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IFAC PapersOnLine 55-26 (2022) 107–112

A comparative study of reduced model based boundary control design for linear port Hamiltonian systems ⋆

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Abstract: A comparative study of passivity based boundary control design for a class of infinite dimensional port-Hamiltonian system using two different model reduction approaches is presented. The first approach is based on a direct low order structure preserving discretization while the second approach arise from the structure preserving model reduction of a high order discretzed model. Two passivity-based control techniques, namely control by interconnection and damping injection, are used to change the equilibrium point and the convergence rate of the closed-loop system. An Euler-Bernoulli beam example is used to illustrate the findings by means of discussion and numerical simulations.

Keywords: Infinite dimensional system, port-Hamiltonian systems, passivity-based control, spatial discretization, model reduction.

1. INTRODUCTION

The port-Hamiltonian (PH) formalism has shown to have nice features when dealing with passivity based control (PBC) (van der Schaft, 2000) and the modeling of complex systems (Duindam et al., 2009). A special class are linear Boundary-Controlled port-Hamiltonian Systems (BC-PHS) defined on 1D spatial domains (Jacob and Zwart, 2012). Many structure preserving spatial discretizaction techniques have been reported for BC-PHS, as for instance the one based on mixed finite elements (Golo et al., 2004), partitioned finite elements (Cardoso-Ribeiro et al., 2018), finite differences (Trenchant et al., 2018), or pseudo-spectral methods (Moulla et al., 2012). In addition, standard finite element method (sFEM) (Cook et al., 2007) can be applied to mechanical or electrical systems by means of Hamilton's principle (Reddy, 2017) such the discretized model preserves the Lagrangian structure and therefore can be written as a finite-dimensional PHS. The design of PBC uses the PHS structure to shape the closed-loop energy and modify the interconnection and dissipation structure in order to obtain a system with a desired dynamic behavior (Ortega et al., 2002). Moreover, when dealing with BC of infinite dimensional systems, it has been shown that the power-preserving interconnection of a BC-PHS with a strictly passive or a strictly input passive controller assures asymptotic, respectively exponential stability of the closed-loop system (Villegas, 2007; Ramirez et al., 2014). Techniques such as interconnection and damping assignment (IDA)-PBC or control by interconnection (CbI) are conditioned by the solution of a set of partial differential equations that depend on the PHS structure (van der Schaft, 2000; Macchelli et al., 2017). In this sense a high dimensional model is in general less convenient than a low dimensional model for control design. Several model order reduction techniques can be found in the literature (Moore, 1981; Varga, 1995), some of which preserve the PH structure (Polyuga and van der Schaft, 2010; Ionescu and Astolfi, 2013; Kawano and Scherpen, 2018). The objective of this work is to compare and comment on some main differences when using a low order finite dimensional model arising from a direct spatial approximation or the model reduction of a high dimensional spatial approximation. The performed comparison is qualitative and considered the simplicity of the control design and closed-loop performances evaluated by means of numerical simulations. The considered class encompass a large class of mechanical systems. The paper is organized as follows. In Section 2 the structure preserving discretization and model order reduction techniques are presented. Section 3 presents the CbI and DI techniques. Section 4, presents the boundary control design based on the low order finite dimensional PHS. Section 5 shows numerical simulations when an Euler-Bernoulli beam model is considered as example. Section 6 gives some final remarks and discussion on future work.

2. DISCRETIZED AND/OR REDUCED PHS

The control objective consists in changing the equilibrium and closed-loop behavior of a class of mechanical systems

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⋆ The first author acknowledges financial support from ANID/Becas/Doctorado Nacional/2021-21211290 (Chile). The second author acknowledges Chilean FONDECYT 1191544 and CONICYT BASAL FB0008 projects.

10.1016/j.ifacol.2022.10.385

modeled as BC-PHS by means of a linear finite dimensional boundary controller. To this end a finite dimensional low order linear PHS is used to perform the BC design. The low order models are obtained from either direct spatial discretization of the infinite dimensional system or from a structure preserving model reduction of a higher order discretized model.

2.1 Discretized mechanical PHS

Consider a mechanical system which admits linear BC-PHS representation and the following finite dimensional approximation

$$\begin{aligned}\underbrace{\begin{bmatrix}\dot p\\ \dot q\end{bmatrix}}_{\dot x} &= \underbrace{\begin{bmatrix}0 & -I\\ I & 0\end{bmatrix}}_{J}\underbrace{\begin{bmatrix}M^{-1} & 0\\ 0 & K\end{bmatrix}}_{Q}\underbrace{\begin{bmatrix}p\\ q\end{bmatrix}}_{x} + \underbrace{\begin{bmatrix}B\\ 0\end{bmatrix}}_{g}u\\ y &= g^\top Q\, x\end{aligned} \tag{1}$$

where n is the order of the spatial approximation, $q \in \mathbb{R}^n$ is the vector of generalized displacements, $p = M\,\dot q \in \mathbb{R}^n$ is the vector of generalized momentum, $M = M^\top > 0 \in \mathbb{R}^{n\times n}$ is the mass matrix, $K = K^\top > 0 \in \mathbb{R}^{n\times n}$ is the stiffness matrix, $B \in \mathbb{R}^{n\times m}$ is the input matrix, with I the identity matrix of appropriate dimensions, $u \in \mathbb{R}^m$ is the input vector and $y \in \mathbb{R}^m$ is the output vector. Assume that B has the following form

$$B = [0\ \ I]^\top \tag{2}$$

The PHS (1) can be obtained for instance by using sFEM (Cook et al., 2007), and B according to (2) is always possible if the discretization mesh is properly structured and the inputs are applied directly at the nodes. The Hamiltonian of (1) is given by

$$H(x) = \frac{1}{2}x^\top Q x = \frac{1}{2}p^\top M^{-1}\,p + \frac{1}{2}q^\top K\,q \geq 0$$

which is the sum of total kinetic and elastic potential energy. Note from (1) that $y = g^\top \nabla_x H$, where $\nabla_x H = Q\,x$ is the Hamiltonian gradient respect to x, so the power of the system is $\dot H = y^\top u$, so the system is passive (van der Schaft, 2000).

2.2 Structure-preserving model reduction

The PHS structure is convenient for control design (Ortega et al., 2002), so it is important to preserve it when performing model reduction. In this section a structure-preserving model reduction technique based on balanced truncation is briefly introduced. For details see Kawano and Scherpen (2018). Define a transformation matrix $T_B \in \mathbb{R}^{N\times N}$, with $N = 2n$, such that the controllability and observability Gramians of the system are equal and diagonal, so the original state $x \in \mathbb{R}^N$ is related to the new state $x_B \in \mathbb{R}^N$ by $x = T_B\,x_B$. The balanced realization of (1) can then be written as

$$\begin{aligned}\dot x_B &= J_B\,Q_B\,x_B + g_B\,u\\ y &= g_B^\top\,Q_B\,x_B\end{aligned} \tag{3}$$

with $J_B = T_B^{-1} J\,T_B^{-\top}$, $Q_B = T_B^\top Q\,T_B = I$, and $g_B = T_B^{-1}g$. Then, by means of a suitable partition of the state x_B, (3) can be written as

$$\begin{aligned}\underbrace{\begin{bmatrix}\dot x_r\\ \dot x_{N-r}\end{bmatrix}}_{\dot x_B} &= \underbrace{\begin{bmatrix}J_r & 0\\ 0 & J_{N-r}\end{bmatrix}}_{J_B}\underbrace{\begin{bmatrix}Q_r & 0\\ 0 & Q_{N-r}\end{bmatrix}}_{Q_B}\underbrace{\begin{bmatrix}x_r\\ x_{N-r}\end{bmatrix}}_{x_B} + \underbrace{\begin{bmatrix}g_r\\ g_{N-r}\end{bmatrix}}_{g_B}u\\ y &= \underbrace{\begin{bmatrix}g_r^\top & g_{N-r}^\top\end{bmatrix}}_{g_B^\top}\underbrace{\begin{bmatrix}Q_r & 0\\ 0 & Q_{N-r}\end{bmatrix}}_{Q_B}\underbrace{\begin{bmatrix}x_r\\ x_{N-r}\end{bmatrix}}_{x_B}\end{aligned}$$

and $T_B = [T_r\ \ T_{N-r}]$, $T_B^{-1} = \begin{bmatrix}T_{ri}\\ T_{(N-r)i}\end{bmatrix}$. Then, the reduced model denoted as Σ_r is given by

$$\begin{aligned}\dot x_r &= J_r\,Q_r\,x_r + g_r\,u\\ y_r &= g_r^\top Q_r\,x_r\end{aligned} \tag{4}$$

where $x_r \in \mathbb{R}^r$, with $r < N$, is the reduced state which preserves the most controllable and observable states of x_B, which are associated with the highest Hankel singular values; and $T_r \in \mathbb{R}^{N\times r}$ and $T_{ri} \in \mathbb{R}^{r\times N}$ are transformation matrices that relate the original state x with the reduced state x_r by

$$x_r = T_{ri}\,x, \quad x \approx T_r\,x_r \tag{5}$$

3. CONTROL OF PORT-HAMILTONIAN SYSTEMS

Two PBC techniques that shall be used in this work are shortly revised in this section, namely control by interconnection (CbI) and damping injection (DI) (van der Schaft, 2000).

3.1 Control by interconnection (CbI)

Consider the PHS controller

$$\begin{aligned}\dot x_c &= (J_c - R_c)\nabla_{x_c} H_c + g_c\,u_c\\ y_c &= g_c^\top \nabla_{x_c} H_c\end{aligned} \tag{6}$$

with $x_c \in \mathbb{R}^{n_c}$ the controller state. CbI consists in interconnecting, in a power-preserving fashion, a PHS controller with the plant in order to change the closed-loop energy function so that it has its minimum at the desired equilibrium and such that it has desired dynamical properties. The power-preserving interconnection is given by

$$u = -y_c, \quad y = u_c \tag{7}$$

an the closed-loop system is

$$\begin{aligned}\begin{bmatrix}\dot x\\ \dot x_c\end{bmatrix} &= \left(\begin{bmatrix}J & -g\,g_c^\top\\ g_c\,g & J_c\end{bmatrix} - \begin{bmatrix}0 & 0\\ 0 & R_c\end{bmatrix}\right)\begin{pmatrix}Q\,x\\ \nabla_{x_c} H_c\end{pmatrix}\\ \begin{bmatrix}y\\ y_c\end{bmatrix} &= \begin{bmatrix}g^\top & 0\\ 0 & g_c^\top\end{bmatrix}\begin{pmatrix}Q\,x\\ \nabla_{x_c} H_c\end{pmatrix}\end{aligned} \tag{8}$$

which is again a PHS. The Hamiltonian of the closed-loop system is given by $H_d(x, x_c) = H(x) + H_c(x_c)$. For control design and stability analysis it is convenient to express the closed-loop Hamiltonian as a function of the states of the plant only. i.e., $H_d = H_d(x)$. To this end the dynamics of the closed-loop system are restricted to a submanifold parameterized by x only (van der Schaft, 2000). This means that we look for structural invariants, named Casimir functions, that relate each state of the controller with the states of the plant. Defining the vector function $F(x) = [F_1(x)\ \ F_2(x)\ \ \ldots\ \ F_{n_c}(x)]^\top$, the Casimirs are defined as follow

$$C(x, x_c) = \sum_{i=1}^{n_c}(F_i(x) - x_{c_i}) = \sum_{i=1}^{n_c} C_i(x, x_{c_i}) \tag{9}$$

where $C(x, x_c) \in \mathbb{R}^{n_c}$ is a structural invariant of the system. The conditions for the existence of Casimir functions are given by the following matching equations (ME) (van der Schaft, 2000)

$$\frac{\partial F^\top}{\partial x} J \frac{\partial F}{\partial x} = J_c$$

$$R \frac{\partial F}{\partial x} = 0$$

$$R_c = 0$$

$$\frac{\partial F^\top}{\partial x} J = g_c \, g^\top$$

Solving the ME the closed-loop Hamiltonian becomes

$$H_d(x) = H(x) + H_c(F(x) - C) \tag{10}$$

Denote the desired equilibrium point by x^* and consider a quadratic desired closed-loop Hamiltonian, then

$$H_d(x) = \frac{1}{2} (x - x^*)^\top Q_d \, (x - x^*) \tag{11}$$

and the closed-loop system (8) restricted to the invariant submanifold is

$$\begin{aligned} \dot{x} &= J \, Q_d \, (x - x^*) + gu \\ y_d &= g^\top Q_d \, (x - x^*) \end{aligned}$$

where y_d is the closed-loop conjugated output and $Q_d = Q_d^\top > 0 \in \mathbb{R}^{N \times N}$ is the desired energy matrix, which is a control design parameter.

3.2 Damping injection (DI)

DI allows to drive the system to the equilibrium point by means of the following output feedback

$$u = -K_{DI} \, y_d = -K_{DI} g^\top Q_d \, (x - x^*)$$

where $K_{DI} = K_{DI}^\top \geq 0$ is a design parameter. The closed-loop system then becomes

$$\dot{x} = (J - g \, K_{DI} \, g^\top) \, Q_d \, (x - x^*)$$

and $\dot{H}_d = -(x - x^*)^\top Q_d^\top g \, K_{DI} \, g^\top Q_d \, (x - x^*) < 0$, i.e., H_d qualifies as Lyapunov function and consequently asymptotic stability can be concluded (van der Schaft, 2000).

3.3 A comment on the stabilization of BC-PHS

The stabilization and BC design for BC-PHS has been given an increasing attention in recent years (Rashad et al., 2020). Depending on the specific parameters of the BC (6) it allows to guarantee exponential (Ramirez et al., 2014; Macchelli et al., 2020) or asymptotic stability (Villegas, 2007; Macchelli et al., 2017). In this paper we focus on the model approximation/reduction, hence, we do not study which particular class of stability the resulting controller achieves.

4. REDUCED MODEL CONTROL DESIGN

The equilibrium point of (1) is denoted as $x^* = [p^* \;\; q^*]^\top$, with

$$\begin{aligned} p^* &= 0 \\ q^* &= K^{-1} B \, u^* \end{aligned} \tag{12}$$

where u^* is the stationary input. As the model considers m inputs, only m state variables can be directly imposed and the rest will be a consequence of these. If we write $q^* = [q_u^* \;\; q_k^*]^\top$, where $q_k^* \in \mathbb{R}^m$ is the vector of generalized displacements imposed at the nodes where the inputs are applied, and $q_u^* \in \mathbb{R}^{n-m}$ is the vector of generalized displacements that are a consequence of q_k^*. Then, writing the stiffness matrix as

$$K = \begin{bmatrix} K_{uu} & K_{uk} \\ K_{ku} & K_{kk} \end{bmatrix}$$

(12) can be written equivalently as

$$\begin{bmatrix} K_{uu} & K_{uk} \\ K_{ku} & K_{kk} \end{bmatrix} \begin{bmatrix} q_u^* \\ q_k^* \end{bmatrix} = \begin{bmatrix} 0 \\ I \end{bmatrix} u^*$$

leading to

$$q_u^* = - K_{uu}^{-1} K_{uk} \, q_k^* \tag{13}$$

$$u^* = (K_{kk} - K_{ku} K_{uu}^{-1} K_{uk}) \, q_k^* \tag{14}$$

On the other hand, using (13) we have that

$$q^* = \begin{bmatrix} -K_{uu}^{-1} \, K_{uk} \\ I \end{bmatrix} q_k^* \tag{15}$$

4.1 Controller based on direct discretization

Since the model (1) has m inputs, only m state variables can be directly imposed on the system. Therefore, we seek to design a PH controller with $n_c = m$ states. Choosing the input map of the controller as $g_c = I$, it follows from the ME that $R_c = 0$, $J_c = 0$, and

$$\frac{\partial F}{\partial x} = (g^\top J^{-1})^\top = \begin{bmatrix} 0 \\ B \end{bmatrix} \tag{16}$$

then, given the structure of the matrix B we have

$$F(x) = q_k + C \tag{17}$$

from the Casimir function in (9) we can write that

$$q_k = x_c + C \tag{18}$$

Note that from (18) and (15) we can write q as

$$q = \begin{bmatrix} -K_{uu}^{-1} \, K_{uk} \\ I \end{bmatrix} (x_c + C) \tag{19}$$

Choosing the desired closed-loop energy function as

$$H_d = \frac{1}{2} p^\top M^{-1} \, p + \frac{1}{2} (q - q^*)^\top K_{ds} \, (q - q^*)$$

with $K_{ds} = K_{ds}^\top > 0$ a controller design parameter that can be interpreted as a desired stiffness matrix in closed-loop. According to (10), the energy function of the controller H_c is given by $H_c = H_d - H$. So, for H_c it follows that

$$\begin{aligned} H_c = \; & \frac{1}{2} q^\top (K_{ds} - K) \, q + \frac{1}{2} q^{*\top} K_{ds} \, q^* - \\ & \frac{1}{2} q^\top K_{ds} \, q^* - \frac{1}{2} q^{*\top} K_{ds} \, q \end{aligned} \tag{20}$$

Replacing (19) in (20) gives

$$\begin{aligned} H_c = \; & \frac{1}{2} (x_c + C)^\top Q_0 \, (x_c + C) + \frac{1}{2} q^{*\top} K^* \, q^* - \\ & \frac{1}{2} (x_c + C)^\top Q_1 \, q^* - \frac{1}{2} q^{*\top} Q_1^\top \, (x_c + C) \end{aligned}$$

where

$$\begin{aligned} Q_0 = \; & K_{uk}^\top K_{uu}^{-\top} K_{ds_{uu}} K_{uu}^{-1} K_{uk} - K_{uk}^\top K_{uu}^{-\top} K_{ds_{uk}} - \\ & K_{ds_{ku}} K_{uu}^{-1} K_{uk} + K_{ds_{kk}} + K_{ku} K_{uu}^{-1} K_{uk} - K_{kk} \end{aligned} \tag{21}$$

$$Q_1 = \begin{bmatrix} -K_{uk}^\top K_{uu}^{-\top} K_{ds_{uu}} + K_{ds_{ku}} \\ -K_{uk}^\top K_{uu}^{-\top} K_{ds_{uk}} + K_{ds_{kk}} \end{bmatrix}^\top \tag{22}$$

Then, the gradient of H_c with respect to the controller states x_c is given by

$$\nabla_{x_c} H_c = Q_0 \, (x_c + C) - Q_1 \, q^* \tag{23}$$

According to (6) the PH controller, denoted by Σ_{c_1}, is described by the matrices $J_c = 0$, $R_c = 0$, $g_c = I$, and $\nabla_{x_c} H_c$ in (23), that is

$$\begin{aligned} \dot{x}_c &= u_c \\ y_c &= Q_0\,(x_c + C) - Q_1\, q^* \end{aligned} \tag{24}$$

Note that if we only want to change the equilibrium point without changing the energy function of the system, from (21) we have $Q_0 = 0$ because $K_{ds} = K$. Then, according to (7) and taking into account Q_1 defined in (22), the stationary input u^* that brings the plant (1) to the desired equilibrium point is given by

$$u^* = -y_c = (K_{kk} - K_{ku} K_{uu}^{-1} K_{uk})\, q_k^* \tag{25}$$

Remark 1: Due to the structure of the model (1), in particular the input map g, the functions in (17) are Casimir functions for no matter the dimension of the discretized model. Also, from (17) note that the componets of $F(x)$ depend on only one state of the model, so it was possible to establish an invertible relationship between the imposed states of the plant and those of the controller.

4.2 Controller based on a reduced model

Consider the balanced model in (3) with state $x_B \in \mathbb{R}^N$. Since the transformation matrix T_B yields $Q_B = T_B^\top Q T_B = I$, the parameters of the system, originally encoded in the energy function, are now encoded in the interconnection matrix J_B. The resulting interconnection matrix J_B has a block diagonal structure, that is

$$J_B = \begin{bmatrix} \ddots & & 0 \\ & [J_{B_k}] & \\ 0 & & \ddots \end{bmatrix}, \quad J_{B_k} = \begin{bmatrix} 0 & \Im\{\lambda_k\} \\ -\Im\{\lambda_k\} & 0 \end{bmatrix}$$

with $\Im\{\lambda_k\}$ the imaginary part of the k-th eigenvalue of JQ. With the above, each block J_{B_k} can be associated with an eigenvalue of the system, and the state x_B can be interpreted as modal state. Rearranging the matrix J_B in such a way that the first two states of x_B are associated with the first eigenvalue, and so on, it is possible to reduce the model by taking modal criteria. Note that although the reduced state vector $x_r \in \mathbb{R}^r$ no longer represents the generalized momentum and positions due to the balanced transformation, using (5) and (12) the desired equilibrium point x_r^* of Σ_r can be found, which is given by the desired equilibrium point x^*. That is,

$$x_r^* = T_{ri}\, x^*$$

Choosing the input map of the controller as $g_c = I$, it follows from the ME that $R_c = 0$ and

$$\begin{aligned} J_c &= g_r^\top J_r^{-\top} g_r \\ \frac{\partial F}{\partial x_r} &= (g_r^\top J_r^{-1})^\top \end{aligned} \tag{26}$$

Integrating (26) respect to x_r we get

$$F(x_r) = \left(\frac{\partial F^\top}{\partial x_r}\right) x_r + C \tag{27}$$

from the Casimir function in (9) we can write that

$$\begin{aligned} x_c &= \left(\frac{\partial F^\top}{\partial x_r}\right) x_r - C \\ x_r &\approx \left(\frac{\partial F^\top}{\partial x_r}\right)^\dagger (x_c + C) \end{aligned} \tag{28}$$

where $()^\dagger$ denotes the Moore-Penrose pseudoinverse. Note that unlike the previous case, since the input map g_r does not have a simple structure like that given by B, the states of the reduced model cannot be expressed explicitly as a function of the states of the controller, so it is approximated by a pseudoinverse. The implications of this on the controller and the closed-loop system will be discussed in Remark 2. Now, consider a quadratic desired energy function as (11) described by a desired energy matrix $Q_d \in \mathbb{R}^{N \times N}$ given by

$$Q_d = \begin{bmatrix} M^{-1} & 0 \\ 0 & K_d \end{bmatrix}$$

where $K_d = K_d^\top > 0$ is a controller design parameter that can be interpreted as a desired stiffness matrix in closed-loop. Then, a reduced order energy matrix is given by

$$Q_{dr} = T_r^\top Q_d T_r$$

and the desired closed-loop energy function is given by

$$H_d = \frac{1}{2}(x_r - x_r^*)^\top Q_{dr}\,(x_r - x_r^*)$$

According to (10), the energy function of the controller H_c is given by $H_c = H_d - H_r$, where H_r is the open-loop energy function of the reduced system. For H_c it follows that

$$\begin{aligned} H_c = {} & \frac{1}{2} x_r^\top (Q_{dr} - Q_r)\, x_r + \frac{1}{2} x_r^{*\top} Q_{dr}\, x_r^* - \\ & \frac{1}{2} x_r^\top Q_{dr}\, x_r^* - \frac{1}{2} x_r^{*\top} Q_{dr}\, x_r \end{aligned} \tag{29}$$

Replacing (28) in (29) gives

$$\begin{aligned} H_c = {} & \frac{1}{2}(x_c + C)^\top Q_2\,(x_c + C) + \frac{1}{2} x_r^{*\top} Q_{dr}\, x_r^* - \\ & \frac{1}{2}(x_c + C)^\top Q_3\, x_r^* - \frac{1}{2} x_r^{*\top} Q_3^\top (x_c + C) \end{aligned}$$

where

$$\left(\frac{\partial F}{\partial x_r}\right)^\dagger = \left(g_r^\top J_r^{-1} J_r^{-\top} g_r\right)^{-1} g_r^\top J_r^{-1} \tag{30}$$

$$Q_2 = \left(\frac{\partial F}{\partial x_r}\right)^\dagger (Q_{dr} - Q_r) \left(\frac{\partial F^\top}{\partial x_r}\right)^\dagger \tag{31}$$

$$Q_3 = \left(\frac{\partial F}{\partial x_r}\right)^\dagger Q_{dr} \tag{32}$$

Then, the gradient of H_c with respect to the controller states x_c is given by

$$\nabla_{x_c} H_c = Q_2\,(x_c + C) - Q_3\, x_r^* \tag{33}$$

According to (6), the PH controller, denoted by Σ_{c_2}, is described by the matrices $J_c = g_r^\top J_r^{-\top} g_r$, $R_c = 0$, $g_c = I$, and $\nabla_{x_c} H_c$ in (33), that is

$$\begin{aligned} \dot{x}_c &= (g_r^\top J_r^{-\top} g_r)(Q_2\,(x_c + C) - Q_3\, x_r^*) + u_c \\ y_c &= Q_2\,(x_c + C) - Q_3\, x_r^* \end{aligned} \tag{34}$$

As in the previous case, if we only want to change the equilibrium point without changing the energy function of the system, from (31) we have $Q_2 = 0$ because $Q_{dr} = Q_r$. Then, according to (7) and taking into account (30), (32) and $Q_r = I$, the stationary input u_r^* that brings the plant Σ_r to the desired equilibrium point is given by

$$u_r^* = -y_c = \left(g_r^\top J_r^{-1} J_r^{-\top} g_r\right)^{-1} g_r^\top J_r^{-1}\, x_r^* \tag{35}$$

Remark 2: From (27), note that each component of $F(x_r)$ depend on all the states of the reduced model, so it was not possible to establish an invertible relationship between the states of the plant and the controller. This is attributed to the effect of applying the order reduction technique, which despite preserving the PH structure, changes the simple structure of the input map. As a result of using (28) in (29) to design the PH controller, it is expected that there will be differences between the desired equilibrium point and the one to which the closed-loop system converges.

5. EXAMPLE: THE EULER-BERNOULLI BEAM

This section shows by means of numerical simulations the behavior of controllers Σ_{c_1} and Σ_{c_2} when applied to a BC-PHS corresponding to a linear Euler-Bernoulli beam defined on a 1D spatial domain. The BC-PHS defined along the z axis is given by (Brugnoli et al., 2019)

$$\begin{aligned} \underbrace{\begin{pmatrix}\dot p\\ \dot q\end{pmatrix}}_{\dot x} &= \underbrace{\begin{bmatrix}0 & -\partial_z^2\\ \partial_z^2 & 0\end{bmatrix}}_{\mathcal{J}} \underbrace{\begin{pmatrix}e_p\\ e_q\end{pmatrix}}_{\delta_x H} \\ u_\partial &= [e_p(0)\ -\partial_z e_p(0)\ e_q(L)\ \partial_z e_q(L)]^\top \\ y_\partial &= [\partial_z e_q(0)\ e_q(0)\ \partial_z e_p(L)\ -e_p(L)]^\top \end{aligned} \tag{36}$$

with the Hamiltonian H given by

$$H(x) = \frac{1}{2}\int_0^L \left(\frac{p^2}{\mu(z)} + EI(z)q^2\right) dz$$

where $\partial_z^2 = \partial^2/\partial z^2$, $\mathcal{J}$ is a skew-adjoint and second order differential operator (Le Gorrec et al., 2005), $p(z,t) = \mu(z)\frac{\partial w}{\partial t}$ is the linear momentum, $q(z,t) = \frac{\partial^2 w}{\partial z^2}$ is the deformation due to bending, $w(z,t)$ is the deflection at point $z \in (0 \le z \le L)$ and time t, L is the beam length, $\mu(z)$ is the mass per unit length, and $EI(z)$ is the bending stiffness of the beam. Note that $e_p = \delta_p H = p/\mu(z)$, and $e_q = \delta_q H = EIq$, where δ_x denote a variational derivative respect to x. The cantilever boundary conditions with one end fixed at $z = 0$ and the other actuated at $z = L$ are given by $u_\partial = [0\ 0\ u_1\ u_2]^\top$, where $u_1(t)$ and $u_2(t)$ are the imposed internal shearing force and internal bending moment at the boundary $z = L$, respectively. For simulations purposes (36) the following parameter are considered, $L = 1.36\ (m)$, $\mu = 2.37\ (kg/m)$, $EI = 150\ (N \cdot m^2)$, and discretized using sFEM with ten finite elements (Cook et al., 2007), implying that $p, q \in \mathbb{R}^{22}$, $K, M \in \mathbb{R}^{22\times 22}$, $B \in \mathbb{R}^{22\times 2}$, $u \in \mathbb{R}^2$, and $x = [p\ q]^\top \in \mathbb{R}^{44}$. The low order model for the design of Σ_{c_1} is obtained by discretizing (36) using sFEM with only one finite element, so this corresponds to the lowest order model that can be obtained by sFEM. The state $x = [p\ q]^\top \in \mathbb{R}^8$ is such that p contains the linear and angular momentum of the nodes at $z = 0$ and $z = L$, and q the linear and angular displacements at the same nodes. In order to design controllers from comparable models, the reduced model Σ_r is such that $x_r \in \mathbb{R}^8$. Then, for controllers Σ_{c_1} and Σ_{c_1}, their design parameters are the matrices $K_{ds} = K_{ds}^\top > 0 \in \mathbb{R}^{4\times 4}$ and $K_d = K_d^\top > 0 \in \mathbb{R}^{22\times 22}$, respectively. K_{ds} and K_d are chosen as the stiffness matrices resulting from discretizing (36) using one and ten finite elements, respectively; L, μ remain at the indicated values, and $EI = 75$, $EI = 150$, $EI = 300$. Note that when $EI = 150$ we are in the case where $K_{ds} = K_s$, and $K_d = K$, so both controllers only change the desired equilibrium point without affecting the dynamic characteristics of the beam. Finally, as the beam model does not consider dissipation, damping injection is applied with $K_{DI} = \text{diag}\{2{,}2\}$. *Case 1:* Consider a desired equilibrium point imposing $q_k^* = [q_{21}^*\ \ q_{22}^*]^\top = [0\ \ -0.01]^\top$. *Case 2:* Consider a desired equilibrium point imposing $q_k^* = [q_{21}^*\ \ q_{22}^*]^\top = [-0.03\ \ -0.01]^\top$. Note that the equilibrium point given by q^* represents a desired equilibrium shape for the beam. Figures 1 and 3 show the equilibrium shape reached by the beam when it is interconnected with both controllers. It is observed that the controller Σ_{c_1} achieves smaller variations with respect to the desired equilibrium shape when both, the equilibrium point and the dynamics of the beam are changed, that is, for the cases with $EI = 75$; $EI = 300$. On the other hand, if we only want to change the equilibrium point the controller Σ_{c_2} achieves a better result. Figures 2 and 4 show the dynamic response of q_{21}, which represents the linear displacement of the beam at $z = L$. It can be seen that both controllers affect the dynamic response of the beam. Note that increasing EI makes the dynamics more oscillatory, which is consistent with physics since EI is a measure of bending stiffness of the beam.

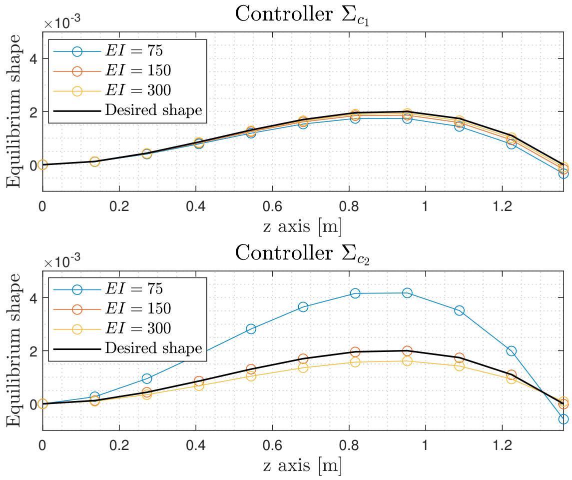

Fig. 1. Equilibrium shape, Case 1.

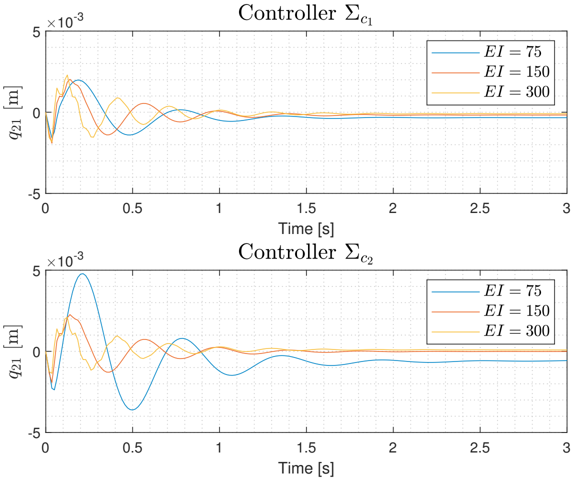

Fig. 2. Dynamic response, Case 1.

6. CONCLUSION

Two passive controllers were designed from different low-order models. The structure of the input map of the plant used in the design of the controller Σ_{c_1} allowed to find

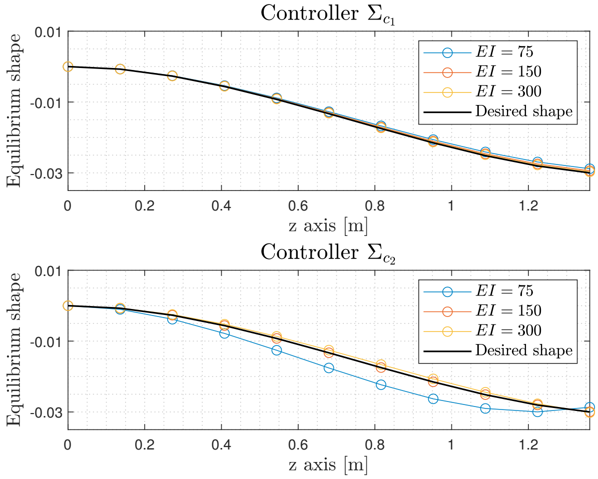

Fig. 3. Equilibrium shape, Case 2.

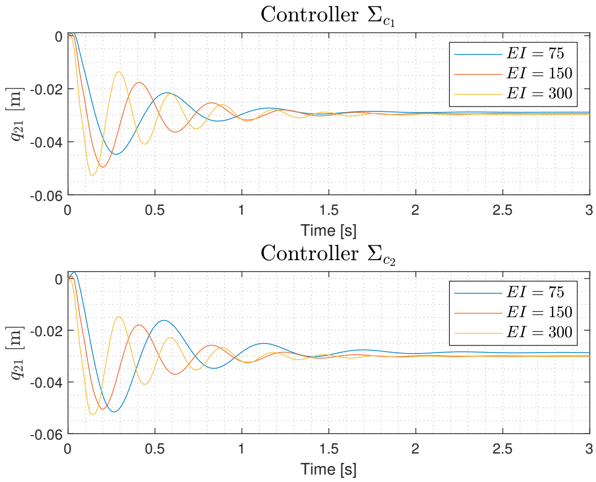

Fig. 4. Dynamic response, Case 2.

Casimir functions for any dimension of the discretized model. Furthermore, it was possible to find an invertible relationship between the states of the controller and the imposed states of the plant, which allowed to characterize the controller energy function accurately. In the case of the controller Σ_{c_2}, since the input map of the reduced model does not have a simple structure, the Casimir functions depend on all the states of the model, so the relation between the controller states and those of the plant is given by means of a pseudoinverse, leading to an approximate controller energy function. Hence it is expected to be differences between the desired equilibrium point and the one at which the closed-loop system converges. The latter was observed on the example of a linear Euler-Bernoulli beam. As future work it remains to study in a quantitative fashion these differences. The extension of the results to mechanical systems defined on 2D spatial domains and that include damping and actuation within the domain are currently under consideration.

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