{R}{2}(\ell+M)y_t^2(0,t) - Ry(0,t)(ay_x)(0,t). \tag{3.4.9}$$

Step 3: Differentiation and bounding of the Lyapunov functional.
Having established the upper bounds for the derivatives of both perturbation components, we now differentiate the complete Lyapunov functional $V(t)$ defined in (3.3.2). By invoking (3.4.7) and (3.4.9), the time derivative satisfies the following inequality:

$$\begin{aligned} V'(t) \leq{}& -\varepsilon RE(t) + \alpha(yy_t)(0,t) + \frac{\varepsilon}{2}R\alpha y^2(0,t) - \varepsilon\phi(0)y_t^2(0,t) + \frac{\varepsilon}{2}R(\ell+M)y_t^2(0,t) \\ &+ (ay_x)(0,t)\left\{-y_t(0,t) - \varepsilon Ry(0,t) - \varepsilon\phi(0)y_x(0,t)\right\}. \end{aligned} \tag{3.4.10}$$

At this stage, our goal is to design a control law $u(t)$ capable of canceling all boundary terms on the right-hand side of inequality (3.4.10), except for the energy term $-\varepsilon RE(t)$. To achieve this, we introduce the feedback control law $u(t)$ defined by

$$u(t) = y_t(0,t) + \varepsilon Ry(0,t) + v(t), \tag{3.4.11}$$

where $v(t)$ denotes an auxiliary control input. Substituting the control design (3.4.11) into the boundary condition (1.0.1b) yields:

$$(ay_x)(0,t) = y_t(0,t) + \varepsilon Ry(0,t) + v(t) + d(t). \tag{3.4.12}$$

Now we focus on expanding the last product on the right-hand side of (3.4.10), given by

$$(ay_x)(0,t)\left\{-y_t(0,t) - \varepsilon Ry(0,t) - \varepsilon\phi(0)y_x(0,t)\right\}. \tag{3.4.13}$$

From (3.4.12) and (3.4.13) we get:

$$\begin{aligned} &(ay_x)(0,t)\left(-y_t(0,t) - \varepsilon Ry(0,t) - \varepsilon\phi(0)y_x(0,t)\right) \\ &\quad = \left\{(y_t(0,t) + \varepsilon Ry(0,t)) + v(t) + d(t)\right\}\left\{-\left(y_t(0,t) + \varepsilon Ry(0,t)\right) - \varepsilon\phi(0)y_x(0,t)\right\}. \end{aligned}$$

We expand this expression into the following form:

$$\begin{aligned} &(ay_x)(0,t)\left\{-y_t(0,t) - \varepsilon Ry(0,t) - \varepsilon\phi(0)y_x(0,t)\right\} \\ &\quad = -\left(y_t(0,t) + \varepsilon Ry(0,t)\right)^2 - \varepsilon\phi(0)y_x(0,t)\left(y_t(0,t) + \varepsilon Ry(0,t)\right) \\ &\qquad - \left(v(t) + d(t)\right)\left\{y_t(0,t) + \varepsilon Ry(0,t) + \varepsilon\phi(0)y_x(0,t)\right\}. \end{aligned} \tag{3.4.14}$$

Next, we rewrite $-\phi(0)y_x(0,t)$ and consider (3.4.12) to obtain:

$$\begin{aligned}-\varepsilon\phi(0)y_x(0,t) &= -\frac{\varepsilon\phi(0)}{a(0)}(ay_x)(0,t)\\ &= -\frac{\varepsilon\phi(0)}{a(0)}\left(y_t(0,t)+\varepsilon Ry(0,t)\right)-\frac{\varepsilon\phi(0)}{a(0)}\left(v(t)+d(t)\right).\end{aligned} \tag{3.4.15}$$

Substituting (3.4.15) back into the second product of (3.4.14) yields:

$$\begin{aligned}&(ay_x)(0,t)\left\{-y_t(0,t)-\varepsilon Ry(0,t)-\varepsilon\phi(0)y_x(0,t)\right\}\\ &\quad= -\left(y_t(0,t)+\varepsilon Ry(0,t)\right)^2\\ &\qquad+\left\{-\frac{\varepsilon\phi(0)}{a(0)}\left(y_t(0,t)+\varepsilon Ry(0,t)\right)-\frac{\varepsilon\phi(0)}{a(0)}\left(v(t)+d(t)\right)\right\}\left(y_t(0,t)+\varepsilon Ry(0,t)\right)\\ &\qquad-\left(v(t)+d(t)\right)\left\{y_t(0,t)+\varepsilon Ry(0,t)+\varepsilon\phi(0)y_x(0,t)\right\}.\end{aligned}$$

By rearranging terms, we arrive at:

$$\begin{aligned}&(ay_x)(0,t)\left\{-y_t(0,t)-\varepsilon Ry(0,t)-\varepsilon\phi(0)y_x(0,t)\right\}\\ &\quad= -\left(1+\frac{\varepsilon\phi(0)}{a(0)}\right)\left(y_t(0,t)+\varepsilon Ry(0,t)\right)^2\\ &\qquad-\left(v(t)+d(t)\right)\left\{\left(1+\frac{\varepsilon\phi(0)}{a(0)}\right)\left(y_t(0,t)+\varepsilon Ry(0,t)\right)+\varepsilon\phi(0)y_x(0,t)\right\}.\end{aligned} \tag{3.4.16}$$

To simplify the notation, we define the function $S(t)$ as

$$S(t) := \left(1+\frac{\varepsilon\phi(0)}{a(0)}\right)\left(y_t(0,t)+\varepsilon Ry(0,t)\right)+\varepsilon\phi(0)y_x(0,t). \tag{3.4.17}$$

Therefore, by considering (3.4.17) in (3.4.16), we obtain:

$$\begin{aligned}&(ay_x)(0,t)\left\{-y_t(0,t)-\varepsilon Ry(0,t)-\varepsilon\phi(0)y_x(0,t)\right\}\\ &\quad= -\left(1+\frac{\varepsilon\phi(0)}{a(0)}\right)\left\{y_t^2(0,t)+2\varepsilon R(yy_t)(0,t)+(\varepsilon R)^2y^2(0,t)\right\}-\left(v(t)+d(t)\right)S(t).\end{aligned} \tag{3.4.18}$$

Finally, substituting (3.4.18) back into (3.4.10), and gathering the coefficients associated with the common state variables $(yy_t)(0,t)$, $y^2(0,t)$, and $y_t^2(0,t)$, we establish the following upper bound:

$$
\begin{aligned}
V'(t) \leq & -\varepsilon R E(t) + \left(\alpha - 2\left(1 + \frac{\varepsilon\phi(0)}{a(0)}\right)\varepsilon R\right)(yy_t)(0,t) \\
& + \left(\frac{\varepsilon R\alpha}{2} - \left(1 + \frac{\varepsilon\phi(0)}{a(0)}\right)(\varepsilon R)^2\right) y^2(0,t) \\
& + \left(-\varepsilon\phi(0) + \frac{\varepsilon R(\ell + M)}{2} - \left(1 + \frac{\varepsilon\phi(0)}{a(0)}\right)\right) y_t^2(0,t) \\
& - (v(t) + d(t))S(t).
\end{aligned} \tag{3.4.19}
$$

Step 4: Parameter tuning.

To eliminate the indefinite cross-term $(yy_t)(0,t)$ of (3.4.19), we choose α as given in (3.4.1).

Next, we proceed to choose $\varepsilon > 0$ such that $\alpha > 0$. To that end, we introduce $\varepsilon_1 > 0$, defined by

$$
\varepsilon_1 := \left(-\phi(0) - \frac{\phi(0)}{a(0)} + \frac{R(\ell + M)}{2}\right)^{-1}. \tag{3.4.20}
$$

Since $\phi(0) < 0$, $a(0) > 0$, and the parameters R and M are strictly positive, it is clear that $\varepsilon_1 > 0$ is well-defined and satisfies the upper bound $\varepsilon_1 < -\frac{a(0)}{\phi(0)}$. Consequently, if $0 < \varepsilon \leq \varepsilon_1$, then we preserve the strict positivity of the parameter α as

$$
0 < \varepsilon \leq \varepsilon_1 < -\frac{a(0)}{\phi(0)} \implies -\varepsilon\frac{\phi(0)}{a(0)} < 1 \implies 1 + \frac{\varepsilon\phi(0)}{a(0)} > 0 \implies \alpha > 0.
$$

By evaluating the chosen definition of α (3.4.1) into the remaining coefficients of (3.4.19), the factor multiplying $y^2(0,t)$ simplifies to

$$
\frac{\varepsilon R\alpha}{2} - \left(1 + \frac{\varepsilon\phi(0)}{a(0)}\right)(\varepsilon R)^2 = \frac{\varepsilon R}{2}(\alpha - \alpha) = 0.
$$

Consequently, we observe that this term vanishes too. On the other hand, by exploiting (3.4.20), the coefficient of $y_t^2(0,t)$ can be rewritten and bounded as follows:

$$
\begin{aligned}
& -\varepsilon\phi(0) + \frac{\varepsilon R(\ell + M)}{2} - \left(1 + \frac{\varepsilon\phi(0)}{a(0)}\right) \\
& \quad = \varepsilon\left(-\phi(0) - \frac{\phi(0)}{a(0)} + \frac{R(\ell + M)}{2}\right) - 1 \\
& \quad = \frac{\varepsilon}{\varepsilon_1} - 1 \leq 0.
\end{aligned}
$$

Therefore, by implementing the control choice $v(t) = D\,\mathrm{sign}(S(t))$, with $\mathrm{sign}(\cdot)$ defined in (2.5.1), and assuming that the unknown disturbance satisfies the uniform bound (A2), we use (2.5.2) to obtain:

$$-(v(t)+d(t))S(t) = -D|S(t)| - d(t)S(t) \leq -D|S(t)| + D|S(t)| = 0.$$

Applying all these parameter choices and inequalities to (3.4.19), the derivative reduces to the dissipation estimate

$$V'(t) \leq -\varepsilon R E(t), \quad \forall t \geq 0. \tag{3.4.21}$$

Step 5: Asymptotic consistency of the threshold ε_0.
To complete the proof, we must verify the consistency of the smallness condition on ε required in Lemma 3.3.2:

$$0 < \varepsilon \leq \varepsilon_0(\varepsilon) := C_1^{-1}\left(1 - \frac{1}{\sqrt{K}}\right), \tag{3.4.22}$$

where

$$C_1 = 2\left\|\frac{\phi}{\sqrt{a}}\right\|_{L^\infty(0,\ell)} + \left(\frac{\sqrt{M}}{\sqrt{\alpha}} + \frac{\sqrt{\ell}}{\sqrt{\alpha}}\right)R, \quad \alpha = 2\varepsilon R\left(1 + \frac{\varepsilon\phi(0)}{a(0)}\right).$$

Substituting this expression, $\varepsilon_0(\varepsilon)$ takes the form

$$\varepsilon_0(\varepsilon) = \frac{1 - \frac{1}{\sqrt{K}}}{2\left\|\frac{\phi}{\sqrt{a}}\right\|_{L^\infty(0,\ell)} + \frac{R(\sqrt{M}+\sqrt{\ell})}{\sqrt{2\varepsilon R\left(1+\frac{\varepsilon\phi(0)}{a(0)}\right)}}}.$$

Thus, (3.4.22) is equivalent to

$$2\varepsilon\left\|\frac{\phi}{\sqrt{a}}\right\|_{L^\infty(0,\ell)} + \frac{\sqrt{\varepsilon}(\sqrt{M}+\sqrt{\ell})}{\sqrt{2R\left(1+\frac{\varepsilon\phi(0)}{a(0)}\right)}} < 1 - \frac{1}{\sqrt{K}}. \tag{3.4.23}$$

Taking the limit as $\varepsilon \to 0^+$, we find that

$$\lim_{\varepsilon\to 0^+} 2\varepsilon\left\|\frac{\phi}{\sqrt{a}}\right\|_{L^\infty(0,\ell)} + \frac{\sqrt{\varepsilon}(\sqrt{M}+\sqrt{\ell})}{\sqrt{2R\left(1+\frac{\varepsilon\phi(0)}{a(0)}\right)}} = 0.$$

Consequently, there exists $\varepsilon_2 > 0$ such that for all $\varepsilon \in (0, \varepsilon_2)$, (3.4.23) holds, which in turn implies (3.4.22).

Step 6: Completing the proof of the exponential decay.

Let us define the final parameter $\varepsilon_* := \min\{\varepsilon_1, \varepsilon_2\}$. By combining (3.4.21) with the right-hand side of (3.3.5) established in Lemma 3.3.2, we achieve the following differential inequality for any $0 < \varepsilon \leq \varepsilon_*$:

$$V'(t) \leq -\frac{\varepsilon R}{\sqrt{K}} V(t), \quad \forall t \geq 0. \tag{3.4.24}$$

Multiplying (3.4.24) by the positive integrating factor $e^{\frac{\varepsilon R}{\sqrt{K}}t}$ yields:

$$\frac{d}{dt}\left(V(t)e^{\frac{\varepsilon R}{\sqrt{K}}t}\right) \leq 0.$$

Integrating this expression over $[0, t]$, we find:

$$V(t) \leq V(0)e^{-\frac{\varepsilon R}{\sqrt{K}}t}. \tag{3.4.25}$$

Finally, by utilizing the left-hand side of the equivalence lemma relation (3.3.5) on $V(t)$ and the right-hand side on the initial condition $V(0)$, inequality (3.4.25) can be written as

$$\frac{1}{\sqrt{K}}E(t) \leq V(t) \leq V(0)e^{-\frac{\varepsilon R}{\sqrt{K}}t} \leq \sqrt{K}E(0)e^{-\frac{\varepsilon R}{\sqrt{K}}t},$$

From this, we conclude (3.4.2), with $\omega := \frac{\varepsilon R}{\sqrt{K}} > 0$. This completes the proof of the theorem.

□

Chapter 4

Well-posedness

When we incorporate the robust control law designed in the previous section, given by:

$$u(t) = y_t(0,t) + \varepsilon R y(0,t) + D\,\mathrm{sign}(S(t)), \tag{4.0.1}$$

the hybrid system (1.0.1) changes. Due to the presence of the multivalued sign operator, the closed-loop system becomes a differential inclusion, where this inclusion is located at the boundary condition $x = 0$:

$$\begin{cases} y_{tt} - (a y_x)_x = 0, & 0 < x < \ell,\ t > 0 & \text{(4.0.2a)} \\ (a y_x)(0,t) \in y_t(0,t) + \varepsilon R y(0,t) + d(t) + D\,\mathrm{sign}(S(t)), & t > 0 & \text{(4.0.2b)} \\ (a y_x)(\ell,t) + M y_{tt}(\ell,t) = 0, & t > 0 & \text{(4.0.2c)} \\ y(x,0) = y_0(x), \quad y_t(x,0) = y_1(x), & 0 < x < \ell & \text{(4.0.2d)} \end{cases}$$

where the function $S(t)$ is defined as

$$S(t) := \left(1 + \frac{\varepsilon\phi(0)}{a(0)}\right)(y_t(0,t) + \varepsilon R y(0,t)) + \varepsilon\phi(0) y_x(0,t). \tag{4.0.3}$$

This closed-loop system presents two main challenges for mathematical analysis. First, the unknown disturbance $d(t)$ persists directly in the boundary condition. Second, the control law has a non-linear nature due to the multivalued sign operator. To prove the well-posedness of this system, we adopt an approach based on the theory of nonlinear semigroups and maximal monotone operators.

First, we perform a change of variables to shift the disturbance $d(t)$ from the boundary condition into the PDE. After that, we formulate the transformed system as an abstract

differential inclusion of the form:

$$\frac{d}{dt}X(t) + \mathcal{A}X(t) \ni F(t),$$

defined on an appropriate Hilbert space H. Consequently, the problem of well-posedness reduces to proving the monotonicity and maximality of the operator $\mathcal{A}$.

4.1 Change of Variables

To study the well-posedness of the closed-loop system using classical semigroup theory, the associated operator must be independent of time. In our system, the presence of the disturbance $d(t)$ at the boundary $x = 0$ introduces a time dependence in the domain of the operator. To avoid the technical complexities of working with maximal monotone operators defined on time-dependent domains, we introduce a change of variables. The goal of this transformation is to move the disturbance $d(t)$ from the boundary condition into the PDE.

We define the new state variable $w(x,t)$ as follows:

$$w(x,t) = y(x,t) - z(x,t),$$

where we consider a translation variable $z(x,t)$ given by the combination of two pairs of separable variables

$$z(x,t) = \varphi^1(x)\psi^1(t) + \varphi^2(x)\psi^2(t). \tag{4.1.1}$$

The spatial functions $\varphi^i(x)$ and the time-dependent functions $\psi^i(t)$ must be chosen so that the new state $w(x,t)$ satisfies the following target system:

$$\begin{cases} w_{tt} - (aw_x)_x = f(x,t), & 0 < x < \ell,\ t > 0 & \text{(4.1.2a)} \\ (aw_x)(0,t) \in w_t(0,t) + \varepsilon R w(0,t) + D\,\mathrm{sign}(S_w(t)), & t > 0 & \text{(4.1.2b)} \\ (aw_x)(\ell,t) + M w_{tt}(\ell,t) = 0, & t > 0 & \text{(4.1.2c)} \\ w(x,0) = w_0(x), \quad w_t(x,0) = w_1(x), & 0 < x < \ell & \text{(4.1.2d)} \end{cases}$$

where the source term $f(x,t)$ and the new initial conditions $w_0(x), w_1(x)$ are given by:

$$f(x,t) = -z_{tt} + (az_x)_x, \tag{4.1.3}$$

$$w_0(x) = y_0(x) - z(x,0), \tag{4.1.4}$$

$$w_1(x) = y_1(x) - z_t(x,0), \tag{4.1.5}$$

and $S_w(t)$ is formulated exactly as $S(t)$ in (4.0.3), but using the new variable $w(x,t)$ instead of $y(x,t)$.

To define this change of variables explicitly, we substitute $y = w + z$ into the original closed-loop system (4.0.2) and from this formulation, we determine the conditions for $\varphi^i(x)$ and $\psi^i(t)$ that eliminate the residual terms at the boundaries.

First, we analyze the dynamic boundary condition at $x = \ell$:

$$\begin{aligned}
0 &= (ay_x)(\ell,t) + My_{tt}(\ell,t) \\
&= (aw_x)(\ell,t) + (az_x)(\ell,t) + Mw_{tt}(\ell,t) + Mz_{tt}(\ell,t) \\
&= (aw_x)(\ell,t) + \left\{a(\ell)\varphi_x^1(\ell)\psi^1(t) + a(\ell)\varphi_x^2(\ell)\psi^2(t)\right\} + Mw_{tt}(\ell,t) + \left\{M\varphi^1(\ell)\psi_{tt}^1(t) + M\varphi^2(\ell)\psi_{tt}^2(t)\right\}.
\end{aligned}$$

If we require the spatial functions to satisfy

$$\varphi^1(\ell) = \varphi_x^1(\ell) = \varphi^2(\ell) = \varphi_x^2(\ell) = 0, \tag{4.1.6}$$

the extra terms vanish, and we successfully recover the desired boundary condition for $w(x,t)$:

$$(aw_x)(\ell,t) + Mw_{tt}(\ell,t) = 0.$$

Next, we substitute $y = w + z$ into the control boundary condition at $x = 0$, which yields:

$$\begin{aligned}
&(aw_x)(0,t) - w_t(0,t) - \varepsilon Rw(0,t) + \{(az_x)(0,t) - z_t(0,t) - \varepsilon Rz(0,t) - d(t)\} \\
&\in D\,\mathrm{sign}\Big(S_w(t) + S_z(t)\Big).
\end{aligned}$$

To recover our target homogeneous system, we must simultaneously vanish the residual terms on the left side of the inclusion, and the extra terms inside the argument of the multivalued sign operator. Let us denote the z-dependent terms on the left side as $L_z(t)$. Our problem reduces to finding $z(x,t)$ such that

$$\begin{cases}
L_z(t) := (az_x)(0,t) - z_t(0,t) - \varepsilon Rz(0,t) = d(t), & \text{(4.1.7a)} \\
S_z(t) := \left(1 + \dfrac{\varepsilon\phi(0)}{a(0)}\right)\Big(z_t(0,t) + \varepsilon Rz(0,t)\Big) + \varepsilon\phi(0)z_x(0,t) = 0. & \text{(4.1.7b)}
\end{cases}$$

Remark 4.1.1. In [16, 17], separable variables $z(x,t) = \varphi(x)\psi(t)$ are preferred to construct the translation variable because of the simplicity of the method, as it avoids the need to solve an auxiliary partial differential equation. Typically, a single pair of separable variables is sufficient to eliminate boundary disturbances. However, in our framework, the boundary

condition at $x = 0$ generates two simultaneous algebraic constraints: one to cancel the residual terms on the left-hand side of the inclusion and absorb $d(t)$ (4.1.7a), and another to eliminate the residual terms $S_z(t)$ inside the argument of the multivalued sign operator (4.1.7b). Since a single pair $\varphi(x)\psi(t)$ does not provide enough degrees of freedom to satisfy both constraints at $x = 0$ simultaneously, we introduce two pairs of separable variables.

By adding and subtracting the term $(a(0)+\varepsilon\phi(0))z_x(0,t)$ in equation (4.1.7b), we can rewrite the system in the following equivalent form:

$$\begin{cases} L_z(t) = d(t), & \text{(4.1.8a)} \\ -\left(1 + \dfrac{\varepsilon\phi(0)}{a(0)}\right) L_z(t) + \big(a(0) + 2\varepsilon\phi(0)\big) z_x(0,t) = 0. & \text{(4.1.8b)} \end{cases}$$

Substituting $L_z(t) = d(t)$ into the second equation implies that $z_x(0,t)$ must satisfy

$$-\left(1 + \frac{\varepsilon\phi(0)}{a(0)}\right) d(t) + \big(a(0) + 2\varepsilon\phi(0)\big) z_x(0,t) = 0. \tag{4.1.9}$$

Recalling that $z_x(0,t) = \varphi_x^1(0)\psi^1(t)+\varphi_x^2(0)\psi^2(t)$, we can simplify equation (4.1.9) by choosing

$$\varphi_x^1(0) = 0 \quad \text{and} \quad \varphi_x^2(0) = 1. \tag{4.1.10}$$

With this strategic choice, the term involving $\psi^1(t)$ vanishes from $z_x(0,t)$, and equation (4.1.9) simplifies directly to

$$-\left(1 + \frac{\varepsilon\phi(0)}{a(0)}\right) d(t) + \big(a(0) + 2\varepsilon\phi(0)\big) \psi^2(t) = 0. \tag{4.1.11}$$

Assuming that $\varepsilon < \varepsilon_3 := -\frac{a(0)}{2\phi(0)}$, we can explicitly define the second time function as

$$\psi^2(t) = \frac{a(0) + \varepsilon\phi(0)}{a(0)\big(a(0) + 2\varepsilon\phi(0)\big)} d(t). \tag{4.1.12}$$

Finally, we ensure the satisfaction of the condition $L_z(t) = d(t)$. Expanding $L_z(t)$ using the definition of $z(x,t)$ yields:

$$\begin{aligned} L_z(t) = {} & a(0)\varphi_x^1(0)\psi^1(t) - \varphi^1(0)\psi_t^1(t) - \varepsilon R\varphi^1(0)\psi^1(t) \\ & + a(0)\varphi_x^2(0)\psi^2(t) - \varphi^2(0)\psi_t^2(t) - \varepsilon R\varphi^2(0)\psi^2(t). \end{aligned}$$

Using the previously assigned values $\varphi_x^1(0) = 0$, $\varphi_x^2(0) = 1$, and setting:

$$\varphi^1(0) = 1, \quad \varphi^2(0) = 0, \tag{4.1.13}$$

the expression for $L_z(t)$ reduces to

$$L_z(t) = -\psi_t^1(t) - \varepsilon R\psi^1(t) + a(0)\psi^2(t).$$

Substituting the definition of $\psi^2(t)$ from (4.1.12) into $L_z(t) = d(t)$, we obtain:

$$\psi_t^1(t) + \varepsilon R\psi^1(t) = -\left(1 - \frac{a(0) + \varepsilon\phi(0)}{a(0) + 2\varepsilon\phi(0)}\right) d(t) = -\left(\frac{\varepsilon\phi(0)}{a(0) + 2\varepsilon\phi(0)}\right) d(t). \tag{4.1.14}$$

As we did for $\psi^2(t)$, we can obtain $\psi^1(t)$ explicitly. Indeed, thanks to the regularity of $d(t)$ stated in (A1.1) or (A1.2), there exists a unique global solution $\psi^1(t)$ for this ODE satisfying the initial condition $\psi^1(0) = 0$:

$$\psi^1(t) = -\left(\frac{\varepsilon\phi(0)}{a(0) + 2\varepsilon\phi(0)}\right) \int_0^t e^{\varepsilon R(s-t)} d(s)\, ds. \tag{4.1.15}$$

With $\psi^1(t)$ explicitly determined, we complete the change of variables by constructing the spatial functions $\varphi^1(x)$ and $\varphi^2(x)$ as cubic polynomials satisfying the interpolation conditions (4.1.6), (4.1.10), and (4.1.13). These functions are defined by

$$\varphi^1(x) = \left(1 - \frac{x}{\ell}\right)^2 \left(1 + \frac{2x}{\ell}\right), \tag{4.1.16}$$

$$\varphi^2(x) = x\left(1 - \frac{x}{\ell}\right)^2. \tag{4.1.17}$$

Assuming the conditions $d(0) = 0$ and $d_t(0) = 0$ stated in (A1.1) and (A1.2), together with $\psi^1(0) = 0$, ensures that $z(x, 0) = 0$ and $z_t(x, 0) = 0$. This implies that the initial conditions of the transformed system remain identical to the original ones, that is, $w_0(x) = y_0(x)$ and $w_1(x) = y_1(x)$.

Finally, since the change of variables is invertible, establishing the well-posedness of the transformed system directly implies the well-posedness of the original system. Therefore, in the next sections, we will prove the existence and uniqueness of solutions for the transformed system (4.1.2).

4.2 Abstract Formulation and Maximal Monotonicity

In this section, we apply the theory of maximal monotone operators and non-linear semigroups to prove the well-posedness of the transformed system (4.1.2), which is formulated as an abstract differential inclusion.

4.2.1 Hilbert Space and Operator Definition

We first define the state vector $X = (w, z, \xi, \eta)$, where we identify the auxiliary variables as

$$z = w_t, \quad \xi = w(\ell), \quad \text{and} \quad \eta = w_t(\ell).$$

Next, we define the energy Hilbert space H as

$$H := \left\{(w, z, \xi, \eta) \in H^1(0,\ell) \times L^2(0,\ell) \times \mathbb{R} \times \mathbb{R} : w(\ell) = \xi\right\}, \tag{4.2.1}$$

endowed with the energy inner product defined by

$$\Big\langle (w, z, \xi, \eta), (\widetilde{w}, \widetilde{z}, \widetilde{\xi}, \widetilde{\eta}) \Big\rangle_E := \int_0^\ell \big(a w_x \widetilde{w}_x + z\widetilde{z}\big)dx + \alpha w(0)\widetilde{w}(0) + M\eta\widetilde{\eta}. \tag{4.2.2}$$

Remark 4.2.1. The energy norm $\|X\|_E := \sqrt{\langle X, X\rangle_E}$ induced by (4.2.2) is equivalent to the standard norm of $H^1(0,\ell) \times L^2(0,\ell) \times \mathbb{R}^2$. Consequently, $(H, \langle\cdot,\cdot\rangle_E)$ is a Hilbert space.

Remark 4.2.2. For a state vector $X(t) = (w, w_t, w(\ell), w_t(\ell))$, we have $\|X(t)\|_E^2 = 2E(t)$, where $E(t)$ denotes the energy of the system, given by (3.1.1).

We now define the non-linear multivalued operator $\mathcal{A} : D(\mathcal{A}) \subset H \to H$ as

$$\mathcal{A}(w, z, \xi, \eta) := \left(-z,\ -(aw_x)_x,\ -\eta,\ \frac{1}{M}(aw_x)(\ell)\right), \tag{4.2.3}$$

with domain $D(\mathcal{A})$ given by:

$$D(\mathcal{A}) := \left\{ (w, z, \xi, \eta) \in \big(H^2(0,\ell) \times H^1(0,\ell) \times \mathbb{R}^2\big) : \begin{array}{l} w(\ell) = \xi, \quad z(\ell) = \eta, \\ (aw_x)(0) \in z(0) + \varepsilon R w(0) + D\,\mathrm{sign}(S) \end{array} \right\}, \tag{4.2.4}$$

where the argument S inside the multivalued sign operator is defined as

$$S := \left(1 + \frac{\varepsilon\phi(0)}{a(0)}\right)\big(z(0) + \varepsilon R w(0)\big) + \varepsilon\phi(0) w_x(0). \tag{4.2.5}$$

Remark 4.2.3. Without the change of variables in Section 4.1, the perturbation $d(t)$ would appear explicitly in the boundary condition. This would make the domain time-dependent, $D(\mathcal{A}(t))$, which would prevent us from using the standard theory of semigroups.

Using this framework, the transformed system (4.1.2) can be compactly written as the abstract Cauchy problem for a differential inclusion:

$$\begin{cases} \dfrac{d}{dt}X(t) + \mathcal{A}X(t) \ \ni\ F(t), \quad t > 0, \\ X(0) = X_0, \end{cases} \tag{4.2.6}$$

where $X_0 = (w_0, z_0, \xi_0, \eta_0)$ is the initial state and $F(t) = \big(0,\ f(\cdot,t),\ 0,\ 0\big)$.

4.2.2 Modified Inner Product

The energy inner product $\langle \cdot, \cdot \rangle_E$ defined in (4.2.2) is not convenient for proving the monotonicity of $\mathcal{A}$, because it does not naturally yield the boundary terms involving the sign operator S. To overcome this difficulty, we introduce an equivalent inner product $\langle \cdot, \cdot \rangle_V$ associated with the Lyapunov functional $V(t)$ given by (3.3.1) and (3.3.2).

Let $X = (w, z, \xi, \eta) \in H$. We define the inner product $\langle \cdot, \cdot \rangle_V$ as

$$\langle X, \widetilde{X} \rangle_V := \langle X, \widetilde{X} \rangle_E + N(X, \widetilde{X}), \tag{4.2.7}$$

where $N(X, \widetilde{X})$ is a symmetric bilinear form defined through its associated quadratic form $N(X, X)$ given by

$$N(X, X) = 2\varepsilon \left\{ 2 \int_0^\ell \phi z w_x \, dx + Rw(0) \int_0^\ell z \, dx + Rw(0) M \eta \right\}. \tag{4.2.8}$$

Remark 4.2.4. For a state vector $X(t) = (w, w_t, w(\ell), w_t(\ell))$, the norm induced by $\langle \cdot, \cdot \rangle_V$ directly relates to the modified Lyapunov energy function through $\|X(t)\|_V^2 = 2V(t)$.

To express $N(X, \widetilde{X})$ in an explicit symmetric bilinear form, we complete squares

$$\begin{aligned} N(X, X) &= 4\varepsilon \int_0^\ell \phi z w_x \, dx \pm 2\varepsilon \int_0^\ell (\phi^2 z^2 + w_x^2) \, dx \\ &\quad + 2\varepsilon \int_0^\ell Rw(0) z \, dx \pm \varepsilon \int_0^\ell \left(R^2 w(0)^2 + z^2 \right) dx \\ &\quad + 2\varepsilon M R w(0) \eta \pm \varepsilon M \left(R^2 w(0)^2 + \eta^2 \right) \\ &= 2\varepsilon \int_0^\ell (\phi z + w_x)^2 \, dx - 2\varepsilon \int_0^\ell \phi^2 z^2 \, dx - 2\varepsilon \int_0^\ell w_x^2 \, dx \\ &\quad + \varepsilon \int_0^\ell (Rw(0) + z)^2 \, dx - \varepsilon \ell R^2 w(0)^2 - \varepsilon \int_0^\ell z^2 \, dx \\ &\quad + \varepsilon M (Rw(0) + \eta)^2 - \varepsilon M R^2 w(0)^2 - \varepsilon M \eta^2. \end{aligned}$$

Deriving the symmetric bilinear form associated with this quadratic form yields $N(X, \widetilde{X})$:

$$\begin{aligned} N(X, \widetilde{X}) &= 2\varepsilon \int_0^\ell (\phi z + w_x)(\phi \widetilde{z} + \widetilde{w}_x) \, dx - 2\varepsilon \int_0^\ell \phi^2 z \widetilde{z} \, dx - 2\varepsilon \int_0^\ell w_x \widetilde{w}_x \, dx \\ &\quad + \varepsilon \int_0^\ell (Rw(0) + z)(R\widetilde{w}(0) + \widetilde{z}) \, dx - \varepsilon \ell R^2 w(0) \widetilde{w}(0) - \varepsilon \int_0^\ell z \widetilde{z} \, dx \\ &\quad + \varepsilon M (Rw(0) + \eta)(R\widetilde{w}(0) + \widetilde{\eta}) - \varepsilon M R^2 w(0) \widetilde{w}(0) - \varepsilon M \eta \widetilde{\eta}. \end{aligned} \tag{4.2.9}$$

Adding $N(X, \widetilde{X})$ to the standard energy inner product $\langle X, \widetilde{X} \rangle_E$, we obtain the full expression for $\langle X, \widetilde{X} \rangle_V$:

$$\begin{aligned}\langle X, \widetilde{X} \rangle_V = &\int_0^\ell (a(x) - 2\varepsilon) w_x \widetilde{w}_x \, dx + \int_0^\ell (1 - 2\varepsilon\phi(x)^2 - \varepsilon) z\widetilde{z} \, dx \\ &+ (\alpha - \varepsilon\ell R^2 - \varepsilon M R^2) w(0)\widetilde{w}(0) + M(1-\varepsilon)\eta\widetilde{\eta} \\ &+ 2\varepsilon \int_0^\ell (\phi z + w_x)(\phi\widetilde{z} + \widetilde{w}_x) \, dx + \varepsilon \int_0^\ell (Rw(0) + z)(R\widetilde{w}(0) + \widetilde{z}) \, dx \\ &+ \varepsilon M (Rw(0) + \eta)(R\widetilde{w}(0) + \widetilde{\eta}). \end{aligned} \tag{4.2.10}$$

To guarantee that $\langle \cdot, \cdot \rangle_V$ defines a valid, coercive inner product on H, we must ensure that all coefficient weights in (4.2.10) remain strictly positive:

1. We require $a(x) - 2\varepsilon > 0$ for all $x \in [0, \ell]$. Defining $\varepsilon_4 := \frac{a_0}{2}$, for all $\varepsilon \in (0, \varepsilon_4)$, we have:

$$0 < a_0 - 2\varepsilon < a(x) - 2\varepsilon, \quad \forall x \in [0, \ell].$$

2. We require $1 - 2\varepsilon\phi(x)^2 - \varepsilon > 0$ for all $x \in [0, \ell]$. Since $\phi(x)$ is a negative and strictly increasing function with $\phi(0) < \phi(x) < \phi(\ell) < 0$, it follows that $\phi(0)^2 > \phi^2(x) > \phi(\ell)^2 > 0$. Thus,

$$\frac{1}{1 + 2\phi^2(x)} > \frac{1}{1 + 2\phi(0)^2}, \quad \forall x \in (0, \ell).$$

   Therefore, defining $\varepsilon_5 := (1 + 2\phi(0)^2)^{-1}$ and choosing $\varepsilon \in (0, \varepsilon_5)$, it guarantees that $1 - 2\varepsilon\phi^2(x) - \varepsilon > 0$ for all $x \in [0, \ell]$.

3. We require $\alpha - \varepsilon\ell R^2 - \varepsilon M R^2 > 0$. Recalling the definition of α in (3.4.1), this condition can be rewritten as

$$\varepsilon R \left\{ 2 \left(1 + \frac{\varepsilon\phi(0)}{a(0)} \right) - R(\ell + M) \right\} > 0.$$

   To ensure that the coefficient is strictly positive, we must guarantee the positivity of the expression inside the curly braces. This is equivalent to requiring that

$$\varepsilon < \frac{-a(0)}{2\phi(0)} \big(2 - R(\ell + M) \big).$$

   By taking $0 < R(\ell + M) < 1$, the required condition is satisfied because

$$\frac{-a(0)}{2\phi(0)} \big(2 - R(\ell + M) \big) > \frac{-a(0)}{2\phi(0)} = \varepsilon_3 > \varepsilon,$$

4. We require $M(1-\varepsilon) > 0$, which is trivially satisfied by defining $\varepsilon_6 := 1$ and choosing $\varepsilon < \varepsilon_6$.

Having established the positivity of all necessary coefficients under the conditions on R and ε, we are now ready to state and prove the equivalence between the modified inner product $\langle \cdot, \cdot \rangle_V$ and the energy inner product.

Lemma 4.2.5 (Equivalence of Norms). *Let* $0 < R < (\ell + M)^{-1}$ *and* $0 < \varepsilon < \min\{\varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6\}$. *Then, there exist positive constants* $c, C > 0$ *such that*

$$c\|X\|_E^2 \leq \|X\|_V^2 \leq C\|X\|_E^2, \quad \forall X \in H. \tag{4.2.11}$$

Consequently, $\langle \cdot, \cdot \rangle_V$ *induces an equivalent norm on* H, *and* $(H, \langle \cdot, \cdot \rangle_V)$ *is a Hilbert space.*

Proof. The lower bound $\|X\|_V^2 \geq c\|X\|_E^2$ follows directly by dropping the non-negative completed square terms and using the strict positivity of the coefficients, guaranteed by the choices of R and ε. Conversely, the upper bound $\|X\|_V^2 \leq C\|X\|_E^2$ is obtained by applying Young's inequalities to control the cross-terms. □

4.2.3 Monotonicity of the Operator

We now establish that the operator $\mathcal{A}$ defined in (4.2.3) and (4.2.4) is monotone. To achieve this, we utilize the modified inner product introduced in (4.2.10).

Proposition 4.2.6 (Monotonicity of $\mathcal{A}$). *Assume that* $0 < R < (\ell + M)^{-1}$, $\alpha > 0$ *defined by* (3.4.1) *and* $0 < \varepsilon < \min\{\varepsilon_1, \varepsilon_3, \ldots, \varepsilon_7\}$, *where* $\varepsilon_7 > 0$ *is defined by*

$$\varepsilon_7 := \left(-\phi(0) + \frac{R(\ell + M)}{4} - \frac{\phi(0)}{a(0)} \right)^{-1}. \tag{4.2.12}$$

Then, the operator $\mathcal{A}$ *is monotone in* H, *that is,*

$$\langle \mathcal{A}X - \mathcal{A}\widetilde{X}, X - \widetilde{X} \rangle_V \geq 0, \quad \forall\, X, \widetilde{X} \in D(\mathcal{A}), \tag{4.2.13}$$

or more precisely,

$$\langle Y - \widetilde{Y}, X - \widetilde{X} \rangle_V \geq 0, \quad \forall\, Y \in \mathcal{A}X \text{ and } \widetilde{Y} \in \mathcal{A}\widetilde{X}.$$

Proof. In order to easy the notation, we use (4.2.13) instead of the other one.

Since the operator $\mathcal{A}$ is nonlinear only at the boundary condition $x = 0$ of its domain, we can first compute $\langle \mathcal{A}X, X \rangle_V$ for a single state $X \in D(\mathcal{A})$, as if $\mathcal{A}$ were linear, before

evaluating the multivalued boundary condition at $x = 0$. This makes the exposition clearer and simplifies the notation.

Recalling that

$$\mathcal{A}X = \left(-z, -(aw_x)_x, -\eta, \frac{1}{M}(aw_x)(\ell)\right),$$

and using the definition of the modified inner product $\langle X, \widetilde{X}\rangle_V := \langle X, \widetilde{X}\rangle_E + N(X, \widetilde{X})$ given in (4.2.7) (where N is defined by (4.2.9)), we evaluate $\langle \mathcal{A}X, X\rangle_V$ by decomposing the expression into four distinct group terms:

$$\langle \mathcal{A}X, X\rangle_V = \langle \mathcal{A}X, X\rangle_E + N(\mathcal{A}X, X) = I_1 + (I_2 + I_3 + I_4),$$

Step 1: Computation of the terms I_1

Applying integration by parts to the second term and utilizing the domain boundary identity $z(\ell) = \eta$, we obtain:

$$\begin{aligned} I_1 &= -\int_0^\ell a z_x w_x \, dx - \int_0^\ell (aw_x)_x z \, dx - \alpha z(0)w(0) + M\left(\frac{1}{M}(aw_x)(\ell)\right)\eta \\ &= -\int_0^\ell a z_x w_x \, dx + \int_0^\ell a w_x z_x \, dx - a(\ell)w_x(\ell)z(\ell) + a(0)w_x(0)z(0) \\ &\quad - \alpha z(0)w(0) + a(\ell)w_x(\ell)\eta \\ &= a(0)w_x(0)z(0) - \alpha z(0)w(0). \end{aligned} \tag{4.2.14}$$

Step 2: Computation of the terms I_2

Expanding the integrands and simplifying common terms:

$$\begin{aligned} I_2 &= -2\varepsilon \int_0^\ell \big(\phi(aw_x)_x + z_x\big)(\phi z + w_x)\, dx + 2\varepsilon \int_0^\ell \phi^2 (aw_x)_x z \, dx + 2\varepsilon \int_0^\ell z_x w_x \, dx \\ &= -2\varepsilon \int_0^\ell \phi^2 (aw_x)_x z \, dx - 2\varepsilon \int_0^\ell \phi (aw_x)_x w_x \, dx - 2\varepsilon \int_0^\ell \phi z z_x \, dx - 2\varepsilon \int_0^\ell z_x w_x \, dx \\ &\quad + 2\varepsilon \int_0^\ell \phi^2 (aw_x)_x z \, dx + 2\varepsilon \int_0^\ell z_x w_x \, dx \\ &= -2\varepsilon \int_0^\ell \phi (aw_x)_x w_x \, dx - 2\varepsilon \int_0^\ell \phi z z_x \, dx. \end{aligned}$$

Re-interpreting the integrands as exact derivatives, we have:

$$\phi(aw_x)_x w_x = \left(\frac{\phi}{a}\right)(aw_x)(aw_x)_x = \frac{1}{2}\left(\frac{\phi}{a}\right)\frac{d}{dx}\left((aw_x)^2\right),$$

$$\phi z z_x = \frac{1}{2}\phi\frac{d}{dx}(z^2).$$

Integrating by parts yields:

$$\begin{aligned} I_2 &= -\varepsilon\int_0^\ell \left(\frac{\phi}{a}\right)\frac{d}{dx}\left((aw_x)^2\right)\,dx - \varepsilon\int_0^\ell \phi\frac{d}{dx}(z^2)\,dx \\ &= \varepsilon\int_0^\ell \left(\frac{\phi}{a}\right)_x (aw_x)^2\,dx - \frac{\varepsilon\phi(\ell)}{a(\ell)}(aw_x)^2(\ell) + \frac{\varepsilon\phi(0)}{a(0)}(aw_x)^2(0) \\ &\quad + \varepsilon\int_0^\ell \phi_x z^2\,dx - \varepsilon\phi(\ell)z(\ell)^2 + \varepsilon\phi(0)z(0)^2. \end{aligned} \tag{4.2.15}$$

Step 3: Computation of the terms I_3

Expanding the algebraic product, solving the integrals and simplifying common terms:

$$\begin{aligned} I_3 &= -\varepsilon\int_0^\ell \big(Rz(0) + (aw_x)_x\big)(Rw(0)+z)\,dx + \varepsilon R^2\ell z(0)w(0) + \varepsilon\int_0^\ell (aw_x)_x z\,dx \\ &= -\varepsilon R^2\ell z(0)w(0) - \varepsilon Rz(0)\int_0^\ell z\,dx - \varepsilon Rw(0)a(\ell)w_x(\ell) + \varepsilon Rw(0)a(0)w_x(0) \\ &\quad - \varepsilon\int_0^\ell (aw_x)_x z\,dx + \varepsilon R^2\ell z(0)w(0) + \varepsilon\int_0^\ell (aw_x)_x z\,dx \\ &= -\varepsilon Rz(0)\int_0^\ell z\,dx - \varepsilon Rw(0)a(\ell)w_x(\ell) + \varepsilon Rw(0)a(0)w_x(0). \end{aligned} \tag{4.2.16}$$

Step 4: Computation of the terms I_4

Expanding the algebraic product and simplifying common terms:

$$\begin{aligned} I_4 &= \varepsilon M\left(-Rz(0) + \frac{1}{M}(aw_x)(\ell)\right)(Rw(0)+\eta) + \varepsilon MR^2 z(0)w(0) - \varepsilon M\frac{1}{M}(aw_x)(\ell)\eta \\ &= -\varepsilon MR^2 z(0)w(0) - \varepsilon MRz(0)\eta + \varepsilon Ra(\ell)w_x(\ell)w(0) + \varepsilon a(\ell)w_x(\ell)\eta \\ &\quad + \varepsilon MR^2 z(0)w(0) - \varepsilon a(\ell)w_x(\ell)\eta \\ &= \varepsilon Ra(\ell)w_x(\ell)w(0) - \varepsilon MRz(0)\eta. \end{aligned} \tag{4.2.17}$$

Step 5: Combination and factorization

Summing $I_1+I_2+I_3+I_4$, the boundary terms containing $a(\ell)w_x(\ell)w(0)$ cancel out. Factoring $(aw_x)(0)$ from the remaining terms, we obtain:

$$\begin{aligned}\langle \mathcal{A}X, X\rangle_V = &-\alpha w(0)z(0) + \varepsilon \int_0^\ell \left(\frac{\phi}{a}\right)_x (aw_x)^2\,dx - \frac{\varepsilon\phi(\ell)}{a(\ell)}(aw_x)^2(\ell)\\ &+ \varepsilon\int_0^\ell \phi_x z^2\,dx - \varepsilon\phi(\ell)z(\ell)^2 + \varepsilon\phi(0)z(0)^2\\ &- \varepsilon Rz(0)\int_0^\ell z\,dx - \varepsilon MRz(0)\eta\\ &+ (aw_x)(0)\left\{z(0) + \varepsilon Rw(0) + \varepsilon\phi(0)w_x(0)\right\}. \end{aligned} \tag{4.2.18}$$

Step 6: Extension to state differences and the multivalued boundary condition
We now drop the abuse of notation and explicitly analyze the monotonicity expression $\langle \mathcal{A}X - \mathcal{A}\widetilde{X}, X - \widetilde{X}\rangle_V$. From the nonlinear condition on the domain $D(\mathcal{A})$, we have:

$$a(0)(w-\widetilde{w})_x(0) \in (z-\widetilde{z})(0) + \varepsilon R(w-\widetilde{w})(0) + D\,\mathrm{sign}(S) - D\,\mathrm{sign}(\widetilde{S}), \tag{4.2.19}$$

where S is given by (4.2.5). Let us recall that we are using an alternative notation for multivalued operators.

Substituting inclusion (4.2.19) into the factored boundary term of $\langle \mathcal{A}(X-\widetilde{X}), X-\widetilde{X}\rangle_V$ (4.2.18) yields:

$$\begin{aligned}&a(0)(w-\widetilde{w})_x(0)\left\{(z-\widetilde{z})(0) + \varepsilon R(w-\widetilde{w})(0) + \varepsilon\phi(0)(w-\widetilde{w})_x(0)\right\}\\ &= \left\{(z-\widetilde{z})(0) + \varepsilon R(w-\widetilde{w})(0) + D\,\mathrm{sign}(S) - D\,\mathrm{sign}(\widetilde{S})\right\}\left\{(z-\widetilde{z})(0) + \varepsilon R(w-\widetilde{w})(0) + \varepsilon\phi(0)(w-\widetilde{w})_x(0)\right\}\\ &= \left\{(z-\widetilde{z})(0) + \varepsilon R(w-\widetilde{w})(0)\right\}^2 + \frac{\varepsilon\phi(0)}{a(0)}a(0)(w-\widetilde{w})_x(0)\left\{(z-\widetilde{z})(0) + \varepsilon R(w-\widetilde{w})(0)\right\}\\ &\quad + \left(D\,\mathrm{sign}(S) - D\,\mathrm{sign}(\widetilde{S})\right)\left\{(z-\widetilde{z})(0) + \varepsilon R(w-\widetilde{w})(0) + \varepsilon\phi(0)(w-\widetilde{w})_x(0)\right\}.\end{aligned}$$

Re-substituting (4.2.19) into the second term, we find:

$$\begin{aligned}&a(0)(w-\widetilde{w})_x(0)\left\{(z-\widetilde{z})(0) + \varepsilon R(w-\widetilde{w})(0) + \varepsilon\phi(0)(w-\widetilde{w})_x(0)\right\}\\ &= \left(1 + \frac{\varepsilon\phi(0)}{a(0)}\right)\left\{(z-\widetilde{z})(0) + \varepsilon R(w-\widetilde{w})(0)\right\}^2\\ &\quad + D\big(\mathrm{sign}(S) - \mathrm{sign}(\widetilde{S})\big)(S-\widetilde{S}).\end{aligned}$$

Since the multivalued sign operator $\mathrm{sign}(\cdot)$ is monotone by Proposition 2.5.2, we have:

$$D\big(\mathrm{sign}(S) - \mathrm{sign}(\widetilde{S})\big)(S-\widetilde{S}) \geq 0.$$

Substituting back into the expression, we arrive at the following lower bound for (4.2.18):

$$\begin{aligned}\langle \mathcal{A}X - \mathcal{A}\widetilde{X}, X - \widetilde{X}\rangle_V \geq & -\alpha(w-\widetilde{w})(0)(z-\widetilde{z})(0) \\ & + \varepsilon \int_0^\ell \left(\frac{\phi}{a}\right)_x (a(w-\widetilde{w})_x)^2\,dx - \varepsilon\phi(\ell)a(\ell)((w-\widetilde{w})_x(\ell))^2 \\ & + \varepsilon \int_0^\ell \phi_x(z-\widetilde{z})^2\,dx - \varepsilon\phi(\ell)(\eta-\widetilde{\eta})^2 + \varepsilon\phi(0)((z-\widetilde{z})(0))^2 \\ & - \varepsilon R(z-\widetilde{z})(0)\int_0^\ell (z-\widetilde{z})\,dx - \varepsilon MR(z-\widetilde{z})(0)(\eta-\widetilde{\eta}) \\ & + \left(1+\frac{\varepsilon\phi(0)}{a(0)}\right)\left\{(z-\widetilde{z})(0) + \varepsilon R(w-\widetilde{w})(0)\right\}^2.\end{aligned}$$

Step 7: Bounding the monotonicity expression
Substituting $\alpha = 2\varepsilon R\left(1+\frac{\varepsilon\phi(0)}{a(0)}\right)$ and utilizing the properties for ϕ given by (3.2.3), we lower bound the monotonicity expression as

$$\begin{aligned}\langle \mathcal{A}X - \mathcal{A}\widetilde{X}, X - \widetilde{X}\rangle_V \geq & -2\varepsilon R\left(1+\frac{\varepsilon\phi(0)}{a(0)}\right)(w-\widetilde{w})(0)(z-\widetilde{z})(0) \\ & + \varepsilon \int_0^\ell (a(w-\widetilde{w})_x)^2\,dx + \varepsilon MRa(\ell)((w-\widetilde{w})_x(\ell))^2 \\ & + \varepsilon R\int_0^\ell (z-\widetilde{z})^2\,dx + \varepsilon MR(\eta-\widetilde{\eta})^2 + \varepsilon\phi(0)((z-\widetilde{z})(0))^2 \\ & - \varepsilon R(z-\widetilde{z})(0)\int_0^\ell (z-\widetilde{z})\,dx - \varepsilon MR(z-\widetilde{z})(0)(\eta-\widetilde{\eta}) \\ & + \left(1+\frac{\varepsilon\phi(0)}{a(0)}\right)\left\{(z-\widetilde{z})(0) + \varepsilon R(w-\widetilde{w})(0)\right\}^2.\end{aligned}$$

Expanding the final square term, we observe:

$$\begin{aligned}\langle \mathcal{A}X - \mathcal{A}\widetilde{X}, X - \widetilde{X}\rangle_V \geq & \ \varepsilon \int_0^\ell (a(w-\widetilde{w})_x)^2\,dx + \varepsilon MRa(\ell)((w-\widetilde{w})_x(\ell))^2 \\ & + \varepsilon R\int_0^\ell (z-\widetilde{z})^2\,dx + \varepsilon MR(\eta-\widetilde{\eta})^2 + \varepsilon\phi(0)((z-\widetilde{z})(0))^2 \\ & - \varepsilon R(z-\widetilde{z})(0)\int_0^\ell (z-\widetilde{z})\,dx - \varepsilon MR(z-\widetilde{z})(0)(\eta-\widetilde{\eta}) \\ & + \left(1+\frac{\varepsilon\phi(0)}{a(0)}\right)((z-\widetilde{z})(0))^2 + (\varepsilon R)^2\left(1+\frac{\varepsilon\phi(0)}{a(0)}\right)((w-\widetilde{w})(0))^2.\end{aligned} \tag{4.2.20}$$

Next, we apply Young's inequality $uv \leq \frac{1}{2\gamma}u^2 + \frac{\gamma}{2}v^2$ to the cross-terms, with $\gamma = 2$:

$$-\varepsilon R(z-\widetilde{z})(0)\int_0^\ell (z-\widetilde{z})\,dx \geq -\frac{\varepsilon R\ell}{4}(z-\widetilde{z})^2(0) - \varepsilon R\int_0^\ell (z-\widetilde{z})^2\,dx, \tag{4.2.21}$$

$$-\varepsilon MR(z-\widetilde{z})(0)(\eta-\widetilde{\eta}) \geq -\frac{\varepsilon MR}{4}(z-\widetilde{z})^2(0) - \varepsilon MR(\eta-\widetilde{\eta})^2. \tag{4.2.22}$$

Substituting inequalities (4.2.21) and (4.2.22) back into (4.2.20), the integral term $\varepsilon R\int_0^\ell (z-\widetilde{z})^2\,dx$ and the boundary term $\varepsilon MR(\eta-\widetilde{\eta})^2$ are completely canceled out. Grouping the remaining terms for $((z-\widetilde{z})(0))^2$, we obtain:

$$\begin{aligned}\langle \mathcal{A}X - \mathcal{A}\widetilde{X}, X-\widetilde{X}\rangle_V \geq \varepsilon \int_0^\ell (a(w-\widetilde{w})_x)^2\,dx + \varepsilon MRa(\ell)((w-\widetilde{w})_x(\ell))^2 \\ + \left\{1+\varepsilon\left(\phi(0) - \frac{R(\ell+M)}{4} + \frac{\phi(0)}{a(0)}\right)\right\}(z-\widetilde{z})^2(0).\end{aligned}$$

Since $0<\varepsilon\leq\varepsilon_7$, the bracketed coefficient is non-negative. Furthermore, all remaining terms on the right-hand side are non-negative (due to the strict positivity of ε, M, R, and $a(\ell)$, along with the squared components), which directly proves (4.2.13). This completes the proof of the proposition. □

4.2.4 Maximality of the Operator

In this section, we study the maximality of the operator associated with our closed-loop system (4.0.2).

Since $\mathcal{A}$ is monotone, its maximality is equivalent to the range condition

$$R(I+\mathcal{A}) = H,$$

by Proposition 2.3.4. Thus, it suffices to prove that, for every $Y\in H$, there exists $X\in D(\mathcal{A})$ such that $(I+\mathcal{A})X = Y$.

Proposition 4.2.7 (Maximality of $\mathcal{A}$)**.** *The operator $\mathcal{A}$ defined in (4.2.3) and (4.2.4) satisfies the range condition*

$$R(I+\mathcal{A}) = H. \tag{4.2.23}$$

Proof. Let $Y=(f_1,f_2,\mu_1,\mu_2)$ in the Hilbert space H. This means

$$\begin{cases}(f_1,f_2,\mu_1,\mu_2)\in H^1(0,\ell)\times L^2(0,\ell)\times\mathbb{R}\times\mathbb{R} & \text{(4.2.24a)}\\ f_1(\ell)=\mu_1. & \text{(4.2.24b)}\end{cases}$$

We need to prove the existence of $X = (w, z, \xi, \eta) \in D(\mathcal{A})$ such that $(I + \mathcal{A})X = Y$. This means

$$\begin{cases} w - z = f_1 & \text{(4.2.25a)} \\ z - (aw_x)_x = f_2 & \text{(4.2.25b)} \\ \xi - \eta = \mu_1 & \text{(4.2.25c)} \\ \eta + \dfrac{1}{M}(aw_x)(\ell) = \mu_2 & \text{(4.2.25d)} \end{cases}$$

and

$$\begin{cases} (w, z, \xi, \eta) \in H^2(0, \ell) \times H^1(0, \ell) \times \mathbb{R} \times \mathbb{R} & \text{(4.2.26a)} \\ w(\ell) = \xi & \text{(4.2.26b)} \\ z(\ell) = \eta & \text{(4.2.26c)} \\ (aw_x)(0) - z(0) - \varepsilon R w(0) \in D\,\mathrm{sign}(S) & \text{(4.2.26d)} \end{cases}$$

The proof is divided into four steps. First, the problem in (4.2.25) and (4.2.26) is reduced to an elliptic problem for the first component w. Second, the multivalued boundary condition is handled by introducing an auxiliary linear problem depending on a parameter $b \in [-1, 1]$. Third, exploiting the linearity of this auxiliary problem, its solution is written in a form that isolates the dependence on b. Finally, this representation is used to choose a parameter b satisfying the multivalued sign condition, thereby recovering a solution to the original problem.

Step 1: Reduction to an elliptic problem.
Adding equations (4.2.25a), (4.2.25b) and considering the regularity established in (4.2.24a), it follows that

$$w - (aw_x)_x = f_2 + f_1.$$

Combining equations (4.2.25c) and (4.2.25d) with (4.2.26b) yields:

$$w(\ell) + \frac{1}{M}(aw_x)(\ell) = \mu_1 + \mu_2.$$

Next, isolating z from (4.2.25a) and substituting it into (4.2.26d) results in

$$\begin{aligned} &(aw_x)(0) - (1 + \varepsilon R)w(0) + f_1(0) \\ &\quad \in D\,\mathrm{sign}\left(\left(1 + \frac{\varepsilon\phi(0)}{a(0)}\right)\left\{(1 + \varepsilon R)w(0) - f_1(0)\right\} + \varepsilon\phi(0)w_x(0)\right). \end{aligned}$$

Thus, our problem reduces to proving the existence of $w \in H^2(0,\ell)$ satisfying the following system:

$$\begin{cases} w - (aw_x)_x = g, & \text{(4.2.27a)} \\ (aw_x)(\ell) + Mw(\ell) = m, & \text{(4.2.27b)} \\ (aw_x)(0) - A_1 w(0) + f_1(0) \in D\,\mathrm{sign}(\widetilde{S}), & \text{(4.2.27c)} \end{cases}$$

where $g = f_2 + f_1 \in L^2(0,\ell)$, $m = M(\mu_1 + \mu_2)$, $A_1 = 1 + \varepsilon R$, $A_2 = 1 + \frac{\varepsilon\phi(0)}{a(0)}$, $A_3 = \varepsilon\phi(0)$, and

$$\widetilde{S} = A_2\,(A_1 w(0) - f_1(0)) + A_3 w_x(0). \tag{4.2.28}$$

Step 2: Introduction of the auxiliary linear problem via a parameter b.
Because the boundary inclusion (4.2.27c) involves the multivalued sign operator whose argument $\widetilde{S}$ depends on $w(0)$ and $w_x(0)$, we decouple the nonlinearity by introducing a parameter $b \in [-1,1]$.

We replace the multivalued term $D\,\mathrm{sign}(\widetilde{S})$ by Db, which leads to the auxiliary linear problem:

$$\begin{cases} w - (aw_x)_x = g, & \text{(4.2.29a)} \\ (aw_x)(\ell) + Mw(\ell) = m, & \text{(4.2.29b)} \\ (aw_x)(0) - A_1 w(0) = Db - f_1(0). & \text{(4.2.29c)} \end{cases}$$

To prove the existence and uniqueness of a solution $w_b \in H^2(0,\ell)$ for this auxiliary problem, we construct its weak formulation. Let $h \in H^1(0,\ell)$ be an arbitrary test function. Multiplying equation (4.2.29a) by h and integrating by parts over the interval $(0,\ell)$, we get:

$$\int_0^\ell wh\,dx + \int_0^\ell aw_x h_x\,dx - a(\ell)w_x(\ell)h(\ell) + a(0)w_x(0)h(0) = \int_0^\ell gh\,dx. \tag{4.2.30}$$

Using the boundary condition (4.2.29b) at $x = \ell$, which gives $a(\ell)w_x(\ell) = m - Mw(\ell)$, and the boundary condition (4.2.29c) at $x = 0$, which gives $a(0)w_x(0) = A_1 w(0) + Db - f_1(0)$, we substitute these expressions into equation (4.2.30):

$$\int_0^\ell wh\,dx + \int_0^\ell aw_x h_x\,dx - (m - Mw(\ell))h(\ell) + (A_1 w(0) + Db - f_1(0))h(0) = \int_0^\ell gh\,dx.$$

Rearranging the terms by moving the known data to the right-hand side, we arrive at the equation:

$$\mathfrak{a}(w,h) = \mathfrak{L}(h), \quad \forall h \in H^1(0,\ell), \tag{4.2.31}$$

where the bilinear form $\mathfrak{a}(\cdot,\cdot)$ and the linear functional $\mathfrak{L}(\cdot)$ are defined respectively by

$$\mathfrak{a}(w,h) = \int_0^\ell wh\,dx + \int_0^\ell aw_x h_x\,dx + Mw(\ell)h(\ell) + A_1 w(0)h(0), \tag{4.2.32}$$

$$\mathfrak{L}(h) = \int_0^\ell gh\,dx + mh(\ell) + (f_1(0) - Db)h(0). \tag{4.2.33}$$

Since the coefficient $a(x) \geq a_0 > 0$, $M > 0$, and $A_1 > 0$, the bilinear form $\mathfrak{a}(\cdot,\cdot)$ is continuous and coercive on the Sobolev space $H^1(0,\ell)$. Since $g \in L^2(0,\ell)$, the linear functional $\mathfrak{L}(h)$ is continuous on $H^1(0,\ell)$.

Therefore, by a direct application of the Lax–Milgram theorem [4, Corollary 5.8], there exists a unique weak solution $w_b \in H^1(0,\ell)$ satisfying (4.2.31). Since $w_b \in H^1(0,\ell)$ and $g \in L^2(0,\ell)$, equation (4.2.31) implies, in the distributional sense,

$$-(aw_{bx})_x = g - w_b \in L^2(0,\ell). \tag{4.2.34}$$

Hence $aw_{bx} \in H^1(0,\ell)$. Since $a \in H^1(0,\ell)$ and $a(x) \geq a_0 > 0$, we have $a^{-1} \in H^1(0,\ell)$. Furthermore, due to the continuous embedding of $H^1(0,\ell)$ into continuous functions, the space is closed under multiplication, and therefore

$$w_{bx} = a^{-1}(aw_{bx}) \in H^1(0,\ell).$$

Consequently,

$$w_b \in H^2(0,\ell).$$

It remains to verify that w_b satisfies the boundary conditions in (4.2.29). Since $w_b \in H^2(0,\ell)$ and $a \in H^1(0,\ell)$, integration by parts is justified in (4.2.31). Using (4.2.34) we obtain:

$$\big\{(aw_{bx})(0) - A_1 w_b(0) - Db + f_1(0)\big\}h(0) + \big\{-(aw_{bx})(\ell) - Mw_b(\ell) + m\big\}h(\ell) = 0, \qquad \forall h \in H^1(0,\ell).$$

Since the traces $h(0)$ and $h(\ell)$ can be prescribed independently, it follows that

$$(aw_{bx})(0) - A_1 w_b(0) = Db - f_1(0),$$

and

$$(aw_{bx})(\ell) + Mw_b(\ell) = m.$$

Thus, $w_b \in H^2(0,\ell)$ is the unique solution of the auxiliary problem (4.2.29).

Step 3: Linear superposition of the auxiliary solutions.
Since the auxiliary problem (4.2.29) is linear in the data g, m, $f_1(0)$, and b, its solution depends affinely on the parameter b. Thus, we can write:

$$w_b = \bar{w} + b\widetilde{w}. \tag{4.2.35}$$

The functions $\bar{w}$ and $\widetilde{w}$ satisfy their own independent problems:

$$\begin{cases} \bar{w} - (a\bar{w}_x)_x = g, & \text{(4.2.36a)} \\ (a\bar{w}_x)(\ell) + M\bar{w}(\ell) = m, & \text{(4.2.36b)} \\ (a\bar{w}_x)(0) - A_1\bar{w}(0) = -f_1(0). & \text{(4.2.36c)} \end{cases} \qquad \begin{cases} \widetilde{w} - (a\widetilde{w}_x)_x = 0, & \text{(4.2.37a)} \\ (a\widetilde{w}_x)(\ell) + M\widetilde{w}(\ell) = 0, & \text{(4.2.37b)} \\ (a\widetilde{w}_x)(0) - A_1\widetilde{w}(0) = D. & \text{(4.2.37c)} \end{cases}$$

Step 4: Verification of the sign condition and fixed-point argument.
To complete the proof of maximality, we must find a value of $b \in [-1, 1]$ such that $b \in \operatorname{sign}(\widetilde{S}(b))$, where the function $\widetilde{S}(b)$ is evaluated using the corresponding solution w_b.

Substituting the linear decomposition $w_b = \bar{w} + b\widetilde{w}$ into (4.2.28), we obtain:

$$\begin{aligned} \widetilde{S}(b) &= A_2\left(A_1 w_b(0) - f_1(0)\right) + A_3 w_{bx}(0) \\ &= A_2\left\{A_1\big(\bar{w}(0) + b\widetilde{w}(0)\big) - f_1(0)\right\} + A_3\left(\bar{w}_x(0) + b\widetilde{w}_x(0)\right) \\ &= \Big\{A_2\left(A_1\bar{w}(0) - f_1(0)\right) + A_3\bar{w}_x(0)\Big\} + b\Big\{A_2A_1\widetilde{w}(0) + A_3\widetilde{w}_x(0)\Big\} \\ &= B_1 + B_2 b, \end{aligned} \tag{4.2.38}$$

where B_1 and B_2 are real constants independent of the auxiliary parameter b that depend on the data Y, defined explicitly by the evaluations of $\bar{w}$ and $\widetilde{w}$ at $x = 0$.

Thus, $\widetilde{S}(b)$ is an affine function of b. It remains to choose $b \in [-1, 1]$ so that the solution w_b satisfies the original multivalued boundary condition, namely,

$$b \in \operatorname{sign}(\widetilde{S}(b)).$$

The existence of such a parameter follows from the following three cases.

1. If $\widetilde{S}(1) = B_1 + B_2 \geq 0$, we can choose $b = 1$, which satisfies $1 \in \operatorname{sign}(\widetilde{S}(1))$.
2. If $\widetilde{S}(-1) = B_1 - B_2 \leq 0$, we can choose $b = -1$, which satisfies $-1 \in \operatorname{sign}(\widetilde{S}(-1))$.
3. If $\widetilde{S}(1) < 0$ and $\widetilde{S}(-1) > 0$, then by the intermediate value theorem, since $\widetilde{S}(b)$ is continuous with respect to b, there exists a value $b^* \in (-1, 1)$ such that $\widetilde{S}(b^*) = 0$.

For this root b^*, we have $\operatorname{sign}(\widetilde{S}(b^*)) = \operatorname{sign}(0) = [-1,1]$, which trivially contains b^*. Therefore, the relation $b^* \in \operatorname{sign}(\widetilde{S}(b^*))$ holds.

In all possible cases, we successfully construct a valid parameter $b^* \in [-1,1]$ satisfying the inclusion $b^* \in \operatorname{sign}(\widetilde{S}(b^*))$. Consequently, the function $w_{b^*} \in H^2(0,\ell)$ is a solution to the system

$$\begin{cases} w - (aw_x)_x = g, & \text{(4.2.39a)} \\ (aw_x)(\ell) + Mw(\ell) = m, & \text{(4.2.39b)} \\ (aw_x)(0) - A_1 w(0) + f_1(0) = Db^* \in D\operatorname{sign}(\widetilde{S}(b^*)). & \text{(4.2.39c)} \end{cases}$$

Hence, the corresponding solution w_{b^*} solves the elliptic problem (4.2.27). This confirms that the range condition (4.2.23) is satisfied and concludes the proof. □

4.2.5 Density of the Domain

We now prove that the domain of the operator $\mathcal{A}$ is dense in H. Recall that

$$H = \left\{(w,z,\xi,\eta) \in H^1(0,\ell) \times L^2(0,\ell) \times \mathbb{R}^2 : w(\ell) = \xi\right\},$$

and that $D(\mathcal{A})$ is given by (4.2.4).

To prove the density of $D(\mathcal{A})$, we introduce the auxiliary set

$$D_0 := \left\{ (w,z,\xi,\eta) \in C^\infty([0,\ell]) \times C^\infty([0,\ell]) \times \mathbb{R}^2 : \begin{array}{ll} w(\ell) = \xi, & z(\ell) = \eta, \\ w_x(0) = 0, & z(0) + \varepsilon R w(0) = 0 \end{array} \right\}.$$

We first note that $D_0 \subset D(\mathcal{A})$. Indeed, $C^\infty([0,\ell]) \subset H^2(0,\ell) \subset H^1(0,\ell)$, and the first two boundary conditions in the definition of D_0 are also satisfied in $D(\mathcal{A})$. Moreover, if $w_x(0) = 0$ and $z(0) + \varepsilon R w(0) = 0$, then the third condition in the definition of $D(\mathcal{A})$ is satisfied, since by substituting the conditions defined at $x = 0$ we obtain:

$$0 \in D\operatorname{sign}(0) = [-D, D],$$

It is therefore enough to prove that D_0 is dense in H.

Let

$$X = (w, z, w(\ell), \eta) \in H.$$

We construct a sequence $\{X^n\}_{n\in\mathbb{N}} \subset D_0$ that converges to X in H. Since $C_0^\infty(0,\ell)$ is dense in $L^2(0,\ell)$, there exists a sequence $\{\widetilde{w}^n\}_{n\in\mathbb{N}} \subset C_0^\infty(0,\ell)$ such that

$$\widetilde{w}^n \longrightarrow w_x \quad \text{in } L^2(0,\ell).$$

Then, we define

$$w^n(x) := w(0) + \int_0^x \widetilde{w}^n(s)\, ds.$$

Thus, $w^n \in C^\infty([0,\ell])$ and

$$w^n_x = \widetilde{w}^n, \qquad w^n_x(0) = \widetilde{w}^n(0) = 0, \qquad w^n(0) = w(0).$$

Moreover, by the fundamental theorem of calculus and since $w^n(0) = w(0)$, we have:

$$w(x) - w^n(x) = \int_0^x (w_x(s) - w^n_x(s))\, ds = \int_0^x (w_x(s) - \widetilde{w}^n(s))\, ds.$$

Then, squaring both sides and applying the Cauchy–Schwarz inequality on the interval $[0,x]$:

$$\begin{aligned}
|w(x) - w^n(x)|^2 &= \left(\int_0^x (w_x(s) - \widetilde{w}^n(s))\, ds \right)^2 \\
&\leq \left(\int_0^x 1^2\, ds \right) \left(\int_0^x |w_x(s) - \widetilde{w}^n(s)|^2\, ds \right) \\
&\leq x \int_0^\ell |w_x(s) - \widetilde{w}^n(s)|^2\, ds \\
&= x\, \|w_x - \widetilde{w}^n\|^2_{L^2(0,\ell)}.
\end{aligned}$$

Integrating this inequality over $x \in (0,\ell)$ yields:

$$\begin{aligned}
\|w - w^n\|^2_{L^2(0,\ell)} &= \int_0^\ell |w(x) - w^n(x)|^2\, dx \\
&\leq \|w_x - \widetilde{w}^n\|^2_{L^2(0,\ell)} \int_0^\ell x\, dx \\
&= \frac{\ell^2}{2}\, \|w_x - \widetilde{w}^n\|^2_{L^2(0,\ell)}.
\end{aligned}$$

Therefore,

$$w^n \longrightarrow w \quad \text{in } H^1(0,\ell).$$

In particular, by the continuity of the trace operator,

$$w^n(\ell) \longrightarrow w(\ell).$$

It remains to approximate the second component while satisfying the boundary conditions in D_0. Let $q \in C^\infty([0,\ell])$ satisfy

$$q(0) = -\varepsilon R w(0), \qquad q(\ell) = \eta.$$

Such a function can be obtained, for example, by choosing a polynomial satisfying these two interpolation conditions.

Since $z - q \in L^2(0,\ell)$ and $C_0^\infty(0,\ell)$ is dense in $L^2(0,\ell)$, there exists a sequence $\{\widetilde{z}^n\}_{n\in\mathbb{N}} \subset C_0^\infty(0,\ell)$ such that

$$\widetilde{z}^n \longrightarrow z - q \quad \text{in } L^2(0,\ell).$$

Define

$$z^n := \widetilde{z}^n + q.$$

Then $z^n \in C^\infty([0,\ell])$ and

$$z^n \longrightarrow z \quad \text{in } L^2(0,\ell).$$

Since $\widetilde{z}^n(0) = \widetilde{z}^n(\ell) = 0$, we also have:

$$z^n(0) = -\varepsilon R w(0), \qquad z^n(\ell) = \eta.$$

We can now define

$$X^n := \left(w^n, z^n, w^n(\ell), z^n(\ell)\right).$$

The previous properties show that $X^n \in D_0$ for every $n \in \mathbb{N}$. Indeed,

$$z^n(0) + \varepsilon R w^n(0) = -\varepsilon R w(0) + \varepsilon R w(0) = 0 \qquad \text{and} \qquad w_x^n(0) = 0.$$

Furthermore,

$$w^n \longrightarrow w \quad \text{in } H^1(0,\ell), \qquad z^n \longrightarrow z \quad \text{in } L^2(0,\ell),$$

and

$$w^n(\ell) \longrightarrow w(\ell), \qquad z^n(\ell) = \eta.$$

Hence,

$$X^n \longrightarrow X \quad \text{in } H.$$

Thus, D_0 is dense in H. Since $D_0 \subset D(\mathcal{A})$, we conclude that $D(\mathcal{A})$ is dense in H.

4.3 Well-posedness of the closed-loop system

Based on the results established in the previous sections, we are now ready to prove the well-posedness of the closed-loop system.

Due to Proposition 4.2.6 and Proposition 4.2.7, we can conclude from Proposition 2.3.4 that the operator $\mathcal{A}$ is maximal monotone on the Hilbert space H. Therefore, the abstract problem can be studied using the theory of maximal monotone operators. In the case of regular initial data and disturbance, we use Theorem 2.4.1 to obtain a strong solution. For less regular initial data and disturbance, we use an approximation argument to obtain a generalized solution.

We first consider the case of regular initial data and disturbance.

Theorem 4.3.1 (Strong solution). *Assume that* $\varepsilon \in (0, \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6, \varepsilon_7\})$, *that* α *is defined by* (3.4.1), *and that* $0 < R(\ell + M) < 1$.

Assume that the disturbance satisfies (A1.1) *and* (A2). *Let* $X_0 \in D(\mathcal{A})$. *Then, there exists a unique strong solution* $y = y(x,t)$ *of the closed-loop system* (4.0.2). *Moreover,*

$$y \in W^{1,1}_{loc}(0,\infty; H^1(0,\ell)), \qquad y_t \in W^{1,1}_{loc}(0,\infty; L^2(0,\ell)),$$

and

$$y(\cdot,t) \in H^2(0,\ell) \qquad \text{for every } t \geq 0.$$

In addition, for every $K > 1$, *the energy satisfies*

$$E[y](t) \leq KE[y](0)e^{-\omega t}, \qquad t \geq 0,$$

where $\omega = \frac{\varepsilon R}{\sqrt{K}} > 0$.

Remark 4.3.2. For a strong solution y, the notation $E[y](t)$ denotes the energy defined in (3.1.1) evaluated at y at time t.

Proof. We first consider a fixed $T > 0$. By the regularity assumption on the disturbance, given by (A1.1), the functions ψ^1 and ψ^2 introduced in Section 4.1 belong to $W^{3,1}(0,T)$. Since the spatial functions φ^1 and φ^2 are polynomials, the translation function z defined in (4.1.1) satisfies

$$z_{tt} \in W^{1,1}(0,T; H^2(0,\ell)).$$

Moreover, since $a \in H^1(0,\ell)$, we have $(az_x)_x \in L^2(0,\ell)$ for every $t \in [0,T]$. Hence, the function

$$f(x,t) = -z_{tt}(x,t) + (a(x)z_x(x,t))_x$$

satisfies

$$f \in W^{1,1}(0,T;L^2(0,\ell)).$$

Consequently, the source term

$$F(t) = \big(0, f(\cdot,t), 0, 0\big)$$

belongs to $W^{1,1}(0,T;H)$. In particular, F is absolutely continuous as a function from $[0,T]$ into H.

Let

$$X_0 = (w_0, w_1, \xi_0, \eta_0) \in D(\mathcal{A}).$$

Then, by Theorem 2.4.1, there exists a unique absolutely continuous function $X : [0,T] \to H$ satisfying

$$\begin{cases} \dfrac{d}{dt}X(t) + \mathcal{A}X(t) \ni F(t), & \text{for almost every } t \in (0,T), \\ X(t) \in D(\mathcal{A}), & \text{for every } t \in [0,T], \\ X(0) = X_0. \end{cases} \tag{4.3.1}$$

We now recover the solution of the transformed problem from the abstract solution. Since the state vector is identified as

$$X(t) = \big(w(\cdot,t), w_t(\cdot,t), w(\ell,t), w_t(\ell,t)\big),$$

we have:

$$w \in W^{1,1}(0,T;H^1(0,\ell)), \qquad w_t \in W^{1,1}(0,T;L^2(0,\ell)).$$

In addition, since $X(t) \in D(\mathcal{A})$ for every $t \in [0,T]$,

$$w(\cdot,t) \in H^2(0,\ell), \qquad w_t(\cdot,t) \in H^1(0,\ell).$$

By (4.3.1), the function w satisfies the transformed system (4.1.2) introduced in Section 4.1. Thus, X is the abstract solution of the transformed problem, and w is the corresponding solution of the transformed PDE system.

We can now recover the solution of the original closed-loop system (4.0.2). Using the change of variables introduced in Section 4.1, we define

$$y(x,t) = w(x,t) + z(x,t).$$

Since z has the regularity obtained above, we have:

$$y \in W^{1,1}(0,T;H^1(0,\ell)), \qquad y_t \in W^{1,1}(0,T;L^2(0,\ell)).$$

Moreover, for every $t \in [0, T]$,

$$y(\cdot, t) \in H^2(0, \ell).$$

Therefore, y has the regularity required to apply the estimates established in Chapter 3. Under the assumptions on ε, α, and R stated in the theorem, the corresponding solution of the original closed-loop system satisfies

$$E[y](t) \leq KE[y](0)e^{-\omega t}, \qquad 0 \leq t \leq T, \tag{4.3.2}$$

where

$$\omega = \frac{\varepsilon R}{\sqrt{K}} > 0.$$

Since $T > 0$ was arbitrary, the previous arguments can be applied on every finite time interval. Therefore, the strong solution satisfies

$$y \in W^{1,1}_{\text{loc}}(0, \infty; H^1(0, \ell)), \qquad y_t \in W^{1,1}_{\text{loc}}(0, \infty; L^2(0, \ell)),$$

and

$$y(\cdot, t) \in H^2(0, \ell), \qquad t \geq 0.$$

Moreover, the exponential decay holds for all $t \geq 0$. This concludes the proof. □

We now consider less regular initial data and disturbance. In this case, the regularity required to apply Theorem 2.4.1 directly is not available. We therefore approximate the initial data and the disturbance by regular ones and use the corresponding strong solutions to construct the generalized solution.

Theorem 4.3.3 (Generalized solution)**.** *Assume that* $\varepsilon \in (0, \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6, \varepsilon_7\})$, *that* α *is defined by* (3.4.1), *and that* $0 < R(\ell + M) < 1$.

Assume that the disturbance satisfies (A1.2) *and* (A2). *Let* $X_0 \in H$. *Then, there exists a unique generalized solution* $y = y(x, t)$ *of the closed-loop system* (4.0.2) *such that*

$$y \in C([0, \infty); H^1(0, \ell)), \qquad y_t \in C([0, \infty); L^2(0, \ell)).$$

In addition, let $\eta_y(t)$ *denote the scalar component of the generalized state associated with the velocity at* $x = \ell$. *We define the generalized energy by*

$$E_{\text{gen}}[y, \eta_y](t) := \frac{1}{2}\left\{\int_0^\ell \left(y_t^2(x, t) + a(x)y_x^2(x, t)\right) dx + \alpha y^2(0, t) + M\eta_y^2(t)\right\}. \tag{4.3.3}$$

Then, for every $K > 1$, the energy satisfies

$$E_{\text{gen}}[y, \eta_y](t) \leq K E_{\text{gen}}[y, \eta_y](0) e^{-\omega t}, \qquad t \geq 0,$$

where $\omega = \frac{\varepsilon R}{\sqrt{K}} > 0$.

Proof. We first consider a fixed $T > 0$. Let $X_0 \in H$ and assume that d satisfies (A1.2) and (A2).

Then, by the definition of the source term in Section 4.2.1, the corresponding source term satisfies

$$F \in L^1(0, T; H).$$

Since $D(\mathcal{A})$ is dense in H, there exists a sequence $\{X_0^n\}_{n \in \mathbb{N}} \subset D(\mathcal{A})$ such that

$$X_0^n \longrightarrow X_0 \qquad \text{in } H. \tag{4.3.4}$$

Similarly, since $W^{3,1}(0, T)$ is dense in $W^{2,1}(0, T)$, we can choose a sequence

$$\{d^n\}_{n \in \mathbb{N}} \subset W^{3,1}(0, T)$$

such that

$$d^n \longrightarrow d \qquad \text{in } W^{2,1}(0, T). \tag{4.3.5}$$

The approximations can be chosen so that $d^n(0) = d_t^n(0) = 0$ and that they satisfy (A2) for every $n \in \mathbb{N}$.

Let F^n be the source term associated with d^n. From (4.3.5) and the definition of F^n, we obtain:

$$F^n \longrightarrow F \qquad \text{in } L^1(0, T; H). \tag{4.3.6}$$

For each $n \in \mathbb{N}$, let X^n be the strong solution corresponding to X_0^n and F^n, given by Theorem 2.4.1. Then, for $n, m \in \mathbb{N}$, we have:

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|X^n(t) - X^m(t)\|_V^2 = &\langle F^n(t) - F^m(t), X^n(t) - X^m(t) \rangle_V \\ &- \langle \mathcal{A}X^n(t) - \mathcal{A}X^m(t), X^n(t) - X^m(t) \rangle_V . \end{aligned}$$

By the monotonicity of $\mathcal{A}$ and the Cauchy–Schwarz inequality,

$$\frac{1}{2} \frac{d}{dt} \|X^n(t) - X^m(t)\|_V^2 \leq \|F^n(t) - F^m(t)\|_V \|X^n(t) - X^m(t)\|_V.$$

Hence,

$$\frac{d}{dt}\|X^n(t) - X^m(t)\|_V \leq \|F^n(t) - F^m(t)\|_V,$$

and, after integration,

$$\|X^n(t) - X^m(t)\|_V \leq \|X_0^n - X_0^m\|_V + \int_0^t \|F^n(s) - F^m(s)\|_V \, ds. \tag{4.3.7}$$

Therefore,

$$\sup_{t\in[0,T]} \|X^n(t) - X^m(t)\|_V \leq \|X^n(0) - X^m(0)\|_V + \|F^n - F^m\|_{L^1(0,T;H)}.$$

By (4.3.4) and (4.3.6), it follows that $\{X^n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $C([0,T];H)$. Hence, there exists

$$X \in C([0,T];H)$$

such that

$$X^n \longrightarrow X \qquad \text{in } C([0,T];H). \tag{4.3.8}$$

By Definition 2.4.3, X is the generalized solution of (4.3.1) with initial data X_0 and source term F.

We now translate these solutions back to the original variables. For each $n \in \mathbb{N}$, let w^n be the solution of the transformed problem associated with X^n, and let z^n be the translation function associated with d^n. We define

$$y^n = w^n + z^n.$$

Since $X^n \to X$ in $C([0,T];H)$, we have:

$$w^n \longrightarrow w \qquad \text{in } C([0,T];H^1(0,\ell)), \qquad w_t^n \longrightarrow w_t \qquad \text{in } C([0,T];L^2(0,\ell)).$$

Moreover, the scalar components of X^n converge uniformly in time. In particular,

$$w^n(\ell,t) \longrightarrow w(\ell,t), \qquad w_t^n(\ell,t) \longrightarrow \eta(t), \qquad 0 \leq t \leq T,$$

where η denotes the fourth component of the limit state X.

On the other hand, from (4.3.5) and the definition of the translation function, we have

$$z^n \longrightarrow z \qquad \text{in } C([0,T];H^1(0,\ell)), \qquad z_t^n \longrightarrow z_t \qquad \text{in } C([0,T];L^2(0,\ell)).$$

Since the spatial functions defining z^n are fixed and smooth, their boundary values also converge uniformly in time. Hence,

$$z^n(\ell,t) \longrightarrow z(\ell,t), \qquad z_t^n(\ell,t) \longrightarrow z_t(\ell,t), \qquad 0 \le t \le T.$$

Therefore,

$$y^n \longrightarrow y \quad \text{in } C([0,T];H^1(0,\ell)), \qquad y_t^n \longrightarrow y_t \quad \text{in } C([0,T];L^2(0,\ell)), \tag{4.3.9}$$

where

$$y = w + z.$$

Moreover, if

$$\eta_y(t) := \eta(t) + z_t(\ell,t),$$

then

$$y_t^n(\ell,t) \longrightarrow \eta_y(t) \qquad \text{uniformly for } t \in [0,T]. \tag{4.3.10}$$

Thus, the generalized solution of the transformed problem gives a solution of the original closed-loop system (4.0.2), with

$$y \in C([0,T];H^1(0,\ell)), \qquad y_t \in C([0,T];L^2(0,\ell)).$$

For a generalized solution, the trace $y_t(\ell,t)$ is not necessarily defined. Therefore, the energy (3.1.1) cannot be used directly. Instead, we introduce the generalized energy as (4.3.3).

The convergence in (4.3.9), together with (4.3.10), implies that

$$E_{\text{gen}}[y^n, y_t^n(\ell,\cdot)](t) \longrightarrow E_{\text{gen}}[y,\eta_y](t), \qquad 0 \le t \le T. \tag{4.3.11}$$

On the other hand, for every $n \in \mathbb{N}$, the solution y^n is a strong solution of the original closed-loop system. Therefore, by Theorem 3.4.1,

$$E[y^n](t) \le KE[y^n](0)e^{-\omega t}, \qquad 0 \le t \le T. \tag{4.3.12}$$

Since each y^n is a strong solution, $\eta_{y^n}(t) = y_t^n(\ell,t)$, and therefore

$$E_{\text{gen}}[y^n,\eta_{y^n}](t) = E[y^n](t).$$

Hence, by (4.3.11), we can pass to the limit in (4.3.12) and obtain

$$E_{\text{gen}}[y,\eta_y](t) \le KE_{\text{gen}}[y,\eta_y](0)e^{-\omega t}, \qquad 0 \le t \le T. \tag{4.3.13}$$

Therefore, the exponential decay established for strong solutions is also satisfied by the generalized solution, without requiring the existence of the trace $y_t(\ell, t)$.

Since $T > 0$ was arbitrary, the previous arguments can be applied on every finite time interval. Therefore, the generalized solution satisfies

$$y \in C([0,\infty); H^1(0,\ell)), \qquad y_t \in C([0,\infty); L^2(0,\ell)).$$

Moreover, the source term satisfies

$$F \in L^1_{\text{loc}}(0,\infty; H),$$

and the exponential decay estimate (4.3.13) holds for all $t \geq 0$. This concludes the proof.

□

Remark 4.3.4. The change of variables does not modify the initial data. Indeed, from (A1.1) and (A1.2), we have

$$d(0) = d_t(0) = 0.$$

Therefore, by the definition of the translation function z in (4.1.1),

$$z(\cdot, 0) = z_t(\cdot, 0) = 0.$$

Hence,

$$w(\cdot, 0) = y(\cdot, 0), \qquad w_t(\cdot, 0) = y_t(\cdot, 0).$$

Thus, the initial data of the transformed and original problems are the same.

Chapter 5

Conclusions

In this thesis, we studied the stabilization and well-posedness of a hybrid system with an unknown boundary disturbance. The main objective was to design a feedback control that guarantees exponential decay of the energy without requiring the knowledge of the disturbance.

The stabilization result was obtained in Chapter 3 using Lyapunov techniques. We were inspired by the work of Rao [28] for the undisturbed system and considered the case where we have an unknown disturbance $d(t)$, but its bound is known. The main difficulty was to construct a feedback law that could dissipate the effect of the disturbance while still providing exponential decay of the energy.

To overcome this difficulty, we introduced a feedback law containing a multivalued sign operator. The key point is that the boundary condition at $x = 0$, the sign property (2.5.2), and the bound (A2) can be used together to dissipate the disturbance without knowing its value. Moreover, they allow us to control the remaining terms in the derivative of the Lyapunov functional and obtain the dissipation estimate (3.4.21).

After choosing the parameters according to the required assumptions, the Lyapunov functional was shown to be equivalent to the energy, as stated in (3.3.5), and to satisfy the differential inequality (3.4.24). Therefore, we obtained the exponential decay of the energy given in Theorem 3.4.1. This provides a robust feedback stabilization result for the perturbed system without requiring knowledge of the disturbance.

Chapter 4 was devoted to the well-posedness of the closed-loop system (4.0.2). The presence of the disturbance in the boundary condition and the multivalued sign operator make the problem more difficult to treat directly. We first introduced a change of variables that

moves the disturbance from the boundary condition to the PDE. This allows the transformed problem to be written using a time-independent operator. The resulting system was then written as the abstract differential inclusion

$$\frac{d}{dt}X(t) + \mathcal{A}X(t) \ni F(t).$$

The source term $F(t)$ contains the disturbance through the transformed PDE, while the operator $\mathcal{A}$ contains the spatial and boundary parts of the problem.

The change of variables itself also requires some care. As noted in Remark 4.1.1, one pair of separable variables does not provide enough freedom to satisfy both conditions at $x = 0$ associated with the residual terms. Moreover, the disturbance appears in only one of the two boundary conditions. We therefore introduced two pairs of separable variables, which allow the disturbance to be moved from the boundary condition to the PDE while preserving the required boundary conditions.

The operator analysis also required a modification of the inner product. With the energy inner product, the boundary terms at $x = 0$ prevented us from obtaining the desired monotonicity estimate. We therefore introduced an equivalent inner product inspired by the Lyapunov functional. Its associated norm is directly related to the Lyapunov functional through

$$\|X(t)\|_V^2 = 2V(t).$$

With this choice, the operator was shown to be monotone. The multivalued sign term was handled using its maximal monotonicity property established in Proposition 2.5.2, which is also consistent with the sign estimate used in the stabilization analysis.

The main difficulty in proving maximality was the nonlinear and multivalued boundary condition. Instead of treating this condition directly, we reduced the range problem to an auxiliary linear elliptic problem depending on a scalar parameter $b \in [-1, 1]$. For each fixed b, the linear problem has a unique solution, and its dependence on b can be written as an affine function, given by (4.2.35). The parameter can then be chosen according to the three possible cases described in Section 4.2.4, so that the original multivalued boundary condition is satisfied. This proves the range condition

$$R(I + \mathcal{A}) = H$$

and, together with monotonicity, gives the maximal monotonicity of $\mathcal{A}$.

Once the maximal monotonicity of $\mathcal{A}$ was established, we used Theorem 2.4.1 to obtain the well-posedness of the transformed problem. More precisely, for regular initial data $X_0 \in D(\mathcal{A})$ and a disturbance $d \in W^{3,1}_{\text{loc}}(0,\infty)$, the corresponding source term is sufficiently regular to apply Theorem 2.4.1. We obtain a unique strong solution

$$X \in W^{1,1}_{\text{loc}}(0,\infty;H), \qquad X(t) \in D(\mathcal{A}),$$

and, by identifying the components of X, a solution $w(x,t)$ of the transformed PDE system. Using the inverse change of variables $y = w + z$, we then obtain a solution of the original closed-loop system with

$$y \in W^{1,1}_{\text{loc}}(0,\infty;H^1(0,\ell)) \qquad \text{and} \qquad y(\cdot,t) \in H^2(0,\ell), \qquad y_t(\cdot,t) \in H^1(0,\ell),$$

for every $t \geq 0$. This regularity is sufficient to apply the exponential decay result established in Chapter 3.

The result can also be extended to less regular data. For $X_0 \in H$ and $d \in W^{2,1}_{\text{loc}}(0,\infty)$, we use the density of $D(\mathcal{A})$ in H and the density of $W^{3,1}(0,T)$ in $W^{2,1}(0,T)$ for each $T > 0$ to approximate the initial data and the disturbance by regular sequences. The corresponding strong solutions are then used to construct a generalized solution

$$X \in C([0,\infty);H).$$

After translating back to the original variable, we obtain:

$$y \in C([0,\infty);H^1(0,\ell)), \qquad y_t \in C([0,\infty);L^2(0,\ell)).$$

Since the strong approximating solutions satisfy the exponential decay estimate and converge to the generalized solution, the energy estimate is preserved in the limit. Therefore, the generalized solution also satisfies the exponential decay estimate.

Therefore, the two main objectives of the thesis are achieved. First, we designed a feedback control that is robust with respect to an unknown boundary disturbance and proved exponential decay of the energy. Second, we established the well-posedness of the resulting nonlinear and multivalued closed-loop system by means of an abstract formulation based on maximal monotone operator theory. Moreover, the well-posedness result is obtained both for regular and less regular data, while in both cases the exponential decay obtained in Chapter 3 is preserved.

Bibliography

[1] G. Arias, *Stabilization of a microbeam model with distributed disturbance*, System and Control Letters, **173** (2023), Article 105466.

[2] V. Barbu, *Nonlinear Differential Equations of Monotone Types in Banach Spaces*, Springer, Springer Monographs in Mathematics, 2010.

[3] H. Brezis, *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert*, North-Holland, Mathematics Studies, Vol. 5, 1973.

[4] H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer, Universitext, 2010.

[5] H. Brezis and A. Pazy, *Accretive sets and differential equations in Banach spaces*, Israel Journal of Mathematics, **8** (1970), pp. 367–383.

[6] M. G. Crandall and A. Pazy, *Nonlinear evolution equations in Banach spaces*, Israel Journal of Mathematics, **11** (1972), pp. 57–94.

[7] R. Curtain and H. Zwart, *An Introduction to Infinite-Dimensional Linear Systems Theory*, Springer, Texts in Applied Mathematics, Vol. 21, 1995.

[8] C. M. Dafermos and M. Slemrod, *Asymptotic behavior of nonlinear contraction semigroups*, Journal of Functional Analysis, **13** (1973), No. 1, pp. 97–106.

[9] Q.-H. Fu and G.-Q. Xu, *Exponential stabilization of 1-d wave equation with distributed disturbance*, WSEAS Transactions on Mathematics, **14** (2015), pp. 192–201.

[10] B.-Z. Guo and F.-F. Jin, *Sliding mode and active disturbance rejection control to stabilization of one-dimensional anti-stable wave equations subject to disturbance in boundary input*, IEEE Transactions on Automatic Control, **58** (2013), No. 5, pp. 1269–1274.

[11] B.-Z. Guo and W. Kang, *Lyapunov approach to the boundary stabilisation of a beam equation with boundary disturbance*, International Journal of Control, **87** (2014), No. 5, pp. 925–939.

[12] B.-Z. Guo and J.-J. Liu, *Sliding mode control and active disturbance rejection control to the stabilization of one-dimensional Schrödinger equation subject to boundary control matched disturbance*, International Journal of Robust and Nonlinear Control, **24** (2014), No. 16, pp. 2194–2212.

[13] B.-Z. Guo and W. Kang, *The Lyapunov approach to boundary stabilization of an anti-stable one-dimensional wave equation with boundary disturbance*, International Journal of Robust and Nonlinear Control, **24** (2014), No. 1, pp. 54–69.

[14] P. Guzmán and C. Prieur, *Rapid stabilization of a reaction-diffusion equation with distributed disturbance*, 59th IEEE Conference on Decision and Control, Jeju Island, Korea, 2020.

[15] P. Guzmán and E. Hernández, *Stabilization of the heat equation with disturbance at the flux boundary condition*, Mathematical Methods in the Applied Sciences, **46** (2023), No. 17, pp. 18035–18043.

[16] P. Guzmán and E. Hernández, *Rapid stabilization of an unstable heat equation with disturbance at the flux boundary condition*, System and Control Letters, **196** (2025), Article 105973.

[17] P. Guzmán, A. Huerta and H. Parada, *Rapid stabilization for a wave equation with boundary disturbance*, arXiv preprint arXiv:2510.04893, 2025.

[18] F.-F. Jin and B.-Z. Guo, *Lyapunov approach to output feedback stabilization for the Euler-Bernoulli beam equation with boundary input disturbance*, Automatica, **52** (2015), pp. 95–102.

[19] W. Kang and B.-Z. Guo, *Lyapunov approach to boundary stabilization of an anti-stable heat equation with boundary disturbance*, 33rd Chinese Control Conference, Nanjing, China, 2014.

[20] W. Kang and B.-Z. Guo, *Arbitrary decay for boundary stabilization of Schrödinger equation subject to unknown disturbance by Lyapunov approach*, IFAC Journal of Systems and Control, **7** (2019), Article 100033.

[21] H. Khalil, *Nonlinear Systems*, Prentice Hall, 2002.

[22] Y. Li and G. Xu, *Stabilization of an Euler-Bernoulli beam with a tip mass under the unknown boundary external disturbances*, Journal of Systems Science and Complexity, **30** (2017), No. 4, pp. 803–817.

[23] J.-J. Liu and J.-M. Wang, *Active disturbance rejection control and sliding mode control of one-dimensional unstable heat equation with boundary uncertainties*, IMA Journal of Mathematical Control and Information, **32** (2015), No. 1, pp. 97–117.

[24] W. Liu, *Elementary Feedback Stabilization of the Linear Reaction-Convection-Diffusion Equation and the Wave Equation*, Springer, Mathémathiques et Applications, Vol. 66, 2010.

[25] Z.-H. Luo, B.-Z. Guo and O. Morgul, *Stability and Stabilization of Infinite Dimensional Systems with Applications*, Springer, Communications and Control Engineering, 1999.

[26] S. Misra, G. Gorain and S. Kar, *Stability of wave equation with a tip mass under unknown boundary external disturbance*, Applied Mathematics E-Notes, **19** (2019), pp. 128–140.

[27] J. Peypouquet and S. Sorin, *Evolution equations for maximal monotone operators: Asymptotic analysis in continuous and discrete time*, Journal of Convex Analysis, **17** (2010), No. 3–4, pp. 1113–1163.

[28] B. Rao, *Decay estimates of solutions for a hybrid system of flexible structures*, Euro. Jl. of Applied Mathematics, **4** (1993), pp. 303–319.

[29] R. Showalter, *Monotone Operators in Banach Space and Nonlinear Partial Differential Equations*, American Mathematical Society, Mathematical Surveys and Monographs, Vol. 49, 1997.