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PDE-constrained optimization for nonlinear fluid mechanics and fractional diffusion

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Abstract

This work addresses three optimal control problems with constraints given by partial differential equations (PDEs) and a coupled system of nonlocal PDEs. For each control problem, we perform an exhaustive study that includes: existence of optimal solutions, analysis of first order necessary and second order necessary and sufficient optimality conditions, as well as the study of regularity properties of the optimal solutions. After this, we perform a finite element discretization analysis. We consider two discretization schemes: a fully discrete scheme, where the control variable is approximated by piecewise constant functions, and a semidiscrete scheme, where the control variable is not discretized. For both discretization schemes, we analyze the convergence of discrete solutions to continuous solutions. Finally, we derive a priori error estimates.

For the coupled system of nonlocal PDEs, we investigate existence, uniqueness, and regularity properties of solutions. Finally, we study the finite element discretization and the approximation of the discrete scheme using the Schwarz method are analyzed, providing associated a priori error estimates and demonstrating the convergence of this method.

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Chapter 1

Introduction

The development of accurate mathematical models and efficient numerical algorithms is crucial for overcoming numerous challenges in science and engineering. However, the inherent complexity of real world problems often requires models to be simplified. As a result, model parameters may be inaccurate, unknown or subject to uncertainty. To overcome these possible scenarios, PDE constrained optimization problems enable optimal parameter adjustments for accurate modeling (parameter identification) or the desired system control (optimal control).

The theory of PDE constrained optimization has progressed considerably in the past decades, with several monographs concerning its mathematical treatment in terms of first and second order optimality conditions for existence and uniqueness of an optimal solution, together with numerical optimization algorithms [41, 68, 69, 104]. The applications include fluid dynamics, material sciences, control of heating and cooling processes, nonlocal diffusion phenomena, among others.

In this thesis, we are interested in PDE-constrained optimization problems that admit the form

$$\min\{J(y, u) : y \in \mathbb{Y}, u \in \mathbb{U}\} \quad \text{subject to} \quad e(y, u) = 0, \quad u \in \mathbb{U}_{ad},$$

where

- \mathbb{Y} and \mathbb{U} are function spaces associated with the *state variable* y and the *control variable* u , respectively.
- $J : \mathbb{Y} \times \mathbb{U} \rightarrow \mathbb{R}$ denotes a suitable *cost functional*,
- $e(y, u)$ describes the so-called *state equation*, and finally,
- $\mathbb{U}_{ad} \subset \mathbb{U}$ denotes the *admissible set* and accounts for suitable constraints imposed on the control variable (control constraints).

The goal of a PDE-constrained optimization problem is to determine an admissible control–state pair that minimizes the value of the cost functional J . The basic problematic is thus twofold. First, study the optimal control problem, that is: provide existence of optimal solutions, together with first and, since nonlinear PDEs are considered, second order optimality conditions. Secondly, the design of efficient solution techniques for the computation of numerical approximations. We notice that first order optimality conditions are thus required to analyze regularity properties of the involved variables, while second order optimality conditions allow us to provide error estimates.

In the study of an optimal control problem, we have to overcome a number of challenges. One of them is a system of nonlinear equations resulting from the derivation of optimality conditions that strongly depend on the state problem. From an optimization point of view, several algorithms allow progress in this study, such as descent methods, Newton methods, sequential quadratic programming, active set strategies, to name a few. In this context, it is natural to consider Galerkin discretizations.

Tesis outline and contributions

In the following we outline the problems we will study in this thesis.

Chapter 2: An optimal control problem for the Navier–Stokes equations with point sources

We analyze, in two dimensions, an optimal control problem for the Navier–Stokes equations where the control variable corresponds to the amplitude of forces modeled as point sources; control constraints are also considered. This particular setting leads to solutions to the state equation exhibiting reduced regularity properties. We operate under the framework of Muckenhoupt weights, Muckenhoupt-weighted Sobolev spaces, and the corresponding weighted norm inequalities and derive the existence of optimal solutions and first and, necessary and sufficient, second order optimality conditions.

The discussion in this chapter is based on the reference [57].

Chapter 3: Bilinear optimal control for the fractional Laplacian: analysis and discretization

We adopt the *integral* definition of the fractional Laplace operator and study an optimal control problem on *Lipschitz domains* that involves a fractional elliptic partial differential equation (PDE) as state equation and a control variable that enters the state equation as a coefficient; pointwise constraints on the control variable are considered as well. We establish the existence of optimal solutions and analyze first and, necessary and sufficient, second order optimality conditions. Regularity estimates for optimal variables are also analyzed. We develop two finite element discretization strategies: a

semidiscrete scheme in which the control variable is not discretized, and a fully discrete scheme in which the control variable is discretized with piecewise constant functions. For both schemes, we analyze the convergence properties of discretizations and derive error estimates.

The discussion in this chapter is based on the reference [13].

Chapter 4: Fractional, semilinear, and sparse optimal control: a priori error bounds

In this work, we use the integral definition of the fractional Laplace operator and study a sparse optimal control problem involving a fractional, semilinear, and elliptic partial differential equation as state equation; control constraints are also considered. We establish the existence of optimal solutions and first- and second-order optimality conditions. We also analyze regularity properties for optimal variables. We propose and analyze two finite element strategies of discretization: a fully discrete scheme, where the control variable is discretized with piecewise constant functions, and a semidiscrete scheme, where the control variable is not discretized. For both discretization schemes, we analyze convergence properties and a priori error bounds.

The discussion in this chapter is based on the reference [14].

Chapter 5: A nonlocal coupled system: analysis and discretization

We consider a nonlocal coupled system resulting from the minimization of a suitable energy functional. This system involves two nonlocal operators related to the *restricted* fractional Laplacian posed on different domains and another nonlocal term representing the coupling. We prove that the considered energy functional admits a unique minimizer and investigate regularity estimates for such a minimizer. We also propose a suitable discretization scheme based on finite elements and derive error estimates. Finally, we propose an alternating Schwarz-type method, both for the continuous and the discrete settings, and prove its convergence.

The paper associated to this work is in submission process.

Chapter 2

An optimal control problem for the Navier–Stokes equations with point sources

2.1 Introduction

The purpose of this paper is to study the existence of optimal solutions and first and, necessary and sufficient, second order optimality conditions for an optimal control problem that involves the stationary Navier–Stokes equations. The control variable corresponds to the amplitude of forces modeled as point sources supported at some prescribed points of the underlying spatial domain (Dirac measures); control constraints are also considered. The thus singular control forcing appears in the right-hand side of the momentum equation. We notice that, since Dirac measures are supported at points, and points have Lebesgue measure zero, the aforementioned optimization setting can be seen as an instance of *sparse* PDE-constrained optimization [26, 37, 100, 108] and finds relevance in applications where one can specify the position of actuators at finitely many prespecified points. We mention references [12] and [66] for applications within the context of the active control of sound and vibrations, respectively. Regarding analysis, we mention references [7, 10, 63], where the corresponding PDE-constrained optimization problem for when the state equation is a Poisson problem is considered. These references also design and analyze some suitable finite element discretizations. Extensions of the theory to the Stokes and semilinear elliptic equations have been recently investigated in [59] and [88], respectively. To the best of our knowledge, the only work available in the literature that considers an optimal control problem for the stationary Navier–Stokes equations with a control that is measure valued is

[27]. Under the assumption that the underlying domain $\Omega \subset \mathbb{R}^2$ is of class C^2 , the authors derive the existence of local solutions for the corresponding optimal control problem and derive necessary and sufficient conditions for local optimality of controls. In addition, on the basis of a suitable second order condition, the authors prove the stability of optimal states with respect to perturbations of the optimal control problem data.

In our work we analyze an optimal control problem for the stationary Navier–Stokes equations with a control variable that corresponds to the amplitude of forces modeled as point sources. This setting leads to the first difficulty within our analysis: standard energy arguments do not apply to obtain suitable estimates and solutions to the Navier–Stokes equations exhibit reduced regularity properties. In order to deal with such a singular setting, we operate under the framework developed in [92, 93], which is based on the theory of Muckenhoupt weights, Muckenhoupt-weighted Sobolev spaces, and weighted norm inequalities. A second difficulty within our analysis is the nonuniqueness of solutions to the Navier–Stokes equations. An assumption guaranteeing local uniqueness of the state equation around optimal controls is thus needed to derive first and second order optimality conditions [27, 29]. We thus operate under the framework of *regular solutions* (see Definition 2.4.1) [27, 29, 94]. Note that this framework is satisfied whenever a suitable smallness assumption on controls is fulfilled. We provide a complete analysis for our optimal control problem that includes existence of optimal solutions (Theorem 2.5.1), first order optimality conditions (Theorem 2.6.3), and necessary and sufficient second order optimality conditions (Theorems 2.6.6 and 2.6.7). As instrumental results, we analyze a suitable linearization of the Navier–Stokes equations and the corresponding adjoint state equations in weighted spaces. We also analyze regularity properties for the solution to the adjoint equations. In addition to the difficulties that were previously mentioned, we have to deal with the fact that solutions to the state and adjoint equations lie in different function spaces. The analysis that we provide thus requires fine properties of Muckenhoupt weights and embeddings between weighted and non-weighted spaces. This subtle intertwining of ideas is one of the highlights of our contribution.

The contents of our manuscript are organized as follows. In Sect. 2.2 we introduce the PDE-constrained optimization problem that is under consideration. We collect background information and the main assumptions under which we shall operate in Sect. 2.3. Here, we also introduce the concept of *regular solution* for the Navier–Stokes equations and prove that an operator associated to the linearization of such a system is an isomorphism on suitable weighted spaces. In Sect. 2.5 we introduce a weak formulation for our optimal control problem and prove the existence of solutions. Sect. 2.6 is dedicated to the analysis of optimality conditions: we derive first and, necessary and sufficient, second order optimality conditions.

2.2 Statement of the Problem

To describe our problem, we let $\Omega \subset \mathbb{R}^2$ be an open and bounded domain with Lipschitz boundary $\partial\Omega$ and let $\emptyset \neq \mathcal{D} \subset \Omega$ be a finite ordered set with cardinality $\#\mathcal{D} =: \ell$. Given a desired velocity field $\mathbf{y}_\Omega \in \mathbf{L}^2(\Omega)$ and a regularization parameter $\eta > 0$, we introduce the cost functional

$$J(\mathbf{y}, \mathcal{U}) := \frac{1}{2} \|\mathbf{y} - \mathbf{y}_\Omega\|_{\mathbf{L}^2(\Omega)}^2 + \frac{\eta}{2} \sum_{t \in \mathcal{D}} |\mathbf{u}_t|^2, \quad \mathcal{U} = (\mathbf{u}_1, \dots, \mathbf{u}_\ell), \quad \mathbf{u}_t \in \mathbb{R}^2. \quad (2.2.1)$$

The PDE-constrained optimization problem under consideration reads as follows: Find $\min J(\mathbf{y}, \mathcal{U})$ subject to the stationary Navier–Stokes equations

$$-\nu \Delta \mathbf{y} + (\mathbf{y} \cdot \nabla) \mathbf{y} + \nabla p = \sum_{t \in \mathcal{D}} \mathbf{u}_t \delta_t \text{ in } \Omega, \quad \operatorname{div} \mathbf{y} = 0 \text{ in } \Omega, \quad \mathbf{y} = \mathbf{0} \text{ on } \partial\Omega, \quad (2.2.2)$$

and the control constraints

$$\mathcal{U} \in \mathcal{U}_{ad}, \quad \mathcal{U}_{ad} := \{\mathcal{V} = (\mathbf{v}_1, \dots, \mathbf{v}_\ell) \in [\mathbb{R}^2]^\ell : \mathbf{a}_t \leq \mathbf{v}_t \leq \mathbf{b}_t \text{ for all } t \in \mathcal{D}\}, \quad (2.2.3)$$

with $\mathbf{a}_t, \mathbf{b}_t \in \mathbb{R}^2$ satisfying $\mathbf{a}_t < \mathbf{b}_t$ for every $t \in \mathcal{D}$. We immediately comment that, throughout this work, vector inequalities must be understood componentwise and that $|\cdot|$ denotes the euclidean norm in \mathbb{R}^2 . In (2.2.2), \mathbf{y} represents the velocity of the fluid, p represents the pressure, $\nu > 0$ denotes the kinematic viscosity, and δ_t corresponds to the Dirac delta supported at the interior point $t \in \mathcal{D}$.

2.3 Notation and Preliminaries

The main purpose of this section is to introduce the main notation and recall basic results which we shall use later on.

2.3.1 Notation

Let \mathfrak{X} be a Banach function space. We denote by \mathfrak{X}' , \mathfrak{X}'' , and $\|\cdot\|_{\mathfrak{X}}$ the dual, the bidual, and the norm of \mathfrak{X} , respectively. Let $\{x_n\}_{n \in \mathbb{N}}$ be a sequence in \mathfrak{X} . We denote by $x_n \rightarrow x$ and $x_n \rightharpoonup x$ the strong and weak convergence, respectively, of $\{x_n\}_{n \in \mathbb{N}}$ to x in \mathfrak{X} . We denote by $\langle \cdot, \cdot \rangle_{\mathfrak{X}', \mathfrak{X}}$ the duality pairing between \mathfrak{X}' and \mathfrak{X} and simply write $\langle \cdot, \cdot \rangle$ when \mathfrak{X}' and \mathfrak{X} are clear from the context. We write $\mathfrak{X} \hookrightarrow \mathfrak{Y}$ to denote that \mathfrak{X} is continuously embedded in the Banach function space \mathfrak{Y} .

Let $E \subset \mathbb{R}^2$ be a Lebesgue measurable set. We denote the Lebesgue measure of such a set by $|E|$. For

$f : E \rightarrow \Omega$, we set

$$\int_E f = \frac{1}{|E|} \int_E f.$$

By $a \lesssim b$ we mean $a \leq Cb$, with a positive constant C that does not depend on either a or b . The value of C might change at each occurrence. If the particular value of the constant C is of relevance for our analysis, we will thus assign it a name.

2.3.2 Muckenhoupt Weights

By a weight, we shall mean a locally integrable function ω on \mathbb{R}^2 such that $\omega(x) > 0$ for a.e. $x \in \mathbb{R}^2$. A special class of weights that will be of importance for our analysis is the so-called Muckenhoupt class A_2 [50, 53, 85, 105].

Definition 2.3.1 (Muckenhoupt Class A_2). *A weight ω belongs to the Muckenhoupt class A_2 if*

$$[\omega]_{A_2} := \sup_B \left(\int_B \omega \right) \left(\int_B \omega^{-1} \right) < \infty,$$

where the supremum is taken over all balls B in \mathbb{R}^2 . We call $[\omega]_{A_2}$ the Muckenhoupt characteristic of ω .

We refer the interested reader to [50, 53, 85, 105] for basic facts about the Muckenhoupt class A_2 . To present prototypical examples of Muckenhoupt weights, we let \mathcal{K} be a smooth compact submanifold of dimension $k \in \{0, 1\}$ and define $d_{\mathcal{K}}^\alpha(x) := \text{dist}(x, \mathcal{K})^\alpha$. The weight $d_{\mathcal{K}}^\alpha$ belongs to the Muckenhoupt class A_2 provided $\alpha \in (-(2-k), (2-k))$; see [5] and [54, Lemma 2.3(vi)]. We thus identify the following two particular cases:

1. Let $z \in \Omega$. Then, the weight $d_z^\alpha \in A_2$ if $\alpha \in (-2, 2)$.
2. Let $\gamma \subset \Omega$ be a smooth closed curve without self-intersections. Then, the weight $d_\gamma^\alpha \in A_2$ if $\alpha \in (-1, 1)$.

As a consequence of the fact that the lower dimensional objects z and γ are strictly contained in Ω , there are neighborhoods of $\partial\Omega$ where the weights d_z^α and d_γ^α have no degeneracies or singularities. This simple observation motivates the following restricted class of Muckenhoupt weights [54, Definition 2.5].

Definition 2.3.2 (Class $A_2(G)$). *Let $G \subset \mathbb{R}^2$ be a Lipschitz domain. We say that $\omega \in A_2$ belongs to $A_2(G)$ if there is an open set $\mathcal{G} \subset G$ and $\varepsilon, \omega_l > 0$ such that $\{x \in G : \text{dist}(x, \partial G) < \varepsilon\} \subset \mathcal{G}$, $\omega \in C(\bar{\mathcal{G}})$, and $\omega(x) \geq \omega_l$ for all $x \in \bar{\mathcal{G}}$.*

2.3.3 Muckenhoupt-weighted Sobolev Spaces

Let $\omega \in A_2$ and $G \subset \mathbb{R}^2$ be an open set. We define

- $L^2(\omega, G)$ as the space of measurable functions f on G such that

$$\|f\|_{L^2(\omega, G)} := \left(\int_G |f|^2 \omega \right)^{\frac{1}{2}} < \infty,$$

- $H^1(\omega, G) := \{f \in L^2(\omega, G) : D^\alpha f \in L^2(\omega, G) \text{ for } |\alpha| \leq 1\}$, endowed with the norm $\|f\|_{H^1(G)} := (\|f\|_{L^2(G)}^2 + \|\nabla f\|_{L^2(G)}^2)^{\frac{1}{2}}$, and
- $H_0^1(\omega, G)$ as the closure of $C_0^\infty(G)$ in $H^1(\omega, G)$.

For basic properties of these spaces, such as approximation by smooth functions, extensions theorems, and interpolation inequalities, we refer the interested reader to [105, Chapter 2].

Spaces of vector valued functions will be denoted by boldface uppercase letters whereas lowercase bold letters will be used to denote vector valued functions. In particular, we introduce $\mathbf{H}_0^1(\omega, G)$ and $|\cdot|_{\mathbf{H}^1(\omega, G)}$ as follows:

$$\mathbf{H}_0^1(\omega, G) := [H_0^1(\omega, G)]^2, \quad |\mathbf{v}|_{\mathbf{H}^1(\omega, G)} := \|\nabla \mathbf{v}\|_{\mathbf{L}^2(\omega, G)} = \sum_{i=1}^2 \|\nabla v_i\|_{L^2(\omega, G)},$$

for every $\mathbf{v} \in \mathbf{H}_0^1(\omega, G)$.

2.3.4 Weighted Inequalities and Embeddings

The following fundamental result, which is known as *reverse Hölder inequality*, will be essential for our analysis; see [50, Theorem 7.4].

Proposition 2.3.1 (Reverse Hölder Inequality). *If $\omega \in A_2$, then there exists a positive constant ϵ such that, for every ball $B \subset \mathbb{R}^2$, we have*

$$\int_B \omega^{1+\epsilon} \lesssim \left(\int_B \omega \right)^{1+\epsilon}.$$

The hidden constant only depends on the Muckenhoupt characteristic $[\omega]_{A_2}$.

We now present the following embedding results.

Theorem 2.3.3 (Continuous Embeddings). *If $\omega \in A_2$, then there exists $\epsilon > 0$ such that $\mathbf{H}_0^1(\omega, \Omega) \hookrightarrow \mathbf{L}^{2+\epsilon}(\Omega)$ and there exists $\kappa > 1$ such that $\mathbf{H}_0^1(\omega, \Omega) \hookrightarrow \mathbf{W}_0^{1, \kappa}(\Omega)$.*

Proof. We prove the first embedding result; the second one follows from similar considerations. Let $\omega \in A_2$ and $\Phi \in \mathbf{H}_0^1(\omega, \Omega)$. An application of [53, Theorem 1.3] implies that $\Phi \in \mathbf{L}^4(\omega, \Omega)$. We thus invoke Hölder's inequality to obtain

$$\int_{\Omega} |\Phi|^{2+\varepsilon} = \int_{\Omega} |\Phi|^{2+\varepsilon} \omega^{\frac{2+\varepsilon}{4}} \omega^{-\frac{2+\varepsilon}{4}} \leq \left(\int_{\Omega} |\Phi|^4 \omega \right)^{\frac{2+\varepsilon}{4}} \left(\int_{\Omega} \omega^{-\frac{2+\varepsilon}{2-\varepsilon}} \right)^{\frac{2-\varepsilon}{4}}, \quad (2.3.1)$$

for some $\varepsilon > 0$. We observe that, since $\omega \in A_2$ and $\frac{2+\varepsilon}{2-\varepsilon} = 1 + \delta$, with $\delta = \frac{2\varepsilon}{2-\varepsilon}$, the reverse Hölder inequality of Proposition 2.3.1 allows us to obtain

$$\int_{\Omega} \omega^{-\frac{2+\varepsilon}{2-\varepsilon}} = \int_{\Omega} \omega^{-(1+\delta)} \lesssim |\Omega|^{-\delta} \left(\int_{\Omega} \omega^{-1} \right)^{1+\delta} = |\Omega| \left(\int_{\Omega} \omega^{-1} \right)^{\frac{2+\varepsilon}{2-\varepsilon}}.$$

Here, $\varepsilon > 0$ is sufficiently small such that the previously defined parameter δ is less or equal that the one dictated by the reverse Hölder inequality. Since $\int_{\Omega} \omega^{-1}$ is uniformly bounded, the previous bound combined with estimate (2.3.1) allow us to conclude. \square

2.3.4.1 A Particular Weight

In this section, we introduce a particular weight in the class A_2 that will be of fundamental importance. With the finite set $\mathcal{D} \subset \Omega$ at hand, we define

$$d_{\mathcal{D}} := \begin{cases} \text{dist}(\mathcal{D}, \partial\Omega), & \text{if } \ell = 1, \\ \min \{ \text{dist}(\mathcal{D}, \partial\Omega), \min \{ |t - t'| : t, t' \in \mathcal{D}, t \neq t' \} \}, & \text{otherwise.} \end{cases} \quad (2.3.2)$$

We recall that $\ell = \#\mathcal{D}$. Since $\mathcal{D} \subset \Omega$ is finite, we immediately conclude that $d_{\mathcal{D}} > 0$. With this notation at hand, we define the weight ρ as follows:

$$\text{If } \ell = 1, \rho(x) = \mathbf{d}_t^\alpha(x), \text{ otherwise, } \rho(x) = \begin{cases} \mathbf{d}_t^\alpha(x), & \exists t \in \mathcal{D} : \mathbf{d}_t(x) < \frac{d_{\mathcal{D}}}{2}, \\ 1, & \mathbf{d}_t(x) \geq \frac{d_{\mathcal{D}}}{2} \forall t \in \mathcal{D}, \end{cases} \quad (2.3.3)$$

where $\mathbf{d}_t(x) := |x - t|$ and $\alpha \in (0, 2)$. Since $(0, 2) \subset (-2, 2)$, owing to [5, Theorem 6] and [54, Lemma 2.3 (vi)], $\rho \in A_2$. The extra restriction on α , namely, $\alpha > 0$, is needed in order to guarantee that for $t \in \mathcal{D}$ and $\mathbf{v}_t \in \mathbb{R}^2$, $\mathbf{v}_t \delta_t \in \mathbf{H}_0^1(\rho^{-1}, \Omega)'$; see [65, Remark 21.19] and [51, Proposition 5.2] for details.

The following lemma provides instrumental embedding and density results.

Lemma 2.3.4 (Embedding and Density Results). *Let ρ be the weight defined in (2.3.3). If $\alpha \in (0, 2)$, then*

$$(i) \quad \mathbf{H}_0^1(\rho^{-1}, \Omega) \hookrightarrow \mathbf{H}_0^1(\Omega) \hookrightarrow \mathbf{H}_0^1(\rho, \Omega),$$

(ii) $\mathbf{H}_0^1(\rho, \Omega)' \hookrightarrow \mathbf{H}^{-1}(\Omega) \hookrightarrow \mathbf{H}_0^1(\rho^{-1}, \Omega)'$, and

(iii) $\mathbf{H}^{-1}(\Omega)$ is dense in $\mathbf{H}_0^1(\rho^{-1}, \Omega)'$.

Proof. (i) We prove that $\mathbf{H}_0^1(\rho^{-1}, \Omega) \hookrightarrow \mathbf{H}_0^1(\Omega)$; the other embedding follows from similar considerations. If $\mathbf{v} \in \mathbf{H}_0^1(\rho^{-1}, \Omega)$, then

$$\|\nabla \mathbf{v}\|_{\mathbf{L}^2(\Omega)} = \|\rho^{\frac{1}{2}} \rho^{-\frac{1}{2}} \nabla \mathbf{v}\|_{\mathbf{L}^2(\Omega)} \leq \|\rho^{\frac{1}{2}}\|_{L^\infty(\Omega)} \|\nabla \mathbf{v}\|_{\mathbf{L}^2(\rho^{-1}, \Omega)}.$$

Notice that, since $\alpha > 0$, the weight ρ is uniformly bounded in Ω .

(ii) We prove that $\mathbf{H}^{-1}(\Omega) \hookrightarrow \mathbf{H}_0^1(\rho^{-1}, \Omega)'$; the other embedding follows from similar considerations. Let \mathbf{L} be an arbitrary element in $\mathbf{H}^{-1}(\Omega)$. In view of the embedding $\mathbf{H}_0^1(\rho^{-1}, \Omega) \hookrightarrow \mathbf{H}_0^1(\Omega)$, we immediately deduce that

$$\begin{aligned} \|\mathbf{L}\|_{\mathbf{H}_0^1(\rho^{-1}, \Omega)'} &:= \sup_{\mathbf{v} \in \mathbf{H}_0^1(\rho^{-1}, \Omega)} \frac{\langle \mathbf{L}, \mathbf{v} \rangle}{\|\nabla \mathbf{v}\|_{\mathbf{L}^2(\rho^{-1}, \Omega)}} \\ &\leq \sup_{\mathbf{v} \in \mathbf{H}_0^1(\Omega)} \frac{\langle \mathbf{L}, \mathbf{v} \rangle}{\|\nabla \mathbf{v}\|_{\mathbf{L}^2(\rho^{-1}, \Omega)}} \leq \|\rho^{\frac{1}{2}}\|_{L^\infty(\Omega)} \|\mathbf{L}\|_{\mathbf{H}^{-1}(\Omega)}. \end{aligned}$$

This implies that $\mathbf{L} \in \mathbf{H}_0^1(\rho^{-1}, \Omega)'$, as we intended to show.

(iii) For completeness, we provide a proof based on [19, Corollary 1.8 and Remark 5]: Let $\mathbf{F} \in \mathbf{H}_0^1(\rho^{-1}, \Omega)''$ be such that

$$\langle \mathbf{F}, \mathbf{L} \rangle_{\mathbf{H}_0^1(\rho^{-1}, \Omega)'', \mathbf{H}_0^1(\rho^{-1}, \Omega)'} = 0 \quad \forall \mathbf{L} \in \mathbf{H}^{-1}(\Omega). \quad (2.3.4)$$

We have to prove that $\mathbf{F} = \mathbf{0}$ in $\mathbf{H}_0^1(\rho^{-1}, \Omega)''$. Since $\mathbf{H}_0^1(\rho^{-1}, \Omega)$ is a reflexive space, there exists $\mathbf{f} \in \mathbf{H}_0^1(\rho^{-1}, \Omega)$ such that

$$\langle \mathbf{F}, \mathbf{Q} \rangle_{\mathbf{H}_0^1(\rho^{-1}, \Omega)'', \mathbf{H}_0^1(\rho^{-1}, \Omega)'} = \langle \mathbf{Q}, \mathbf{f} \rangle_{\mathbf{H}_0^1(\rho^{-1}, \Omega)', \mathbf{H}_0^1(\rho^{-1}, \Omega)}, \quad (2.3.5)$$

for all $\mathbf{Q} \in \mathbf{H}_0^1(\rho^{-1}, \Omega)'$. In view of $\mathbf{H}^{-1}(\Omega) \hookrightarrow \mathbf{H}_0^1(\rho^{-1}, \Omega)'$, relation (2.3.5) is also valid for all $\mathbf{Q} \in \mathbf{H}^{-1}(\Omega)$. On the other hand, the Riesz representation theorem immediately yields that for every $\mathbf{u} \in \mathbf{H}_0^1(\Omega)$ there exists $\mathbf{L} \in \mathbf{H}^{-1}(\Omega)$ satisfying

$$\langle \mathbf{L}, \mathbf{w} \rangle_{\mathbf{H}^{-1}(\Omega), \mathbf{H}_0^1(\Omega)} = \int_{\Omega} \nabla \mathbf{u} : \nabla \mathbf{w} \quad \forall \mathbf{w} \in \mathbf{H}_0^1(\Omega). \quad (2.3.6)$$

Since $\mathbf{H}_0^1(\rho^{-1}, \Omega) \hookrightarrow \mathbf{H}_0^1(\Omega)$, identity (2.3.6) holds for $\mathbf{u} \in \mathbf{H}_0^1(\rho^{-1}, \Omega)$. In particular, for $\mathbf{f} \in$

$\mathbf{H}_0^1(\rho^{-1}, \Omega)$, there exists $\mathbf{L}_f \in \mathbf{H}^{-1}(\Omega)$ such that

$$\langle \mathbf{L}_f, \mathbf{w} \rangle_{\mathbf{H}^{-1}(\Omega), \mathbf{H}_0^1(\Omega)} = \int_{\Omega} \nabla \mathbf{f} : \nabla \mathbf{w} \quad \forall \mathbf{w} \in \mathbf{H}_0^1(\Omega).$$

Set $\mathbf{w} = \mathbf{f}$ into the previous relation and invoke (2.3.4) and (2.3.5) to conclude that $\|\nabla \mathbf{f}\|_{\mathbf{L}^2(\Omega)} = 0$. This implies that $\mathbf{f} = \mathbf{0}$ a.e. in Ω . Consequently, $\mathbf{f} = \mathbf{0}$ in $\mathbf{H}_0^1(\rho^{-1}, \Omega)$ and hence $\mathbf{F} = \mathbf{0}$ in $\mathbf{H}_0^1(\rho^{-1}, \Omega)''$. □

2.4 The Navier–Stokes Equations Under Singular Forcing

In this section, we follow the weighted approach developed in [93] and review existence results for a suitable variational formulation of the stationary Navier–Stokes equations under singular forcing. By singular, we mean that the forcing term of the momentum equation is allowed to belong to the space $\mathbf{H}_0^1(\omega^{-1}, \Omega)'$ with $\omega \in A_2$. To be precise, given $\mathbf{f} \in \mathbf{H}_0^1(\omega^{-1}, \Omega)'$, we consider the following weak problem: Find $(\Phi, \zeta) \in \mathbf{H}_0^1(\omega, \Omega) \times L^2(\omega, \Omega)/\mathbb{R}$ such that

$$\begin{aligned} \int_{\Omega} (\nu \nabla \Phi : \nabla \mathbf{v} - \Phi \otimes \Phi : \nabla \mathbf{v} - \zeta \operatorname{div} \mathbf{v}) &= \langle \mathbf{f}, \mathbf{v} \rangle_{\mathbf{H}_0^1(\omega^{-1}, \Omega)', \mathbf{H}_0^1(\omega^{-1}, \Omega)}, \\ \int_{\Omega} q \operatorname{div} \Phi &= 0, \end{aligned} \tag{2.4.1}$$

for all $(\mathbf{v}, q) \in \mathbf{H}_0^1(\omega^{-1}, \Omega) \times L^2(\omega^{-1}, \Omega)/\mathbb{R}$. Here, $\nu > 0$ and $\omega \in A_2$.

Existence of solutions without smallness conditions is as follows [93, Theorem 1]: Let Ω be Lipschitz, $\omega \in A_2(\Omega)$, $\nu > 0$, and $\mathbf{f} \in \mathbf{H}_0^1(\omega^{-1}, \Omega)'$. Thus, (2.4.1) has at least one solution $(\Phi, \zeta) \in \mathbf{H}_0^1(\omega, \Omega) \times L^2(\omega, \Omega)/\mathbb{R}$, which satisfies

$$\|\nabla \Phi\|_{\mathbf{L}^2(\omega, \Omega)} + \|\zeta\|_{L^2(\omega, \Omega)} \lesssim \|\mathbf{f}\|_{\mathbf{H}_0^1(\omega^{-1}, \Omega)'}. \tag{2.4.2}$$

A similar result can be obtained on L^p -based spaces. In an abuse of notation, we denote by $(\Phi, \zeta) \in \mathbf{W}_0^{1,p}(\Omega) \times L^p(\Omega)/\mathbb{R}$ the solution to

$$\begin{aligned} \int_{\Omega} (\nu \nabla \Phi : \nabla \mathbf{v} - \Phi \otimes \Phi : \nabla \mathbf{v} - \zeta \operatorname{div} \mathbf{v}) &= \langle \mathbf{f}, \mathbf{v} \rangle_{\mathbf{W}^{-1,p}(\Omega), \mathbf{W}^{1,p'}(\Omega)}, \\ \int_{\Omega} q \operatorname{div} \Phi &= 0, \end{aligned} \tag{2.4.3}$$

for all $(\mathbf{v}, q) \in \mathbf{W}_0^{1,p'}(\Omega) \times L^{p'}(\Omega) \setminus \mathbb{R}$. Here, $\mathbf{f} \in \mathbf{W}^{-1,p}(\Omega)$ and p' is such that $1/p + 1/p' = 1$. Let us assume that Ω is Lipschitz and that $\nu > 0$. Within this setting at hand, we have the following existence

result: If $p \in (4/3 - \epsilon, 2)$, where $\epsilon = \epsilon(\Omega) > 0$ denotes a constant that depends on Ω , then problem (2.4.3) has at least one solution $(\Phi, \zeta) \in \mathbf{W}_0^{1,p}(\Omega) \times L^p(\Omega)/\mathbb{R}$. In addition, we have the following stability bound:

$$\|\nabla \Phi\|_{\mathbf{L}^p(\Omega)} + \|\zeta\|_{L^p(\Omega)} \lesssim \|\mathbf{f}\|_{\mathbf{W}^{-1,p}(\Omega)}. \quad (2.4.4)$$

The proof of such an existence result and the stability bound (2.4.4) follows from the arguments elaborated in [77, Section 3].

2.4.1 Regular Solutions

In this section, we follow [27] and introduce the concept of *regular solutions* for the Navier–Stokes equations.

Definition 2.4.1 (Regular Solution). *Let $(\Phi, \zeta) \in \mathbf{H}_0^1(\rho, \Omega) \times L^2(\rho, \Omega)/\mathbb{R}$ be a weak solution to (2.2.2) associated to a control $\mathcal{U} = (\mathbf{u}_1, \dots, \mathbf{u}_\ell) \in \mathbb{U}_{ad}$. We say that the velocity field Φ is regular if for every $\mathbf{g} \in \mathbf{H}^{-1}(\Omega)$ the weak problem: Find $(\boldsymbol{\theta}, \xi) \in \mathbf{H}_0^1(\Omega) \times L^2(\Omega)/\mathbb{R}$ such that*

$$\begin{aligned} \int_{\Omega} [(\nu \nabla \boldsymbol{\theta} - \Phi \otimes \boldsymbol{\theta} - \boldsymbol{\theta} \otimes \Phi) : \nabla \mathbf{w} - \xi \operatorname{div} \mathbf{w}] &= \langle \mathbf{g}, \mathbf{w} \rangle_{\mathbf{H}^{-1}(\Omega), \mathbf{H}_0^1(\Omega)}, \\ \int_{\Omega} \operatorname{sdiv} \boldsymbol{\theta} &= 0, \end{aligned} \quad (2.4.5)$$

for all $(\mathbf{w}, s) \in \mathbf{H}_0^1(\Omega) \times L^2(\Omega)/\mathbb{R}$, is well-posed.

Let us introduce the linear map

$$\begin{aligned} T : \mathbf{V}(\Omega) \times L^2(\Omega)/\mathbb{R} &\rightarrow \mathbf{H}^{-1}(\Omega), \\ (\boldsymbol{\theta}, \xi) &\mapsto -\nu \Delta \boldsymbol{\theta} + \operatorname{div}(\Phi \otimes \boldsymbol{\theta}) + \operatorname{div}(\boldsymbol{\theta} \otimes \Phi) + \nabla \xi, \end{aligned} \quad (2.4.6)$$

where $\mathbf{V}(\Omega) = \{\mathbf{w} \in \mathbf{H}_0^1(\Omega) : \operatorname{div} \mathbf{w} = 0 \text{ in } \Omega\}$. We notice that, as a consequence of Definition 2.4.1, if the velocity field Φ is regular, then the map T is an isomorphism from $\mathbf{V}(\Omega) \times L^2(\Omega)/\mathbb{R}$ into $\mathbf{H}^{-1}(\Omega)$ [27, Section 2].

In the following result, we show a well-posedness result that is crucial for the upcoming analysis.

Theorem 2.4.2 (Well-Posedness in Weighted Spaces). *Let (Φ, ζ) be a solution to (2.2.2) associated to $\mathcal{U} \in \mathbb{U}_{ad}$ such that Φ is regular. Then, for every $\mathbf{g} \in \mathbf{H}_0^1(\rho^{-1}, \Omega)'$ the problem: Find $(\boldsymbol{\theta}, \xi) \in \mathbf{H}_0^1(\rho, \Omega) \times L^2(\rho, \Omega)/\mathbb{R}$ such that*

$$\begin{aligned} \int_{\Omega} [(\nu \nabla \boldsymbol{\theta} - \Phi \otimes \boldsymbol{\theta} - \boldsymbol{\theta} \otimes \Phi) : \nabla \mathbf{w} - \xi \operatorname{div} \mathbf{w}] \\ = \langle \mathbf{g}, \mathbf{w} \rangle_{\mathbf{H}_0^1(\rho^{-1}, \Omega)', \mathbf{H}_0^1(\rho^{-1}, \Omega)}, \quad \int_{\Omega} \operatorname{sdiv} \boldsymbol{\theta} = 0, \end{aligned} \quad (2.4.7)$$

for all $(\mathbf{w}, s) \in \mathbf{H}_0^1(\rho^{-1}, \Omega) \times L^2(\rho^{-1}, \Omega)/\mathbb{R}$, admits a unique solution. In addition, we have the stability bound

$$\|\nabla \boldsymbol{\theta}\|_{\mathbf{L}^2(\rho, \Omega)} + \|\xi\|_{L^2(\rho, \Omega)} \lesssim (1 + \|\nabla \Phi\|_{\mathbf{L}^p(\Omega)}) \|\mathbf{g}\|_{\mathbf{H}_0^1(\rho^{-1}, \Omega)'}, \quad (2.4.8)$$

where $p \in (4/3 - \epsilon, 2)$ and $\epsilon = \epsilon(\Omega) > 0$.

Proof. We adapt the duality argument elaborated in the proof of [27, Theorem 2.9] to our weighted setting. To accomplish this task, we introduce the map

$$\begin{aligned} T_\rho : \mathbf{V}(\rho, \Omega) \times L^2(\rho, \Omega)/\mathbb{R} &\rightarrow \mathbf{H}_0^1(\rho^{-1}, \Omega)', \\ (\boldsymbol{\theta}, \xi) &\mapsto -\nu \Delta \boldsymbol{\theta} + \operatorname{div}(\Phi \otimes \boldsymbol{\theta}) + \operatorname{div}(\boldsymbol{\theta} \otimes \Phi) + \nabla \xi, \end{aligned} \quad (2.4.9)$$

where $\mathbf{V}(\rho, \Omega) = \{\mathbf{v} \in \mathbf{H}_0^1(\rho, \Omega) : \operatorname{div} \mathbf{v} = 0 \text{ in } \Omega\}$, and prove that T_ρ is an isomorphism on the basis of three steps.

Step 1. Well-posedness of the adjoint problem in $\mathbf{H}_0^1(\Omega) \times L^2(\Omega)/\mathbb{R}$. Given $\boldsymbol{\psi} \in \mathbf{H}^{-1}(\Omega)$, we introduce the adjoint problem: Find (\mathbf{z}, r) such that

$$-\nu \Delta \mathbf{z} - (\Phi \cdot \nabla) \mathbf{z} + (\nabla \Phi)^\top \mathbf{z} + \nabla r = \boldsymbol{\psi} \text{ in } \Omega, \quad \operatorname{div} \mathbf{z} = 0 \text{ in } \Omega, \quad \mathbf{z} = \mathbf{0} \text{ on } \partial\Omega. \quad (2.4.10)$$

We also introduce a suitable linear map associated to the system (2.4.10):

$$\begin{aligned} S : \mathbf{V}(\Omega) \times L^2(\Omega)/\mathbb{R} &\rightarrow \mathbf{H}^{-1}(\Omega), \\ (\mathbf{z}, r) &\mapsto -\nu \Delta \mathbf{z} - (\Phi \cdot \nabla) \mathbf{z} + (\nabla \Phi)^\top \mathbf{z} + \nabla r. \end{aligned} \quad (2.4.11)$$

In what follows, we prove that S is an isomorphism. As a first step, we derive the bound

$$\|\nabla \mathbf{z}\|_{\mathbf{L}^2(\Omega)} + \|r\|_{L^2(\Omega)} \lesssim \|S(\mathbf{z}, r)\|_{\mathbf{H}^{-1}(\Omega)} \quad \forall (\mathbf{z}, r) \in \mathbf{V}(\Omega) \times L^2(\Omega)/\mathbb{R}. \quad (2.4.12)$$

Let $\mathbf{g} \in \mathbf{H}^{-1}(\Omega)$ and let $(\mathbf{z}, r) \in \mathbf{V}(\Omega) \times L^2(\Omega)/\mathbb{R}$. Since Φ is regular, the map T , defined in (2.4.6), is an isomorphism. Consequently, there exists $(\boldsymbol{\theta}, \xi) \in \mathbf{V}(\Omega) \times L^2(\Omega)/\mathbb{R}$ such that $T(\boldsymbol{\theta}, \xi) = \mathbf{g}$. Invoke the definitions of T and S , given by (2.4.6) and (2.4.11), respectively, integration by parts, and the fact that $\operatorname{div} \boldsymbol{\theta} = \operatorname{div} \Phi = \operatorname{div} \mathbf{z} = 0$ to arrive at

$$\begin{aligned} \langle \mathbf{g}, \mathbf{z} \rangle &= \langle T(\boldsymbol{\theta}, \xi), \mathbf{z} \rangle = \langle -\nu \Delta \boldsymbol{\theta} + \operatorname{div}(\Phi \otimes \boldsymbol{\theta}) + \operatorname{div}(\boldsymbol{\theta} \otimes \Phi) + \nabla \xi, \mathbf{z} \rangle \\ &= \langle -\nu \Delta \mathbf{z} + (\nabla \Phi)^\top \mathbf{z} - (\Phi \cdot \nabla) \mathbf{z} + \nabla r, \boldsymbol{\theta} \rangle = \langle S(\mathbf{z}, r), \boldsymbol{\theta} \rangle \\ &\lesssim \|S(\mathbf{z}, r)\|_{\mathbf{H}^{-1}(\Omega)} \|\mathbf{g}\|_{\mathbf{H}^{-1}(\Omega)}, \end{aligned}$$

where, in the last step, we have utilized the bound $\|\nabla\boldsymbol{\theta}\|_{\mathbf{L}^2(\Omega)} \lesssim \|\mathbf{g}\|_{\mathbf{H}^{-1}(\Omega)}$; the latter follows from the fact that $\boldsymbol{\Phi}$ is regular (cf. Definition 2.4.1). Since \mathbf{g} and (\mathbf{z}, r) are arbitrary, we can thus conclude that, for every $(\mathbf{z}, r) \in \mathbf{V}(\Omega) \times L^2(\Omega)/\mathbb{R}$, we have

$$\|\nabla\mathbf{z}\|_{\mathbf{L}^2(\Omega)} \lesssim \|S(\mathbf{z}, r)\|_{\mathbf{H}^{-1}(\Omega)}. \quad (2.4.13)$$

It remains to bound $\|r\|_{L^2(\Omega)}$. Invoke the definition of S given in (2.4.11) to obtain

$$\begin{aligned} \|\nabla r\|_{\mathbf{H}^{-1}(\Omega)} &\leq \|S(\mathbf{z}, r)\|_{\mathbf{H}^{-1}(\Omega)} + \|\nu\Delta\mathbf{z}\|_{\mathbf{H}^{-1}(\Omega)} \\ &\quad + \|(\boldsymbol{\Phi} \cdot \nabla)\mathbf{z}\|_{\mathbf{H}^{-1}(\Omega)} + \|(\nabla\boldsymbol{\Phi})^\top\mathbf{z}\|_{\mathbf{H}^{-1}(\Omega)}. \end{aligned} \quad (2.4.14)$$

It is clear that $\|\Delta\mathbf{z}\|_{\mathbf{H}^{-1}(\Omega)} \leq \|\nabla\mathbf{z}\|_{\mathbf{L}^2(\Omega)}$. To bound the first convective term on the right-hand side of (2.4.14), we invoke Hölder's inequality, the standard Sobolev embedding $\mathbf{H}_0^1(\Omega) \hookrightarrow \mathbf{L}^\beta(\Omega)$, which holds for every $\beta < \infty$, and the first embedding result of Theorem 2.3.3. These arguments reveal that

$$\|(\boldsymbol{\Phi} \cdot \nabla)\mathbf{z}\|_{\mathbf{H}^{-1}(\Omega)} \lesssim \|\boldsymbol{\Phi}\|_{\mathbf{L}^{2+\varepsilon}(\Omega)} \|\nabla\mathbf{z}\|_{\mathbf{L}^2(\Omega)} \lesssim \|\nabla\boldsymbol{\Phi}\|_{\mathbf{L}^2(\rho, \Omega)} \|\nabla\mathbf{z}\|_{\mathbf{L}^2(\Omega)}.$$

Here, $\varepsilon > 0$ is as in the statement of Theorem 2.3.3. The second convective term on the right-hand side of (2.4.14) can be controlled as follows:

$$\begin{aligned} \|(\nabla\boldsymbol{\Phi})^\top\mathbf{z}\|_{\mathbf{H}^{-1}(\Omega)} &\leq \sup_{\mathbf{v} \in \mathbf{H}_0^1(\Omega)} \frac{\|\nabla\boldsymbol{\Phi}\|_{\mathbf{L}^2(\rho, \Omega)} \|\mathbf{z}\|_{\mathbf{L}^\kappa(\Omega)} \|\mathbf{v}\|_{\mathbf{L}^\mu(\Omega)} \|\rho^{-\frac{1}{2}}\|_{\mathbf{L}^\varsigma(\Omega)}}{\|\nabla\mathbf{v}\|_{\mathbf{L}^2(\Omega)}} \\ &\lesssim \|\nabla\boldsymbol{\Phi}\|_{\mathbf{L}^2(\rho, \Omega)} \|\nabla\mathbf{z}\|_{\mathbf{L}^2(\Omega)}, \quad \kappa^{-1} + \mu^{-1} + \varsigma^{-1} = 2^{-1}, \end{aligned} \quad (2.4.15)$$

upon utilizing $\mathbf{H}_0^1(\Omega) \hookrightarrow \mathbf{L}^\beta(\Omega)$ ($\beta < \infty$). To bound $\|\rho^{-\frac{1}{2}}\|_{\mathbf{L}^\varsigma(\Omega)}$ we invoke Proposition 2.3.1 and the fact that ς can be written as $\varsigma = 2 + \delta$ for $\delta > 0$ arbitrarily small. In fact, we have

$$\|\rho^{-\frac{1}{2}}\|_{\mathbf{L}^\varsigma(\Omega)}^\varsigma = \left(\int_\Omega \rho^{-1-\frac{\delta}{2}} \right) \lesssim |\Omega|^{-\frac{\delta}{2}} \left(\int_\Omega \rho^{-1} \right)^{1+\frac{\delta}{2}}. \quad (2.4.16)$$

Replace (2.4.13) and the estimates previously obtained into (2.4.14) to obtain

$$\begin{aligned} \|\nabla r\|_{\mathbf{H}^{-1}(\Omega)} &\lesssim \|S(\mathbf{z}, r)\|_{\mathbf{H}^{-1}(\Omega)} + (\nu + \|\nabla\boldsymbol{\Phi}\|_{\mathbf{L}^2(\rho, \Omega)}) \|\nabla\mathbf{z}\|_{\mathbf{L}^2(\Omega)} \\ &\lesssim (1 + \nu + \|\nabla\boldsymbol{\Phi}\|_{\mathbf{L}^2(\rho, \Omega)}) \|S(\mathbf{z}, r)\|_{\mathbf{H}^{-1}(\Omega)}. \end{aligned}$$

This bound, together with (2.4.13), allows us to obtain (2.4.12). With estimate (2.4.12) at hand, we can thus deduce that the linear and bounded operator S is injective with a closed range in $\mathbf{H}^{-1}(\Omega)$.

The surjectivity of S can be obtained with similar arguments to the ones developed in the *Step 1* of the proof of [27, Theorem 2.9].

Step 2. Well-posedness of the adjoint problem in $\mathbf{H}_0^1(\rho^{-1}, \Omega) \times L^2(\rho^{-1}, \Omega)/\mathbb{R}$. The purpose of this step is to prove that problem (2.4.10) is well-posed in the space $\mathbf{H}_0^1(\rho^{-1}, \Omega) \times L^2(\rho^{-1}, \Omega)/\mathbb{R}$ whenever $\psi \in \mathbf{H}_0^1(\rho, \Omega)'$. To accomplish this task, we introduce the map

$$\begin{aligned} S_\rho : \mathbf{V}(\rho^{-1}, \Omega) \times L^2(\rho^{-1}, \Omega)/\mathbb{R} &\rightarrow \mathbf{H}_0^1(\rho, \Omega)', \\ (\mathbf{z}, r) &\mapsto -\nu \Delta \mathbf{z} - (\Phi \cdot \nabla) \mathbf{z} + (\nabla \Phi)^\top \mathbf{z} + \nabla r. \end{aligned} \quad (2.4.17)$$

In what follows, we prove that the linear map S_ρ is an isomorphism.

Step 2.1. S_ρ is surjective: Let $\psi \in \mathbf{H}_0^1(\rho, \Omega)'$. Since $\mathbf{H}_0^1(\rho, \Omega)' \subset \mathbf{H}^{-1}(\Omega)$ (cf. Lemma 2.3.4), we immediately deduce that $\psi \in \mathbf{H}^{-1}(\Omega)$. As a consequence, the well-posedness results obtained in *Step 1* yield the existence of a unique solution $(\mathbf{z}, r) \in \mathbf{H}_0^1(\Omega) \times L^2(\Omega)/\mathbb{R}$ to system (2.4.10) together with a suitable stability bound. We now prove that (\mathbf{z}, r) belongs to $\mathbf{H}_0^1(\rho^{-1}, \Omega) \times L^2(\rho^{-1}, \Omega)/\mathbb{R}$. To accomplish this task, we first observe that (\mathbf{z}, r) can be seen as the solution to the Stokes problem

$$-\nu \Delta \mathbf{z} + \nabla r = \psi + (\Phi \cdot \nabla) \mathbf{z} - (\nabla \Phi)^\top \mathbf{z} \text{ in } \Omega, \quad \operatorname{div} \mathbf{z} = 0 \text{ in } \Omega, \quad \mathbf{z} = \mathbf{0} \text{ on } \partial\Omega,$$

and notice that the forcing term of the momentum equation belongs to $\mathbf{H}_0^1(\rho, \Omega)'$. In fact, the control of the convective term $(\Phi \cdot \nabla) \mathbf{z}$ in $\mathbf{H}_0^1(\rho, \Omega)'$ is as follows:

$$\begin{aligned} \|(\Phi \cdot \nabla) \mathbf{z}\|_{\mathbf{H}_0^1(\rho, \Omega)'} &\leq \sup_{\mathbf{v} \in \mathbf{H}_0^1(\rho, \Omega)} \frac{\|\Phi\|_{\mathbf{L}^\mu(\Omega)} \|\nabla \mathbf{z}\|_{\mathbf{L}^2(\Omega)} \|\mathbf{v}\|_{\mathbf{L}^\varsigma(\Omega)}}{\|\nabla \mathbf{v}\|_{\mathbf{L}^2(\rho, \Omega)}} \\ &\lesssim \|\nabla \Phi\|_{\mathbf{L}^p(\Omega)} \|\nabla \mathbf{z}\|_{\mathbf{L}^2(\Omega)}, \quad \mu^{-1} + \varsigma^{-1} = 2^{-1}, \end{aligned}$$

upon setting $\varsigma = 2 + \varepsilon$ with ε being dictated by Theorem 2.3.3. Notice that, we have also utilized the fact that $\Phi \in \mathbf{W}_0^{1,p}(\Omega)$ for every $p \in (4/3 - \varepsilon, 2)$, where $\varepsilon = \varepsilon(\Omega) > 0$; see estimate (2.4.4). The second convective term can be bounded in light of similar arguments:

$$\begin{aligned} \|(\nabla \Phi)^\top \mathbf{z}\|_{\mathbf{H}_0^1(\rho, \Omega)'} &\leq \sup_{\mathbf{v} \in \mathbf{H}_0^1(\rho, \Omega)} \frac{\|\nabla \Phi\|_{\mathbf{L}^p(\Omega)} \|\mathbf{z}\|_{\mathbf{L}^\mu(\Omega)} \|\mathbf{v}\|_{\mathbf{L}^\varsigma(\Omega)}}{\|\nabla \mathbf{v}\|_{\mathbf{L}^2(\rho, \Omega)}} \\ &\lesssim \|\nabla \Phi\|_{\mathbf{L}^p(\Omega)} \|\nabla \mathbf{z}\|_{\mathbf{L}^2(\Omega)}, \quad p^{-1} + \mu^{-1} + \varsigma^{-1} = 1, \end{aligned}$$

upon setting, again, $\varsigma = 2 + \varepsilon$ with ε being dictated by Theorem 2.3.3. Notice that we have also utilized the standard Sobolev embedding $\mathbf{H}_0^1(\Omega) \hookrightarrow \mathbf{L}^\beta(\Omega)$, which holds for every $\beta < \infty$. Having proved that the forcing term of the momentum equation belongs to $\mathbf{H}_0^1(\rho, \Omega)'$, we invoke [92, Theorem

17] to conclude that $(\mathbf{z}, r) \in \mathbf{V}(\rho^{-1}, \Omega) \times L^2(\rho^{-1}, \Omega)/\mathbb{R}$ together with the bound

$$\|\nabla \mathbf{z}\|_{\mathbf{L}^2(\rho^{-1}, \Omega)} + \|r\|_{L^2(\rho^{-1}, \Omega)} \lesssim \|\boldsymbol{\psi}\|_{\mathbf{H}_0^1(\rho, \Omega)'} (1 + \|\nabla \Phi\|_{\mathbf{L}^p(\Omega)}), \quad (2.4.18)$$

where we have utilized the bounds $\|\nabla \mathbf{z}\|_{\mathbf{L}^2(\Omega)} \lesssim \|\boldsymbol{\psi}\|_{\mathbf{H}^{-1}(\Omega)} \lesssim \|\boldsymbol{\psi}\|_{\mathbf{H}_0^1(\rho, \Omega)'}$. The first bound follows from the results of *Step 1* while the second one follows from the item (ii) in Lemma 2.3.4. We have thus proved that S_ρ is surjective.

Step 2.2. S_ρ is injective: Let $(\mathbf{z}, r) \in \mathbf{V}(\rho^{-1}, \Omega) \times L^2(\rho^{-1}, \Omega)/\mathbb{R}$ be such that $S_\rho(\mathbf{z}, r) = \mathbf{0}$. Since $\mathbf{V}(\rho^{-1}, \Omega) \times L^2(\rho^{-1}, \Omega)/\mathbb{R} \subset \mathbf{V}(\Omega) \times L^2(\Omega)/\mathbb{R}$, we have that $S(\mathbf{z}, r) = \mathbf{0}$. The fact that S is an isomorphism allows us to conclude that $(\mathbf{z}, r) = (\mathbf{0}, 0)$.

Step 3. Well-posedness of problem (2.4.7). We prove that problem (2.4.7) is well-posed. To accomplish this task, we proceed on the basis of a density argument. Let $\mathbf{g} \in \mathbf{H}_0^1(\rho^{-1}, \Omega)'$. Since $\mathbf{H}^{-1}(\Omega)$ is dense in $\mathbf{H}_0^1(\rho^{-1}, \Omega)'$ (cf. Lemma 2.3.4), there exists a sequence $\{\mathbf{g}_k\}_{k \in \mathbb{N}} \subset \mathbf{H}^{-1}(\Omega)$ such that $\mathbf{g}_k \rightarrow \mathbf{g}$ in $\mathbf{H}_0^1(\rho^{-1}, \Omega)'$ as $k \uparrow \infty$. On the other hand, since Φ is regular, the map T is an isomorphism. Consequently, for every $k \in \mathbb{N}$, there exists a unique pair $(\boldsymbol{\theta}_k, \xi_k) \in \mathbf{H}_0^1(\Omega) \times L^2(\Omega)/\mathbb{R}$ that solves problem (2.4.5) with \mathbf{g} being replaced by \mathbf{g}_k .

Step 3.1. $\{(\boldsymbol{\theta}_k, \xi_k)\}_{k \in \mathbb{N}}$ is bounded in $\mathbf{H}_0^1(\rho, \Omega) \times L^2(\rho, \Omega)/\mathbb{R}$. Let $\boldsymbol{\psi} \in \mathbf{H}_0^1(\rho, \Omega)'$. The results obtained in *Step 2* guarantee that S_ρ , which is defined in (2.4.17), is an isomorphism. As a consequence, there exists a pair $(\mathbf{z}, r) \in \mathbf{V}(\rho^{-1}, \Omega) \times L^2(\rho^{-1}, \Omega)/\mathbb{R}$ such that $\boldsymbol{\psi} = S_\rho(\mathbf{z}, r)$. Let us now observe that

$$\langle \boldsymbol{\psi}, \boldsymbol{\theta}_k \rangle = \langle S_\rho(\mathbf{z}, r), \boldsymbol{\theta}_k \rangle = \langle \mathbf{z}, T(\boldsymbol{\theta}_k, \xi_k) \rangle = \langle \mathbf{g}_k, \mathbf{z} \rangle. \quad (2.4.19)$$

With the previous identity at hand, the stability bound (2.4.18) reveals that

$$\begin{aligned} |\langle \boldsymbol{\psi}, \boldsymbol{\theta}_k \rangle| &\leq \|\mathbf{g}_k\|_{\mathbf{H}_0^1(\rho^{-1}, \Omega)'} \|\nabla \mathbf{z}\|_{\mathbf{L}^2(\rho^{-1}, \Omega)} \\ &\lesssim \|\mathbf{g}_k\|_{\mathbf{H}_0^1(\rho^{-1}, \Omega)'} \|\boldsymbol{\psi}\|_{\mathbf{H}_0^1(\rho, \Omega)'} (1 + \|\nabla \Phi\|_{\mathbf{L}^p(\Omega)}). \end{aligned}$$

The arbitrariness of $\boldsymbol{\psi}$ allows us to deduce the following bound for $\boldsymbol{\theta}_k$ and $k \in \mathbb{N}$: $\|\nabla \boldsymbol{\theta}_k\|_{\mathbf{L}^2(\rho, \Omega)} \lesssim \|\mathbf{g}_k\|_{\mathbf{H}_0^1(\rho^{-1}, \Omega)'} (1 + \|\nabla \Phi\|_{\mathbf{L}^p(\Omega)})$. This estimate and the inf-sup condition on weighted spaces of [51, Lemma 6.1] yield the boundedness of the sequence $\{\|\xi_k\|_{L^2(\rho, \Omega)}\}_{k \in \mathbb{N}}$ in \mathbb{R} .

Step 3.2. Existence of solutions for (2.4.7). Since the sequence $\{(\boldsymbol{\theta}_k, \xi_k)\}_{k \in \mathbb{N}}$ is bounded in $\mathbf{H}_0^1(\rho, \Omega) \times L^2(\rho, \Omega)/\mathbb{R}$, we deduce the existence of a nonrelabeled subsequence $\{(\boldsymbol{\theta}_k, \xi_k)\}_{k \in \mathbb{N}}$ such that

$$\boldsymbol{\theta}_k \rightharpoonup \boldsymbol{\theta} \text{ in } \mathbf{H}_0^1(\rho, \Omega), \quad \xi_k \rightarrow \xi \text{ in } L^2(\rho, \Omega)/\mathbb{R}, \quad k \uparrow \infty.$$

In what follows, we show that $(\boldsymbol{\theta}, \xi) \in \mathbf{H}_0^1(\rho, \Omega) \times L^2(\rho, \Omega)/\mathbb{R}$ solves the system (2.4.7). To accom-

plish this task, we let (\mathbf{w}, s) be an arbitrary pair in $\mathbf{H}_0^1(\rho^{-1}, \Omega) \times L^2(\rho^{-1}, \Omega)/\mathbb{R}$ and observe that $|\int_{\Omega} \nabla(\boldsymbol{\theta} - \boldsymbol{\theta}_k) : \nabla \mathbf{w}| \rightarrow 0$ and that

$$\left| \int_{\Omega} (\boldsymbol{\Phi} \otimes \boldsymbol{\theta} - \boldsymbol{\Phi} \otimes \boldsymbol{\theta}_k) : \nabla \mathbf{w} \right| \leq \|\boldsymbol{\Phi}\|_{\mathbf{L}^4(\rho, \Omega)} \|\boldsymbol{\theta} - \boldsymbol{\theta}_k\|_{\mathbf{L}^4(\rho, \Omega)} \|\nabla \mathbf{w}\|_{\mathbf{L}^2(\rho^{-1}, \Omega)} \rightarrow 0,$$

as $k \uparrow \infty$; the second convergence result being a consequence of the weighted compact Sobolev embedding $\mathbf{H}_0^1(\rho, \Omega) \hookrightarrow \mathbf{L}^4(\rho, \Omega)$ [64, Theorem 4.12] (see also [93, Proposition 2]). Similarly, we have

$$\left| \int_{\Omega} (\boldsymbol{\theta} - \boldsymbol{\theta}_k) \otimes \boldsymbol{\Phi} : \nabla \mathbf{w} \right|, \quad \left| \int_{\Omega} (\xi - \xi_k) \operatorname{div} \mathbf{w} \right|, \quad \left| \int_{\Omega} \operatorname{sdiv} (\boldsymbol{\theta} - \boldsymbol{\theta}_k) \right| \rightarrow 0,$$

as $k \uparrow \infty$. We have thus proved that $(\boldsymbol{\theta}, \xi) \in \mathbf{H}_0^1(\rho, \Omega) \times L^2(\rho, \Omega)/\mathbb{R}$ is a solution to problem (2.4.7).

Step 3.3. Stability bound. Let $\boldsymbol{\psi} \in \mathbf{H}_0^1(\rho, \Omega)'$ and let $(\mathbf{z}, r) \in \mathbf{H}_0^1(\rho^{-1}, \Omega) \times L^2(\rho^{-1}, \Omega)/\mathbb{R}$ be the unique solution to (2.4.10). Similar arguments to the ones utilized to obtain (2.4.19) combined with the stability bound (2.4.18) yield

$$\begin{aligned} \langle \boldsymbol{\psi}, \boldsymbol{\theta} \rangle &= \langle \mathbf{g}, \mathbf{z} \rangle \leq \|\mathbf{g}\|_{\mathbf{H}_0^1(\rho^{-1}, \Omega)'} \|\nabla \mathbf{z}\|_{\mathbf{L}^2(\rho^{-1}, \Omega)} \\ &\lesssim \|\mathbf{g}\|_{\mathbf{H}_0^1(\rho^{-1}, \Omega)'} \|\boldsymbol{\psi}\|_{\mathbf{H}_0^1(\rho, \Omega)'} (1 + \|\nabla \boldsymbol{\Phi}\|_{\mathbf{L}^p(\Omega)}). \end{aligned}$$

Since $\boldsymbol{\psi}$ is an arbitrary element of $\mathbf{H}_0^1(\rho, \Omega)'$, we can thus deduce that

$$\|\nabla \boldsymbol{\theta}\|_{\mathbf{L}^2(\rho, \Omega)} \lesssim \|\mathbf{g}\|_{\mathbf{H}_0^1(\rho^{-1}, \Omega)'} (1 + \|\nabla \boldsymbol{\Phi}\|_{\mathbf{L}^p(\Omega)}).$$

We now utilize the inf-sup condition on weighted spaces of [51, Lemma 6.1] to control the pressure:

$$\|\xi\|_{L^2(\rho, \Omega)} \lesssim \|\mathbf{g}\|_{\mathbf{H}_0^1(\rho^{-1}, \Omega)'} (1 + \|\nabla \boldsymbol{\Phi}\|_{\mathbf{L}^p(\Omega)}).$$

Step 3.4. The map T_ρ is injective. Let $(\boldsymbol{\theta}, \xi) \in \mathbf{H}_0^1(\rho, \Omega) \times L^2(\rho, \Omega)/\mathbb{R}$ be such that $T_\rho(\boldsymbol{\theta}, \xi) = \mathbf{0}$. This immediately implies that $\langle T_\rho(\boldsymbol{\theta}, \xi), \mathbf{z} \rangle = 0$ for every $\mathbf{z} \in \mathbf{H}_0^1(\rho^{-1}, \Omega)$. An argument based on integration by parts thus reveals that $\langle \boldsymbol{\theta}, S_\rho(\mathbf{z}, r) \rangle = 0$ for every $(\mathbf{z}, r) \in \mathbf{H}_0^1(\rho^{-1}, \Omega) \times L^2(\rho^{-1}, \Omega)/\mathbb{R}$. This implies that $\boldsymbol{\theta} = \mathbf{0}$. Since $T_\rho(\boldsymbol{\theta}, \xi) = \mathbf{0}$, we invoke the definition of T_ρ to deduce that $\nabla \xi = \mathbf{0}$ and thus that $\xi = 0$. This concludes the proof. \square

Remark 2.4.3 (Well-Posedness in $\mathbf{W}_0^{1,p}(\Omega) \times L^p(\Omega)/\mathbb{R}$). Let $\mathbf{g} \in \mathbf{W}^{-1,p}(\Omega)$ and p' be such that $1/p + 1/p' = 1$. Let us denote by $(\boldsymbol{\theta}, \xi)$ the weak solution to the system (2.4.7) with

$$\langle \mathbf{g}, \mathbf{w} \rangle_{\mathbf{W}^{-1,p}(\Omega), \mathbf{W}_0^{1,p'}(\Omega)},$$

as a forcing term. An adaptation of the arguments elaborated in the proof of Theorem 2.4.2, that are in turn inspired by the ones in [27, Theorem 2.9], show that problem 2.4.7 is well posed in $\mathbf{W}_0^{1,p}(\Omega) \times$

$L^p(\Omega)/\mathbb{R}$ whenever p belongs to $(4/3, 2)$. This result holds under the assumption that Ω is Lipschitz and therefore improves on [27, Theorem 2.9] where $\partial\Omega \in C^2$. We notice that the only place in the proof of [27, Theorem 2.9] where such a regularity on Ω is needed is [27, estimate (2.19)]. Since $p \in (4/3, 2)$ and thus $p' \in (2, 4)$, [27, estimate (2.19)] on Lipschitz domains can be obtained upon utilizing [82, Theorem 1.6, (1.52)].

2.4.1.1 Differentiability Properties of a Solution Operator

In this section, we investigate differentiability properties for a solution map associated to system (2.2.2) around a regular velocity field \mathbf{y} . We present some of these properties in the following theorem.

Theorem 2.4.4 (Differentiability of $\mathcal{U} \mapsto (\mathbf{y}, p)$). *Let $\bar{\mathcal{U}} \in \mathcal{U}_{ad}$ and let $(\bar{\mathbf{y}}, \bar{p}) \in \mathbf{H}_0^1(\rho, \Omega) \times L^2(\rho, \Omega)/\mathbb{R}$ be a solution to (2.2.2). If $\bar{\mathbf{y}}$ is regular, then there exist open neighborhoods $\mathcal{O}(\bar{\mathcal{U}}) \subset [\mathbb{R}^2]^\ell$, $\mathcal{O}(\bar{\mathbf{y}}) \subset \mathbf{V}(\rho, \Omega)$, and $\mathcal{O}(\bar{p}) \subset L^2(\rho, \Omega)/\mathbb{R}$ of $\bar{\mathcal{U}}$, $\bar{\mathbf{y}}$, and \bar{p} , respectively, and a map of class C^2 ,*

$$Q : \mathcal{O}(\bar{\mathcal{U}}) \rightarrow \mathcal{O}(\bar{\mathbf{y}}) \times \mathcal{O}(\bar{p}), \quad (2.4.20)$$

such that $Q(\bar{\mathcal{U}}) = (\bar{\mathbf{y}}, \bar{p})$. In addition, the neighborhood $\mathcal{O}(\bar{\mathcal{U}})$ can be taken such that, for each $\mathcal{U} \in \mathcal{O}(\bar{\mathcal{U}})$,

- (i) the pair $(\mathbf{y}_{\mathcal{U}}, p_{\mathcal{U}}) = Q(\mathcal{U})$ uniquely solves (2.2.2) in $\mathcal{O}(\bar{\mathbf{y}}) \times \mathcal{O}(\bar{p})$,
- (ii) the map $Q'(\mathcal{U}) : [\mathbb{R}^2]^\ell \rightarrow \mathbf{V}(\rho, \Omega) \times L^2(\rho, \Omega)/\mathbb{R}$ is an isomorphism,
- (iii) if $\mathcal{V} \in [\mathbb{R}^2]^\ell$, then $(\boldsymbol{\theta}, \xi) := Q'(\mathcal{U})\mathcal{V}$ corresponds to the unique solution to (2.4.7) with $\boldsymbol{\Phi}$ and \mathbf{g} being replaced by $\mathbf{y}_{\mathcal{U}}$ and $\sum_{t \in \mathcal{D}} \mathbf{v}_t \delta_t$, respectively, and
- (iv) if $\mathcal{V}_1, \mathcal{V}_2 \in [\mathbb{R}^2]^\ell$, then $(\boldsymbol{\psi}, \gamma) := Q''(\mathcal{U})\mathcal{V}_1\mathcal{V}_2$ corresponds to the unique solution to

$$\begin{aligned} & \int_{\Omega} ([\nu \nabla \boldsymbol{\psi} - \mathbf{y}_{\mathcal{U}} \otimes \boldsymbol{\psi} - \boldsymbol{\psi} \otimes \mathbf{y}_{\mathcal{U}}] : \nabla \mathbf{v} - \gamma \operatorname{div} \mathbf{v}) \\ & = \int_{\Omega} (\boldsymbol{\theta}_{\mathcal{V}_1} \otimes \boldsymbol{\theta}_{\mathcal{V}_2} + \boldsymbol{\theta}_{\mathcal{V}_2} \otimes \boldsymbol{\theta}_{\mathcal{V}_1}) : \nabla \mathbf{v}, \quad \int_{\Omega} q \operatorname{div} \boldsymbol{\psi} = 0, \end{aligned} \quad (2.4.21)$$

for all $(\mathbf{v}, q) \in \mathbf{H}_0^1(\rho^{-1}, \Omega) \times L^2(\rho^{-1}, \Omega)/\mathbb{R}$. Here, $(\boldsymbol{\theta}_{\mathcal{V}_i}, \xi_{\mathcal{V}_i}) = Q'(\mathcal{U})\mathcal{V}_i$, with $i \in \{1, 2\}$, corresponds to the unique solution to (2.4.7) with $\boldsymbol{\Phi}$ and \mathbf{g} being replaced by $\mathbf{y}_{\mathcal{U}}$ and $\sum_{t \in \mathcal{D}} \mathbf{v}_t \delta_t$, respectively.

Proof. The proof follows from slight modifications of the arguments elaborated in the proof of [27, Theorem 2.10 and Corollary 2.11] upon utilizing the results of Theorem 2.4.2. For brevity, we skip the details. \square

We conclude this section with the following Lipschitz property for Q , which will be of importance to study second order conditions in Section 2.6.3.

Lemma 2.4.5 (Lipschitz Property). *In the framework of Theorem 2.4.4, we have the following Lipschitz property for the map Q :*

$$\|\nabla(\mathbf{y} - \bar{\mathbf{y}})\|_{\mathbf{L}^2(\rho, \Omega)} + \|p - \bar{p}\|_{L^2(\rho, \Omega)} \lesssim \|\mathcal{U} - \bar{\mathcal{U}}\|_{[\mathbb{R}^2]^\ell} \quad \forall \mathcal{U} \in \mathcal{O}(\bar{\mathcal{U}}), \quad (2.4.22)$$

with a hidden constant that depends on Q' and $\mathcal{O}(\bar{\mathcal{U}})$.

Proof. In view of the results of Theorem 2.4.4, we can choose an open, bounded, and convex neighborhood $\mathcal{O}(\bar{\mathcal{U}})$ of $\bar{\mathcal{U}}$ such that $Q'(\mathcal{U}) : [\mathbb{R}^2]^\ell \rightarrow \mathbf{V}(\rho, \Omega) \times L^2(\rho, \Omega)/\mathbb{R}$ is an isomorphism and $\|Q'(\mathcal{U})\| \leq \mathcal{M}_Q$ for every $\mathcal{U} \in \mathcal{O}(\bar{\mathcal{U}})$. Here, $\mathcal{M}_Q > 0$ and $\|\cdot\|$ denotes the norm in the space of linear and continuous operators from $[\mathbb{R}^2]^\ell$ into $\mathbf{V}(\rho, \Omega) \times L^2(\rho, \Omega)/\mathbb{R}$. Thus, as a consequence of the mean value theorem for operators [11, Proposition 5.3.11], we have

$$\|Q(\mathcal{U}) - Q(\bar{\mathcal{U}})\|_{\mathbf{H}_0^1(\rho, \Omega) \times L^2(\rho, \Omega)/\mathbb{R}} \leq \sup_{t \in [0, 1]} \|Q'((1-t)\mathcal{U} + t\bar{\mathcal{U}})\| \|\mathcal{U} - \bar{\mathcal{U}}\|_{[\mathbb{R}^2]^\ell},$$

□

for every $\mathcal{U} \in \mathcal{O}(\bar{\mathcal{U}})$. Invoke the fact that $\|Q'(\mathcal{U})\| \leq \mathcal{M}_Q$, for every $\mathcal{U} \in \mathcal{O}(\bar{\mathcal{U}})$, to immediately arrive at the desired bound.

2.5 The Optimal Control Problem

In this section, we propose and analyze the following weak formulation for the optimal control problem (2.2.1)–(2.2.3): Find

$$\min\{J(\mathbf{y}, \mathcal{U}) : (\mathbf{y}, \mathcal{U}) \in \mathbf{H}_0^1(\rho, \Omega) \times \mathbb{U}_{ad}\}, \quad (2.5.1)$$

subject to the weak formulation of the stationary Navier–Stokes equations

$$\int_{\Omega} (\nu \nabla \mathbf{y} : \nabla \mathbf{v} - \mathbf{y} \otimes \mathbf{y} : \nabla \mathbf{v} - p \operatorname{div} \mathbf{v}) = \sum_{t \in \mathcal{D}} \langle \mathbf{u}_t \delta_t, \mathbf{v} \rangle, \quad \int_{\Omega} q \operatorname{div} \mathbf{y} = 0, \quad (2.5.2)$$

for all $(\mathbf{v}, q) \in \mathbf{H}_0^1(\rho^{-1}, \Omega) \times L^2(\rho^{-1}, \Omega)/\mathbb{R}$. The weight ρ is defined as in (2.3.3), where the parameter α belongs to $(0, 2)$. We comment that, since the velocity component \mathbf{y} of a solution to the state equation is sought in $\mathbf{H}_0^1(\rho, \Omega)$, an application of Theorem 2.3.3 guarantees that $\mathbf{y} \in \mathbf{L}^2(\Omega)$. Consequently, all the terms involved in the definition of the cost functional J are well defined.

2.5.1 Existence of Optimal Solutions

The existence of an optimal solution $(\bar{\mathbf{y}}, \bar{\mathcal{U}})$ is as follows.

Theorem 2.5.1 (Existence). *The control problem (2.5.1)–(2.5.2) admits at least one global solution $(\bar{\mathbf{y}}, \bar{\mathcal{U}}) \in \mathbf{H}_0^1(\rho, \Omega) \times \mathcal{U}_{ad}$.*

Proof. Let $\{(\mathbf{y}_k, \mathcal{U}_k)\}_{k \in \mathbb{N}}$ be a minimizing sequence, i.e., for $k \in \mathbb{N}$, the pair $(\mathbf{y}_k, p_k) \in \mathbf{H}_0^1(\rho, \Omega) \times L^2(\rho, \Omega)/\mathbb{R}$ solves

$$\int_{\Omega} (\nu \nabla \mathbf{y}_k : \nabla \mathbf{v} - \mathbf{y}_k \otimes \mathbf{y}_k : \nabla \mathbf{v} - p_k \operatorname{div} \mathbf{v}) = \sum_{t \in \mathcal{D}} \langle \mathbf{u}_t^k \delta_t, \mathbf{v} \rangle, \quad \int_{\Omega} q \operatorname{div} \mathbf{y}_k = 0,$$

for all $(\mathbf{v}, q) \in \mathbf{H}_0^1(\rho^{-1}, \Omega) \times L^2(\rho^{-1}, \Omega)/\mathbb{R}$, and the pair $(\mathbf{y}_k, \mathcal{U}_k)$ is such that $J(\mathbf{y}_k, \mathcal{U}_k) \rightarrow \mathfrak{i} := \inf\{J(\mathbf{y}, \mathcal{U}) : (\mathbf{y}, \mathcal{U}) \in \mathbf{H}_0^1(\rho, \Omega) \times \mathcal{U}_{ad}\}$ as $k \uparrow \infty$. Here, for $k \in \mathbb{N}$, we denote $\mathcal{U}_k := \{\mathbf{u}_t^k\}_{t \in \mathcal{D}}$. We notice that the existence of solutions for the previously stated problem follows from the results of Section 2.4. Since \mathcal{U}_{ad} is compact, we immediately conclude the existence of a nonrelabeled subsequence $\{\mathcal{U}_k\}_{k \in \mathbb{N}}$ such that $\mathcal{U}_k \rightarrow \bar{\mathcal{U}}$ in $[\mathbb{R}^2]^\ell$ with $\bar{\mathcal{U}} \in \mathcal{U}_{ad}$. On the other hand, in view of the stability bound (2.4.2), we conclude that $\{(\mathbf{y}_k, p_k)\}_{k \in \mathbb{N}}$ is uniformly bounded in $\mathbf{H}_0^1(\rho, \Omega) \times L^2(\rho, \Omega)/\mathbb{R}$. Consequently, we deduce the existence of a nonrelabeled subsequence $\{(\mathbf{y}_k, p_k)\}_{k \in \mathbb{N}}$ such that $(\mathbf{y}_k, p_k) \rightharpoonup (\bar{\mathbf{y}}, \bar{p})$ in $\mathbf{H}_0^1(\rho, \Omega) \times L^2(\rho, \Omega)/\mathbb{R}$ as $k \uparrow \infty$; $(\bar{\mathbf{y}}, \bar{p})$ being the natural candidate for an optimal state. The rest of the proof is dedicated to prove that $(\bar{\mathbf{y}}, \bar{p})$ solves (2.5.2) with \mathbf{u}_t being replaced by $\bar{\mathbf{u}}_t$ for $t \in \mathcal{D}$.

With the weak convergence $(\mathbf{y}_k, p_k) \rightharpoonup (\bar{\mathbf{y}}, \bar{p})$ in $\mathbf{H}_0^1(\rho, \Omega) \times L^2(\rho, \Omega)/\mathbb{R}$ as $k \uparrow \infty$, at hand, we obtain

$$\int_{\Omega} \nu \nabla (\mathbf{y}_k - \bar{\mathbf{y}}) : \nabla \mathbf{v} \rightarrow 0, \quad \int_{\Omega} (p_k - \bar{p}) \operatorname{div} \mathbf{v} \rightarrow 0, \quad \int_{\Omega} q \operatorname{div} (\mathbf{y}_k - \bar{\mathbf{y}}) \rightarrow 0,$$

as $k \uparrow \infty$, for every $\mathbf{v} \in \mathbf{H}_0^1(\rho^{-1}, \Omega)$ and $q \in L^2(\rho^{-1}, \Omega)/\mathbb{R}$. On the other hand, the convergence $\mathcal{U}_k \rightarrow \bar{\mathcal{U}}$ in $[\mathbb{R}^2]^\ell$ yields $\sum_{t \in \mathcal{D}} \langle \mathbf{u}_t^k \delta_t, \mathbf{v} \rangle \rightarrow \sum_{t \in \mathcal{D}} \langle \bar{\mathbf{u}}_t \delta_t, \mathbf{v} \rangle$ as $k \uparrow \infty$. It thus suffices to analyze the convective term. To accomplish this task, we invoke Hölder's inequality to arrive at

$$\left| \int_{\Omega} (\mathbf{y}_k \otimes \mathbf{y}_k - \bar{\mathbf{y}} \otimes \bar{\mathbf{y}}) : \nabla \mathbf{v} \right| \leq (\|\mathbf{y}_k\|_{\mathbf{L}^4(\rho, \Omega)} + \|\bar{\mathbf{y}}\|_{\mathbf{L}^4(\rho, \Omega)}) \|\bar{\mathbf{y}} - \mathbf{y}_k\|_{\mathbf{L}^4(\rho, \Omega)} \|\nabla \mathbf{v}\|_{\mathbf{L}^2(\rho^{-1}, \Omega)}.$$

The compact embedding $\mathbf{H}_0^1(\rho, \Omega) \hookrightarrow \mathbf{L}^4(\rho, \Omega)$, which follows from [64, Theorem 4.12] (see also [93, Proposition 2]), combined with $\mathbf{y}_k \rightharpoonup \bar{\mathbf{y}}$ in $\mathbf{H}_0^1(\rho, \Omega)$ as $k \uparrow \infty$, allow us to conclude that $(\bar{\mathbf{y}}, \bar{p})$ solves (2.5.2) with \mathbf{u}_t being replaced by $\bar{\mathbf{u}}_t$ for $t \in \mathcal{D}$; $\bar{\mathcal{U}} = \{\bar{\mathbf{u}}_t\}_{t \in \mathcal{D}}$.

To conclude the proof, we must prove the optimality of $\bar{\mathcal{U}}$. Observe that $\mathcal{U}_k \rightarrow \bar{\mathcal{U}}$ in $[\mathbb{R}^2]^\ell$ as $k \uparrow \infty$,

and that $\mathbf{y}_k \rightarrow \bar{\mathbf{y}}$ in $\mathbf{L}^2(\Omega)$ as $k \uparrow \infty$. The latter follows from

$$\|\bar{\mathbf{y}} - \mathbf{y}_k\|_{\mathbf{L}^2(\Omega)} \leq \|\bar{\mathbf{y}} - \mathbf{y}_k\|_{\mathbf{L}^4(\rho, \Omega)} \left(\int_{\Omega} \rho^{-1} \right)^{\frac{1}{4}} \lesssim \|\bar{\mathbf{y}} - \mathbf{y}_k\|_{\mathbf{L}^4(\rho, \Omega)} \rightarrow 0, \quad k \uparrow \infty.$$

With these convergence properties at hand, we thus conclude the optimality of $\bar{\mathcal{U}}$: $J(\bar{\mathbf{y}}, \bar{\mathcal{U}}) = \lim_{k \rightarrow \infty} J(\mathbf{y}_k, \mathcal{U}_k) =$
i. □

2.6 First and Second Order Optimality Conditions

In this section, we analyze first and second order optimality conditions for the optimal control problem (2.5.1)–(2.5.2). We must immediately mention that, since (2.5.1)–(2.5.2) is not convex, we distinguish between local and global solutions and present optimality conditions in the context of local solutions [27, 29].

Definition 2.6.1 (Local Solutions). *We say that $(\bar{\mathbf{y}}, \bar{\mathcal{U}})$ is a local solution for problem (2.5.1)–(2.5.2) if there exist neighborhoods $\mathcal{A} \subset \mathbf{H}_0^1(\rho, \Omega)$ and $\mathcal{B} \subset [\mathbb{R}^2]^\ell \cap \mathcal{U}_{ad}$ of $\bar{\mathbf{y}}$ and $\bar{\mathcal{U}}$, respectively, such that $J(\bar{\mathbf{y}}, \bar{\mathcal{U}}) \leq J(\mathbf{y}, \mathcal{U})$ for all $(\mathbf{y}, \mathcal{U}) \in \mathcal{A} \times \mathcal{B}$. If the inequality is strict for every $(\mathbf{y}, \mathcal{U}) \in \mathcal{A} \times \mathcal{B} \setminus \{(\bar{\mathbf{y}}, \bar{\mathcal{U}})\}$, we say that $(\bar{\mathbf{y}}, \bar{\mathcal{U}})$ is a strict local solution.*

From now on, we will assume that $(\bar{\mathbf{y}}, \bar{\mathcal{U}})$ is a local solution to (2.5.1)–(2.5.2) such that $\bar{\mathbf{y}}$ is regular. Within this setting, the results of Theorem 2.4.4 guarantee the existence of neighborhoods $\mathcal{O}(\bar{\mathcal{U}}) \subset [\mathbb{R}^2]^\ell$, $\mathcal{O}(\bar{\mathbf{y}}) \subset \mathbf{V}(\rho, \Omega)$, and $\mathcal{O}(\bar{p}) \subset L^2(\rho, \Omega)/\mathbb{R}$ of $\bar{\mathcal{U}}$, $\bar{\mathbf{y}}$, and \bar{p} , respectively, and a map of class C^2 ,

$$Q : \mathcal{O}(\bar{\mathcal{U}}) \rightarrow \mathcal{O}(\bar{\mathbf{y}}) \times \mathcal{O}(\bar{p}),$$

such that $(\bar{\mathbf{y}}, \bar{p}) = Q(\bar{\mathcal{U}})$. In addition, for each $\mathcal{U} \in \mathcal{O}(\bar{\mathcal{U}})$, the pair $(\mathbf{y}_{\mathcal{U}}, p_{\mathcal{U}}) := Q(\mathcal{U})$ corresponds to the unique solution of (2.2.2) in $\mathcal{O}(\bar{\mathbf{y}}) \times \mathcal{O}(\bar{p})$.

2.6.1 Adjoint Equation

We begin the section by introducing the *adjoint problem* as follows: Find $(\mathbf{z}, r) \in \mathbf{H}_0^1(\rho^{-1}, \Omega) \times L^2(\rho^{-1}, \Omega)/\mathbb{R}$ such that

$$\begin{aligned} \int_{\Omega} (\nu \nabla \mathbf{z} : \nabla \mathbf{w} - (\mathbf{y}_{\mathcal{U}} \cdot \nabla) \mathbf{z} \mathbf{w} + \nabla \mathbf{y}_{\mathcal{U}}^T \mathbf{z} \cdot \mathbf{w} - r \operatorname{div} \mathbf{w}) \\ = \int_{\Omega} (\mathbf{y}_{\mathcal{U}} - \mathbf{y}_{\Omega}) \cdot \mathbf{w}, \quad \int_{\Omega} s \operatorname{div} \mathbf{z} = 0, \end{aligned} \quad (2.6.1)$$

for all $(\mathbf{w}, s) \in \mathbf{H}_0^1(\rho, \Omega) \times L^2(\rho, \Omega)/\mathbb{R}$. Here, \mathbf{y}_U denotes the velocity component of the unique solution $(\mathbf{y}_U, p_U) \in \mathbf{H}_0^1(\rho, \Omega) \times L^2(\rho, \Omega)/\mathbb{R}$ to problem (2.5.2), associated to $U \in \mathcal{O}(\bar{U})$, in the neighborhood $\mathcal{O}(\bar{\mathbf{y}}) \times \mathcal{O}(\bar{p})$.

The well-posedness of the adjoint problem in weighted spaces is as follows: Since $(\bar{\mathbf{y}}, \bar{U})$ is a local solution to (2.5.1)–(2.5.2) such that $\bar{\mathbf{y}}$ is regular, a direct application of item (ii) in Theorem 2.4.4 reveals that

$$Q'(U) : [\mathbb{R}^2]^\ell \rightarrow \mathbf{V}(\rho, \Omega) \times L^2(\rho, \Omega)/\mathbb{R},$$

is an isomorphism for every $U \in \mathcal{O}(\bar{U})$; the characterization of $Q'(U)$ being available in the item (iii) of Theorem 2.4.4. On the basis of this fact, the duality argument elaborated within the proof of Theorem 2.4.2 reveals that problem (2.6.1) admits a unique solution $(\mathbf{z}, r) \in \mathbf{H}_0^1(\rho^{-1}, \Omega) \times L^2(\rho^{-1}, \Omega)/\mathbb{R}$. In addition, in view of (2.4.18), we have the following stability bound in weighted spaces:

$$\begin{aligned} \|\nabla \mathbf{z}\|_{\mathbf{L}^2(\rho^{-1}, \Omega)} + \|r\|_{L^2(\rho^{-1}, \Omega)} &\lesssim \|\mathbf{y}_U - \mathbf{y}_\Omega\|_{\mathbf{H}_0^1(\rho, \Omega)}, \\ &\lesssim \|\mathbf{y}_U - \mathbf{y}_\Omega\|_{\mathbf{L}^2(\Omega)}. \end{aligned} \quad (2.6.2)$$

The following result guarantees that point evaluations of the velocity component \mathbf{z} of the adjoint pair (\mathbf{z}, r) are well-defined.

Theorem 2.6.2 (Regularity Estimates). *If (\mathbf{z}, r) solves (2.6.1), then \mathbf{z} belongs to $\mathbf{W}^{1,q}(\Omega)$ for some $q > 2$. Consequently, $\mathbf{z} \in \mathbf{C}(\bar{\Omega})$.*

Proof. We begin the proof by rewriting the adjoint equation as the following Stokes problem:

$$\begin{aligned} \int_{\Omega} (\nu \nabla \mathbf{z} : \nabla \mathbf{w} - r \operatorname{div} \mathbf{w}) &= \int_{\Omega} (\mathbf{y}_U - \mathbf{y}_\Omega) \cdot \mathbf{w} + \int_{\Omega} [(\mathbf{y}_U \cdot \nabla) \mathbf{z} \mathbf{w} - \nabla \mathbf{y}_U^T \mathbf{z} \cdot \mathbf{w}], \\ \int_{\Omega} s \operatorname{div} \mathbf{z} &= 0, \end{aligned}$$

for all $(\mathbf{w}, s) \in \mathbf{H}_0^1(\rho, \Omega) \times L^2(\rho, \Omega)/\mathbb{R}$.

Denote $\mathbf{W}^{-1,q}(\Omega) = \mathbf{W}_0^{1,q'}(\Omega)'$ and define the linear functional $\mathfrak{F} := \mathfrak{F}_1 - \mathfrak{F}_2$, where $\mathfrak{F}_1, \mathfrak{F}_2 : \mathbf{H}_0^1(\rho, \Omega) \rightarrow \mathbb{R}$ are defined by $\mathfrak{F}_1(\mathbf{w}) := \int_{\Omega} (\mathbf{y}_U \cdot \nabla) \mathbf{z} \mathbf{w}$ and $\mathfrak{F}_2(\mathbf{w}) := \int_{\Omega} \nabla \mathbf{y}_U^T \mathbf{z} \cdot \mathbf{w}$. Let us prove that $\mathfrak{F} \in \mathbf{W}^{-1,q}(\Omega)$ for some $q > 2$. To accomplish this task, we first study \mathfrak{F}_1 on the basis of Hölder's inequality:

$$\begin{aligned} \|\mathfrak{F}_1\|_{\mathbf{W}^{-1,q}(\Omega)} &\leq \sup_{\mathbf{w} \in \mathbf{W}_0^{1,q'}(\Omega)} \frac{\|\rho^{\frac{1}{4}}\|_{L^\infty(\Omega)} \|\mathbf{y}_U\|_{\mathbf{L}^4(\rho, \Omega)} \|\nabla \mathbf{z}\|_{\mathbf{L}^2(\rho^{-1}, \Omega)} \|\mathbf{w}\|_{\mathbf{L}^4(\Omega)}}{\|\nabla \mathbf{w}\|_{\mathbf{L}^{q'}(\Omega)}} \\ &\lesssim \|\rho^{\frac{1}{4}}\|_{L^\infty(\Omega)} \|\nabla \mathbf{y}_U\|_{\mathbf{L}^2(\rho, \Omega)} \|\nabla \mathbf{z}\|_{\mathbf{L}^2(\rho^{-1}, \Omega)}, \end{aligned}$$

where we have used $\mathbf{H}_0^1(\rho, \Omega) \hookrightarrow \mathbf{L}^4(\rho, \Omega)$, $\mathbf{W}_0^{1,q'}(\Omega) \hookrightarrow \mathbf{L}^4(\Omega)$, which holds for $q' \geq 4/3$ ($q \leq 4$), and $\mathbf{z} \in \mathbf{H}_0^1(\rho^{-1}, \Omega)$. We thus deduce the existence of $q > 2$ such that $\mathfrak{F}_1 \in \mathbf{W}^{-1,q}(\Omega)$ and $\|\mathfrak{F}_1\|_{\mathbf{W}^{-1,q}(\Omega)} \lesssim$

$$\|\nabla \mathbf{y}_U\|_{\mathbf{L}^2(\rho, \Omega)} \|\nabla \mathbf{z}\|_{\mathbf{L}^2(\rho^{-1}, \Omega)}.$$

We now control the term \mathfrak{F}_2 . To accomplish this task, we invoke Hölder's inequality combined with the fact that there exists $\epsilon = \epsilon(\Omega) > 0$ such that $\mathbf{y}_U \in \mathbf{W}_0^{1,p}(\Omega)$ for every $p \in (4/3 - \epsilon, 2)$:

$$\|\mathfrak{F}_2\|_{\mathbf{W}^{-1,q}(\Omega)} \leq \sup_{\mathbf{w} \in \mathbf{W}_0^{1,q'}(\Omega)} \frac{\|\nabla \mathbf{y}_U\|_{\mathbf{L}^p(\Omega)} \|\mathbf{z}\|_{\mathbf{L}^\mu(\Omega)} \|\mathbf{w}\|_{\mathbf{L}^v(\Omega)}}{\|\nabla \mathbf{w}\|_{\mathbf{L}^{q'}(\Omega)}},$$

with $p^{-1} + \mu^{-1} + v^{-1} = 1$. Invoke now that $\mathbf{W}_0^{1,q'}(\Omega) \hookrightarrow \mathbf{L}^\sigma(\Omega)$, which holds for every $\sigma \leq 2q'/(2-q')$, that $\mathbf{z} \in \mathbf{H}_0^1(\rho^{-1}, \Omega)$, and the Sobolev embeddings $\mathbf{H}_0^1(\rho^{-1}, \Omega) \hookrightarrow \mathbf{H}_0^1(\Omega) \hookrightarrow \mathbf{L}^\beta(\Omega)$, which hold for every $\beta < \infty$, to arrive at the existence of $q > 2$ such that $\|\mathfrak{F}_2\|_{\mathbf{W}^{-1,q}(\Omega)} \lesssim \|\nabla \mathbf{y}_U\|_{\mathbf{L}^p(\Omega)} \|\nabla \mathbf{z}\|_{\mathbf{L}^2(\rho^{-1}, \Omega)}$. Having obtained the existence of $q > 2$ such that $\mathfrak{F} \in \mathbf{W}^{-1,q}(\Omega)$, it suffices to invoke [83, (1.52)] to conclude that $\mathbf{z} \in \mathbf{W}^{1,q}(\Omega)$. \square

2.6.2 First Order Optimality Conditions

In this section, we derive first order optimality conditions for the optimal control problem (2.5.1)–(2.5.2). To accomplish this task, we begin this section by introducing some preliminary ingredients. Before presenting them, we recall that we are operating under the assumption that $(\bar{\mathbf{y}}, \bar{\mathcal{U}})$ is a local solution to (2.5.1)–(2.5.2), which is such that $\bar{\mathbf{y}}$ is regular. The first ingredient is the operator \mathcal{G} , which is defined as follows:

$$\mathcal{G} : \mathcal{O}(\bar{\mathcal{U}}) \subset [\mathbb{R}^2]^\ell \rightarrow \mathcal{O}(\bar{\mathbf{y}}) \subset \mathbf{H}_0^1(\rho, \Omega), \quad \mathcal{U} \mapsto \mathbf{y}, \quad (2.6.3)$$

where \mathbf{y} corresponds to the velocity component of the pair $(\mathbf{y}, p) = Q(\mathcal{U})$. The second ingredient is the reduced cost functional:

$$j : \mathcal{O}(\bar{\mathcal{U}}) \rightarrow \mathbb{R}, \quad j(\mathcal{U}) := \frac{1}{2} \|\mathcal{G}(\mathcal{U}) - \mathbf{y}_\Omega\|_{\mathbf{L}^2(\Omega)}^2 + \frac{\eta}{2} \sum_{t \in \mathcal{D}} |\mathbf{u}_t|^2. \quad (2.6.4)$$

Having defined the reduced cost functional, we present the following standard result: If $\bar{\mathcal{U}} \in \mathcal{U}_{ad}$ denotes a locally optimal control for problem (2.5.1)–(2.5.2), then it satisfies the following variational inequality [104, Lemma 4.18]:

$$j'(\bar{\mathcal{U}})(\mathcal{U} - \bar{\mathcal{U}}) \geq 0 \quad \forall \mathcal{U} \in \mathcal{U}_{ad}. \quad (2.6.5)$$

The following result explores the variational inequality (2.6.5).

Theorem 2.6.3 (First Order Optimality Conditions). *If the pair $(\bar{\mathbf{y}}, \bar{\mathcal{U}}) \in \mathbf{H}_0^1(\rho, \Omega) \times \mathcal{U}_{ad}$ denotes a local solution to the optimal control problem (2.5.1)–(2.5.2) such that $\bar{\mathbf{y}}$ is regular, then $\bar{\mathcal{U}} \in \mathcal{U}_{ad}$*

satisfies the variational inequality

$$\sum_{t \in \mathcal{D}} (\bar{\mathbf{z}}(t) + \eta \bar{\mathbf{u}}_t) \cdot (\mathbf{u}_t - \bar{\mathbf{u}}_t) \geq 0 \quad \forall \mathcal{U} = (\mathbf{u}_1, \dots, \mathbf{u}_\ell) \in \mathbb{U}_{ad}, \quad (2.6.6)$$

where $(\bar{\mathbf{z}}, \bar{r}) \in \mathbf{H}_0^1(\rho^{-1}, \Omega) \times L^2(\rho^{-1}, \Omega)/\mathbb{R}$ denotes the optimal adjoint pair, which solves the adjoint problem (2.6.1) with $\mathbf{y}_{\mathcal{U}}$ replaced by $\bar{\mathbf{y}}$.

Proof. We begin the proof by computing the expression $j'(\bar{\mathcal{U}})(\mathcal{U} - \bar{\mathcal{U}})$ and rewriting the basic variational inequality (2.6.5) as follows:

$$\int_{\Omega} (\bar{\mathbf{y}} - \mathbf{y}_{\Omega}) \cdot \mathcal{G}'(\bar{\mathcal{U}})(\mathcal{U} - \bar{\mathcal{U}}) + \eta \sum_{t \in \mathcal{D}} \bar{\mathbf{u}}_t \cdot (\mathbf{u}_t - \bar{\mathbf{u}}_t) \geq 0 \quad \forall \mathcal{U} \in \mathbb{U}_{ad}. \quad (2.6.7)$$

Define $\boldsymbol{\theta} := \mathcal{G}'(\bar{\mathcal{U}})(\mathcal{U} - \bar{\mathcal{U}})$. Observe that $(\boldsymbol{\theta}, \xi) \in \mathbf{H}_0^1(\rho, \Omega) \times L^2(\rho, \Omega)/\mathbb{R}$ solves

$$\begin{aligned} \int_{\Omega} (\nu \nabla \boldsymbol{\theta} : \nabla \mathbf{v} - \bar{\mathbf{y}} \otimes \boldsymbol{\theta} : \nabla \mathbf{v} - \boldsymbol{\theta} \otimes \bar{\mathbf{y}} : \nabla \mathbf{v} - \xi \operatorname{div} \mathbf{v}) \\ = \sum_{t \in \mathcal{D}} \langle (\mathbf{u}_t - \bar{\mathbf{u}}_t) \delta_t, \mathbf{v} \rangle, \quad \int_{\Omega} q \operatorname{div} \boldsymbol{\theta} = 0, \end{aligned} \quad (2.6.8)$$

for all $(\mathbf{v}, q) \in \mathbf{H}_0^1(\rho^{-1}, \Omega) \times L^2(\rho^{-1}, \Omega)/\mathbb{R}$. Having introduced the pair $(\boldsymbol{\theta}, \xi)$, the variational inequality (2.6.7) becomes

$$\int_{\Omega} (\bar{\mathbf{y}} - \mathbf{y}_{\Omega}) \cdot \boldsymbol{\theta} + \eta \sum_{t \in \mathcal{D}} \bar{\mathbf{u}}_t \cdot (\mathbf{u}_t - \bar{\mathbf{u}}_t) \geq 0 \quad \forall \mathcal{U} = (\mathbf{u}_1, \dots, \mathbf{u}_\ell) \in \mathbb{U}_{ad}. \quad (2.6.9)$$

Since the second term on the right-hand side of the previous expression is already present in the desired inequality (2.6.6), we focus on the first term.

Let us set $(\bar{\mathbf{z}}, \bar{r}) \in \mathbf{H}_0^1(\rho^{-1}, \Omega) \times L^2(\rho^{-1}, \Omega)/\mathbb{R}$ as a test pair in problem (2.6.8). This yields

$$\int_{\Omega} (\nu \nabla \boldsymbol{\theta} : \nabla \bar{\mathbf{z}} - \bar{\mathbf{y}} \otimes \boldsymbol{\theta} : \nabla \bar{\mathbf{z}} - \boldsymbol{\theta} \otimes \bar{\mathbf{y}} : \nabla \bar{\mathbf{z}}) = \sum_{t \in \mathcal{D}} (\mathbf{u}_t - \bar{\mathbf{u}}_t) \cdot \bar{\mathbf{z}}(t), \quad (2.6.10)$$

upon utilizing the fact that $\int_{\Omega} \xi \operatorname{div} \bar{\mathbf{z}}$ vanishes and that there exists $q > 2$ such that $\bar{\mathbf{z}} \in \mathbf{W}^{1,q}(\Omega) \hookrightarrow \mathbf{C}(\bar{\Omega})$ (cf. Theorem 2.6.2). We now set $\mathbf{w} = \boldsymbol{\theta}$ as a test function in the first equation of problem (2.6.1) to obtain

$$\int_{\Omega} (\nu \nabla \bar{\mathbf{z}} : \nabla \boldsymbol{\theta} - (\bar{\mathbf{y}} \cdot \nabla) \bar{\mathbf{z}} \boldsymbol{\theta} + \nabla \bar{\mathbf{y}}^T \bar{\mathbf{z}} \cdot \boldsymbol{\theta} - \bar{r} \operatorname{div} \boldsymbol{\theta}) = \int_{\Omega} (\bar{\mathbf{y}} - \mathbf{y}_{\Omega}) \cdot \boldsymbol{\theta}. \quad (2.6.11)$$

We thus utilize (2.6.10), (2.6.11), the fact that $\int_{\Omega} \bar{r} \operatorname{div} \boldsymbol{\theta}$ vanishes, and an integration by parts formula for the convective terms in (2.6.11) to arrive at the needed relation $\sum_{t \in \mathcal{D}} (\mathbf{u}_t - \bar{\mathbf{u}}_t) \cdot \bar{\mathbf{z}}(t) = \int_{\Omega} (\bar{\mathbf{y}} - \mathbf{y}_{\Omega}) \cdot \boldsymbol{\theta}$. In view of (2.6.9), the previously derived identity allows to arrive at (2.6.6). \square

2.6.3 Second Order Optimality Conditions

In this section, we analyze necessary and sufficient second order optimality conditions.

2.6.3.1 Auxiliary Results

We begin this section by recalling that, as stated at the beginning of Section 2.6, we are operating under the assumption that $(\bar{\mathbf{y}}, \bar{\mathcal{U}}) \in \mathbf{H}_0^1(\rho, \Omega) \times \mathcal{U}_{ad}$ is a local solution of (2.5.1)–(2.5.2) such that $\bar{\mathbf{y}}$ is regular.

We begin our studies with the following estimate.

Lemma 2.6.4 (Auxiliary Estimate). *Let $\mathcal{U} \in \mathcal{O}(\bar{\mathcal{U}})$ and $\mathcal{V} \in [\mathbb{R}^2]^\ell$. Let $\mathbf{y} = \mathcal{G}(\mathcal{U})$, $(\boldsymbol{\theta}, \xi) = Q'(\mathcal{U})\mathcal{V}$, and $(\bar{\boldsymbol{\theta}}, \bar{\xi}) = Q'(\bar{\mathcal{U}})\mathcal{V}$. Then, we have the estimate*

$$\|\nabla(\boldsymbol{\theta} - \bar{\boldsymbol{\theta}})\|_{\mathbf{L}^2(\rho, \Omega)} \lesssim \|\mathcal{U} - \bar{\mathcal{U}}\|_{[\mathbb{R}^2]^\ell} \|\mathcal{V}\|_{[\mathbb{R}^2]^\ell}, \quad (2.6.12)$$

where the hidden constant is independent of $(\boldsymbol{\theta}, \xi)$, $(\bar{\boldsymbol{\theta}}, \bar{\xi})$, and \mathcal{V} .

Proof. We begin the proof by noticing that $(\boldsymbol{\theta} - \bar{\boldsymbol{\theta}}, \xi - \bar{\xi})$ solves the following problem: Find $(\boldsymbol{\theta} - \bar{\boldsymbol{\theta}}, \xi - \bar{\xi}) \in \mathbf{H}_0^1(\rho, \Omega) \times L^2(\rho, \Omega)/\mathbb{R}$ such that

$$\begin{aligned} & \int_{\Omega} (\nu \nabla(\boldsymbol{\theta} - \bar{\boldsymbol{\theta}}) : \nabla \mathbf{w} - \bar{\mathbf{y}} \otimes (\boldsymbol{\theta} - \bar{\boldsymbol{\theta}}) : \nabla \mathbf{w} - (\boldsymbol{\theta} - \bar{\boldsymbol{\theta}}) \otimes \bar{\mathbf{y}} : \nabla \mathbf{w} - (\xi - \bar{\xi}) \operatorname{div} \mathbf{w}) \\ &= \int_{\Omega} [(\mathbf{y} - \bar{\mathbf{y}}) \otimes \boldsymbol{\theta} + \boldsymbol{\theta} \otimes (\mathbf{y} - \bar{\mathbf{y}})] : \nabla \mathbf{w}, \quad \int_{\Omega} s \operatorname{div}(\boldsymbol{\theta} - \bar{\boldsymbol{\theta}}) = 0, \end{aligned} \quad (2.6.13)$$

for all $(\mathbf{w}, s) \in \mathbf{H}_0^1(\rho^{-1}, \Omega) \times L^2(\rho^{-1}, \Omega)/\mathbb{R}$. Since $\bar{\mathbf{y}}$ is regular, a direct application of Theorem 2.4.2 reveals that problem (2.6.13) is well-posed upon realizing that the forcing term of the momentum equation belongs to the dual space of $\mathbf{H}_0^1(\rho^{-1}, \Omega)$; see the estimates in (2.6.14) below.

To derive (2.6.12) we invoke the stability estimate (2.4.8), Hölder's inequality, the weighted embeddings $\mathbf{H}_0^1(\rho^\pm, \Omega) \hookrightarrow \mathbf{L}^4(\rho^\pm, \Omega)$, and the Lipschitz property of Lemma 2.4.5. With these arguments and estimates, we have

$$\begin{aligned} \|\nabla(\boldsymbol{\theta} - \bar{\boldsymbol{\theta}})\|_{\mathbf{L}^2(\rho, \Omega)} &\lesssim \|(\mathbf{y} - \bar{\mathbf{y}}) \otimes \boldsymbol{\theta} + \bar{\boldsymbol{\theta}} \otimes (\mathbf{y} - \bar{\mathbf{y}})\|_{\mathbf{H}_0^1(\rho^{-1}, \Omega)'} \\ &\lesssim [\|\nabla \boldsymbol{\theta}\|_{\mathbf{L}^2(\rho, \Omega)} + \|\nabla \bar{\boldsymbol{\theta}}\|_{\mathbf{L}^2(\rho, \Omega)}] \|\nabla(\mathbf{y} - \bar{\mathbf{y}})\|_{\mathbf{L}^2(\rho, \Omega)} \\ &\lesssim \|\mathcal{V}\|_{[\mathbb{R}^2]^\ell} \|\mathcal{U} - \bar{\mathcal{U}}\|_{[\mathbb{R}^2]^\ell}. \end{aligned} \quad (2.6.14)$$

This concludes the proof. \square

We conclude this section with the following result.

Theorem 2.6.5 (Properties of j''). *The reduced cost functional j , defined in (2.6.4), is of class C^2 . In addition, for $\mathcal{U} \in \mathcal{O}(\bar{\mathcal{U}})$ and $\mathcal{V} \in [\mathbb{R}^2]^\ell$, we have*

$$j''(\mathcal{U})\mathcal{V}^2 = \int_{\Omega} |\boldsymbol{\theta}|^2 + 2 \int_{\Omega} \boldsymbol{\theta} \otimes \boldsymbol{\theta} : \nabla \mathbf{z} + \eta \sum_{t \in \mathcal{D}} |\mathbf{v}_t|^2. \quad (2.6.15)$$

Here, $(\boldsymbol{\theta}, \xi) = Q'(\mathcal{U})\mathcal{V}$ and (\mathbf{z}, r) denotes the unique solution of (2.6.1) with $\mathbf{y}_{\mathcal{U}} = \mathcal{G}(\mathcal{U})$. Finally, we have the bound

$$|j''(\mathcal{U})\mathcal{V}^2 - j''(\bar{\mathcal{U}})\bar{\mathcal{V}}^2| \lesssim \|\mathcal{U} - \bar{\mathcal{U}}\|_{[\mathbb{R}^2]^\ell} \|\mathcal{V}\|_{[\mathbb{R}^2]^\ell}^2. \quad (2.6.16)$$

Proof. Since Theorem 2.4.4 guarantees that Q is second order Fréchet differentiable, it is immediate that \mathcal{G} , defined in (2.6.3), is second order Fréchet differentiable as a map from $[\mathbb{R}^2]^\ell$ into $\mathbf{H}_0^1(\rho, \Omega)$. Consequently, j is of class C^2 .

We now derive the identity (2.6.15). To accomplish this task, we begin with a basic computation, which reveals that, for $\mathcal{U} \in \mathcal{O}(\bar{\mathcal{U}})$ and $\mathcal{V} \in [\mathbb{R}^2]^\ell$, we have

$$j''(\mathcal{U})\mathcal{V}^2 = \int_{\Omega} |\mathcal{G}'(\mathcal{U})\mathcal{V}|^2 + \int_{\Omega} (\mathcal{G}(\mathcal{U}) - \mathbf{y}_{\Omega}) \cdot \mathcal{G}''(\mathcal{U})\mathcal{V}^2 + \eta \sum_{t \in \mathcal{D}} |\mathbf{v}_t|^2. \quad (2.6.17)$$

Define $(\boldsymbol{\psi}, \gamma) := Q''(\mathcal{U})\mathcal{V}^2$, i.e., $(\boldsymbol{\psi}, \gamma) \in \mathbf{H}_0^1(\rho, \Omega) \times L^2(\rho, \Omega)/\mathbb{R}$ corresponds to the unique solution to (2.4.21) with both $\boldsymbol{\theta}_{\nu_1}$ and $\boldsymbol{\theta}_{\nu_2}$ being replaced by $\boldsymbol{\theta}$ and $\mathbf{y}_{\mathcal{U}} = \mathcal{G}(\mathcal{U})$. Notice that $\boldsymbol{\psi} = \mathcal{G}''(\mathcal{U})\mathcal{V}^2$. Setting $\mathbf{v} = \mathbf{z}$ in the first equation of the problem that $(\boldsymbol{\psi}, \gamma)$ solves (c.f. (2.4.21) with $\boldsymbol{\theta}_{\nu_1} = \boldsymbol{\theta}_{\nu_2} = \boldsymbol{\theta}$), we arrive at

$$\int_{\Omega} ([\nu \nabla \boldsymbol{\psi} - \mathbf{y}_{\mathcal{U}} \otimes \boldsymbol{\psi} - \boldsymbol{\psi} \otimes \mathbf{y}_{\mathcal{U}}] : \nabla \mathbf{z} - \gamma \operatorname{div} \mathbf{z}) = 2 \int_{\Omega} \boldsymbol{\theta} \otimes \boldsymbol{\theta} : \nabla \mathbf{z}.$$

Here, $(\mathbf{z}, r) \in \mathbf{H}_0^1(\rho^{-1}, \Omega) \times L^2(\rho^{-1}, \Omega)/\mathbb{R}$ corresponds to the unique solution to the adjoint problem (2.6.1). Similarly, we set $\mathbf{w} = \boldsymbol{\psi}$ in the first equation of the adjoint problem (2.6.1). This yields

$$\int_{\Omega} (\nu \nabla \mathbf{z} : \nabla \boldsymbol{\psi} - (\mathbf{y}_{\mathcal{U}} \cdot \nabla) \mathbf{z} \boldsymbol{\psi} + \nabla \mathbf{y}_{\mathcal{U}}^T \mathbf{z} \cdot \boldsymbol{\psi} - r \operatorname{div} \boldsymbol{\psi}) = \int_{\Omega} (\mathbf{y}_{\mathcal{U}} - \mathbf{y}_{\Omega}) \cdot \boldsymbol{\psi}.$$

We now resort to an integration by parts argument based on the fact that $\operatorname{div} \boldsymbol{\psi} = \operatorname{div} \mathbf{y}_{\mathcal{U}} = \operatorname{div} \mathbf{z} = 0$ to obtain (2.6.15).

Let us proceed with the task of deriving (2.6.16). To accomplish this task, we define $(\bar{\boldsymbol{\theta}}, \bar{\xi}) := Q'(\bar{\mathcal{U}})\bar{\mathcal{V}}$ and notice that $(\boldsymbol{\theta}, \xi)$ and $(\bar{\boldsymbol{\theta}}, \bar{\xi})$ solve problem (2.4.7) with $\boldsymbol{\Phi} = \mathcal{G}(\mathcal{U})$ and $\boldsymbol{\Phi} = \mathcal{G}(\bar{\mathcal{U}})$, respectively, and

$\mathbf{g} = \sum_{t \in \mathcal{D}} \mathbf{v}_t \delta_t$. In view of the derived identity (2.6.15), we write the following equality:

$$\begin{aligned} j''(\mathcal{U})\mathcal{V}^2 - j''(\bar{\mathcal{U}})\mathcal{V}^2 &= \int_{\Omega} (\boldsymbol{\theta} - \bar{\boldsymbol{\theta}}) \cdot (\boldsymbol{\theta} + \bar{\boldsymbol{\theta}}) + 2 \left[\int_{\Omega} \boldsymbol{\theta} \otimes (\boldsymbol{\theta} - \bar{\boldsymbol{\theta}}) : \nabla \mathbf{z} \right. \\ &\quad \left. + \int_{\Omega} \boldsymbol{\theta} \otimes \bar{\boldsymbol{\theta}} : \nabla(\mathbf{z} - \bar{\mathbf{z}}) + \int_{\Omega} (\boldsymbol{\theta} - \bar{\boldsymbol{\theta}}) \otimes \bar{\boldsymbol{\theta}} : \nabla \bar{\mathbf{z}} \right]. \end{aligned} \quad (2.6.18)$$

Here, $(\bar{\mathbf{z}}, \bar{r})$ denotes the unique solution to problem (2.6.1) with $\mathbf{y}_{\mathcal{U}}$ being replaced by $\bar{\mathbf{y}} = \mathcal{G}(\bar{\mathcal{U}})$. Invoke Hölder's inequality and the Sobolev embeddings $\mathbf{H}_0^1(\rho, \Omega) \hookrightarrow \mathbf{L}^4(\rho, \Omega)$ (cf. [53, Theorem 1.3]) and $\mathbf{H}_0^1(\rho, \Omega) \hookrightarrow \mathbf{L}^2(\Omega)$ (cf. Theorem 2.3.3) to arrive at

$$\begin{aligned} |j''(\mathcal{U})\mathcal{V}^2 - j''(\bar{\mathcal{U}})\mathcal{V}^2| &\lesssim \|\nabla \boldsymbol{\theta}\|_{\mathbf{L}^2(\rho, \Omega)} \|\nabla \bar{\boldsymbol{\theta}}\|_{\mathbf{L}^2(\rho, \Omega)} \|\nabla(\mathbf{z} - \bar{\mathbf{z}})\|_{\mathbf{L}^2(\rho^{-1}, \Omega)} \\ &\quad + \Lambda \|\nabla(\boldsymbol{\theta} - \bar{\boldsymbol{\theta}})\|_{\mathbf{L}^2(\rho, \Omega)} (\|\nabla \boldsymbol{\theta}\|_{\mathbf{L}^2(\rho, \Omega)} + \|\nabla \bar{\boldsymbol{\theta}}\|_{\mathbf{L}^2(\rho, \Omega)}), \end{aligned}$$

where $\Lambda := 1 + \|\nabla \mathbf{z}\|_{\mathbf{L}^2(\rho^{-1}, \Omega)} + \|\nabla \bar{\mathbf{z}}\|_{\mathbf{L}^2(\rho^{-1}, \Omega)}$. This bound combined with the stability estimate (2.4.8), the auxiliary estimate (2.6.12), and the boundedness of $\bar{\mathbf{z}}, \mathbf{z}$ in $\mathbf{H}_0^1(\rho^{-1}, \Omega)$, which follows from (2.6.2), yield

$$|j''(\mathcal{U})\mathcal{V}^2 - j''(\bar{\mathcal{U}})\mathcal{V}^2| \lesssim \|\mathcal{U} - \bar{\mathcal{U}}\|_{[\mathbb{R}^2]^\ell} \|\mathcal{V}\|_{[\mathbb{R}^2]^\ell}^2 + \|\nabla(\mathbf{z} - \bar{\mathbf{z}})\|_{\mathbf{L}^2(\rho^{-1}, \Omega)} \|\mathcal{V}\|_{[\mathbb{R}^2]^\ell}^2.$$

Therefore, it suffices to bound $\|\nabla(\mathbf{z} - \bar{\mathbf{z}})\|_{\mathbf{L}^2(\rho^{-1}, \Omega)}$. To accomplish this goal, we notice that $(\mathbf{z} - \bar{\mathbf{z}}, r - \bar{r}) \in \mathbf{H}_0^1(\rho^{-1}, \Omega) \times L^2(\rho^{-1}, \Omega)/\mathbb{R}$ solves

$$\begin{aligned} &\int_{\Omega} (\nu \nabla(\mathbf{z} - \bar{\mathbf{z}}) : \nabla \mathbf{w} - (\mathbf{y}_{\mathcal{U}} \cdot \nabla)(\mathbf{z} - \bar{\mathbf{z}}) \mathbf{w} + \nabla \mathbf{y}_{\mathcal{U}}^T (\mathbf{z} - \bar{\mathbf{z}}) \cdot \mathbf{w} - (r - \bar{r}) \operatorname{div} \mathbf{w}) \\ &= \int_{\Omega} (\mathbf{y}_{\mathcal{U}} - \bar{\mathbf{y}}) \cdot \mathbf{w} + \int_{\Omega} ([\mathbf{y}_{\mathcal{U}} - \bar{\mathbf{y}}] \cdot \nabla) \bar{\mathbf{z}} \mathbf{w} + \int_{\Omega} [\nabla(\bar{\mathbf{y}} - \mathbf{y}_{\mathcal{U}})]^T \bar{\mathbf{z}} \cdot \mathbf{w}, \\ &\hspace{25em} \int_{\Omega} s \operatorname{div}(\mathbf{z} - \bar{\mathbf{z}}) = 0, \end{aligned} \quad (2.6.19)$$

for all $(\mathbf{w}, s) \in \mathbf{H}_0^1(\rho, \Omega) \times L^2(\rho, \Omega)/\mathbb{R}$. We now bound the $\mathbf{H}_0^1(\rho, \Omega)'$ -norm of the right hand-side of the first equation of (2.6.19). We begin by noticing that Hölder's inequality combined with the embedding $\mathbf{H}_0^1(\rho, \Omega) \hookrightarrow \mathbf{L}^4(\rho, \Omega)$ yield

$$\|([\mathbf{y}_{\mathcal{U}} - \bar{\mathbf{y}}] \cdot \nabla) \bar{\mathbf{z}}\|_{\mathbf{H}_0^1(\rho, \Omega)'} \lesssim \|\nabla(\mathbf{y}_{\mathcal{U}} - \bar{\mathbf{y}})\|_{\mathbf{L}^2(\rho, \Omega)} \|\nabla \bar{\mathbf{z}}\|_{\mathbf{L}^2(\rho^{-1}, \Omega)}.$$

Similarly, by exploiting the fact that $\bar{\mathbf{y}}, \mathbf{y}_{\mathcal{U}} \in \mathbf{W}_0^{1,p}(\Omega)$ for every $p \in (4/3 - \epsilon, 2)$, where $\epsilon = \epsilon(\Omega) > 0$

(cf. estimate (2.4.4)), we obtain

$$\|[\nabla(\mathbf{y}\mathcal{U} - \bar{\mathbf{y}})]^\top \bar{\mathbf{z}}\|_{\mathbf{H}_0^1(\rho, \Omega)'} \lesssim \sup_{\mathbf{v} \in \mathbf{H}_0^1(\rho, \Omega)} \frac{\|\nabla(\mathbf{y}\mathcal{U} - \bar{\mathbf{y}})\|_{\mathbf{L}^p(\Omega)} \|\bar{\mathbf{z}}\|_{\mathbf{L}^\mu(\Omega)} \|\mathbf{v}\|_{\mathbf{L}^\varepsilon(\Omega)}}{\|\nabla \mathbf{v}\|_{\mathbf{L}^2(\rho, \Omega)}},$$

where $p^{-1} + \mu^{-1} + \varepsilon^{-1} = 1$. In view of the embeddings $\mathbf{H}_0^1(\rho^{-1}, \Omega) \hookrightarrow \mathbf{H}_0^1(\Omega) \hookrightarrow \mathbf{L}^\beta(\Omega)$ for every $\beta < \infty$, we can thus set $\varrho = 2 + \varepsilon$, with ε being dictated by Theorem 2.3.3 and utilize the embedding $\mathbf{H}_0^1(\rho, \Omega) \hookrightarrow \mathbf{L}^{2+\varepsilon}(\Omega)$ to conclude that

$$\|[\nabla(\mathbf{y}\mathcal{U} - \bar{\mathbf{y}})]^\top \bar{\mathbf{z}}\|_{\mathbf{H}_0^1(\rho, \Omega)'} \lesssim \|\nabla(\mathbf{y}\mathcal{U} - \bar{\mathbf{y}})\|_{\mathbf{L}^p(\Omega)} \|\nabla \bar{\mathbf{z}}\|_{\mathbf{L}^2(\rho^{-1}, \Omega)}.$$

Having controlled the $\mathbf{H}_0^1(\rho, \Omega)'$ -norm of the right hand-side of the first equation of (2.6.19), we invoke the weighted stability estimate (2.6.2) twice to obtain

$$\begin{aligned} \|\nabla(\mathbf{z} - \bar{\mathbf{z}})\|_{\mathbf{L}^2(\rho^{-1}, \Omega)} &\lesssim (\|\nabla(\mathbf{y}\mathcal{U} - \bar{\mathbf{y}})\|_{\mathbf{L}^2(\rho, \Omega)} + \|\nabla(\mathbf{y}\mathcal{U} - \bar{\mathbf{y}})\|_{\mathbf{L}^p(\Omega)}) \\ &\quad \cdot \|\nabla \bar{\mathbf{z}}\|_{\mathbf{L}^2(\rho^{-1}, \Omega)} \lesssim (\|\nabla(\mathbf{y}\mathcal{U} - \bar{\mathbf{y}})\|_{\mathbf{L}^2(\rho, \Omega)} + \|\nabla(\mathbf{y}\mathcal{U} - \bar{\mathbf{y}})\|_{\mathbf{L}^p(\Omega)}) \|\bar{\mathbf{y}} - \mathbf{y}\Omega\|_{\mathbf{L}^2(\Omega)}. \end{aligned}$$

We now utilize the Lipschitz property of Lemma 2.4.5 and the one in [27, Lemma 4.4], upon further restricting the neighborhood $\mathcal{O}(\bar{\mathcal{U}})$ if necessary, to obtain the bound $\|\nabla(\mathbf{z} - \bar{\mathbf{z}})\|_{\mathbf{L}^2(\rho^{-1}, \Omega)} \lesssim \|\mathcal{U} - \bar{\mathcal{U}}\|_{[\mathbb{R}^2]^\ell}$. This concludes the proof. \square

2.6.3.2 Second Order Necessary and Sufficient Optimality Conditions

Before presenting necessary and sufficient second order optimality conditions, we introduce a few ingredients. Let us define

$$\Psi := (\Psi_1, \dots, \Psi_\ell) \in [\mathbb{R}^2]^\ell, \quad \Psi_t := \bar{\mathbf{z}}(t) + \eta \bar{\mathbf{u}}_t, \quad t \in \mathcal{D}. \quad (2.6.20)$$

Let $s \in \mathcal{D}$ and $\mathcal{U} \in \mathcal{U}_{ad}$ be such that $\mathbf{u}_t = \bar{\mathbf{u}}_t$ for $t \in \mathcal{D} \setminus \{s\}$. Set \mathcal{U} into the variational inequality (2.6.6). This yields

$$0 \leq (\bar{\mathbf{z}}(s) + \eta \bar{\mathbf{u}}_s) \cdot (\mathbf{u}_s - \bar{\mathbf{u}}_s) = \Psi_s \cdot (\mathbf{u}_s - \bar{\mathbf{u}}_s). \quad (2.6.21)$$

Let $i, j \in \{1, 2\}$ be such that $i \neq j$. Set $(\mathbf{u}_s)_i = (\bar{\mathbf{u}}_s)_i$. If $(\bar{\mathbf{u}}_s)_j = (\mathbf{a}_s)_j$, then inequality (2.6.21) reveals that

$$0 \leq \Psi_s \cdot (\mathbf{u}_s - \bar{\mathbf{u}}_s) = (\Psi_s)_j [(u_s)_j - (\mathbf{a}_s)_j] \implies (\Psi_s)_j \geq 0.$$

Similarly,

- if $(\mathbf{a}_s)_j < (\bar{\mathbf{u}}_s)_j < (\mathbf{b}_s)_j$, then $(\Psi_s)_j = 0$, and

- if $(\bar{\mathbf{u}}_s)_j = (\mathbf{b}_s)_j$, then $(\Psi_s)_j \leq 0$.

Let us also introduce the cone of critical directions at $\bar{\mathcal{U}} \in \mathcal{U}_{ad}$:

$$\mathbf{C}_{\bar{\mathcal{U}}} := \{\mathcal{V} = (\mathbf{v}_1, \dots, \mathbf{v}_\ell) \in [\mathbb{R}^2]^\ell \text{ that satisfies (2.6.22) and (2.6.23)}\},$$

where, for $t \in \mathcal{D}$ and $i \in \{1, 2\}$, conditions (2.6.22) and (2.6.23) read as follows:

$$(\mathbf{v}_t)_i \begin{cases} \geq 0 & \text{if } (\bar{\mathbf{u}}_t)_i = (\mathbf{a}_t)_i, \\ \leq 0 & \text{if } (\bar{\mathbf{u}}_t)_i = (\mathbf{b}_t)_i, \end{cases} \quad (2.6.22)$$

and

$$(\Psi_t)_i \neq 0 \implies (\mathbf{v}_t)_i = 0; \quad (2.6.23)$$

compare with [32, (3.16)].

As stated in [32, Section 3.3], the following result follows from the standard Karush–Kuhn–Tucker theory of mathematical optimization in finite-dimensional spaces (see, for instance, [32, Theorem 3.8] and [80, Section 6.3]) on the basis of the results derived in Theorem 2.6.5.

Theorem 2.6.6 (Second Order Necessary and Sufficient Optimality Conditions). *If $\bar{\mathcal{U}} \in \mathcal{U}_{ad}$ is a local minimum for problem (2.5.1)–(2.5.2), then $j''(\bar{\mathcal{U}})\mathcal{V}^2 \geq 0$ for all $\mathcal{V} \in \mathbf{C}_{\bar{\mathcal{U}}}$. Conversely, if $\bar{\mathcal{U}} \in \mathcal{U}_{ad}$ satisfies the variational inequality (2.6.6) and the second order sufficient condition*

$$j''(\bar{\mathcal{U}})\mathcal{V}^2 > 0 \quad \forall \mathcal{V} \in \mathbf{C}_{\bar{\mathcal{U}}} \setminus \{\mathbf{0}\}, \quad (2.6.24)$$

then there exists $\mu > 0$ and $\sigma > 0$ such that

$$j(\mathcal{U}) \geq j(\bar{\mathcal{U}}) + \frac{\mu}{2} \|\mathcal{U} - \bar{\mathcal{U}}\|_{[\mathbb{R}^2]^\ell}^2 \quad \forall \mathcal{U} \in \mathcal{U}_{ad} : \|\mathcal{U} - \bar{\mathcal{U}}\|_{[\mathbb{R}^2]^\ell} \leq \sigma. \quad (2.6.25)$$

In particular, $\bar{\mathcal{U}}$ is a strict local solution of (2.5.1)–(2.5.2).

To present the following result, we introduce, for $\tau > 0$, the cone

$$\mathbf{C}_{\bar{\mathcal{U}}}^\tau := \{\mathcal{V} = (\mathbf{v}_1, \dots, \mathbf{v}_\ell) \in [\mathbb{R}^2]^\ell \text{ that satisfies (2.6.22) and (2.6.26)}\},$$

where, for $t \in \mathcal{D}$ and $i \in \{1, 2\}$, condition (2.6.26) reads as follows:

$$|(\Psi_t)_i| > \tau \implies (\mathbf{v}_t)_i = 0. \quad (2.6.26)$$

The following result is immediate in view of our finite dimensional setting.

Theorem 2.6.7 (Equivalence). *Let $(\bar{\mathbf{y}}, \bar{p})$, $(\bar{\mathbf{z}}, \bar{r})$, and $\bar{\mathcal{U}}$ satisfy the first order optimality conditions (2.5.2), (2.6.1), and (2.6.6). Then, (2.6.24) is equivalent to*

$$\exists \kappa, \tau > 0 : \quad j''(\bar{\mathcal{U}})\mathcal{V}^2 \geq \kappa \|\mathcal{V}\|_{[\mathbb{R}^2]^\ell}^2 \quad \forall \mathcal{V} \in \mathbf{C}_{\bar{\mathcal{U}}}^\tau. \quad (2.6.27)$$

Chapter 3

Bilinear optimal control for the fractional Laplacian: analysis and discretization

3.1 Introduction

In this paper, we are interested in the analysis and discretization of an optimal control problem involving a fractional elliptic PDE as state equation and a control variable that enters the state equation as a coefficient. To make matters precise, we let Ω be an open and bounded domain in \mathbb{R}^d ($d \geq 2$) with Lipschitz boundary $\partial\Omega$. Given a desired state $u_\Omega \in L^2(\Omega)$ and a regularization parameter $\lambda > 0$, we introduce the cost functional

$$J(u, q) := \frac{1}{2} \|u - u_\Omega\|_{L^2(\Omega)}^2 + \frac{\lambda}{2} \|q\|_{L^2(\Omega)}^2. \quad (3.1.1)$$

Let f be a fixed function in $H^{-s}(\Omega)$. We shall be concerned with the following optimal control problem: Find $\min J(u, q)$ subject to the *fractional* and *elliptic* PDE

$$(-\Delta)^s u + qu = f \text{ in } \Omega, \quad u = 0 \text{ in } \Omega^c, \quad (3.1.2)$$

where $\Omega^c = \mathbb{R}^d \setminus \Omega$, and the *control constraints*

$$q \in \mathbb{Q}_{ad}, \quad \mathbb{Q}_{ad} := \{v \in L^\infty(\Omega) : a \leq v(x) \leq b \text{ a.e. } x \in \Omega\}. \quad (3.1.3)$$

The control bounds a and b are such that $0 < a < b$. The operator $(-\Delta)^s$ corresponds to the *integral definition* of the fractional Laplace operator; its definition can be found in section 3.2.3.

In problem (3.1.1)–(3.1.3) the control variable q enters the state equation as a coefficient and not as a source term. This creates the *nonlinear* coupling qu in (3.1.2), the presence of which has led to this type of problems being referred to as *bilinear optimal control problems* or *control-affine problems* [33]. Mathematically, the coupling qu in (3.1.2) complicates both the analysis and the discretization. In particular, the solution of the state equation depends nonlinearly on the control, so the uniqueness of solutions of (3.1.1)–(3.1.3) cannot be guaranteed [73]; a complete optimization study therefore requires the analysis of second order optimality conditions [33]. In terms of applications, it should be mentioned that several processes in engineering, biology, and ecology, to name just a few, can be modeled by bilinear systems; see, for instance, [22, 38, 84]. We would also like to note that problem (3.1.1)–(3.1.3) can be interpreted as a particular instance of a *coefficient identification problem*. Such problems become particularly relevant in scenarios where coefficients or source terms remain uncertain or unknown. It may be that we have a sparse and/or noisy measurement of the state of the system or an output of interest that we wish to match, and/or that a priori information about some model coefficients is available. In such cases, we can resort to solving a control problem or an inverse problem to recover the unknown parameters and define a more accurate, data-driven mathematical model.

Another important feature of (3.1.1)–(3.1.3) is that it is governed by the *fractional* and *nonlocal* operator $(-\Delta)^s$. Although the use of nonlocal models to describe natural and social phenomena has been of interest for a long time, this interest has only increased significantly in recent years. This is mainly due to the numerous applications of these models, which demonstrate their relevance and versatility. It is therefore only natural that an interest in efficient approximation schemes for nonlocal models arises [15, 43] and that one might be interested in their *control* [8, 9, 45]. By nonlocal models, we mean here model descriptions that at a given point in space depend on the state of the system at points at a far distance from that given point. An important example from the family of nonlocal operators is the *integral* fractional Laplacian $(-\Delta)^s$ ($0 < s < 1$), which corresponds to the infinitesimal generator of a stable Lévy process. It can be shown that $(-\Delta)^s$ converges in a suitable sense to the Laplace operator supplemented with homogeneous Dirichlet boundary conditions and to the identity operator when $s \uparrow 1$ and $s \downarrow 0$, respectively; see [46] for details.

For the *particular case* $s = 1$, there are several works in the literature that provide error estimates for finite element discretizations of (3.1.1)–(3.1.3). To the best of our knowledge, the first work that provides an analysis for suitable finite element discretizations is [73]. In this work, the authors propose two schemes that discretize the admissible control set with piecewise constant and piecewise linear functions, and provide bounds for the error committed within the approximation of a control variable [73, Corollaries 5.6 and 5.10]. These results were later extended to mixed and stabilized finite element

methods in [34] and [56], respectively. Recently, bilinear optimal control problems whose objective functionals do not depend on the controls have been analyzed in [33]. In particular, the authors investigate sufficient second order conditions for bang–bang controls and derive error estimates in L^1 -norms for a suitable finite element discretization. Regarding the analysis of *a posteriori* error estimates for problem (3.1.1)–(3.1.3) with $s = 1$, we refer the reader to [74] and [58]. We conclude this paragraph with a reference to the work [107], in which the authors provide upper bounds for discretization errors with respect to a cost functional and with respect to a given quantity of interest based on a posteriori error estimators.

To the best of our knowledge, the only work available in the literature that provides an advance on the *parabolic* version of (3.1.1)–(3.1.3) is the very recent manuscript [71]. In this paper, the authors provide an analysis for the continuous problem including the existence of solutions and first and second order optimality conditions; the second order sufficiency requires an additional assumption on the problem data (see also [73, Remark 2.21]). In contrast to this work and to the best of our knowledge, this exposition is the first to study approximation techniques for the optimal control problem (3.1.1)–(3.1.3); discretization techniques for the problem where the control variable enters (3.1.2) as a source term are available in [14, 44, 62, 89, 90]. In the following, we list what we consider to be the most important contributions of our work:

- (i) *Existence of optimal solutions*: We prove that the optimal control problem (3.1.1)–(3.1.3) admits at least one optimal solution; see Theorem 3.4.1.
- (ii) *Optimality conditions*: We derive first and necessary and sufficient second order optimality conditions with a minimal gap; see §3.4.2 and §3.4.3.
- (iii) *Regularity estimates*: We analyze regularity properties for optimal variables. In particular, we prove that $\bar{u}, \bar{p}, \bar{q} \in H^{s+\kappa-\epsilon}(\Omega)$; see Theorem 3.4.4 for details.
- (iv) *Finite element discretizations*: We propose two different strategies: a semidiscrete scheme, where the control set is not discretized, and a fully discrete scheme, where such a set is discretized with piecewise constant functions.
- (v) *Convergence of discretizations*: We prove the existence of subsequences of discrete global solutions that converge to global solutions of (3.1.1)–(3.1.3). We also prove that continuous strict local solutions can be approximated by local minima of discrete problems; see section 3.5.3.2 and section 3.5.3.3.
- (vi) *Error estimates*: For each discretization scheme, we provide estimates for the error that occurs when approximating optimal variables; see §3.6. To obtain these results, we have assumed that solutions to suitable finite element discretizations of (3.1.2) are uniformly bounded in $L^\infty(\Omega)$.

We structure our presentation as follows. In section 3.2, we introduce the notation and collect some known facts that shall be useful for our purposes. In section 3.3, we give an overview of regularity and finite element approximation results for problem (3.1.2). Section 3.4 is our first original contribution. We study a weak version of the control problem (3.1.1)–(3.1.3). More precisely, we show the *existence* of optimal solutions and derive *first* and *second* order optimality conditions. The study of numerical schemes begins in section 3.5, where we develop two finite element schemes for (3.1.1)–(3.1.3) and analyze convergence properties. In section 3.6, we derive error bounds. We conclude with section 3.7, where we provide several numerical examples to illustrate our theory.

3.2 Notation and preliminary remarks

Let us establish the notation and recall some facts that will be useful later.

3.2.1 Notation

In the course of this work, let $d \geq 2$ and $\Omega \subset \mathbb{R}^d$ be an open and bounded domain with Lipschitz boundary $\partial\Omega$. We denote by Ω^c the complement of Ω . If \mathcal{X} and \mathcal{Y} are Banach function spaces, we write $\mathcal{X} \hookrightarrow \mathcal{Y}$ to denote that \mathcal{X} is continuously embedded in \mathcal{Y} . We denote by \mathcal{X}' and $\|\cdot\|_{\mathcal{X}}$ the dual and the norm of \mathcal{X} , respectively. We denote by $\langle \cdot, \cdot \rangle_{\mathcal{X}', \mathcal{X}}$ the duality pairing between \mathcal{X}' and \mathcal{X} and simply write $\langle \cdot, \cdot \rangle$ if the spaces \mathcal{X}' and \mathcal{X} are clear from the context. The relation $\mathbf{a} \lesssim \mathbf{b}$ indicates that $\mathbf{a} \leq C\mathbf{b}$, with a positive constant C that does not depend on \mathbf{a} , \mathbf{b} , or the discretization parameters, but may depend on s , d , and Ω . The value of C might change at each occurrence.

3.2.2 Function spaces

Fractional Sobolev spaces provide a natural framework for analyzing the state equation (3.1.2). A family of fractional Sobolev spaces can be defined based on the Fourier transform \mathcal{F} : For any $s \geq 0$, we define $H^s(\mathbb{R}^d)$, a Sobolev space of order s over \mathbb{R}^d , by [102, Definition 15.7], [78, Chapter 1, Section 7]

$$H^s(\mathbb{R}^d) := \{v \in L^2(\mathbb{R}^d) : (1 + |\xi|^2)^{\frac{s}{2}} \mathcal{F}(v) \in L^2(\mathbb{R}^d)\},$$

endowed with the norm $\|v\|_{H^s(\mathbb{R}^d)} := \|(1 + |\xi|^2)^{\frac{s}{2}} \mathcal{F}(v)\|_{L^2(\mathbb{R}^d)}$. We define $\tilde{H}^s(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $H^s(\mathbb{R}^d)$ [81, page 77] and note that it can be equivalently characterized as the following space of zero-extension functions [81, Theorem 3.29]:

$$\tilde{H}^s(\Omega) = \{v|_\Omega : v \in H^s(\mathbb{R}^d), \text{supp } v \subset \bar{\Omega}\}.$$

We endow the space $\tilde{H}^s(\Omega)$ with the following inner product and norm [81, page 75]:

$$(v, w)_{\tilde{H}^s(\Omega)} := \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(v(x) - v(y))(w(x) - w(y))}{|x - y|^{d+2s}} dx dy, \quad |v|_{\tilde{H}^s(\Omega)} := (v, v)_{\tilde{H}^s(\Omega)}^{\frac{1}{2}}.$$

We also introduce $H^{-s}(\Omega)$ as the dual space of the fractional Sobolev space $\tilde{H}^s(\Omega)$.

Let us review a continuity of the product property in fractional Sobolev spaces.

Lemma 3.2.1 (continuity of the product). *Let $\mathfrak{t} \in (0, \infty)$ and let $\varphi, \phi \in H^{\mathfrak{t}}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. Then, the product $\varphi\phi$ belongs to the space $H^{\mathfrak{t}}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$.*

Proof. The result follows from a direct application of the Runst-Sickel lemma [20, Lemma 4.1] ([95, Section 5.3.7]) with $s = \mathfrak{t}$, $p_1 = p_2 = p = q = 2$, and $r_1 = r_2 = \infty$. We note that, as stated in [20, page 390], the Triebel-Lizorkin space $\tilde{\mathcal{F}}_{2,2}^{\mathfrak{t}}$ coincides with $H^{\mathfrak{t}}(\mathbb{R}^d)$; see also [103, Chapter 2, Section 2.3.5]. \square

We conclude this section with the following Sobolev embedding results.

Lemma 3.2.2 (embedding results). *Let $s \in (0, 1)$. If $\mathfrak{r} \in [1, 2d/(d - 2s)]$, then $H^s(\Omega) \hookrightarrow L^{\mathfrak{r}}(\Omega)$. If $\mathfrak{r} \in [1, 2d/(d - 2s))$, then $H^s(\Omega) \hookrightarrow L^{\mathfrak{r}}(\Omega)$ is compact.*

Proof. A proof of $H^s(\Omega) \hookrightarrow L^{\mathfrak{r}}(\Omega)$ can be found in [4, Theorem 7.34]. The fact that the embedding is compact for $\mathfrak{r} < 2d/(d - 2s)$ follows from [46, Corollary 7.2]. \square

3.2.3 The fractional Laplace operator

For $s \in (0, 1)$ and smooth functions $w : \mathbb{R}^d \rightarrow \mathbb{R}$, there are several equivalent definitions of $(-\Delta)^s$ in \mathbb{R}^d . In fact, $(-\Delta)^s$ can be naturally defined by means of the following pointwise formula:

$$(-\Delta)^s w(x) := C(d, s) \text{p.v.} \int_{\mathbb{R}^d} \frac{w(x) - w(y)}{|x - y|^{d+2s}} dy, \quad C(d, s) := \frac{2^{2s} s \Gamma(s + \frac{d}{2})}{\pi^{\frac{d}{2}} \Gamma(1 - s)}, \quad (3.2.1)$$

where p.v. stands for the *Cauchy principal value* and $C(d, s)$ is a normalization constant. The constant $C(d, s)$ is introduced to ensure that the definition (3.2.1) is equivalent to the following one via Fourier transform: $\mathcal{F}((-\Delta)^s w)(\xi) = |\xi|^{2s} \mathcal{F}(w)(\xi)$ for all $\xi \in \mathbb{R}^d$; see [76, chapter 1, section 1] for details. In addition to these two definitions, there are several other equivalent definitions of $(-\Delta)^s$ in \mathbb{R}^d in the literature. For a discussion, we refer the reader to [75].

In bounded domains, for functions supported in $\bar{\Omega}$, we may use the integral representation (3.2.1) to define $(-\Delta)^s$. This gives rise to the so-called *restricted* or *integral* fractional Laplacian, which, from now on, we shall simply refer to as the *integral fractional Laplacian*. Note that we have materialized

a zero Dirichlet condition by restricting the operator to acting only on functions that are zero outside Ω .

To present suitable weak formulations for problems involving $(-\Delta)^s$, we define

$$\mathcal{A} : \tilde{H}^s(\Omega) \times \tilde{H}^s(\Omega) \rightarrow \mathbb{R}, \quad \mathcal{A}(v, w) := \frac{C(d, s)}{2}(v, w)_{\tilde{H}^s(\Omega)}.$$

We note that \mathcal{A} is just a multiple of the inner product in $\tilde{H}^s(\Omega)$ introduced in section 3.2.2; \mathcal{A} is bilinear and bounded. We denote by $\|\cdot\|_s$ the norm induced by \mathcal{A} :

$$\|v\|_s := \sqrt{\mathcal{A}(v, v)} = \mathfrak{C}(d, s)|v|_{\tilde{H}^s(\Omega)}, \quad \mathfrak{C}(d, s) = \sqrt{C(d, s)/2}.$$

3.3 The state equation

In this section, we present a suitable weak formulation for (3.1.2) and give a brief overview of results concerning the well-posedness of such a formulation, regularity estimates for its solution, and finite element approximations.

3.3.1 Weak formulation

Let $s \in (0, 1)$, let $\mathfrak{f} \in H^{-s}(\Omega)$ be a given forcing term, and let \mathfrak{q} be an arbitrary element in \mathbb{Q}_{ad} . Under these conditions, we introduce the following weak formulation of the state equation (3.1.2): Find $\mathbf{u} \in \tilde{H}^s(\Omega)$ such that

$$\mathcal{A}(\mathbf{u}, v) + (\mathfrak{q}\mathbf{u}, v)_{L^2(\Omega)} = \langle \mathfrak{f}, v \rangle \quad \forall v \in \tilde{H}^s(\Omega). \quad (3.3.1)$$

We note that, since $\mathfrak{q} \in \mathbb{Q}_{ad} \subset L^\infty(\Omega)$ and $\mathbf{u}, v \in \tilde{H}^s(\Omega) \subset L^2(\Omega)$, all terms involved in (3.3.1) are well-defined. The well-posedness of problem (3.3.1) follows from the Lax-Milgram lemma. In particular, we have the *stability bound* $\|\mathbf{u}\|_s \lesssim \|\mathfrak{f}\|_{H^{-s}(\Omega)}$.

3.3.2 Regularity estimates

We present the following regularity result.

Theorem 3.3.1 (Sobolev regularity). *Let $s \in (0, 1)$, $\mathfrak{f} \in L^2(\Omega)$, and $\mathfrak{q} \in \mathbb{Q}_{ad}$. Then, the solution \mathbf{u} to problem (3.3.1) belongs to $H^{s+\kappa-\varepsilon}(\Omega)$ for all $0 < \varepsilon < s$, where $\kappa = \frac{1}{2}$ for $\frac{1}{2} < s < 1$ and $\kappa = s - \varepsilon$ for $0 < s \leq \frac{1}{2}$. In addition, we have the bound*

$$\|\mathbf{u}\|_{H^{s+\kappa-\varepsilon}(\Omega)} \leq C\varepsilon^{-\nu}\|\mathfrak{f}\|_{L^2(\Omega)} \quad \forall \varepsilon \in (0, s),$$

where $\nu = \frac{1}{2}$ for $\frac{1}{2} < s < 1$ and $\nu = \frac{1}{2} + \nu_0$ for $0 < s \leq \frac{1}{2}$. Here, ν_0 and C denote positive constants that depend on Ω and d and Ω , d , s , and \mathbf{q} , respectively.

Proof. Since Ω is Lipschitz and $\mathbf{f} - \mathbf{q}\mathbf{u} \in L^2(\Omega)$, the proof follows immediately as an application of [17, Theorem 2.1 and inequality (2.6)]; see also [18, Remark 6]. \square

If we assume a higher integrability assumption for \mathbf{f} , it is possible to obtain an $L^\infty(\Omega)$ -regularity result for the solution \mathbf{u} of problem (3.3.1).

Theorem 3.3.2 ($L^\infty(\Omega)$ -regularity). *Let $s \in (0, 1)$ and $r > d/2s$. If $\mathbf{f} \in L^r(\Omega)$, then the solution \mathbf{u} to problem (3.3.1) belongs to $\tilde{H}^s(\Omega) \cap L^\infty(\Omega)$ and*

$$\|\mathbf{u}\|_s + \|\mathbf{u}\|_{L^\infty(\Omega)} \lesssim \|\mathbf{f}\|_{L^r(\Omega)},$$

with a hidden constant that is independent of \mathbf{u} and the problem data.

Proof. See [89, Theorem 3.1]. \square

3.3.3 Finite element approximation

Under the additional assumption that Ω is a Lipschitz polytope, we now introduce a finite element-like scheme to approximate the solution to (3.3.1). For this purpose, we assume that we have at hand a collection of conforming and quasi-uniform meshes $\{\mathcal{T}_h\}_{h>0}$ of $\bar{\Omega}$ made of closed simplices. We denote by $h := \max\{h_T : T \in \mathcal{T}_h\}$ the mesh-size of $\mathcal{T}_h = \{T\}$, where $h_T := \text{diam}(T)$.

Given a mesh \mathcal{T}_h , we introduce the finite element space

$$\mathbb{V}_h := \{v_h \in C(\bar{\Omega}) : v_h|_T \in \mathbb{P}_1(T) \ \forall T \in \mathcal{T}_h, v_h = 0 \text{ on } \partial\Omega\}. \quad (3.3.2)$$

The following comments are now appropriate. First, for every $s \in (0, 1)$, $\mathbb{V}_h \subset \tilde{H}^s(\Omega)$. Second, we enforce a classical homogeneous Dirichlet boundary condition at $\partial\Omega$. Note that discrete functions are trivially extended by zero to Ω^c .

With \mathbb{V}_h at hand, we introduce an approximation of the solution to (3.3.1):

$$\mathbf{u}_h \in \mathbb{V}_h : \quad \mathcal{A}(\mathbf{u}_h, v_h) + (\mathbf{q}\mathbf{u}_h, v_h)_{L^2(\Omega)} = \langle \mathbf{f}, v_h \rangle \quad \forall v_h \in \mathbb{V}_h. \quad (3.3.3)$$

The existence and uniqueness of $\mathbf{u}_h \in \mathbb{V}_h$ follows from the Lax-Milgram lemma. In particular, for every $h > 0$, we have the *discrete stability bound* $\|\mathbf{u}_h\|_s \lesssim \|\mathbf{f}\|_{H^{-s}(\Omega)}$.

We present the following a priori error estimates.

Theorem 3.3.3 (error estimates). *Let $s \in (0, 1)$, $f \in L^2(\Omega)$, and $q \in \mathbb{Q}_{ad}$. Let Ω be a Lipschitz polytope. Let $u \in \tilde{H}^s(\Omega)$ be the solution to (3.3.1) and let $u_h \in \mathbb{V}_h$ be its finite element approximation obtained as in (3.3.3). Then, we have the error bounds*

$$\|u - u_h\|_s \lesssim h^\gamma |\log h|^\varphi \|f\|_{L^2(\Omega)}, \quad \gamma = \min\{s, \frac{1}{2}\}, \quad (3.3.4)$$

$$\|u - u_h\|_{L^2(\Omega)} \lesssim h^{2\gamma} |\log h|^{2\varphi} \|f\|_{L^2(\Omega)}, \quad (3.3.5)$$

where $\varphi = \nu$ if $s \neq \frac{1}{2}$, $\varphi = 1 + \nu$ if $s = \frac{1}{2}$, and $\nu \geq \frac{1}{2}$ is the constant in Theorem 3.3.1.

Proof. The proof follows from the regularity estimates for u given in Theorem 3.3.1 and the arguments developed in the proofs of [17, Theorem 3.5] and [17, Proposition 3.8]. For the sake of brevity, we omit the details. \square

3.4 The optimal control problem

In this section, we analyze the following weak formulation of the optimal control problem (3.1.1)–(3.1.3): Find

$$\min\{J(u, q) : (u, q) \in \tilde{H}^s(\Omega) \times \mathbb{Q}_{ad}\} \quad (3.4.1)$$

subject to the *fractional* and *elliptic* state equation

$$\mathcal{A}(u, v) + (qu, v)_{L^2(\Omega)} = (f, v)_{L^2(\Omega)} \quad \forall v \in \tilde{H}^s(\Omega), \quad (3.4.2)$$

where \mathbb{Q}_{ad} is defined in (3.1.3) and $f \in L^2(\Omega)$.

3.4.1 Existence of optimal controls

The existence of at least one optimal solution follows from the direct method of calculus of variations [40, Chapter 1].

Theorem 3.4.1 (existence of an optimal solution). *The optimal control problem (3.4.1)–(3.4.2) admits at least one global solution $(\bar{u}, \bar{q}) \in \tilde{H}^s(\Omega) \times \mathbb{Q}_{ad}$.*

Proof. Let $i := \inf\{J(u, q) : (u, q) \in \tilde{H}^s(\Omega) \times \mathbb{Q}_{ad}\}$. Let, for $k \in \mathbb{N}$, $q_k \in \mathbb{Q}_{ad}$ and $u_k \in \tilde{H}^s(\Omega)$ be such that $J(u_k, q_k) \rightarrow i$ as $k \uparrow \infty$ and

$$\mathcal{A}(u_k, v) + (q_k u_k, v)_{L^2(\Omega)} = (f, v)_{L^2(\Omega)} \quad \forall v \in \tilde{H}^s(\Omega), \quad (3.4.3)$$

i.e., $\{(u_k, q_k)\}_{k \in \mathbb{N}} \subset \tilde{H}^s(\Omega) \times \mathbb{Q}_{ad}$ is a minimizing sequence. Let us now invoke the fact that \mathbb{Q}_{ad} is

bounded in $L^\infty(\Omega)$ to deduce the existence of a nonrelabeled subsequence $\{q_k\}_{k \in \mathbb{N}} \subset \mathbb{Q}_{ad}$ such that $q_k \xrightarrow{*} \bar{q}$ in $L^\infty(\Omega)$ as $k \uparrow \infty$; $\bar{q} \in \mathbb{Q}_{ad}$. On the other hand, since for every $k \in \mathbb{N}$ $q_k \in \mathbb{Q}_{ad}$, the well-posedness of (3.4.3) reveals that $\{u_k\}_{k \in \mathbb{N}}$ is uniformly bounded in $\tilde{H}^s(\Omega)$. More precisely: $\|u_k\|_s \lesssim \|f\|_{L^2(\Omega)}$ for every $k \in \mathbb{N}$. We can thus conclude the existence of a nonrelabeled subsequence such that $u_k \rightharpoonup \bar{u}$ in $\tilde{H}^s(\Omega)$ as $k \uparrow \infty$; \bar{u} being the natural candidate for an optimal state. We now prove that \bar{u} solves problem (3.4.2), where q is replaced by \bar{q} , and that (\bar{u}, \bar{q}) is optimal.

Since $u_k \rightharpoonup \bar{u}$ in $\tilde{H}^s(\Omega)$ as $k \uparrow \infty$, we immediately obtain that $\mathcal{A}(u_k, v) \rightarrow \mathcal{A}(\bar{u}, v)$ as $k \uparrow \infty$ for every $v \in \tilde{H}^s(\Omega)$. To analyze the convergence of the bilinear term, we utilize that $u_k \rightharpoonup \bar{u}$ in $\tilde{H}^s(\Omega)$ as $k \uparrow \infty$, the compact embedding $H^s(\Omega) \hookrightarrow L^2(\Omega)$ of Lemma 3.2.2, which holds because $2d/(d-2s) > 2$, the convergence $q_k \xrightarrow{*} \bar{q}$ in $L^\infty(\Omega)$ as $k \uparrow \infty$, and the fact that $\bar{u}v \in L^1(\Omega)$. These arguments allow us to obtain

$$\begin{aligned} |(\bar{q}\bar{u}, v)_{L^2(\Omega)} - (q_k u_k, v)_{L^2(\Omega)}| &\leq |([\bar{q} - q_k]\bar{u}, v)_{L^2(\Omega)}| + |(q_k[\bar{u} - u_k], v)_{L^2(\Omega)}| \\ &\leq |(\bar{q} - q_k, \bar{u}v)_{L^2(\Omega)}| + b\|\bar{u} - u_k\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} \rightarrow 0, \quad k \uparrow \infty. \end{aligned}$$

Finally, we prove that (\bar{u}, \bar{q}) is optimal. To do so, we utilize the strong convergence $u_k \rightarrow \bar{u}$ in $L^2(\Omega)$ as $k \uparrow \infty$ and the fact that the square of $\|\cdot\|_{L^2(\Omega)}$ is weakly lower semicontinuous in $L^2(\Omega)$. With these results, we can therefore conclude that $J(\bar{u}, \bar{q}) \leq \liminf_k J(u_k, q_k) = i$. Consequently, (\bar{u}, \bar{q}) is optimal. \square

Since the control problem (3.4.1)–(3.4.2) is not convex, in the following analysis we will discuss optimality conditions in the context of local solutions in $L^2(\Omega)$ [104, page 207]: We say that $\bar{q} \in \mathbb{Q}_{ad}$ is *locally optimal* in the sense of $L^2(\Omega)$ for (3.4.1)–(3.4.2) if there exists $\eta > 0$ such that $J(\bar{u}, \bar{q}) \leq J(u, q)$ for all $(u, q) \in \tilde{H}^s(\Omega) \times \mathbb{Q}_{ad}$ such that $\|q - \bar{q}\|_{L^2(\Omega)} \leq \eta$. Here, u solves (3.4.2) and \bar{u} solves (3.4.2) with q replaced by \bar{q} .

3.4.2 First order optimality conditions

In this section we formulate first order necessary optimality conditions. We begin our analysis by introducing the set

$$\mathcal{Q} := \{q \in L^\infty(\Omega) : \exists c > 0 \text{ such that } q(x) > c > 0 \text{ for a.e. } x \in \Omega\}$$

[73, page 783] and the control to state map $\mathcal{S} : \mathcal{Q} \rightarrow \tilde{H}^s(\Omega)$, which given a control q associates to it the unique state $u = \mathcal{S}q$ that solves (3.4.2).

The following result provides differentiability properties for the operator \mathcal{S} .

Theorem 3.4.2 (differentiability properties of \mathcal{S}). *The control to state operator $\mathcal{S} : \mathcal{Q} \rightarrow \tilde{H}^s(\Omega)$ is of class C^2 with respect to the $L^\infty(\Omega)$ -topology. In addition, if $w \in L^\infty(\Omega)$, then $z = \mathcal{S}'(q)w \in \tilde{H}^s(\Omega)$ corresponds to the unique solution to*

$$\mathcal{A}(z, v) + (qz, v)_{L^2(\Omega)} = -(wu, v)_{L^2(\Omega)} \quad \forall v \in \tilde{H}^s(\Omega), \quad (3.4.4)$$

where $u = \mathcal{S}q$. Moreover, if $w_1, w_2 \in L^\infty(\Omega)$, then $\mathfrak{z} = \mathcal{S}''(q)(w_1, w_2) \in \tilde{H}^s(\Omega)$ is the unique solution to

$$\mathcal{A}(\mathfrak{z}, v) + (q\mathfrak{z}, v)_{L^2(\Omega)} = -(w_2 z_{w_1}, v)_{L^2(\Omega)} - (w_1 z_{w_2}, v)_{L^2(\Omega)} \quad \forall v \in \tilde{H}^s(\Omega), \quad (3.4.5)$$

where $z_{w_i} = \mathcal{S}'(q)w_i$ and $i \in \{1, 2\}$.

Proof. Let $q \in \mathcal{Q}$. To prove the first order Fréchet differentiability of \mathcal{S} we proceed as in [104, Theorem 4.17] and show the existence of a continuous linear operator $D : L^\infty(\Omega) \rightarrow \tilde{H}^s(\Omega)$ and a mapping $r(q, \cdot) : L^\infty(\Omega) \rightarrow \tilde{H}^s(\Omega)$ such that for all $w \in L^\infty(\Omega)$ satisfying $q + w \in \mathcal{Q}$, we have $\mathcal{S}(q + w) - \mathcal{S}(q) = D(w) + r(q, w)$ and

$$\|r(q, w)\|_s / \|w\|_{L^\infty(\Omega)} \rightarrow 0 \quad (3.4.6)$$

as $\|w\|_{L^\infty(\Omega)} \rightarrow 0$. We immediately note that the mapping $w \mapsto z$ defined by the problem (3.4.4) is linear and continuous. On the other hand, we define $\tilde{u} := \mathcal{S}(q + w)$ and $u := \mathcal{S}(q)$. It is therefore sufficient to prove that $r = \tilde{u} - u - z$ satisfies (3.4.6). To do so, we first note that r solves uniquely the problem: Find $r \in \tilde{H}^s(\Omega)$ such that

$$\mathcal{A}(r, v) + (qr, v)_{L^2(\Omega)} = -(w[\tilde{u} - u], v)_{L^2(\Omega)} \quad \forall v \in \tilde{H}^s(\Omega).$$

A basic stability bound for this problem yields the estimate

$$\|r\|_s \lesssim \|\tilde{u} - u\|_{L^2(\Omega)} \|w\|_{L^\infty(\Omega)} \lesssim \|\tilde{u}\|_{L^2(\Omega)} \|w\|_{L^\infty(\Omega)}^2 \lesssim \|f\|_{L^2(\Omega)} \|w\|_{L^\infty(\Omega)}^2,$$

where we used a stability estimate for the problem that $\tilde{u} - u$ solves and $\|\tilde{u}\|_s \lesssim \|f\|_{L^2(\Omega)}$. Consequently, r satisfies (3.4.6). This proves that \mathcal{S} is Fréchet differentiable.

The second order Fréchet differentiability of \mathcal{S} follows similar considerations. The fact that \mathfrak{z} solves the problem (3.4.5) follows from the arguments in [104, Theorem 4.24(ii)]. Note that the problem (3.4.5) is well posed because $-w_2 z_{w_1} - w_1 z_{w_2} \in L^2(\Omega)$. \square

In order to provide first order optimality conditions, we introduce the reduced cost functional $j : \mathcal{Q} \rightarrow \mathbb{R}$ by $j(q) = J(\mathcal{S}q, q)$ and present the following basic result: If $\bar{q} \in \mathbb{Q}_{ad}$ denotes a locally optimal control

for (3.4.1)–(3.4.2), then [104, Lemma 4.18]

$$j'(\bar{q})(q - \bar{q}) \geq 0 \quad \forall q \in \mathbb{Q}_{ad}. \quad (3.4.7)$$

In (3.4.7), $j'(\bar{q})$ denotes the Gateâux derivative of j at \bar{q} . To investigate (3.4.7), we introduce the *adjoint variable* $p \in \tilde{H}^s(\Omega)$ as the unique solution to the *adjoint equation*

$$\mathcal{A}(v, p) + (qp, v)_{L^2(\Omega)} = (u - u_\Omega, v)_{L^2(\Omega)} \quad \forall v \in \tilde{H}^s(\Omega). \quad (3.4.8)$$

Here, $u = \mathcal{S}q$. The well-posedness of (3.4.8) follows from the Lax-Milgram lemma. In particular, we have the *stability bound* $\|p\|_s \lesssim \|f\|_{L^2(\Omega)} + \|u_\Omega\|_{L^2(\Omega)}$.

We now have all the ingredients to present first order optimality conditions.

Theorem 3.4.3 (first order necessary optimality conditions). *Every locally optimal control $\bar{q} \in \mathbb{Q}_{ad}$ for problem (3.4.1)–(3.4.2) satisfies the variational inequality*

$$(\lambda\bar{q} - \bar{u}\bar{p}, q - \bar{q})_{L^2(\Omega)} \geq 0 \quad \forall q \in \mathbb{Q}_{ad}, \quad (3.4.9)$$

where $\bar{p} \in \tilde{H}^s(\Omega)$ solves (3.4.8) with q and u replaced by \bar{q} and $\bar{u} = \mathcal{S}\bar{q}$, respectively.

Proof. We begin the proof with simple calculations that show that the variational inequality (3.4.7) can be rewritten as follows:

$$(\bar{u} - u_\Omega, \mathcal{S}'(\bar{q})(q - \bar{q}))_{L^2(\Omega)} + \lambda(\bar{q}, q - \bar{q})_{L^2(\Omega)} \geq 0 \quad \forall q \in \mathbb{Q}_{ad}, \quad (3.4.10)$$

where $\bar{u} = \mathcal{S}\bar{q}$. Since the second term on the left-hand side of (3.4.10) is already contained in the desired inequality (3.4.9), we thus concentrate on the first term. Define $z := \mathcal{S}'(\bar{q})(q - \bar{q})$ and observe that z solves problem (3.4.4) with q replaced by \bar{q} and $w = q - \bar{q}$. Setting $v = \bar{p}$ in this problem yields

$$\mathcal{A}(z, \bar{p}) + (\bar{q}z, \bar{p})_{L^2(\Omega)} = -((q - \bar{q})\bar{u}, \bar{p})_{L^2(\Omega)}, \quad (3.4.11)$$

where $\bar{u} = \mathcal{S}\bar{q}$. On the other hand, we set $v = z$ as a test function in (3.4.8) to obtain

$$\mathcal{A}(z, \bar{p}) + (\bar{q}\bar{p}, z)_{L^2(\Omega)} = (\bar{u} - u_\Omega, z)_{L^2(\Omega)}. \quad (3.4.12)$$

The desired variational inequality (3.4.9) thus follows from (3.4.10), (3.4.11), and (3.4.12). \square

The following formula is essential for the derivation of regularity estimates: If \bar{q} is a locally optimal

control for the problem (3.4.1)–(3.4.2), then [104, section 4.6.1]

$$\bar{q}(x) := \Pi_{[a,b]}(\lambda^{-1}\bar{u}(x)\bar{p}(x)) \text{ a.e. } x \in \Omega, \quad (3.4.13)$$

where $\Pi_{[a,b]} : L^1(\Omega) \rightarrow \mathbb{Q}_{ad}$ is defined by $\Pi_{[a,b]}(v) := \min\{b, \max\{v, a\}\}$ a.e. in Ω .

We conclude this section with a regularity result for an optimal control variable, which will be important for the analysis of the schemes in §3.5. To present it, we define

$$\Lambda(f, u_\Omega) := \|f\|_{L^2(\Omega)} (\|f\|_{L^r(\Omega)} + \|u_\Omega\|_{L^r(\Omega)}) + \|f\|_{L^r(\Omega)} \|u_\Omega\|_{L^2(\Omega)}. \quad (3.4.14)$$

Theorem 3.4.4 (regularity of \bar{q}). *Let $s \in (0, 1)$ and $r > d/2s$. Let \bar{q} be a locally optimal control for (3.4.1)–(3.4.2). If $f, u_\Omega \in L^2(\Omega) \cap L^r(\Omega)$, then $\bar{q} \in H^{s+\kappa-\varepsilon}(\Omega)$ for all $0 < \varepsilon < s$, where $\kappa = \frac{1}{2}$ for $\frac{1}{2} < s < 1$ and $\kappa = s - \varepsilon$ for $0 < s \leq \frac{1}{2}$. In addition,*

$$\|\bar{q}\|_{H^{s+\kappa-\varepsilon}(\Omega)} \leq C\varepsilon^{-\nu}(1 + \Lambda(f, u_\Omega)), \quad (3.4.15)$$

where ν is as in the statement of Theorem 3.3.1.

Proof. Let $\bar{u} = \mathcal{S}\bar{q}$ and let \bar{p} be the solution to (3.4.8) with u and q replaced by \bar{u} and \bar{q} , respectively. Since $f, u_\Omega \in L^2(\Omega)$, an application of Theorem 3.3.1 immediately shows that \bar{u} and \bar{p} belong to $H^{s+\kappa-\varepsilon}(\Omega)$, for all $0 < \varepsilon < s$, where κ is as in the statement of the theorem. On the other hand, since $f \in L^r(\Omega)$, for some $r > d/2s$, we can conclude from the results of Theorem 3.3.2 that $\bar{u} \in L^\infty(\Omega)$. Note that, in particular, $\bar{u} \in L^r(\Omega)$, for some $r > d/2s$. Consequently, $\bar{u} - u_\Omega \in L^r(\Omega)$. We can therefore again refer to Theorem 3.3.2 to conclude that $\bar{p} \in L^\infty(\Omega)$. Observe that we have obtained that $\bar{u}, \bar{p} \in H^{s+\kappa-\varepsilon}(\Omega) \cap L^\infty(\Omega)$. This, in light of the continuity of the product property stated in Lemma 3.2.1, allows us to conclude that $\bar{u}\bar{p}$ belongs to $H^{s+\kappa-\varepsilon}(\Omega) \cap L^\infty(\Omega)$ together with the bound [20, estimate (25)] $\|\bar{u}\bar{p}\|_{H^{s+\kappa-\varepsilon}(\Omega)} \lesssim C\varepsilon^{-\nu}\Lambda(f, u_\Omega)$. The desired regularity property for \bar{q} and the estimate (3.4.15) follow from the projection formula (3.4.13), the fact that $\max\{0, \tau\} = (\tau + |\tau|)/2$ for all $\tau \in \mathbb{R}$, and [86, Theorem 1], which applies because $s + \kappa - \varepsilon = s + \frac{1}{2} - \varepsilon < \frac{3}{2}$ if $\frac{1}{2} < s < 1$ and $s + \kappa - \varepsilon = 2s - 2\varepsilon \leq 1 - 2\varepsilon < \frac{3}{2}$ if $0 < s \leq \frac{1}{2}$. This concludes the proof. \square

3.4.3 Second order optimality conditions

In this section we formulate necessary and sufficient second order optimality conditions.

3.4.3.1 Auxiliary results

We begin our studies with the following result.

Theorem 3.4.5 (*j is of class C^2 and j'' is Lipschitz*). *The reduced cost functional $j : \mathbb{Q}_{ad} \subset \mathcal{Q} \rightarrow \mathbb{R}$ is of class C^2 . Moreover, for every $q \in \mathbb{Q}_{ad}$ and $w \in L^\infty(\Omega)$, we have*

$$j''(q)w^2 = \lambda \|w\|_{L^2(\Omega)}^2 - 2(wz, p)_{L^2(\Omega)} + \|z\|_{L^2(\Omega)}^2, \quad (3.4.16)$$

where p solves (3.4.8) and $z = \mathcal{S}'(q)w$. If, in addition, we assume that $f, u_\Omega \in L^r(\Omega)$, with $r > d/2s$, then, for $q_1, q_2 \in \mathbb{Q}_{ad}$ and $w \in L^\infty(\Omega)$, we have

$$|j''(q_1)w^2 - j''(q_2)w^2| \lesssim \|w\|_{L^2(\Omega)}^2 \|q_1 - q_2\|_{L^r(\Omega)}. \quad (3.4.17)$$

Proof. The fact that j is of class C^2 is a direct consequence of the differentiability properties of the control to state map \mathcal{S} given in Theorem 3.4.2. It is therefore sufficient to derive the identity (3.4.16) and the inequality (3.4.17). To accomplish this task, we proceed in two steps.

Step 1. We first obtain (3.4.16). We begin with simple calculations showing that for every $q \in \mathbb{Q}_{ad}$ and $w \in L^\infty(\Omega)$ we have

$$j''(q)w^2 = \lambda \|w\|_{L^2(\Omega)}^2 + (u - u_\Omega, \mathfrak{z})_{L^2(\Omega)} + \|z\|_{L^2(\Omega)}^2, \quad (3.4.18)$$

where $z, \mathfrak{z} \in \tilde{H}^s(\Omega)$ are as in the statement of Theorem 3.4.2 with $w_1 = w_2 = w$ and $z_w = z$. To obtain the desired identity (3.4.16), we proceed as follows. We set $v = \mathfrak{z}$ in (3.4.8) and $v = p$ in (3.4.5). This results in the relation $(u - u_\Omega, \mathfrak{z})_{L^2(\Omega)} = -2(wz, p)_{L^2(\Omega)}$. Substituting this identity into (3.4.18), we obtain (3.4.16).

Step 2. We now prove (3.4.17). Let $q_1, q_2 \in \mathbb{Q}_{ad}$ and $w \in L^\infty(\Omega)$. Define $z_1 = \mathcal{S}'(q_1)w$ and $z_2 = \mathcal{S}'(q_2)w$. Note that z_1 and z_2 solve (3.4.4) with $u_1 := \mathcal{S}q_1$ and $u_2 := \mathcal{S}q_2$, respectively. We now use the derived identity (3.4.16) to obtain

$$\begin{aligned} j''(q_1)w^2 - j''(q_2)w^2 &= 2(w[z_2 - z_1], p_2)_{L^2(\Omega)} + 2(wz_1, p_2 - p_1)_{L^2(\Omega)} \\ &\quad + (z_1 - z_2, z_1 + z_2)_{L^2(\Omega)} =: \mathbf{I} + \mathbf{II} + \mathbf{III}, \end{aligned} \quad (3.4.19)$$

where $p_i \in \tilde{H}^s(\Omega)$ is the solution of (3.4.8) where u and q are replaced by u_i and q_i , respectively ($i \in \{1, 2\}$). In the following, **I**, **II**, and **III** are estimated separately.

We start with the estimation of **I**. In light of Theorem 3.3.2, the fact that $f, u_\Omega \in L^r(\Omega)$, for $r > d/2s$, yields the following bounds

$$\|u_2\|_{L^\infty(\Omega)} \lesssim \|f\|_{L^r(\Omega)}, \quad \|p_2\|_{L^\infty(\Omega)} \lesssim \|f\|_{L^r(\Omega)} + \|u_\Omega\|_{L^r(\Omega)}. \quad (3.4.20)$$

Consequently, $\mathbf{I} \lesssim \|w\|_{L^2(\Omega)} \|z_2 - z_1\|_{L^2(\Omega)}$, with a hidden constant that depends on $\|f\|_{L^r(\Omega)}$ and $\|u_\Omega\|_{L^r(\Omega)}$. To bound $\|z_2 - z_1\|_{L^2(\Omega)}$, we observe that $z_2 - z_1$ solves

$$\mathcal{A}(z_2 - z_1, v) + (q_2(z_2 - z_1), v)_{L^2(\Omega)} = (z_1[q_1 - q_2] - w[u_2 - u_1], v)_{L^2(\Omega)}$$

for all $v \in \tilde{H}^s(\Omega)$. A stability estimate for this problem in conjunction with Hölder's inequality and Lemma 3.2.2 allows us to obtain that

$$\|z_2 - z_1\|_s \lesssim \|z_1\|_{L^{\frac{2d}{d-2s}}(\Omega)} \|q_1 - q_2\|_{L^r(\Omega)} + \|w\|_{L^2(\Omega)} \|u_2 - u_1\|_{L^\infty(\Omega)}. \quad (3.4.21)$$

An estimate for the term $\|z_1\|_{L^{2d/(d-2s)}(\Omega)}$ follows directly from the well-posedness of problem (3.4.4) and the Sobolev embedding of Lemma 3.2.2. In fact, we have

$$\|z_1\|_{L^{\frac{2d}{d-2s}}(\Omega)} \lesssim \|z_1\|_s \lesssim \|w\|_{L^2(\Omega)} \|u_1\|_{L^\infty(\Omega)} \lesssim \|w\|_{L^2(\Omega)} \|f\|_{L^r(\Omega)}. \quad (3.4.22)$$

On the other hand, we notice that $u_2 - u_1 \in \tilde{H}^s(\Omega)$ solves the following problem:

$$\mathcal{A}(u_2 - u_1, v) + (q_2(u_2 - u_1), v)_{L^2(\Omega)} = (u_1[q_1 - q_2], v)_{L^2(\Omega)} \quad \forall v \in \tilde{H}^s(\Omega).$$

Since $u_1(q_1 - q_2) \in L^r(\Omega)$, an application of the results of Theorem 3.3.2 reveals that $\|u_2 - u_1\|_{L^\infty(\Omega)} \lesssim \|u_1\|_{L^\infty(\Omega)} \|q_2 - q_1\|_{L^r(\Omega)} \lesssim \|f\|_{L^r(\Omega)} \|q_2 - q_1\|_{L^r(\Omega)}$. Replacing the estimates obtained for $\|z_1\|_{L^{2d/(d-2s)}(\Omega)}$ and $\|u_2 - u_1\|_{L^\infty(\Omega)}$ into (3.4.21) yields $\|z_2 - z_1\|_s \lesssim \|f\|_{L^r(\Omega)} \|w\|_{L^2(\Omega)} \|q_1 - q_2\|_{L^r(\Omega)}$. From this, we can conclude that

$$\mathbf{I} \lesssim \|w\|_{L^2(\Omega)} \|z_2 - z_1\|_{L^2(\Omega)} \lesssim \|w\|_{L^2(\Omega)} \|z_2 - z_1\|_s \lesssim \|w\|_{L^2(\Omega)}^2 \|q_1 - q_2\|_{L^r(\Omega)}, \quad (3.4.23)$$

with a hidden constant that depends on $\|f\|_{L^r(\Omega)}$ and $\|u_\Omega\|_{L^r(\Omega)}$.

We now control the term \mathbf{II} . To accomplish this task, we utilize Hölder's inequality and (3.4.22) to obtain $\mathbf{II} \lesssim \|w\|_{L^2(\Omega)} \|z_1\|_{L^2(\Omega)} \|p_2 - p_1\|_{L^\infty(\Omega)} \lesssim \|w\|_{L^2(\Omega)}^2 \|p_2 - p_1\|_{L^\infty(\Omega)}$. We now bound the term $\|p_2 - p_1\|_{L^\infty(\Omega)}$ based on the fact that $p_2 - p_1$ solves

$$\mathcal{A}(v, p_2 - p_1) + (q_2(p_2 - p_1), v)_{L^2(\Omega)} = ((u_2 - u_1) + p_1[q_1 - q_2], v)_{L^2(\Omega)} \quad \forall v \in \tilde{H}^s(\Omega).$$

Since the right-hand side of the previous weak formulation belongs to $L^r(\Omega)$, Theorem 3.3.2 immediately implies that $p_2 - p_1 \in \tilde{H}^s(\Omega) \cap L^\infty(\Omega)$ together with the estimate

$$\|p_2 - p_1\|_{L^\infty(\Omega)} \lesssim \|u_2 - u_1\|_{L^\infty(\Omega)} + \|p_1\|_{L^\infty(\Omega)} \|q_2 - q_1\|_{L^r(\Omega)}.$$

We thus obtain, in view of $\|u_2 - u_1\|_{L^\infty(\Omega)} \lesssim \|q_2 - q_1\|_{L^r(\Omega)}$ and an analogue of (3.4.20) for $\|p_1\|_{L^\infty(\Omega)}$, that $\|p_2 - p_1\|_{L^\infty(\Omega)} \lesssim \|q_2 - q_1\|_{L^r(\Omega)}$. Consequently,

$$\mathbf{II} \lesssim \|w\|_{L^2(\Omega)}^2 \|p_2 - p_1\|_{L^\infty(\Omega)} \lesssim \|w\|_{L^2(\Omega)}^2 \|q_2 - q_1\|_{L^r(\Omega)}. \quad (3.4.24)$$

Finally, we control **III**. To do this, we invoke the bound (3.4.22), an analogue of (3.4.22) for z_2 , and $\|z_2 - z_1\|_s \lesssim \|w\|_{L^2(\Omega)} \|q_1 - q_2\|_{L^r(\Omega)}$. These arguments yield

$$\begin{aligned} \mathbf{III} &\lesssim \|z_2 - z_1\|_{L^2(\Omega)} (\|z_1\|_{L^2(\Omega)} + \|z_2\|_{L^2(\Omega)}) \\ &\lesssim \|z_2 - z_1\|_s \|w\|_{L^2(\Omega)} \lesssim \|w\|_{L^2(\Omega)}^2 \|q_2 - q_1\|_{L^r(\Omega)}. \end{aligned} \quad (3.4.25)$$

We conclude (3.4.17) by replacing (3.4.23), (3.4.24), and (3.4.25) into (3.4.19). \square

3.4.3.2 Second order necessary optimality conditions

Let $(\bar{u}, \bar{p}, \bar{q}) \in \tilde{H}^s(\Omega) \times \tilde{H}^s(\Omega) \times \mathbb{Q}_{ad}$ satisfy the first order optimality conditions (3.4.2), (3.4.8), and (3.4.9). Define $\bar{\mathfrak{d}} := \lambda \bar{q} - \bar{u} \bar{p}$. The variational inequality (3.4.9) immediately yields, for a.e. $x \in \Omega$,

$$\bar{\mathfrak{d}}(x) = 0 \text{ if } a < \bar{q}(x) < b, \quad \bar{\mathfrak{d}}(x) \geq 0 \text{ if } \bar{q}(x) = a, \quad \bar{\mathfrak{d}}(x) \leq 0 \text{ if } \bar{q}(x) = b. \quad (3.4.26)$$

In order to formulate second order conditions, we introduce the following *cone of critical directions* inspired by [33, definition (2.7)] and [28, Section 6, page 20]:

$$C_{\bar{q}} := \{w \in L^2(\Omega) \text{ satisfying (3.4.28) and } w(x) = 0 \text{ if } \bar{\mathfrak{d}}(x) \neq 0\}, \quad (3.4.27)$$

where condition (3.4.28) reads as follows:

$$w(x) \geq 0 \text{ a.e. } x \in \Omega \text{ if } \bar{q}(x) = a, \quad w(x) \leq 0 \text{ a.e. } x \in \Omega \text{ if } \bar{q}(x) = b. \quad (3.4.28)$$

In the following, we review the arguments developed in the proof of [28, Theorem 23] and present second order necessary optimality conditions.

Theorem 3.4.6 (second order necessary optimality conditions). *If $\bar{q} \in \mathbb{Q}_{ad}$ denotes a locally optimal control for (3.4.1)–(3.4.2), then $j''(\bar{q})w^2 \geq 0$ for all $w \in C_{\bar{q}}$.*

Proof. Let $w \in C_{\bar{q}}$. Define, for every $k \in \mathbb{N}$ and for a.e. $x \in \Omega$, the function

$$w_k(x) := \begin{cases} 0 & \text{if } x : a < \bar{q}(x) < a + k^{-1}, \quad b - k^{-1} < \bar{q}(x) < b, \\ \Pi_{[-k, k]}(w(x)) & \text{otherwise.} \end{cases}$$

Since $w \in C_{\bar{q}}$, it is immediate that $w_k \in C_{\bar{q}} \cap L^\infty(\Omega)$. On the other hand, we have that $w_k(x) \rightarrow w(x)$ as $k \uparrow \infty$ for a.e. $x \in \Omega$ and that $|w_k(x)| \leq |w(x)|$ for a.e. $x \in \Omega$. Consequently, $w_k \rightarrow w$ in $L^2(\Omega)$ as $k \uparrow \infty$. We now note that, for ρ sufficiently small, or more precisely, for $\rho \in (0, k^{-2}]$, $\bar{q} + \rho w_k$ belongs to \mathbb{Q}_{ad} for every $k \in \mathbb{N}$. Since $\bar{q} + \rho w_k$ is thus admissible, we rely on the fact that \bar{q} is a local minimizer to arrive at the basic inequality $j(\bar{q}) \leq j(\bar{q} + \rho w_k)$ when ρ is sufficiently small. We now use the relation $j'(\bar{q})w_k = 0$, which follows from the fact that $w_k \in C_{\bar{q}}$, and Taylor's theorem for j at \bar{q} to obtain that, for ρ sufficiently small,

$$0 \leq j(\bar{q} + \rho w_k) - j(\bar{q}) = \rho j'(\bar{q})w_k + \frac{\rho^2}{2} j''(\bar{q} + \rho \theta_k w_k) w_k^2 = \frac{\rho^2}{2} j''(\bar{q} + \rho \theta_k w_k) w_k^2,$$

with $\theta_k \in (0, 1)$. We now let $\rho \downarrow 0$ to obtain, on the basis of the estimate (3.4.17),

$$|j''(\bar{q} + \rho \theta_k w_k) w_k^2 - j''(\bar{q}) w_k^2| \lesssim \|w_k\|_{L^2(\Omega)}^2 \rho \theta_k \|w_k\|_{L^r(\Omega)} \rightarrow 0, \quad \rho \downarrow 0,$$

which implies that $j''(\bar{q}) w_k^2 \geq 0$ for every $k \in \mathbb{N}$. Let us now invoke the convergence property $w_k \rightarrow w$ in $L^2(\Omega)$ as $k \uparrow \infty$ and (3.4.16) to conclude that $j''(\bar{q}) w^2 \geq 0$; see the proof of Theorem 3.4.7 below for further details. This concludes the proof. \square

3.4.3.3 Second order sufficient optimality conditions

In this section, we follow the arguments elaborated in the proof of [31, Theorem 2.3] and [28, Theorem 23] and prove a sufficient second order optimality condition with a minimal gap with respect to the necessary condition derived in Theorem 3.4.6.

Theorem 3.4.7 (second order sufficient optimality conditions). *Let $(\bar{u}, \bar{p}, \bar{q}) \in \tilde{H}^s(\Omega) \times \tilde{H}^s(\Omega) \times \mathbb{Q}_{ad}$ satisfy the first order optimality conditions (3.4.2), (3.4.8), and (3.4.9). If $j''(\bar{q}) w^2 > 0$ for all $w \in C_{\bar{q}} \setminus \{0\}$, then there exist $\delta > 0$ and $\sigma > 0$ such that*

$$j(q) \geq j(\bar{q}) + \frac{\delta}{2} \|q - \bar{q}\|_{L^2(\Omega)}^2 \quad \forall q \in \mathbb{Q}_{ad} : \|q - \bar{q}\|_{L^2(\Omega)} \leq \sigma.$$

In particular, \bar{q} is a locally optimal control in the sense of $L^2(\Omega)$.

Proof. We proceed by contradiction and assume that for every natural number k there exists an element

$q_k \in \mathbb{Q}_{ad}$ such that

$$\|\bar{q} - q_k\|_{L^2(\Omega)} < k^{-1}, \quad j(q_k) < j(\bar{q}) + (2k)^{-1}\|\bar{q} - q_k\|_{L^2(\Omega)}^2. \quad (3.4.29)$$

Let us introduce, for $k \in \mathbb{N}$, $\rho_k := \|\bar{q} - q_k\|_{L^2(\Omega)}$ and $w_k := \rho_k^{-1}(q_k - \bar{q})$. Note that, for every $k \in \mathbb{N}$, $\|w_k\|_{L^2(\Omega)} = 1$. We can therefore assume, taking a subsequence if necessary, that $w_k \rightharpoonup w$ in $L^2(\Omega)$ as $k \uparrow \infty$. We now prove that the limit point w belongs to the cone $C_{\bar{q}}$ and then that $w \equiv 0$. We proceed in three steps.

Step 1. $w \in C_{\bar{q}}$. We first note that the set of elements satisfying (3.4.28) is closed and convex in $L^2(\Omega)$; it is thus weakly sequentially closed and therefore w also satisfies (3.4.28). To verify the remaining condition in (3.4.27), we apply the mean value theorem and the estimate of the right-hand side in (3.4.29) to obtain

$$j'(\tilde{q}_k)w_k = \rho_k^{-1}(j(q_k) - j(\bar{q})) < \rho_k(2k)^{-1} \rightarrow 0, \quad k \uparrow \infty, \quad (3.4.30)$$

where $\tilde{q}_k = \bar{q} + \theta_k(q_k - \bar{q})$ and $\theta_k \in (0, 1)$. Define $\tilde{u}_k := \mathcal{S}\tilde{q}_k$ and \tilde{p}_k as the unique solution to (3.4.8) with u and q replaced by \tilde{u}_k and \tilde{q}_k , respectively. Since $\tilde{q}_k \rightarrow \bar{q}$ in $L^2(\Omega)$ as $k \uparrow \infty$, a basic stability bound for the problem that $\bar{u} - \tilde{u}_k$ solves shows that $\tilde{u}_k \rightarrow \bar{u}$ in $\tilde{H}^s(\Omega)$ as $k \uparrow \infty$. With this convergence property at hand, a similar argument yields $\tilde{p}_k \rightarrow \bar{p}$ in $\tilde{H}^s(\Omega)$ as $k \uparrow \infty$. In particular, $\tilde{p}_k \rightarrow \bar{p}$ in $L^2(\Omega)$ as $k \uparrow \infty$. We can thus conclude that $\lambda\tilde{q}_k - \tilde{u}_k\tilde{p}_k =: \tilde{\mathfrak{d}}_k \rightarrow \bar{\mathfrak{d}} = \lambda\bar{q} - \bar{u}\bar{p}$ in $L^2(\Omega)$ as $k \uparrow \infty$, upon using that $\{\tilde{u}_k\}_{k \in \mathbb{N}}$ and $\{\tilde{p}_k\}_{k \in \mathbb{N}}$ are uniformly bounded in $L^\infty(\Omega)$. We now invoke the weak convergence $w_k \rightharpoonup w$ in $L^2(\Omega)$ as $k \uparrow \infty$ and (3.4.30) to deduce that

$$j'(\bar{q})w = \int_{\Omega} \bar{\mathfrak{d}}(x)w(x)dx = \lim_{k \uparrow \infty} \int_{\Omega} \tilde{\mathfrak{d}}_k(x)w_k(x)dx = \lim_{k \uparrow \infty} j'(\tilde{q}_k)w_k \leq 0.$$

On the other hand, from (3.4.9) we have $j'(\bar{q})w = \lim_{k \uparrow \infty} \int_{\Omega} \bar{\mathfrak{d}}(x)w_k(x)dx \geq 0$. Consequently, $\int_{\Omega} \bar{\mathfrak{d}}(x)w(x)dx = 0$. Since w satisfies the condition (3.4.28) and $\bar{\mathfrak{d}}$ satisfies the condition (3.4.26), we infer that $\int_{\Omega} |\bar{\mathfrak{d}}(x)w(x)|dx = \int_{\Omega} \bar{\mathfrak{d}}(x)w(x)dx = 0$. This proves that, a.e. in Ω , $\bar{\mathfrak{d}} \neq 0$ implies that $w = 0$. Consequently, $w \in C_{\bar{q}}$.

Step 2. $w \equiv 0$. With the help of Taylor's theorem, the basic inequality $j'(\bar{q})(q_k - \bar{q}) \geq 0$, and the estimate of the right-hand side in (3.4.29) we arrive at

$$\frac{\rho_k^2}{2}j''(\hat{q}_k)w_k^2 = j(q_k) - j(\bar{q}) - j'(\bar{q})(q_k - \bar{q}) \leq j(q_k) - j(\bar{q}) < \rho_k^2(2k)^{-1}, \quad k \in \mathbb{N},$$

where $\hat{q}_k := \bar{q} + \hat{\theta}_k(q_k - \bar{q})$ and $\hat{\theta}_k \in (0, 1)$. Thus, $j''(\hat{q}_k)w_k^2 < k^{-1} \rightarrow 0$ as $k \uparrow \infty$.

We now show that $j''(\bar{q})w^2 \leq \liminf_{k \uparrow \infty} j''(\hat{q}_k)w_k^2$. As a first step, we invoke the characterization

(3.4.16) to write

$$j''(\hat{q}_k)w_k^2 = \lambda\|w_k\|_{L^2(\Omega)}^2 - 2(w_k\hat{z}_k, \hat{p}_k)_{L^2(\Omega)} + \|\hat{z}_k\|_{L^2(\Omega)}^2.$$

Here, \hat{p}_k denotes the solution to (3.4.8) with q and u replaced by \hat{q}_k and $\hat{u}_k := \mathcal{S}\hat{q}_k$, respectively, and \hat{z}_k solves (3.4.4) with q , w , and u replaced by \hat{q}_k , w_k , and \hat{u}_k , respectively. We note that $\{\hat{u}_k\}_{k \in \mathbb{N}}$ and $\{\hat{p}_k\}_{k \in \mathbb{N}}$ are uniformly bounded in $L^\infty(\Omega)$. In fact, for $k \in \mathbb{N}$, it holds that $\|\hat{u}_k\|_{L^\infty(\Omega)} \lesssim \|f\|_{L^r(\Omega)}$ and $\|\hat{p}_k\|_{L^\infty(\Omega)} \lesssim \|f\|_{L^r(\Omega)} + \|u_\Omega\|_{L^r(\Omega)}$. With the available estimates, we can therefore derive

$$\|\bar{u} - \hat{u}_k\|_s + \|\bar{p} - \hat{p}_k\|_s \lesssim (\|f\|_{L^r(\Omega)} + \|u_\Omega\|_{L^r(\Omega)}) \|\bar{q} - \hat{q}_k\|_{L^2(\Omega)} \rightarrow 0, \quad k \uparrow \infty,$$

upon using basic stability bounds for the problems that $\bar{u} - \hat{u}_k$ and $\bar{p} - \hat{p}_k$ solve.

Let z solve (3.4.4), where q and u are replaced by \bar{q} and \bar{u} , respectively. We now investigate the convergence of $\{\hat{z}_k\}_{k \in \mathbb{N}}$. First, we write the problem that $z - \hat{z}_k$ solves

$$\begin{aligned} A(z - \hat{z}_k, v) + (\bar{q}(z - \hat{z}_k), v)_{L^2(\Omega)} &= -((w - w_k)\bar{u}, v)_{L^2(\Omega)} \\ &\quad + (w_k(\hat{u}_k - \bar{u}), v)_{L^2(\Omega)} + ((\hat{q}_k - \bar{q})\hat{z}_k, v)_{L^2(\Omega)} =: \text{I}_k + \text{II}_k + \text{III}_k \quad \forall v \in \tilde{H}^s(\Omega). \end{aligned}$$

Since $\bar{u} \in L^\infty(\Omega)$ and $w_k \rightharpoonup w$ in $L^2(\Omega)$ as $k \uparrow \infty$, it is immediate that $|\text{I}_k| \rightarrow 0$. To analyze the convergence of II_k , we observe that, for every $\tau \in (2, \infty)$, we have

$$\|\bar{u} - \hat{u}_k\|_{L^\tau(\Omega)}^\tau \leq \| |\bar{u} - \hat{u}_k|^{\tau-2} \|_{L^\infty(\Omega)} \|\bar{u} - \hat{u}_k\|_{L^2(\Omega)}^2 \lesssim \|f\|_{L^r(\Omega)}^{\tau-2} \|\bar{u} - \hat{u}_k\|_{L^2(\Omega)}^2 \rightarrow 0 \quad (3.4.31)$$

as $k \uparrow \infty$, exploiting the fact that $\{\hat{u}_k\}_{k \in \mathbb{N}}$ is uniformly bounded in $L^\infty(\Omega)$ and the bound of Theorem 3.3.2. With this result at hand, we control the term II_k as follows:

$$|\text{II}_k| \leq \|w_k\|_{L^2(\Omega)} \|\bar{u} - \hat{u}_k\|_{L^\tau(\Omega)} \|v\|_{L^\mu(\Omega)}, \quad 2^{-1} + \tau^{-1} + \mu^{-1} = 1.$$

Set $\mu = 2d/(d - 2s)$ (see Lemma 3.2.2) and note that $\mu > 2$. Since $\hat{u}_k \rightarrow \bar{u}$ in $L^\tau(\Omega)$ for every $\tau \in (2, \infty)$, we can thus conclude that $|\text{II}_k| \rightarrow 0$ as $k \uparrow \infty$. We now analyze the convergence of III_k . First, we note that a stability bound for the problem that \hat{z}_k solves, namely $\|\hat{z}_k\|_s \lesssim \|f\|_{L^r(\Omega)}$, for every $k \in \mathbb{N}$, yields

$$\|\hat{z}_k\|_{L^t(\Omega)} \lesssim \|f\|_{L^r(\Omega)}, \quad k \in \mathbb{N}, \quad t \in \left[1, \frac{2d}{d - 2s}\right].$$

Second, as in (3.4.31), $\hat{q}_k \rightarrow \bar{q}$ in $L^2(\Omega)$ as $k \uparrow \infty$ combined with the fact that $\{\hat{q}_k\}$ is uniformly bounded in $L^\infty(\Omega)$ allows us to obtain that $\hat{q}_k \rightarrow \bar{q}$ in $L^\tau(\Omega)$ for every $\tau \in (2, \infty)$. A simple application of

Hölder's inequality thus shows that

$$|\text{III}_k| \leq \|\hat{z}_k\|_{L^t(\Omega)} \|\hat{q}_k - \bar{q}\|_{L^\tau(\Omega)} \|v\|_{L^\mu(\Omega)}, \quad t^{-1} + \tau^{-1} + \mu^{-1} = 1.$$

Since $\hat{q}_k \rightarrow \bar{q}$ in $L^\tau(\Omega)$, for every $\tau < \infty$, we conclude that $|\text{III}_k| \rightarrow 0$ as $k \uparrow \infty$. We have therefore proved that $|\text{I}_k|, |\text{II}_k|, |\text{III}_k| \rightarrow 0$ as $k \uparrow \infty$. Consequently, $\hat{z}_k \rightharpoonup z$ in $\tilde{H}^s(\Omega)$. In view of Lemma 3.2.2, this results in

$$\hat{z}_k \rightarrow z \text{ in } L^\sigma(\Omega), \quad \sigma < 2d/(d-2s) \implies \|\hat{z}_k\|_{L^2(\Omega)}^2 \rightarrow \|z\|_{L^2(\Omega)}^2, \quad k \uparrow \infty. \quad (3.4.32)$$

On the other hand,

$$\begin{aligned} |(wz, \bar{p})_{L^2(\Omega)} - (w_k \hat{z}_k, \hat{p}_k)_{L^2(\Omega)}| &\leq |((w - w_k)z, \bar{p})_{L^2(\Omega)}| \\ &\quad + |(w_k(z - \hat{z}_k), \bar{p})_{L^2(\Omega)}| + |(w_k \hat{z}_k, \bar{p} - \hat{p}_k)_{L^2(\Omega)}| \rightarrow 0, \quad k \uparrow \infty, \end{aligned} \quad (3.4.33)$$

because $w_k \rightharpoonup w$ in $L^2(\Omega)$ as $k \uparrow \infty$, $z\bar{p} \in L^2(\Omega)$, $\|w_k\|_{L^2(\Omega)} = 1$, $\|z - \hat{z}_k\|_{L^2(\Omega)} \rightarrow 0$ as $k \uparrow \infty$, $\bar{p} \in L^\infty(\Omega)$, and $\|\bar{p} - \hat{p}_k\|_{L^\tau(\Omega)} \rightarrow 0$, as $k \uparrow \infty$, for every $\tau \in (2, \infty)$.

Finally, we invoke (3.4.32), (3.4.33), and the fact that the square of the $L^2(\Omega)$ -norm is weakly lower semicontinuous in $L^2(\Omega)$ to deduce that $j''(\bar{q})w^2 \leq \liminf_k j''(\hat{q}_k)w_k^2$.

As a result, since $j''(\bar{q})w^2 \leq \liminf_k j''(\hat{q}_k)w_k^2$, $j''(\hat{q}_k)w_k^2 < k^{-1} \rightarrow 0$ as $k \uparrow \infty$, and $w \in C_{\bar{q}}$, the optimality condition $j''(\bar{q})w^2 > 0$, for all $w \in C_{\bar{q}} \setminus \{0\}$, yields $w \equiv 0$.

Step 3. Contradiction. We finally arrive at the contradiction. Since $w \equiv 0$, it is immediate that $\hat{z}_k \rightarrow 0$ in $\tilde{H}^s(\Omega)$ as $k \uparrow \infty$. Hence, from the equalities

$$\lambda = \lambda \|w_k\|_{L^2(\Omega)}^2 = j''(\hat{q}_k)w_k^2 + 2(w_k \hat{z}_k, \hat{p}_k)_{L^2(\Omega)} - \|\hat{z}_k\|_{L^2(\Omega)}^2,$$

and the fact that $\liminf_k j''(\hat{q}_k)w_k^2 \leq 0$, we conclude that $\lambda \leq 0$. This is a contradiction and concludes the proof. \square

The following result establishes an equivalent representation of the second order optimality condition introduced in Theorem 3.4.7. This equivalence is important for deriving error estimates for the methods proposed in our work. To present it, we introduce $C_{\bar{q}}^\tau := \{w \in L^2(\Omega) \text{ satisfying (3.4.28) and } w(x) = 0 \text{ if } |\bar{\mathfrak{d}}(x)| > \tau\}$.

Theorem 3.4.8 (equivalent optimality conditions). *Let $(\bar{u}, \bar{p}, \bar{q}) \in \tilde{H}^s(\Omega) \times \tilde{H}^s(\Omega) \times \mathbb{Q}_{ad}$ satisfy the first*

order optimality conditions (3.4.2), (3.4.8), and (3.4.9). Then, the following statements are equivalent:

$$j''(\bar{q})w^2 > 0 \forall w \in C_{\bar{q}} \setminus \{0\} \iff \exists \mu, \tau > 0 : j''(\bar{q})w^2 \geq \mu \|w\|_{L^2(\Omega)}^2 \quad \forall w \in C_{\bar{q}}^\tau. \quad (3.4.34)$$

Proof. The proof of the equivalence (3.4.34) follows from a combination of the arguments elaborated in the proofs of [28, Theorem 25], Theorem 3.4.6, and Theorem 3.4.7. For the sake of brevity, we omit details. \square

3.5 Discretization schemes for the control problem

We propose two different finite element discretization schemes to approximate solutions to the optimal control problem (3.4.1)–(3.4.2): a fully discrete one, where the admissible control set is discretized with piecewise constant functions, and a semidiscrete scheme, based on the so-called variational discretization approach, where the control set is not discretized.

3.5.1 A fully discrete scheme

The solution technique reads as follows: Find $\min J(u_h, q_h)$ subject to the *discrete state equation*

$$\mathcal{A}(u_h, v_h) + (q_h u_h, v_h)_{L^2(\Omega)} = (f, v_h)_{L^2(\Omega)} \quad \forall v_h \in \mathbb{V}_h, \quad (3.5.1)$$

and the *control constraints* $q_h \in \mathbb{Q}_{ad,h}$. Here, $\mathbb{Q}_{ad,h} := \mathbb{Q}_h \cap \mathbb{Q}_{ad}$, where $\mathbb{Q}_h = \{q_h \in L^\infty(\Omega) : q_h|_T \in \mathbb{P}_0(T) \forall T \in \mathcal{T}_h\}$. We recall that \mathbb{V}_h , defined in (3.3.2), corresponds to the space of standard continuous and piecewise linear functions that vanish on $\partial\Omega$.

The existence of a discrete solution follows from the compactness of $\mathbb{Q}_{ad,h}$ and the continuity of J . To formulate first order optimality conditions, we introduce the discrete control to state map $\mathcal{S}_h : \mathbb{Q}_{ad,h} \ni q_h \mapsto u_h \in \mathbb{V}_h$, where u_h is the solution to (3.5.1), and the reduced cost functional $j_h(q_h) := J(\mathcal{S}_h q_h, q_h)$.

With these ingredients, the first order conditions read as follows: If $\bar{q}_h \in \mathbb{Q}_{ad,h}$ is a local solution, then

$$j'_h(\bar{q}_h)(q_h - \bar{q}_h) = (\lambda \bar{q}_h - \bar{u}_h \bar{p}_h, q_h - \bar{q}_h)_{L^2(\Omega)} \geq 0 \quad \forall q_h \in \mathbb{Q}_{ad,h}. \quad (3.5.2)$$

Here, $\bar{p}_h \in \mathbb{V}_h$ corresponds to the *discrete adjoint state*, which solves

$$\mathcal{A}(v_h, \bar{p}_h) + (\bar{q}_h \bar{p}_h, v_h)_{L^2(\Omega)} = (\bar{u}_h - u_\Omega, v_h)_{L^2(\Omega)} \quad \forall v_h \in \mathbb{V}_h. \quad (3.5.3)$$

3.5.2 A semidiscrete scheme

In this section, we propose a semidiscrete scheme based on the so-called variational discretization approach [67]. The scheme, in which only the state space is discretized (the control space is not discretized), reads as follows: Find $\min J(u_h, \mathbf{q})$ subject to the *discrete state equation*

$$\mathcal{A}(u_h, v_h) + (\mathbf{q}u_h, v_h)_{L^2(\Omega)} = (f, v_h)_{L^2(\Omega)} \quad \forall v_h \in \mathbb{V}_h,$$

and the *control constraints* $\mathbf{q} \in \mathbb{Q}_{ad}$. The existence of a discrete solution and first order optimality conditions for the semidiscrete scheme follow standard arguments. In particular, if $\bar{\mathbf{q}} \in \mathbb{Q}_{ad}$ denotes a local solution, then

$$j'_h(\bar{\mathbf{q}})(q - \bar{\mathbf{q}}) = (\lambda\bar{\mathbf{q}} - \bar{u}_h\bar{p}_h, q - \bar{\mathbf{q}})_{L^2(\Omega)} \geq 0 \quad \forall q \in \mathbb{Q}_{ad}, \quad (3.5.4)$$

where $\bar{p}_h \in \mathbb{V}_h$ solves problem (3.5.3). We immediately note that, in view of the variational inequality (3.5.4), the following projection formula holds [104, Section 4.6]:

$$\bar{\mathbf{q}}(x) := \Pi_{[a,b]}(\lambda^{-1}\bar{u}_h(x)\bar{p}_h(x)) \text{ a.e. } x \in \Omega.$$

Since $\bar{\mathbf{q}}$ implicitly depends on h , we will use the notation $\bar{\mathbf{q}}_h$ in the following.

3.5.3 Convergence of discretizations

In this section, we analyze the convergence properties of the fully and semidiscrete schemes. To this end, we begin our investigation by providing some auxiliary convergence properties and error bounds for suitable finite element discretizations associated with the state and adjoint equations.

3.5.3.1 Auxiliary error bounds

Let us first provide a convergence property and bounds for the errors $\|u - u_h\|_s$ and $\|u - u_h\|_{L^2(\Omega)}$.

Theorem 3.5.1 (convergence properties). *Let $s \in (0, 1)$ and let $f \in H^{-s}(\Omega)$. Let u and u_h be the unique solutions to problems (3.4.2) and (3.5.1), respectively. Then,*

$$q_h \rightarrow q \text{ in } L^{\frac{d}{2s}}(\Omega), \quad h \rightarrow 0 \implies u_h \rightarrow u \text{ in } \tilde{H}^s(\Omega), \quad h \rightarrow 0. \quad (3.5.5)$$

Let $r > d/2s$. If, in addition, $f \in L^2(\Omega) \cap L^r(\Omega)$, then we have the error bounds

$$\|u - u_h\|_s \lesssim h^\gamma |\log h|^\varphi \|f\|_{L^2(\Omega)} + \|q - q_h\|_{L^2(\Omega)}, \quad \gamma = \min\{s, \frac{1}{2}\}, \quad (3.5.6)$$

$$\|u - u_h\|_{L^2(\Omega)} \lesssim h^{2\gamma} |\log h|^{2\varphi} \|f\|_{L^2(\Omega)} + \|q - q_h\|_{L^2(\Omega)}, \quad (3.5.7)$$

where $\varphi = \nu$ if $s \neq \frac{1}{2}$, $\varphi = 1 + \nu$ if $s = \frac{1}{2}$, and $\nu \geq \frac{1}{2}$ is the constant in Theorem 3.3.1.

Proof. Let us first derive the convergence property (3.5.5). We begin with a simple application of a triangle inequality to write $\|u - u_h\|_s \leq \|u - u_h(q)\|_s + \|u_h(q) - u_h\|_s$. Here, $u_h(q)$ denotes the solution to (3.5.1) with q_h replaced by q . The control of $\|u - u_h(q)\|_s$ follows from a density argument as the one developed in the proof of [35, Theorem 3.2.3] (see also [52, Corollary 1.109]): $\|u - u_h(q)\|_s \rightarrow 0$ as $h \rightarrow 0$. To bound $\|u_h(q) - u_h\|_s$, we invoke the discrete problem that $u_h(q) - u_h$ solves and utilize the fact that $q_h \in \mathbb{Q}_{ad,h} = \mathbb{Q}_{ad} \cap \mathbb{Q}_h$ to obtain

$$\|u_h(q) - u_h\|_s \leq \|u_h(q)(q_h - q)\|_{H^{-s}(\Omega)}.$$

The weak convergence $q_h \rightarrow q$ in $L^{\frac{d}{2s}}(\Omega)$ and the strong one $u_h(q)v \rightarrow uv$ in $L^{\frac{d}{d-2s}}(\Omega)$ as $h \rightarrow 0$, which is valid for every $v \in \tilde{H}^s(\Omega)$, allow us to conclude.

We proceed similarly to derive (3.5.6): $\|u - u_h\|_s \leq \|u - u(q_h)\|_s + \|u(q_h) - u_h\|_s$. Here, $u(q_h)$ denotes the solution to (3.4.2) with q replaced by q_h . Since u_h , the solution to (3.5.1), corresponds to the finite element approximation of $u(q_h)$ within the discrete setting of section 3.3.3, we can invoke the error bound (3.3.4) of Theorem 3.3.3 to deduce

$$\|u(q_h) - u_h\|_s \lesssim h^\gamma |\log h|^\varphi \|f\|_{L^2(\Omega)}. \quad (3.5.8)$$

On the other hand, let us observe that $u - u(q_h) \in \tilde{H}^s(\Omega)$ uniquely solves

$$\mathcal{A}(u - u(q_h), v) + (q(u - u(q_h)), v)_{L^2(\Omega)} = (u(q_h)(q_h - q), v)_{L^2(\Omega)} \quad \forall v \in \tilde{H}^s(\Omega). \quad (3.5.9)$$

Since $f \in L^r(\Omega)$ and $q_h \in \mathbb{Q}_{ad,h}$, Theorem 3.3.2 guarantees $\|u(q_h)\|_{L^\infty(\Omega)} \lesssim \|f\|_{L^r(\Omega)}$, which is a uniform bound with respect to discretization. As a consequence, a basic stability bound for problem (3.5.9) yields $\|u - u(q_h)\|_s \lesssim \|u(q_h)\|_{L^\infty(\Omega)} \|q - q_h\|_{L^2(\Omega)} \lesssim \|f\|_{L^r(\Omega)} \|q - q_h\|_{L^2(\Omega)}$. This bound and estimate (3.5.8) yield the desired bound (3.5.6). The proof of (3.5.7) follows similar arguments. \square

To present the following result, we introduce the variables \mathbf{p} and p_h as follows:

$$\mathbf{p} \in \tilde{H}^s(\Omega) : \quad \mathcal{A}(v, \mathbf{p}) + (q_h \mathbf{p}, v)_{L^2(\Omega)} = (u_h - u_\Omega, v)_{L^2(\Omega)} \quad \forall v \in \tilde{H}^s(\Omega). \quad (3.5.10)$$

$$p_h \in \mathbb{V}_h : \quad \mathcal{A}(v_h, p_h) + (q_h p_h, v_h)_{L^2(\Omega)} = (u_h - u_\Omega, v_h)_{L^2(\Omega)} \quad \forall v_h \in \mathbb{V}_h. \quad (3.5.11)$$

In what follows, we present error bounds for $p - p_h$.

Theorem 3.5.2 (convergence properties). *Let $s \in (0, 1)$ and $r > d/2s$. Let p and p_h be the unique solutions to problems (3.4.8) and (3.5.11), respectively. If $f, u_\Omega \in L^2(\Omega) \cap L^r(\Omega)$ and $\{u_h\}_{h>0}$ is*

uniformly bounded in $L^r(\Omega)$, then

$$\|p - p_h\|_s \lesssim h^\gamma |\log h|^\varphi + \|q - q_h\|_{L^2(\Omega)}, \quad \gamma = \min\{s, \frac{1}{2}\}, \quad (3.5.12)$$

$$\|p - p_h\|_{L^2(\Omega)} \lesssim h^{2\gamma} |\log h|^{2\varphi} + \|q - q_h\|_{L^2(\Omega)}, \quad (3.5.13)$$

where $\varphi = \nu$ if $s \neq \frac{1}{2}$, $\varphi = 1 + \nu$ if $s = \frac{1}{2}$, and $\nu \geq \frac{1}{2}$ is the constant in Theorem 3.3.1.

Proof. We invoke \mathbf{p} and write $\|p - p_h\|_s \leq \|p - \mathbf{p}\|_s + \|\mathbf{p} - p_h\|_s$. To control $\|\mathbf{p} - p_h\|_s$ we apply the error estimate (3.3.4) of Theorem 3.3.3:

$$\|\mathbf{p} - p_h\|_s \lesssim h^\gamma |\log h|^\varphi (\|f\|_{H^{-s}(\Omega)} + \|u_\Omega\|_{L^2(\Omega)}), \quad (3.5.14)$$

upon using $\|u_h\|_s \lesssim \|f\|_{H^{-s}(\Omega)}$, which is uniform with respect to discretization. To control $\|p - \mathbf{p}\|_s$ we observe that $p - \mathbf{p}$ solves

$$\mathcal{A}(v, p - \mathbf{p}) + (q(p - \mathbf{p}), v)_{L^2(\Omega)} = (\mathbf{p}(q_h - q) + (u - u_h), v)_{L^2(\Omega)} \quad \forall v \in \tilde{H}^s(\Omega).$$

Since, by assumption, $\{u_h\}_{h>0}$ is uniformly bounded in $L^r(\Omega)$ ($r > d/2s$), we obtain

$$\begin{aligned} \|p - \mathbf{p}\|_s &\lesssim \|\mathbf{p}\|_{L^{\frac{r}{s}}(\Omega)} \|q - q_h\|_{L^2(\Omega)} + \|u - u_h\|_{L^2(\Omega)} \\ &\lesssim \|q - q_h\|_{L^2(\Omega)} + h^{2\gamma} |\log h|^{2\varphi} \|f\|_{L^2(\Omega)}, \end{aligned} \quad (3.5.15)$$

upon using (3.5.7) and the uniform bound $\|\mathbf{p}\|_{L^\infty(\Omega)} \lesssim \|u_h\|_{L^r(\Omega)} + \|u_\Omega\|_{L^r(\Omega)}$. The bound (3.5.15) in conjunction with (3.5.14) implies (3.5.12). The proof of (3.5.13) follows similar arguments. For the sake of brevity, we omit details. \square

3.5.3.2 Convergence of discretizations: the fully discrete scheme

We begin this section with a convergence result that essentially guarantees that a sequence of discrete global solutions $\{\bar{q}_h\}_{h>0}$ contains subsequences that converge to global solutions of the optimal control problem (3.4.1)–(3.4.2) as $h \rightarrow 0$.

Theorem 3.5.3 (convergence of global solutions). *Let $s \in (0, 1)$, $r > d/2s$, and $f, u_\Omega \in L^2(\Omega) \cap L^r(\Omega)$. Let $h > 0$ and let $\bar{q}_h \in \mathbb{Q}_{ad,h}$ be a global solution of the fully discrete optimal control problem. Then, there exist nonreabeled subsequences of $\{\bar{q}_h\}_{h>0}$ such that $\bar{q}_h \xrightarrow{*} \bar{q}$ in the weak* topology of $L^\infty(\Omega)$ as $h \rightarrow 0$, where \bar{q} is a global solution of the optimal control problem (3.4.1)–(3.4.2). Furthermore, we have*

$$\lim_{h \rightarrow 0} \|\bar{q} - \bar{q}_h\|_{L^2(\Omega)} = 0, \quad \lim_{h \rightarrow 0} j_h(\bar{q}_h) = j(\bar{q}). \quad (3.5.16)$$

Proof. Since $\{\bar{q}_h\}_{h>0} \subset \mathbb{Q}_{ad,h}$ is uniformly bounded in $L^\infty(\Omega)$, there exists a nonrelabeled subsequence such that $\bar{q}_h \overset{*}{\rightharpoonup} \bar{q}$ in $L^\infty(\Omega)$ as $h \rightarrow 0$. In what follows, we prove that $\bar{q} \in \mathbb{Q}_{ad}$ is a global solution to (3.4.1)–(3.4.2) and that (3.5.16) holds.

Let $\mathbf{q} \in \mathbb{Q}_{ad}$ be a global solution to (3.4.1)–(3.4.2). Let $\mathcal{P}_h : L^2(\Omega) \rightarrow \mathbb{Q}_h$ be the orthogonal projection operator. Define $\mathbf{q}_h := \mathcal{P}_h(\mathbf{q})$ and note that $\mathbf{q}_h \in \mathbb{Q}_{ad,h}$. Applying the regularity result from Theorem 3.4.4 and a standard error estimate for \mathcal{P}_h , we obtain $\|\mathbf{q} - \mathbf{q}_h\|_{L^2(\Omega)} \rightarrow 0$ as $h \rightarrow 0$. Therefore, the global optimality of \mathbf{q} , Theorem 3.5.1, the global optimality of \bar{q}_h , and $\mathbf{q}_h \rightarrow \mathbf{q}$ in $L^2(\Omega)$ allow us to obtain that

$$j(\mathbf{q}) \leq j(\bar{q}) \leq \liminf_{h \downarrow 0} j_h(\bar{q}_h) \leq \limsup_{h \downarrow 0} j_h(\bar{q}_h) \leq \limsup_{h \downarrow 0} j_h(\mathbf{q}_h) = j(\mathbf{q}).$$

This shows that \bar{q} is a global solution of (3.4.1)–(3.4.2) and that $j_h(\bar{q}_h) \rightarrow j(\bar{q})$ as $h \rightarrow 0$. To prove the strong convergence of $\{\bar{q}_h\}_{h>0}$ to \bar{q} in $L^2(\Omega)$, we use the convergence result of Theorem 3.5.1 to deduce that $\bar{u}_h \rightarrow \bar{u}$ in $L^t(\Omega)$ for every $t \leq 2d/(d-2s)$. With this convergence result, we use that $j_h(\bar{q}_h) \rightarrow j(\bar{q})$ to obtain $\|\bar{q}_h\|_{L^2(\Omega)} \rightarrow \|\bar{q}\|_{L^2(\Omega)}$ as $h \rightarrow 0$. The fact that $\bar{q}_h \overset{*}{\rightharpoonup} \bar{q}$ in $L^\infty(\Omega)$ as $h \rightarrow 0$ allows us to conclude. \square

Our second convergence result is as follows: strict local solutions of (3.4.1)–(3.4.2) can be approximated by local solutions of the fully discrete optimal control problems.

Theorem 3.5.4 (convergence of local solutions). *Let the assumptions of Theorem 3.5.3 hold. Let $\bar{q} \in \mathbb{Q}_{ad}$ be a strict local minimum of (3.4.1)–(3.4.2). Then, there exists a sequence of local minima $\{\bar{q}_h\}_{h < h_\square}$ of the fully discrete scheme such that (3.5.16) holds.*

Proof. Since \bar{q} is a strict local minimum of (3.4.1)–(3.4.2), there exists $\varepsilon > 0$ such that the problem: Find $\min\{j(q) : q \in \mathbb{Q}_{ad} \cap B_\varepsilon(\bar{q})\}$ admits as a unique solution \bar{q} . Here, $B_\varepsilon(\bar{q}) := \{q \in L^2(\Omega) : \|\bar{q} - q\|_{L^2(\Omega)} \leq \varepsilon\}$. Let us now introduce, for $h > 0$, the discrete problem: Find $\min\{j_h(q_h) : q_h \in \mathbb{Q}_{ad,h} \cap B_\varepsilon(\bar{q})\}$. To conclude that this problem admits at least one solution, we need to verify that the set in which the minimum is sought is nonempty; note that such a set is compact. To do this, we note that there exists $h_\varepsilon > 0$ such that for every $h \leq h_\varepsilon$ the function $\mathcal{P}_h(\bar{q}) \in \mathbb{Q}_{ad,h} \cap B_\varepsilon(\bar{q})$.

Let $h \in (0, h_\varepsilon]$ and let \bar{q}_h be a global solution of the introduced discrete problem. The arguments presented in the proof of Theorem 3.5.3 allow us to conclude the existence of a subsequence of $\{\bar{q}_h\}_{h \leq h_\varepsilon}$ that converges strongly in $L^2(\Omega)$ to a solution of $\min\{j(q) : q \in \mathbb{Q}_{ad} \cap B_\varepsilon(\bar{q})\}$. Since this problem admits a unique solution \bar{q} , we must have $\bar{q}_h \rightarrow \bar{q}$ in $L^2(\Omega)$ as $h \rightarrow 0$. In particular, this guarantees that the constraint $\bar{q}_h \in B_\varepsilon(\bar{q})$ is not active for h sufficiently small. Consequently, \bar{q}_h solves the fully discrete scheme and the convergence properties stated in (3.5.16) hold. \square

3.5.3.3 Convergence of discretizations: the semidiscrete scheme

We present the following results. Let $s \in (0, 1)$, $r > d/2s$, and $f, u_\Omega \in L^2(\Omega) \cap L^r(\Omega)$.

- Let $h > 0$ and let $\bar{q}_h \in \mathbb{Q}_{ad}$ be a global solution of the semidiscrete scheme. Then, there exist nonrelabelled subsequences of $\{\bar{q}_h\}_{h>0}$ such that $\bar{q}_h \xrightarrow{*} \bar{q}$ in $L^\infty(\Omega)$ as $h \rightarrow 0$ and (3.5.16) holds; \bar{q} is a global solution to (3.4.1)–(3.4.2).
- Let $\bar{q} \in \mathbb{Q}_{ad}$ be a strict local minimum of (3.4.1)–(3.4.2). Then, there exists a sequence of local minima $\{\bar{q}_h\}_{h>0}$ of the semidiscrete scheme satisfying (3.5.16).

The proof of these results follows very similar arguments to those developed in the proof of Theorems 3.5.3 and 3.5.4. For the sake of brevity, we will omit further details.

3.6 Error estimates

In this section, we derive error estimates for the fully and semidiscrete schemes introduced in sections 3.5.1 and 3.5.2, respectively.

3.6.1 Error estimates for the fully discrete scheme

Let $\{\bar{q}_h\}_{h>0} \subset \mathbb{Q}_{ad,h}$ be a sequence of local minima of the fully discrete optimal control problems such that $\bar{q}_h \rightarrow \bar{q}$ in $L^2(\Omega)$ as $h \rightarrow 0$; $\bar{q} \in \mathbb{Q}_{ad}$ denotes a local solution of (3.4.1)–(3.4.2) (see Theorems 3.5.3 and 3.5.4). The main goal of this section is to derive the error bound

$$\|\bar{q} - \bar{q}_h\|_{L^2(\Omega)} \lesssim h^{2\gamma} |\log h|^{2\varphi} \quad \forall h < h_\dagger, \quad \gamma = \min\{s, \frac{1}{2}\}, \quad h_\dagger > 0. \quad (3.6.1)$$

Here, $\varphi = \nu$ if $s \neq \frac{1}{2}$, $\varphi = 1 + \nu$ if $s = \frac{1}{2}$, and $\nu \geq \frac{1}{2}$ is the constant in Theorem 3.3.1.

In the following analysis we *assume* that discrete solutions u_h to problem (3.5.1) are uniformly bounded in $L^{d/s}(\Omega)$, i.e.,

$$\exists C > 0 : \quad \|u_h\|_{L^{d/s}(\Omega)} \leq C \quad \forall h > 0. \quad (3.6.2)$$

The following result is helpful for deriving the error bound (3.6.1).

Lemma 3.6.1 (auxiliary error estimate). *Let $s \in (0, 1)$, $r > d/2s$, and $f, u_\Omega \in L^2(\Omega) \cap L^r(\Omega)$. Let us assume that (3.6.2) holds and that $\bar{q} \in \mathbb{Q}_{ad}$ satisfies the second order optimality conditions (3.4.34). If (3.6.1) is false, then there exists $h_\dagger > 0$ such that*

$$\mathbb{C} \|\bar{q} - \bar{q}_h\|_{L^2(\Omega)}^2 \leq [j'(\bar{q}_h) - j'(\bar{q})](\bar{q}_h - \bar{q}) \quad \forall h < h_\dagger, \quad \mathbb{C} = 2^{-1} \min\{\mu, \lambda\}. \quad (3.6.3)$$

Here, λ is the control cost and μ denotes the constant appearing in (3.4.34).

Proof. We follow [30, Section 7] and proceed by contradiction. Since by assumption (3.6.1) is false, there exists a subsequence $\{h_k\}_{k \in \mathbb{N}} \subset \mathbb{R}^+$ such that

$$\lim_{h_k \rightarrow 0} \|\bar{q} - \bar{q}_{h_k}\|_{L^2(\Omega)} = 0, \quad \lim_{h_k \rightarrow 0} \frac{\|\bar{q} - \bar{q}_{h_k}\|_{L^2(\Omega)}}{h_k^{2\gamma} |\log h_k|^{2\varphi}} = +\infty. \quad (3.6.4)$$

To simplify the exposition of the material, we omit the subindex k in the following and denote $\bar{q}_{h_k} = \bar{q}_h$. Note that $h \rightarrow 0$ as $k \uparrow \infty$.

Define $w_h := (\bar{q}_h - \bar{q}) / \|\bar{q}_h - \bar{q}\|_{L^2(\Omega)}$. Since $\{w_h\}_{h>0}$ is uniformly bounded in $L^2(\Omega)$, possibly up to a subsequence, we can assume that $w_h \rightharpoonup w$ in $L^2(\Omega)$ as $h \rightarrow 0$. In what follows, we prove that $w \in C_{\bar{q}}$. Recall that $C_{\bar{q}}$ is defined in (3.4.27). Since, for each $h > 0$, $\bar{q}_h \in \mathbb{Q}_{ad,h} \subset \mathbb{Q}_{ad}$, w_h satisfies the sign conditions (3.4.28). Consequently, w also satisfies (3.4.28). To show that $\bar{\mathfrak{d}}(x) \neq 0$ implies that $w(x) = 0$ for a.e. $x \in \Omega$, we introduce $\bar{\mathfrak{d}}_h := \lambda \bar{q}_h - \bar{u}_h \bar{p}_h$ and recall that $\bar{\mathfrak{d}} = \lambda \bar{q} - \bar{u} \bar{p}$. Let us now invoke $\|\bar{q} - \bar{q}_h\|_{L^2(\Omega)} \rightarrow 0$ as $h \rightarrow 0$, (3.6.2), and estimates (3.5.12) and (3.5.7) to obtain

$$\begin{aligned} \|\bar{\mathfrak{d}} - \bar{\mathfrak{d}}_h\|_{L^2(\Omega)} &\leq \lambda \|\bar{q} - \bar{q}_h\|_{L^2(\Omega)} + \|\bar{u}_h\|_{L^{\frac{d}{s}}(\Omega)} \|\bar{p} - \bar{p}_h\|_{L^{\frac{2d}{d-2s}}(\Omega)} \\ &\quad + \|\bar{p}\|_{L^\infty(\Omega)} \|\bar{u} - \bar{u}_h\|_{L^2(\Omega)} \lesssim \|\bar{q} - \bar{q}_h\|_{L^2(\Omega)} + h^\gamma |\log h|^\varphi \rightarrow 0, \quad h \rightarrow 0. \end{aligned}$$

Hence, we have that $\bar{\mathfrak{d}}_h \rightarrow \bar{\mathfrak{d}}$ in $L^2(\Omega)$ as $h \rightarrow 0$. From this follows

$$\begin{aligned} \int_{\Omega} \bar{\mathfrak{d}}(x) w(x) dx &= \lim_{h \rightarrow 0} \int_{\Omega} \bar{\mathfrak{d}}_h(x) w_h(x) dx \\ &= \lim_{h \rightarrow 0} \frac{1}{\|\bar{q}_h - \bar{q}\|_{L^2(\Omega)}} \left(\int_{\Omega} \bar{\mathfrak{d}}_h(\mathcal{P}_h(\bar{q}) - \bar{q}) dx + \int_{\Omega} \bar{\mathfrak{d}}_h(\bar{q}_h - \mathcal{P}_h(\bar{q})) dx \right), \end{aligned}$$

where \mathcal{P}_h denotes the L^2 -orthogonal projection operator into piecewise constant functions over \mathcal{T}_h . Since $\mathcal{P}_h(\bar{q}) \in \mathbb{Q}_{ad,h}$, the discrete variational inequality (3.5.2) implies that $(\bar{\mathfrak{d}}_h, \bar{q}_h - \mathcal{P}_h(\bar{q}))_{L^2(\Omega)} \leq 0$. The uniform boundedness of $\{\|\bar{\mathfrak{d}}_h\|_{L^2(\Omega)}\}_{h < h_{\triangleright\triangleleft}}$, which follows from $\|\bar{\mathfrak{d}}_h\|_{L^2(\Omega)} \leq \|\bar{\mathfrak{d}}_h - \bar{\mathfrak{d}}\|_{L^2(\Omega)} + \|\bar{\mathfrak{d}}\|_{L^2(\Omega)} \lesssim 1$ for $h < h_{\triangleright\triangleleft}$, thus implies that

$$\int_{\Omega} \bar{\mathfrak{d}}(x) w(x) dx \leq \lim_{h \rightarrow 0} \frac{1}{\|\bar{q}_h - \bar{q}\|_{L^2(\Omega)}} \int_{\Omega} \bar{\mathfrak{d}}_h(\mathcal{P}_h(\bar{q}) - \bar{q}) dx \lesssim \lim_{h \rightarrow 0} \frac{\|\mathcal{P}_h(\bar{q}) - \bar{q}\|_{L^2(\Omega)}}{\|\bar{q}_h - \bar{q}\|_{L^2(\Omega)}} = 0.$$

To obtain the last equality, we have used (3.6.4) and the regularity results for \bar{q} provided in Theorem 3.4.4 in conjunction with standard error estimates for \mathcal{P}_h , which yield

$$\|\bar{q} - \mathcal{P}_h(\bar{q})\|_{L^2(\Omega)} \lesssim h^{2\gamma} |\log h|^\nu (1 + \Lambda(f, u_\Omega)), \quad \gamma = \min\{s, \frac{1}{2}\}. \quad (3.6.5)$$

Here, $\Lambda(f, u_\Omega)$ is defined in (3.4.14). Now, since w satisfies the sign conditions (3.4.28), we have $\bar{\mathfrak{d}}(x)w(x) \geq 0$ for a.e. $x \in \Omega$. Consequently, $\int_\Omega |\bar{\mathfrak{d}}(x)w(x)|dx \leq 0$. As a result, if $\bar{\mathfrak{d}}(x) \neq 0$, then $w(x) = 0$ for a.e. $x \in \Omega$. We have thus obtained that $w \in C_{\bar{q}}$.

Now that we have proven that $w \in C_{\bar{q}}$, let us derive the auxiliary error estimate (3.6.3) by using (3.4.34). To begin, we apply the mean value theorem to obtain

$$[j'(\bar{q}_h) - j'(\bar{q})](\bar{q}_h - \bar{q}) = j''(\hat{q}_h)(\bar{q}_h - \bar{q})^2, \quad \hat{q}_h = \bar{q} + \theta_h(\bar{q}_h - \bar{q}), \quad (3.6.6)$$

where $\theta_h \in (0, 1)$. Let $u(\hat{q}_h)$ be unique solution to (3.4.2) with q replaced by \hat{q}_h and let $p(\hat{q}_h)$ be the unique solution to (3.4.8) with u and q replaced by $u(\hat{q}_h)$ and \hat{q}_h , respectively. Since $\hat{q}_h(x) \geq a$ for a.e. $x \in \Omega$ and $\bar{u} \in L^\infty(\Omega)$, a basic stability bound for the problem that $\bar{u} - u(\hat{q}_h)$ solves combined with the fact that $\bar{q}_h \rightarrow \bar{q}$ in $L^2(\Omega)$ as $h \rightarrow 0$ reveal that $u(\hat{q}_h) \rightarrow \bar{u}$ in $\tilde{H}^s(\Omega)$. Similarly, $p(\hat{q}_h) \rightarrow \bar{p}$ in $\tilde{H}^s(\Omega)$ as $h \rightarrow 0$. We now define $z(w_h)$ as the unique solution to (3.4.4) with q , u and w replaced by \hat{q}_h , $u(\hat{q}_h)$, and w_h , respectively. Proceeding as in the Step 2 of the proof of Theorem 3.4.7, we can show that $w_h \rightarrow w$ in $L^2(\Omega)$ as $h \rightarrow 0$ guarantees that $z(w_h) \rightarrow z$ in $\tilde{H}^s(\Omega)$. With all these convergence properties at hand, we can refer to the characterization (3.4.16), the definition of w_h , and the second order condition (3.4.34) to obtain

$$\begin{aligned} \lim_{h \rightarrow 0} j''(\hat{q}_h)w_h^2 &= \lim_{h \rightarrow 0} \left[\lambda \|w_h\|_{L^2(\Omega)}^2 - 2(w_h z(w_h), p(\hat{q}_h))_{L^2(\Omega)} + \|z(w_h)\|_{L^2(\Omega)}^2 \right] \\ &= \lambda - 2(wz, \bar{p})_{L^2(\Omega)} + \|z\|_{L^2(\Omega)}^2 = \lambda + j''(\bar{q})w^2 - \lambda \|w\|_{L^2(\Omega)}^2 \geq \lambda + (\mu - \lambda) \|w\|_{L^2(\Omega)}^2. \end{aligned}$$

Therefore, since $\|w\|_{L^2(\Omega)} \leq 1$, we obtain $\lim_{h \rightarrow 0} j''(\hat{q}_h)w_h^2 \geq \min\{\mu, \lambda\} > 0$. We can therefore obtain the existence of $h_\dagger > 0$ such that $j''(\hat{q}_h)w_h^2 \geq \min\{\mu, \lambda\}/2$ for every $h < h_\dagger$. Given the definition of w_h and (3.6.6), this allows the conclusion to be drawn. \square

We now derive the error bound (3.6.1). For this purpose, we strengthen the assumption (3.6.2) as follows: solutions to (3.5.1) are uniformly bounded in $L^\infty(\Omega)$, i.e.,

$$\exists C > 0 : \quad \|u_h\|_{L^\infty(\Omega)} \leq C \quad \forall h > 0. \quad (3.6.7)$$

Unfortunately, we are not aware of a proof of (3.6.7) in a general setting. A basic proof can be given for the case where $d = 2$ and $s > 0.5 + \varepsilon$, with $\varepsilon > 0$ as in Theorem 3.3.1. This proof is based on inverse estimates, error estimates for the Lagrange interpolation operator, and the $L^2(\Omega)$ error estimate (3.3.5). Finally, we would like to note that for a fixed h the function u_h is a globally continuous piecewise linear function that is bounded in $L^\infty(\Omega)$ and that $\{u_h\}_{h>0}$ converges strongly in $H^s(\mathbb{R}^n)$ as $h \rightarrow 0$ to a function u belonging to $L^\infty(\Omega)$; see Theorem 3.3.2, §3.5.3.2 and §3.5.3.3, and

Theorem 3.5.1.

Theorem 3.6.2 (error estimate). *Let $s \in (0, 1)$, $r > d/2s$, and $f, u_\Omega \in L^2(\Omega) \cap L^r(\Omega)$. Let us assume that (3.6.7) holds and that $\bar{q} \in \mathbb{Q}_{ad}$ satisfies the second order optimality conditions (3.4.34). Then, there exists $h_\dagger > 0$ such that*

$$\|\bar{q} - \bar{q}_h\|_{L^2(\Omega)} \lesssim h^{2\gamma} |\log h|^{2\varphi} \quad \forall h < h_\dagger, \quad \gamma = \min\{s, \frac{1}{2}\}. \quad (3.6.8)$$

Here, $\varphi = \nu$ if $s \neq \frac{1}{2}$, $\varphi = 1 + \nu$ if $s = \frac{1}{2}$, and $\nu \geq \frac{1}{2}$ is the constant in Theorem 3.3.1.

Proof. We proceed by contradiction and assume that (3.6.8) does not hold. We can therefore refer to the result of Lemma 3.6.1 to conclude that (3.6.3) holds for every $h < h_\dagger$. Let us now set $q = \bar{q}_h$ in (3.4.7) and $q_h = \mathcal{P}_h(\bar{q})$ in (3.5.2) to obtain $-j'(\bar{q})(\bar{q}_h - \bar{q}) \leq 0$ and $j'_h(\bar{q}_h)(\mathcal{P}_h(\bar{q}) - \bar{q}_h) \geq 0$, respectively. Given these two inequalities we invoke the auxiliary error estimate (3.6.3) to obtain

$$\begin{aligned} \|\bar{q} - \bar{q}_h\|_{L^2(\Omega)}^2 &\lesssim j'(\bar{q}_h)(\bar{q}_h - \bar{q}) + j'_h(\bar{q}_h)(\mathcal{P}_h(\bar{q}) - \bar{q}_h) \\ &= j'(\bar{q}_h)(\mathcal{P}_h(\bar{q}) - \bar{q}) + [j'(\bar{q}_h) - j'_h(\bar{q}_h)](\bar{q}_h - \mathcal{P}_h(\bar{q})) =: \mathbf{I}_h + \mathbf{II}_h. \end{aligned} \quad (3.6.9)$$

Let us first bound the term \mathbf{I}_h . For this purpose, we introduce the following auxiliary variables: We define \hat{u} and \hat{p} to be the solutions to the problems

$$\mathcal{A}(\hat{u}, v) + (\bar{q}_h \hat{u}, v)_{L^2(\Omega)} = (f, v)_{L^2(\Omega)} \quad \forall v \in \tilde{H}^s(\Omega), \quad (3.6.10)$$

$$\mathcal{A}(\hat{p}, v) + (\bar{q}_h \hat{p}, v)_{L^2(\Omega)} = (\hat{u} - u_\Omega, v)_{L^2(\Omega)} \quad \forall v \in \tilde{H}^s(\Omega). \quad (3.6.11)$$

With these variables at hand, we control the term \mathbf{I}_h as follows:

$$\begin{aligned} \mathbf{I}_h &= (\lambda \bar{q}_h - \hat{u} \hat{p}, \mathcal{P}_h(\bar{q}) - \bar{q})_{L^2(\Omega)} = (\mathcal{P}_h(\hat{u} \hat{p}) - \hat{u} \hat{p}, \mathcal{P}_h(\bar{q}) - \bar{q})_{L^2(\Omega)} \\ &\leq \|\hat{u} \hat{p} - \mathcal{P}_h(\hat{u} \hat{p})\|_{L^2(\Omega)} \|\mathcal{P}_h(\bar{q}) - \bar{q}\|_{L^2(\Omega)}, \end{aligned}$$

upon using the basic property $(\mathcal{P}_h(\bar{q}) - \bar{q}, q_h) = 0$ for all $q_h \in \mathbb{Q}_h$. The term $\|\mathcal{P}_h(\bar{q}) - \bar{q}\|_{L^2(\Omega)}$ was previously controlled in the proof of Lemma 3.6.1; see (3.6.5). In what follows we control $\|\hat{u} \hat{p} - \mathcal{P}_h(\hat{u} \hat{p})\|_{L^2(\Omega)}$. First, since f and $\hat{u} - u_\Omega$ belong to $L^2(\Omega) \cap L^r(\Omega)$, we deduce from the regularity results of Theorems 3.3.1 and 3.3.2 in conjunction with the continuity of the product property of Lemma 3.2.1 that

$$\hat{u} \hat{p} \in H^{s+\kappa-\varepsilon}(\Omega) \cap L^\infty(\Omega), \quad \|\hat{u} \hat{p}\|_{H^{s+\kappa-\varepsilon}(\Omega)} \lesssim \varepsilon^{-\nu} \Lambda(f, u_\Omega) \quad \forall 0 < \varepsilon < s.$$

Here, κ and ν are as in the statement of Theorem 3.3.1 and $\Lambda(f, u_\Omega)$ is defined in (3.4.14). In view of

this regularity result, a standard approximation property for \mathcal{P}_h shows that $\mathbf{e} := \hat{u}\hat{p} - \mathcal{P}_h(\hat{u}\hat{p})$ satisfies $\|\mathbf{e}\|_{L^2(\Omega)} \lesssim h^{2\gamma} |\log h|^\nu \Lambda(f, u_\Omega)$, where $\gamma = \min\{s, \frac{1}{2}\}$. We can therefore deduce that

$$\mathbf{I}_h \lesssim h^{4\gamma} |\log h|^{2\nu} \Lambda(f, u_\Omega) (1 + \Lambda(f, u_\Omega)), \quad \gamma = \min\{s, \frac{1}{2}\}.$$

We now bound the term \mathbf{II}_h . To accomplish this task, we first notice that $\mathbf{II}_h = (\hat{u}\hat{p} - \bar{u}_h\bar{p}_h, \bar{q}_h - \mathcal{P}_h(\bar{q}))_{L^2(\Omega)} = (\hat{u}\hat{p} - \bar{u}_h\bar{p}_h, \mathcal{P}_h(\bar{q}_h - \bar{q}))_{L^2(\Omega)}$. Secondly, by adding and subtracting $\hat{p}\bar{u}_h$ and using the assumption (3.6.7) and basic inequalities, we arrive at

$$\begin{aligned} \mathbf{II}_h &\leq (\|\hat{p}\|_{L^\infty(\Omega)} \|\hat{u} - \bar{u}_h\|_{L^2(\Omega)} + \|\bar{u}_h\|_{L^\infty(\Omega)} \|\hat{p} - \bar{p}_h\|_{L^2(\Omega)}) \|\bar{q} - \bar{q}_h\|_{L^2(\Omega)} \\ &\leq \mathfrak{C} \left(\|\hat{u} - \bar{u}_h\|_{L^2(\Omega)}^2 + \|\hat{p} - \bar{p}_h\|_{L^2(\Omega)}^2 \right) + \frac{1}{2} \|\bar{q} - \bar{q}_h\|_{L^2(\Omega)}^2, \quad \mathfrak{C} > 0. \end{aligned}$$

A bound for $\|\hat{u} - \bar{u}_h\|_{L^2(\Omega)}$ follows from the fact that \bar{u}_h corresponds to the finite element approximation of \hat{u} in the framework of section 3.3.3. In fact, as an application of Theorem 3.3.3, we have the bound $\|\hat{u} - \bar{u}_h\|_{L^2(\Omega)} \lesssim h^{2\gamma} |\log h|^{2\varphi}$. We now define

$$\tilde{p} \in \tilde{H}^s(\Omega) : \quad \mathcal{A}(\tilde{p}, v) + (\bar{q}_h \tilde{p}, v)_{L^2(\Omega)} = (\bar{u}_h - u_\Omega, v)_{L^2(\Omega)} \quad \forall v \in \tilde{H}^s(\Omega) \quad (3.6.12)$$

and proceed as in the proof of Theorem 3.5.2 to obtain $\|\hat{p} - \bar{p}_h\|_{L^2(\Omega)} \leq \|\hat{p} - \tilde{p}\|_{L^2(\Omega)} + \|\tilde{p} - \bar{p}_h\|_{L^2(\Omega)} \lesssim \|\hat{u} - \bar{u}_h\|_{L^2(\Omega)} + h^{2\gamma} |\log h|^{2\varphi}$. Consequently, we can thus arrive at

$$\mathbf{II}_h \lesssim h^{4\gamma} |\log h|^{4\varphi} + \frac{1}{2} \|\bar{q} - \bar{q}_h\|_{L^2(\Omega)}^2.$$

Finally, we replace the bounds obtained for \mathbf{I}_h and \mathbf{II}_h into (3.6.9) to obtain (3.6.8). This is a contradiction and concludes the proof. \square

Remark 3.6.3 (nearly optimality). The error bound (3.6.8) behaves like $\mathcal{O}(h |\log h|^{2\varphi})$ when $s \geq \frac{1}{2}$. This error bound is nearly-optimal in terms of approximation; nearly due to the presence of the log-term. When $s < \frac{1}{2}$, the derived error bound behaves like $\mathcal{O}(h^{2s} |\log h|^{2\varphi})$. Since the best convergence rate we can expect with piecewise constant approximation is $\mathcal{O}(h)$ and $2s < 1$, the derived error bound is suboptimal; the suboptimality is determined by the regularity properties derived in Theorem 3.4.4.

Remark 3.6.4 (improvement of the error bound (3.6.8)). If $s \leq 1/2$, the regularity results of Theorem 3.4.4 guarantee that $\bar{q} \in H^{2s-2\epsilon}(\Omega)$ for all $0 < \epsilon < s$. This regularity result can be improved for $s \in [1/4, 1/2)$ to $\bar{q} \in H^{s+1/2-\epsilon}(\Omega)$ for all $0 < \epsilon < s + 1/2$ under the assumptions that Ω is a bounded Lipschitz domain satisfying an exterior ball condition and the data f and u_Ω belong to $C^{1/2-s}(\bar{\Omega})$. We also have the following regularity improvement of the state and adjoint state variables:

$\bar{u}, \bar{p} \in H^{s+1/2-\epsilon}(\Omega)$. We leave to the reader the details of how these regularity results can lead to improved error estimates on quasi-uniform meshes. Finally, we note that, as in [90], the use of appropriate graded meshes [15] can also lead to improved convergence rates.

Corollary 3.6.5 (error estimates). *Let the assumptions of Theorem 3.6.2 hold. Then, there exists $h_{\dagger} > 0$ such that, for all $h < h_{\dagger}$,*

$$\|\bar{u} - \bar{u}_h\|_s \lesssim h^\gamma |\log h|^\varphi, \quad \|\bar{p} - \bar{p}_h\|_s \lesssim h^\gamma |\log h|^\varphi, \quad \gamma = \min\{s, \frac{1}{2}\}. \quad (3.6.13)$$

$$\|\bar{u} - \bar{u}_h\|_{L^2(\Omega)} \lesssim h^{2\gamma} |\log h|^{2\varphi}, \quad \|\bar{p} - \bar{p}_h\|_{L^2(\Omega)} \lesssim h^{2\gamma} |\log h|^{2\varphi}. \quad (3.6.14)$$

Here, $\varphi = \nu$ if $s \neq \frac{1}{2}$, $\varphi = 1 + \nu$ if $s = \frac{1}{2}$, and $\nu \geq \frac{1}{2}$ is the constant in Theorem 3.3.1.

Proof. The proof of the estimates in (3.6.13) and (3.6.14) follows directly from the combination of the estimates in Theorems 3.5.1, 3.5.2, and 3.6.2. \square

3.6.2 Error estimates for the semidiscrete scheme

Let $\{\bar{q}_h\}_{h>0} \subset \mathbb{Q}_{ad}$ be a sequence of local minima of the semidiscrete optimal control problems such that $\bar{q}_h \rightarrow \bar{q}$ in $L^2(\Omega)$ as $h \rightarrow 0$; $\bar{q} \in \mathbb{Q}_{ad}$ denotes a local solution to (3.4.1)–(3.4.2) (see section 3.5.3.3).

The main goal of this section is to obtain an estimate for $\|\bar{q} - \bar{q}_h\|_{L^2(\Omega)}$.

The following result, which is instrumental, is analogous to Lemma 3.6.1.

Lemma 3.6.6 (auxiliary error estimate). *Let $s \in (0, 1)$, $r > d/2s$, and $f, u_\Omega \in L^2(\Omega) \cap L^r(\Omega)$. Assume that (3.6.2) holds and that $\bar{q} \in \mathbb{Q}_{ad}$ satisfies the second order optimality conditions (3.4.34). Then, there exists $h_{\dagger} > 0$ such that*

$$\mathbb{C} \|\bar{q} - \bar{q}_h\|_{L^2(\Omega)}^2 \leq [j'(\bar{q}_h) - j'(\bar{q})](\bar{q}_h - \bar{q}) \quad \forall h < h_{\dagger}, \quad \mathbb{C} = 2^{-1} \min\{\mu, \lambda\}. \quad (3.6.15)$$

Here, λ is the control cost and μ denotes the constant appearing in (3.4.34).

Proof. Define $w_h := (\bar{q}_h - \bar{q}) / \|\bar{q}_h - \bar{q}\|_{L^2(\Omega)}$. We assume that (up to a subsequence if necessary) $w_h \rightarrow w$ in $L^2(\Omega)$ as $h \rightarrow 0$. The arguments elaborated in the proof of Lemma 3.6.1 guarantee that w satisfies (3.4.28) and that $\bar{\mathfrak{d}}_h \rightarrow \bar{\mathfrak{d}}$ in $L^2(\Omega)$ as $h \rightarrow 0$. Here, $\bar{\mathfrak{d}} = \lambda \bar{q} - \bar{u} \bar{p}$ and $\bar{\mathfrak{d}}_h = \lambda \bar{q}_h - \bar{u}_h \bar{p}_h$. Invoke (3.5.4) with $q = \bar{q}$ to obtain

$$\int_{\Omega} \bar{\mathfrak{d}}(x) w(x) dx = \lim_{h \rightarrow 0} \frac{1}{\|\bar{q}_h - \bar{q}\|_{L^2(\Omega)}} \int_{\Omega} \bar{\mathfrak{d}}_h(x) (\bar{q}_h(x) - \bar{q}(x)) dx \leq 0.$$

Since $\int_{\Omega} |\bar{\mathfrak{d}}(x) w(x)| dx = \int_{\Omega} \bar{\mathfrak{d}}(x) w(x) dx \leq 0$, we conclude that if $\bar{\mathfrak{d}}(x) \neq 0$, then $w(x) = 0$ for a.e. $x \in \Omega$. Consequently, $w \in C_{\bar{q}}$.

The rest of the proof follows the same arguments used in the proof of Lemma 3.6.1. For the sake of brevity we omit these details. \square

We are now in a position to derive an estimate for the error that arises when approximating a locally optimal control variable \bar{q} .

Theorem 3.6.7 (error estimate). *Let $s \in (0, 1)$, $r > d/2s$, and $f, u_\Omega \in L^2(\Omega) \cap L^r(\Omega)$. Let us assume that (3.6.7) holds and that $\bar{q} \in \mathbb{Q}_{ad}$ satisfies the second order optimality conditions (3.4.34). Then, there exists $h_{\ddagger} > 0$ such that*

$$\|\bar{q} - \bar{q}_h\|_{L^2(\Omega)} \lesssim h^{2\gamma} |\log h|^{2\varphi} \quad \forall h < h_{\ddagger}, \quad \gamma = \min\{s, \frac{1}{2}\}. \quad (3.6.16)$$

Here $\varphi = \nu$ if $s \neq \frac{1}{2}$, $\varphi = 1 + \nu$ if $s = \frac{1}{2}$, and $\nu \geq \frac{1}{2}$ is the constant in Theorem 3.3.1.

Proof. Let us set $q = \bar{q}_h$ in (3.4.7) and $q = \bar{q}$ in (3.5.4) to obtain $-j'(\bar{q})(\bar{q}_h - \bar{q}) \leq 0$ and $j'_h(\bar{q}_h)(\bar{q} - \bar{q}_h) \geq 0$, respectively. Using these estimates in the auxiliary error estimate (3.6.15) we obtain

$$\begin{aligned} \|\bar{q} - \bar{q}_h\|_{L^2(\Omega)}^2 &\lesssim [j'(\bar{q}_h) - j'_h(\bar{q}_h)](\bar{q}_h - \bar{q}) = (\bar{u}_h \bar{p}_h - \hat{u} \hat{p}, \bar{q}_h - \bar{q})_{L^2(\Omega)} \\ &= (\hat{p}(\bar{u}_h - \hat{u}), \bar{q}_h - \bar{q})_{L^2(\Omega)} + (\bar{u}_h(\bar{p}_h - \hat{p}), \bar{q}_h - \bar{q})_{L^2(\Omega)} = \mathfrak{J}_h + \mathfrak{K}_h. \end{aligned}$$

Here, \hat{u} and \hat{p} in $\tilde{H}^s(\Omega)$ denote auxiliary variables that are defined as the solutions to (3.6.10) and (3.6.11), respectively, but where \bar{q}_h is replaced by \bar{q}_h .

Let us estimate \mathfrak{J}_h . Since $f, u_\Omega \in L^r(\Omega)$, by applying the Theorem 3.3.2, we deduce that $\hat{u}, \hat{p} \in L^\infty(\Omega)$.

The error estimate (3.3.5) therefore allows the conclusion that

$$\mathfrak{J}_h \leq \|\hat{p}\|_{L^\infty(\Omega)} \|\hat{u} - \bar{u}_h\|_{L^2(\Omega)} \|\bar{q} - \bar{q}_h\|_{L^2(\Omega)} \lesssim h^{2\gamma} |\log h|^{2\varphi} \|\bar{q} - \bar{q}_h\|_{L^2(\Omega)}. \quad (3.6.17)$$

Here, γ and φ are as in the statement of the theorem. We now focus on the estimation of \mathfrak{K}_h . To do this, we introduce the auxiliary variable \tilde{p} as in (3.6.12), but replacing \bar{q}_h with \bar{q}_h . With this variable at hand, we invoke (3.6.7) to obtain

$$\mathfrak{K}_h \leq \|\bar{u}_h\|_{L^\infty(\Omega)} (\|\hat{p} - \tilde{p}\|_{L^2(\Omega)} + \|\tilde{p} - \bar{p}_h\|_{L^2(\Omega)}) \|\bar{q} - \bar{q}_h\|_{L^2(\Omega)}.$$

A stability estimate for the problem that $\hat{p} - \tilde{p}$ solves yields $\|\hat{p} - \tilde{p}\|_s \lesssim \|\hat{u} - \bar{u}_h\|_{L^2(\Omega)}$. The control of $\|\tilde{p} - \bar{p}_h\|_{L^2(\Omega)}$ follows from noticing that \bar{p}_h corresponds to the finite element approximation of \tilde{p} in the framework of §3.3.3: $\|\tilde{p} - \bar{p}_h\|_{L^2(\Omega)} \lesssim h^{2\gamma} |\log h|^{2\varphi}$. A collection of these estimates allows us to conclude that $\mathfrak{K}_h \lesssim h^{2\gamma} |\log h|^{2\varphi} \|\bar{q} - \bar{q}_h\|_{L^2(\Omega)}$.

The bound derived for \mathfrak{K}_h together with that in (3.6.17) for \mathfrak{J}_h yields the final estimate $\|\bar{q} - \bar{q}_h\|_{L^2(\Omega)} \lesssim h^{2\gamma} |\log h|^{2\varphi}$. This concludes the proof. \square

We conclude this section with the following error estimates.

Corollary 3.6.8 (error estimates). *Let the assumptions of Theorem 3.6.7 hold. Then, there exists $h_{\ddagger} > 0$ such that, for all $h < h_{\ddagger}$,*

$$\begin{aligned} \|\bar{u} - \bar{u}_h\|_s &\lesssim h^\gamma |\log h|^\varphi, & \|\bar{p} - \bar{p}_h\|_s &\lesssim h^\gamma |\log h|^\varphi, & \gamma &= \min\{s, \tfrac{1}{2}\}. \\ \|\bar{u} - \bar{u}_h\|_{L^2(\Omega)} &\lesssim h^{2\gamma} |\log h|^{2\varphi}, & \|\bar{p} - \bar{p}_h\|_{L^2(\Omega)} &\lesssim h^{2\gamma} |\log h|^{2\varphi}. \end{aligned}$$

Here, $\varphi = \nu$ if $s \neq \frac{1}{2}$, $\varphi = 1 + \nu$ if $s = \frac{1}{2}$, and $\nu \geq \frac{1}{2}$ is the constant in Theorem 3.3.1.

3.7 Numerical examples

We present three numerical experiments that illustrate the performance of the fully discrete and semidiscrete methods presented in sections 3.5.1 and 3.5.2, respectively, when used to approximate a solution to the control problem (3.4.1)–(3.4.2). The experiments were performed with a code implemented in MATLAB, and the schemes were solved with a semi-smooth Newton method.

The setting of the experiments is as follows: we let $s \in \{0.1, 0.2, \dots, 0.9\}$, $\lambda = 1$, $d = 2$, and $\Omega = B(0, 1)$; $B(0, 1)$ denotes the unit disc. The exact optimal state and the exact optimal adjoint state are given by

$$\bar{u}(x) = \bar{p}(x) = (2^{2s} \Gamma^2(1+s))^{-1} (1 - |x|^2)_+^s, \quad t_+ = \max\{0, t\}. \quad (3.7.1)$$

3.7.1 Example 1

In this numerical experiment we go beyond the theory presented and consider $a = 0$ and $b = 0.5$.

Figures 3.1–3.3 show the results obtained for both the fully discrete and the semidiscrete schemes. In Figure 3.1, we show for $s \in \{0.1, 0.2, \dots, 0.9\}$ the experimental convergence rates for $\|\bar{u} - \bar{u}_h\|_s$ and $\|\bar{p} - \bar{p}_h\|_s$. We observe that the rates predicted in the Corollaries 3.6.5 and 3.6.8 are achieved when $s \geq 0.5$. However, when $s < 0.5$ the experimental convergence rates exceed those derived in these corollaries. We present experimental convergence rates for $\|\bar{u} - \bar{u}_h\|_{L^2(\Omega)}$, $\|\bar{p} - \bar{p}_h\|_{L^2(\Omega)}$, and $\|\bar{q} - \bar{q}_h\|_{L^2(\Omega)}$ for $s \in \{0.1, \dots, 0.5\}$ and $s \in \{0.6, \dots, 0.9\}$ in Figures 3.2 and 3.3, respectively. It can be observed that the experimental convergence rates for all involved approximation errors are consistent with the error bounds obtained in section 3.6 when $s \geq 0.5$. However, when $s < 0.5$, the reported experimental convergence rates exceed those predicted in our manuscript. These cases are discussed in Remarks 3.7.1 and 3.7.2.

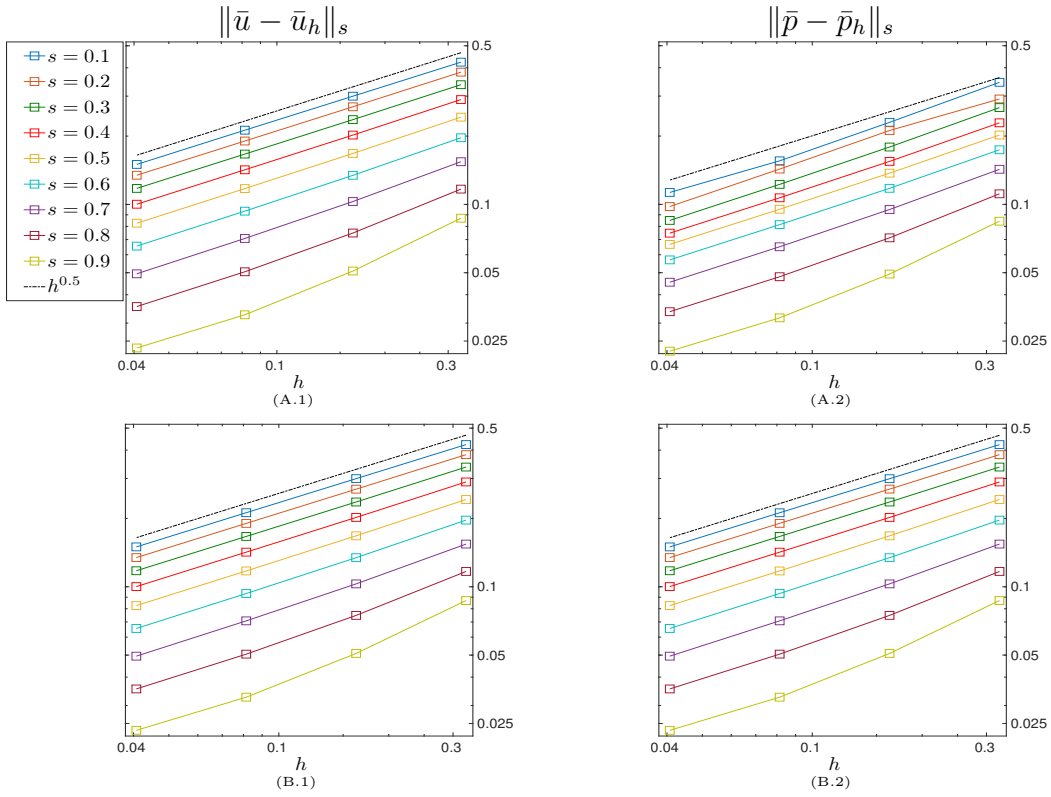


Figure 3.1: Experimental rates of convergence for $\|\bar{u} - \bar{u}_h\|_s$ and $\|\bar{p} - \bar{p}_h\|_s$ considering the fully discrete (A.1)–(A.2) and semidiscrete schemes (B.1)–(B.2) for $s \in \{0.1, 0.2, \dots, 0.9\}$.

Remark 3.7.1 (convergence rates: state and adjoint variables). Figures 3.1 and 3.2 show that the experimental convergence rates for the approximation errors associated to the state and adjoint variables, when $s < 0.5$, exceed the rates predicted in Corollaries 3.6.5 and 3.6.8 but are in agreement with respect to the *maximal regularity*

$$H^{s+\frac{1}{2}-\epsilon}(\Omega), \quad 0 < \epsilon < s + \frac{1}{2}. \tag{3.7.2}$$

The error bounds that we derive in Corollaries 3.6.5 and 3.6.8 are based on the regularity estimates of Theorem 3.4.4, which in turn are inspired by the results in Theorem 3.3.1 ([17, Theorem 2.1]). If $s \in (0, 0.5]$, $u \in H^{2s-2\epsilon}(\Omega)$ for every $\epsilon \in (0, s)$, which is weaker than (3.7.2). As explained in [17, page 1921], one expects the solutions to be smoother than just $H^{2s}(\Omega)$ if the forcing term $f \in H^r(\Omega)$, for some $r > 0$; however, such a result of higher regularity cannot be derived from [17, Theorem 2.1]. Nevertheless, it is important to emphasize that the estimates in [17, Theorem 2.1] hold under the assumption that $\partial\Omega$ is *merely* Lipschitz. Finally, we note that the functions \bar{u} and \bar{p} defined in (3.7.1) satisfy (3.7.2).

Remark 3.7.2 (convergence rates: control variable). Figure 3.2 (subfigures (C.3) and (D.3)) shows that

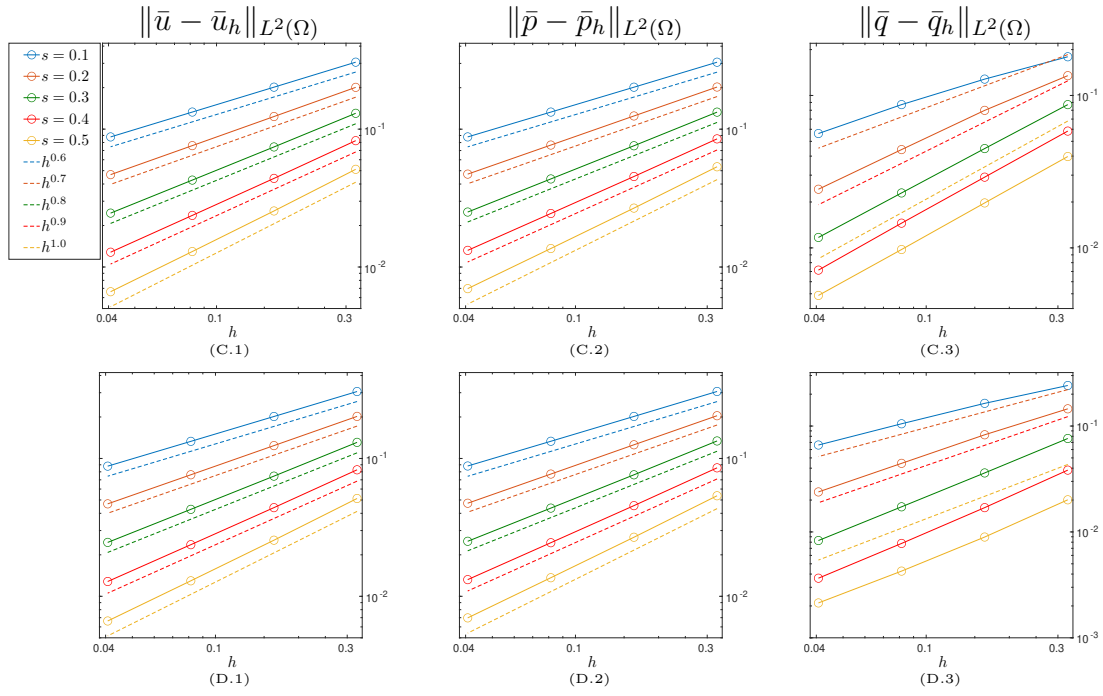


Figure 3.2: Experimental rates of convergence for $\|\bar{u} - \bar{u}_h\|_{L^2(\Omega)}$, $\|\bar{p} - \bar{p}_h\|_{L^2(\Omega)}$, and $\|\bar{q} - \bar{q}_h\|_{L^2(\Omega)}$ considering the fully discrete (C.1)–(C.3) and semidiscrete schemes (D.1)–(D.3) for $s \in \{0.1, 0.2, \dots, 0.5\}$.

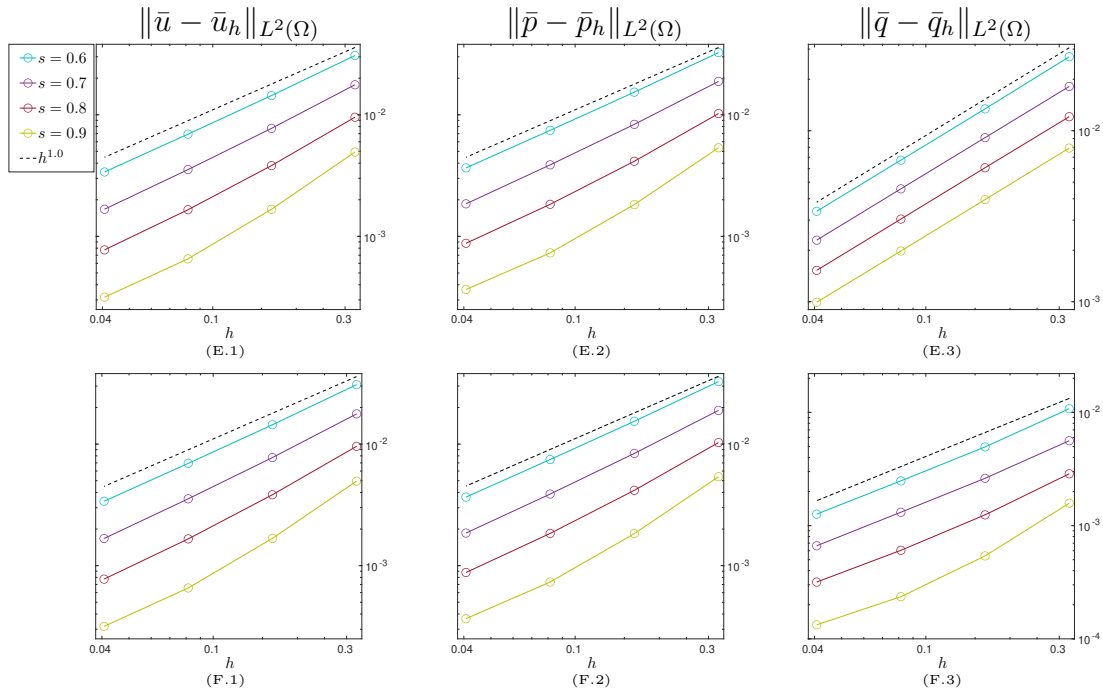


Figure 3.3: Experimental rates of convergence for $\|\bar{u} - \bar{u}_h\|_{L^2(\Omega)}$, $\|\bar{p} - \bar{p}_h\|_{L^2(\Omega)}$, and $\|\bar{q} - \bar{q}_h\|_{L^2(\Omega)}$ considering the fully (E.1)–(E.3) and semidiscrete scheme (F.1)–(F.3) for $s \in \{0.6, 0.7, 0.8, 0.9\}$.

the experimental convergence rates obtained for the control variable are higher than those predicted by (3.6.8) and (3.6.16) when $s < 0.5$. To explain this, we further investigate the regularity properties of \bar{q} in our particular setting. Note that $\bar{u}(x)\bar{p}(x) = \mathfrak{C}(1 - |x|^2)_+^\sigma$, where $\sigma = 2s$ and $\mathfrak{C}^{-1} = (2^{2\sigma}\Gamma^4(1 + s))$. $\bar{u}\bar{p}$ can thus be regarded as the solution to (3.1.2) with s replaced by σ , $q \equiv 0$, and $f \equiv \Gamma^{-2}(1 + s)$. Consequently, the maximal regularity property (3.7.2) in this case reads $\bar{u}\bar{p} \in H^\iota(\Omega)$, where $\iota = \sigma + 0.5 - \epsilon$ and $\epsilon \in (0, \sigma + 0.5)$. In view of the projection formula (3.4.13), we apply [86, Theorem 1] to obtain $\bar{q} \in H^\iota(\Omega)$; observe that $\iota < 1.5$. As a result, in terms of regularity and approximation degree, we would expect, for the fully discrete scheme, $\|\bar{q} - \bar{q}_h\|_{L^2(\Omega)} \lesssim h^\omega$, where $\omega = \min\{2s + 0.5 - \epsilon, 1\}$. This is the behaviour observed in Subfigure (C.3). A similar conclusion holds for the semidiscrete scheme. Since \bar{q}_h is implicitly discretized with piecewise linear functions, we would expect $\|\bar{q} - \bar{q}_h\|_{L^2(\Omega)} \lesssim h^\varpi$, where $\varpi = \min\{2s + 0.5 - \epsilon, 2\}$. However, we observe $\mathcal{O}(h^\omega)$. An important observation in favor of this is the fact that it has been experimentally observed that $\|\bar{u} - \bar{u}_h\|_{L^2(\Omega)}$ and $\|\bar{p} - \bar{p}_h\|_{L^2(\Omega)}$ do not exceed $\mathcal{O}(h)$; see [17, §6.1].

3.7.2 Example 2

We consider $a = 0.001\|\bar{u}\bar{p}\|_{L^\infty(\Omega)}$ and $b = 1.5$.

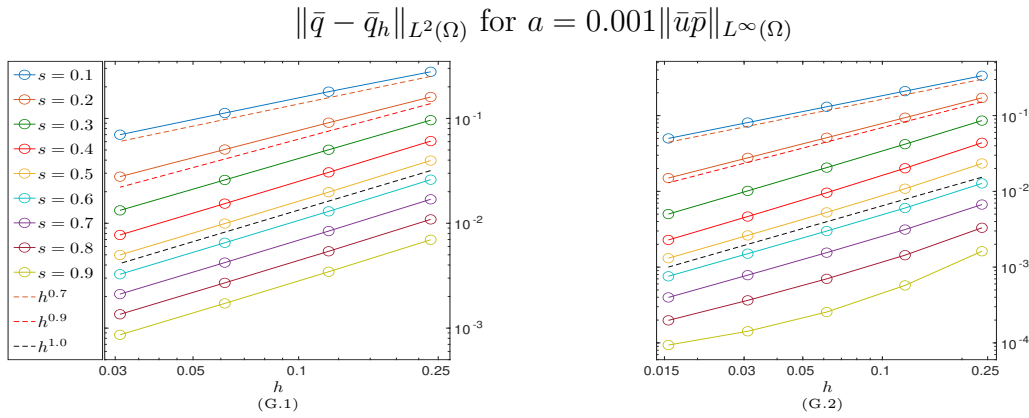


Figure 3.4: Experimental rates of convergence for $\|\bar{q} - \bar{q}_h\|_{L^2(\Omega)}$ considering the fully discrete (G.1) and semidiscrete schemes (G.2) for $a = 0.001\|\bar{u}\bar{p}\|_{L^\infty(\Omega)}$ and $s \in \{0.1, 0.2, \dots, 0.9\}$.

Figure 3.4 shows for $s \in \{0.1, 0.2, \dots, 0.9\}$ the experimental convergence rates for $\|\bar{q} - \bar{q}_h\|_{L^2(\Omega)}$ obtained for both the fully discrete and the semidiscrete schemes. We note that the experimental convergence rates coincide with those reported in Example 1. This particular result can be attributed to the following fact: the value of a is so small that the singular behaviour of both \bar{u} and \bar{p} described in (3.7.1) remains present with the computational resources at our disposal. This singular behaviour is inherited by the projection formula (3.4.13) on \bar{q} .

3.7.3 Example 3

We consider $a = 0.95\|\bar{u}\bar{p}\|_{L^\infty(\Omega)}$ and $b = 1.5$.

$$\|\bar{q} - \bar{q}_h\|_{L^2(\Omega)} \text{ for } a = 0.95\|\bar{u}\bar{p}\|_{L^\infty(\Omega)}$$

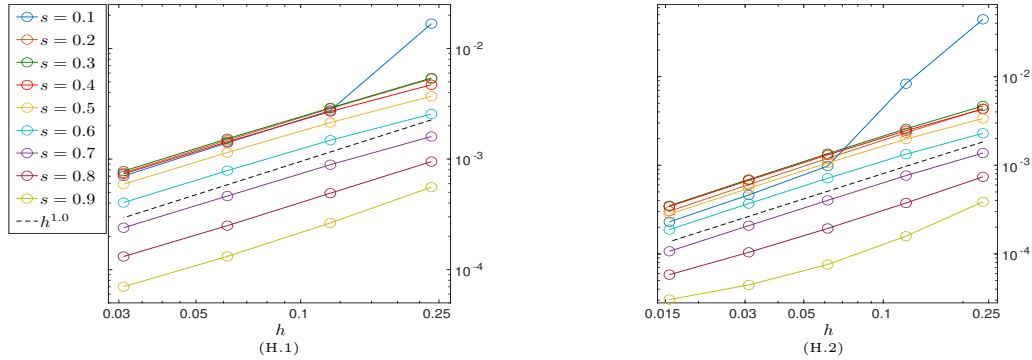


Figure 3.5: Experimental rates of convergence for $\|\bar{q} - \bar{q}_h\|_{L^2(\Omega)}$ considering the fully discrete (H.1) and semidiscrete schemes (H.2) for $a = 0.95\|\bar{u}\bar{p}\|_{L^\infty(\Omega)}$ and $s \in \{0.1, 0.2, \dots, 0.9\}$.

In Figure 3.5 we present for $s \in \{0.1, 0.2, \dots, 0.9\}$ the experimental convergence rates for $\|\bar{q} - \bar{q}_h\|_{L^2(\Omega)}$ obtained for both the fully discrete and the semidiscrete schemes. We note that $\|\bar{q} - \bar{q}_h\|_{L^2(\Omega)}$ achieves the experimental convergence rate $\mathcal{O}(h)$ for both methods and for all considered values of s . In contrast to Example 2, we have here that the singular behaviour near the boundary of Ω disappears when the restriction a is large enough.

Remark 3.7.3 (fully discrete versus semidiscrete approximation). In the following, we present what we consider to be the most important advantages and disadvantages of each discretization scheme.

Advantages(A)/disadvantages(D) of the fully discrete scheme:

- (A1) The scheme provides an explicit discrete control variable.
- (D1) The scheme incurs additional computational costs due to the additional degrees of freedom required to discretize the admissible control set.
- (D2) If the control set is discretized with piecewise constant functions, the expected convergence rate for the control approximation is always limited to $\mathcal{O}(h)$.

Advantages(A)/disadvantages(D) of the semidiscrete scheme:

- (A1) A discrete control is not explicitly used in the computational implementation.
- (A2) If the state and adjoint equations are discretized with piecewise linear functions, the data are smooth, and Ω is convex, then the rate $\mathcal{O}(h^2)$ can be obtained for the control approximation in the case $s = 1$. For the case $s \in (0, 1)$, numerical evidence shows that such a rate is restricted to $\mathcal{O}(h)$.

- (D1) An additional effort has to be made to compute an explicit discrete control by the projection formula. This can be interpreted as a post-processing step.

Chapter 4

Fractional, semilinear, and sparse optimal control: a priori error bounds

4.1 Introduction

Let $d \in \mathbb{N}$ be such that $d \geq 2$. Let $\Omega \subset \mathbb{R}^d$ be an open and bounded domain with Lipschitz boundary $\partial\Omega$. Define the cost functional

$$J(u, q) := \int_{\Omega} L(x, u) dx + \frac{\lambda}{2} \|q\|_{L^2(\Omega)}^2 + \mu \|q\|_{L^1(\Omega)}, \quad (4.1.1)$$

where $L : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ denotes a suitable Carathéodory function, $\lambda > 0$ corresponds to the so-called *regularization* parameter, and $\mu > 0$ denotes a *sparsity* parameter. The necessary assumptions on L are deferred to section 4.2.4. In this paper, we are interested in the analysis and discretization of the following *nonconvex* and *nondifferentiable* optimal control problem for a *fractional, semilinear, and elliptic* partial differential equation (PDE): Find $\min J(u, q)$ subject to the *state equation*

$$(-\Delta)^s u + a(\cdot, u) = q \text{ in } \Omega, \quad u = 0 \text{ in } \Omega^c, \quad (4.1.2)$$

where $\Omega^c = \mathbb{R}^d \setminus \Omega$ and $s \in (0, 1)$, and the *control constraints*

$$q \in \mathbb{Q}_{ad}, \quad \mathbb{Q}_{ad} := \{v \in L^2(\Omega) : \alpha \leq v(x) \leq \beta \text{ a.e. } x \in \Omega\}. \quad (4.1.3)$$

We adopt the integral definition of the fractional Laplace operator $(-\Delta)^s$. For a smooth function $w : \mathbb{R}^d \rightarrow \mathbb{R}$, the operator $(-\Delta)^s$ is defined as follows:

$$(-\Delta)^s w(x) := C(d, s) \text{p.v.} \int_{\mathbb{R}^d} \frac{w(x) - w(y)}{|x - y|^{d+2s}} dy, \quad C(d, s) := \frac{2^{2s} s \Gamma(s + \frac{d}{2})}{\pi^{\frac{d}{2}} \Gamma(1 - s)},$$

where p.v. stands for the *Cauchy principal value* and $C(d, s)$ is a normalization constant. The necessary assumptions on the function a are deferred until section 4.2.4. The control bounds α and β are such that $\alpha < 0 < \beta$; see [26, Remark 2.1] for a discussion.

The optimal control problem under consideration involves a cost functional J that contains the $L^1(\Omega)$ -norm of the control variable. The study of this type of optimal control problems is mainly motivated by the following two observations: First, $\|\cdot\|_{L^1(\Omega)}$ is a natural measure of the control cost for certain applications. Second, $\|\cdot\|_{L^1(\Omega)}$ leads to sparsely supported optimal controls, i.e., optimal controls that are non-zero only in a small region of the domain under consideration. This is a desirable property in applications, for example in the optimal placement of discrete actuators [100]. From a mathematical point of view, the analysis and discretization of the optimal control problem considered here is anything but trivial and interesting, especially due to the following considerations:

1. *Fractional diffusion*: The efficient approximation of problems involving the *integral* fractional Laplacian carries two main difficulties. The first and most important is that $(-\Delta)^s$ is a non-local operator [15, 46]. The second is the lack of boundary regularity, which leads to reduced convergence rates [15, 17].
2. *Non-linearity/Non-convexity*: Since the state equation is a *semilinear* PDE, the control problem is non-convex. Consequently, first-order optimality conditions are necessary conditions for local optimality; sufficiency requires the investigation of second-order optimality conditions [26, 104].
3. *Non-differentiability*: Due to the presence of $\|\cdot\|_{L^1(\Omega)}$ in the cost functional J , the optimal control problem becomes non-differentiable ($\alpha < 0 < \beta$). This leads to some difficulties that do not occur in the differentiable case $\lambda > 0$ and $\mu = 0$ [89, 90], especially when analyzing second-order optimality conditions [26] and in the study of finite element techniques [26, 108].

For the special case $s = 1$, there are several papers in the literature that provide error estimates for finite element discretizations of control problems related to ours. As far as we know, the first paper is [108], in which the authors consider a linear PDE and propose several finite element strategies to discretize the admissible control set. For all strategies considered, the authors obtain bounds for the error that occurs when approximating the optimal control variable in $L^2(\Omega)$. The semilinear scenario was later developed in [26]. In this paper, the authors develop two strategies for discretization: one based on

the variational discretization approach and another where the admissible control set is discretized with piecewise constant functions. Based on a complete study of second-order optimality conditions, the authors obtain error estimates in $L^\infty(\Omega)$ for the error that occurs when approximating all the optimal variables involved. We would also like to mention [25] for the piecewise linear approximation of the admissible control set and [6, 108] for a posteriori error analyses.

For the non-local and linear scenario, $s \in (0, 1)$ and $a \equiv 0$, there are several papers in which finite element strategies are analyzed. We refer the interested reader to [44] for the analysis of a priori error bounds for the differentiable case $\lambda > 0$ and $\mu = 0$ and to the more recent work [109] for the analysis of a posteriori error bounds in the non-differentiable scenario. We also mention [87, 91], where the authors consider the linear version of (4.1.1)–(4.1.3), but with the *spectral* definition of the fractional Laplacian. It is important to mention that the treatment of discretizations of problems involving the spectral and integral definitions of the fractional Laplacian is fundamentally different due to regularity and discretization properties. The present work continues our research in the area of fractional semilinear optimal control. It extends previous work [89, 90] to a non-differentiable and sparse scenario. As far as we know, this paper is the *first* to provide a complete analysis for the semilinear control problem (4.1.1)–(4.1.3), which also includes the development and analysis of finite element strategies.

The structure of this article is as follows. In section 4.2 we introduce the notation, the functional framework and the assumptions that we will use in our work. In section 4.3 we give an overview of (4.1.2) and its discretization by finite elements. In section 4.4 we present a weak formulation of (4.1.1)–(4.1.3), analyze existence results, and derive first- and second-order optimality conditions; furthermore, regularity properties are also analyzed. In section 4.5, we introduce a fully discrete method and provide convergence properties and error bounds. In section 4.6, we develop a semidiscretization scheme and derive error bounds. We conclude our work with section 4.7 in which we perform a numerical experiment that illustrates the performance of the proposed methods.

4.2 Notation and preliminary remarks

Let us set the notation and recall some facts that will be useful later.

4.2.1 Notation

We denote by Ω^c the complement of Ω . For normed spaces \mathcal{X} and \mathcal{Y} , we write $\mathcal{X} \hookrightarrow \mathcal{Y}$ to denote that \mathcal{X} is continuously embedded in \mathcal{Y} . We denote by \mathcal{X}' and $\|\cdot\|_{\mathcal{X}}$ the dual and the norm of \mathcal{X} , respectively. The duality pairing between \mathcal{X} and \mathcal{X}' is denoted by $\langle \cdot, \cdot \rangle_{\mathcal{X}, \mathcal{X}'}$. When the spaces \mathcal{X} and \mathcal{X}' are clear from the context, we simply write $\langle \cdot, \cdot \rangle$. Let $\{x_n\}_{n=1}^\infty$ be a sequence in \mathcal{X} . We denote

by $x_n \rightarrow x$, $x_n \rightharpoonup x$, and $x_n \overset{*}{\rightharpoonup} x$ the strong, weak, and weak* convergence, respectively, of $\{x_n\}_{n=1}^\infty$ to x in \mathcal{X} as $n \uparrow \infty$. Finally, $\mathbf{a} \lesssim \mathbf{b}$ indicates that $\mathbf{a} \leq C\mathbf{b}$, where C is a positive constant that does not depend on either \mathbf{a} or \mathbf{b} . The value of C might change at each occurrence.

4.2.2 Subgradients and subdifferentials

We denote $\mathbb{R} \cup \{+\infty\}$ by \mathbb{R}_∞ . Let $j : Z \rightarrow \mathbb{R}_\infty$ be a given function, where Z is a real normed space, and let z be a point in $\text{Dom } j$, i.e., $z \in Z$ is such that $j(z) < \infty$. An element $\zeta \in Z'$ is called a *subgradient* of j at z if it satisfies the following *subgradient* inequality [36, Chapter 4.1]:

$$j(y) - j(z) \geq \langle \zeta, y - z \rangle_{Z', Z} \quad \forall y \in Z. \quad (4.2.1)$$

The set of all subgradients of j at z is denoted by $\partial j(z)$ and is called the *subdifferential* of j at z . Of particular interest is the case where $Z = L^1(\Omega)$ and $j : Z \rightarrow \mathbb{R}_0^+$ is defined by $j(z) = \|z\|_{L^1(\Omega)}$. Here, \mathbb{R}_0^+ denotes the set of all nonnegative real numbers. In this scenario, it follows that $\zeta \in \partial j(z)$ if and only if

$$\zeta(x) = 1 \text{ if } z(x) > 0, \quad \zeta(x) = -1 \text{ if } z(x) < 0, \quad \zeta(x) \in [-1, 1] \text{ if } z(x) = 0,$$

for a.e. $x \in \Omega$ [96, Proposition 4.6.2]. For $z, v \in L^1(\Omega)$, the directional derivative of j at z in the direction v is given by

$$j'(z; v) = \lim_{\rho \downarrow 0} \frac{j(z + \rho v) - j(z)}{\rho} = \int_{\Omega_z^+} v dx - \int_{\Omega_z^-} v dx + \int_{\Omega_z^0} |v| dx, \quad (4.2.2)$$

where Ω_z^+ , Ω_z^- , and Ω_z^0 denote the sets of points in Ω where z is positive, negative, and zero, respectively. The identity on the left-hand side is the definition of the directional derivative (see e.g. [16, Definition 2.44, Section 2.2.1]), while the identity on the right-hand side follows from elementary calculations in combination with the use of the directional derivative of $|\cdot|$; see also [26, identities (3.2)].

Let $M \subseteq Z$ be nonempty. The functional $\mathbb{1}_M : Z \rightarrow \mathbb{R}_\infty$ defined by $\mathbb{1}_M(z) = 0$ if $z \in M$ and $\mathbb{1}_M(z) = +\infty$ if $z \in Z \setminus M$ is called the *indicator functional* of M . We note that $\mathbb{1}_M$ is proper and convex if and only if M is nonempty and convex. Let $z \in M$. It follows from (4.2.1) that $\zeta \in \partial \mathbb{1}_M(z)$ if and only if $\langle \zeta, y - z \rangle_{Z', Z} \leq 0$ for all $y \in M$.

4.2.3 Function spaces

Let $s \geq 0$ and let $\mathbb{R}^d \ni \xi \mapsto \iota(\xi) = (1 + |\xi|^2)^{\frac{s}{2}} \in \mathbb{R}$. With \mathcal{F} we denote the Fourier transform. We define the fractional Sobolev space $H^s(\mathbb{R}^d) := \{v \in L^2(\mathbb{R}^d) : \iota \mathcal{F}(v) \in L^2(\mathbb{R}^d)\}$, which is endowed with the norm $\|v\|_{H^s(\mathbb{R}^d)} := \|\iota \mathcal{F}(v)\|_{L^2(\mathbb{R}^d)}$; see [102, Definition 15.7] and [78, Chapter 1, Section 7].

We define $\tilde{H}^s(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $H^s(\mathbb{R}^d)$. According to [81, Theorem 3.29], we have the characterization $\tilde{H}^s(\Omega) = \{v \in H^s(\mathbb{R}^d) : \text{supp } v \subset \overline{\Omega}\}$. We endow the fractional Sobolev space $\tilde{H}^s(\Omega)$ with the following inner product and norm:

$$(u, v)_{\tilde{H}^s(\Omega)} := \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{d+2s}} dx dy, \quad \|v\|_{\tilde{H}^s(\Omega)} := (v, v)_{\tilde{H}^s(\Omega)}^{\frac{1}{2}}.$$

We denote by $H^{-s}(\Omega)$ the dual space of $\tilde{H}^s(\Omega)$. Finally, we introduce

$$\mathcal{A} : \tilde{H}^s(\Omega) \times \tilde{H}^s(\Omega) \rightarrow \mathbb{R}, \quad \mathcal{A}(u, v) = 2^{-1}C(d, s)(u, v)_{\tilde{H}^s(\Omega)},$$

and $\|v\|_s := \mathcal{A}(v, v)^{\frac{1}{2}}$.

We will use the following continuous and compact embedding results repeatedly. Let $s \in (0, 1)$. If $\tau \in [1, 2d/(d-2s)]$, then $H^s(\Omega) \hookrightarrow L^\tau(\Omega)$ [4, Theorem 7.34]. If $\tau \in [1, 2d/(d-2s))$, then the embedding is compact [46, Corollary 7.2].

4.2.4 Assumptions

We will operate under the following assumptions on a and L . However, we must mention right away that some of the results obtained in this paper are also valid under less restrictive conditions; when possible, we explicitly mention the assumptions on a and L that are required to obtain a particular result.

(A.1) $a : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function of class C^2 with respect to the second variable and $a(\cdot, 0) \in L^2(\Omega) \cap L^r(\Omega)$ for $r > d/2s$.

(A.2) $\frac{\partial a}{\partial u}(x, u) \geq 0$ for a.e. $x \in \Omega$ and for all $u \in \mathbb{R}$.

(A.3) For all $m > 0$, there exists $C_m > 0$ such that

$$\sum_{i=1}^2 \left| \frac{\partial^i a}{\partial u^i}(x, u) \right| \leq C_m, \quad \left| \frac{\partial^2 a}{\partial u^2}(x, u) - \frac{\partial^2 a}{\partial u^2}(x, v) \right| \leq C_m |u - v|$$

for a.e. $x \in \Omega$ and u, v such that $|u|, |v| \leq m$.

We note that it follows directly from (A.3) and the mean value theorem that a and $\frac{\partial a}{\partial u}$ are locally Lipschitz with respect to the second variable.

(B.1) $L : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function of class C^2 with respect to the second variable and $L(\cdot, 0) \in L^1(\Omega)$.

(B.2) For all $m > 0$, there exist $\psi_m, \phi_m \in L^r(\Omega)$ with $r > d/2s$ such that

$$\left| \frac{\partial L}{\partial u}(x, u) \right| \leq \psi_m(x), \quad \left| \frac{\partial^2 L}{\partial u^2}(x, u) \right| \leq \phi_m(x)$$

for a.e. $x \in \Omega$ and u such that $|u| \leq m$.

The following assumption is necessary to obtain further regularity properties for optimal control variables and to derive error estimates.

(C.1) For all $m > 0$ and $u \in [-m, m]$, $\frac{\partial L}{\partial u}(\cdot, u) \in L^2(\Omega)$ and $\frac{\partial^2 L}{\partial u^2}(\cdot, u) \in L^{\frac{d}{s}}(\Omega)$.

4.3 Fractional semilinear PDEs

Let $s \in (0, 1)$ and let $q \in L^r(\Omega)$ with $r > d/2s$. We introduce the following weak formulation for the *fractional, semilinear, and elliptic* PDE (4.1.2): Find $u \in \tilde{H}^s(\Omega)$ such that

$$\mathcal{A}(u, v) + \int_{\Omega} a(x, u) v dx = \int_{\Omega} q v dx \quad \forall v \in \tilde{H}^s(\Omega). \quad (4.3.1)$$

Here, $a = a(x, u) : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ denotes a Carathéodory function which is monotone increasing in u .

We assume that for every $m > 0$ there exists $\varphi_m \in L^t(\Omega)$ such that

$$|a(x, u)| \leq \varphi_m(x) \text{ a.e. } x \in \Omega, \quad u \in [-m, m], \quad t = 2d/(d + 2s). \quad (4.3.2)$$

If, in addition, $a(\cdot, 0) \in L^r(\Omega)$ with $r > d/2s$, then (4.3.1) has a unique solution $u \in \tilde{H}^s(\Omega) \cap L^\infty(\Omega)$, which satisfies the stability bound $\|u\|_s + \|u\|_{L^\infty(\Omega)} \lesssim \|q - a(\cdot, 0)\|_{L^r(\Omega)}$ [89, Theorem 3.1].

Theorem 4.3.1 (Sobolev regularity). *Let $s \in (0, 1)$ and let $q \in L^2(\Omega) \cap L^r(\Omega)$ with $r > d/2s$. If $a(\cdot, 0) \in L^2(\Omega)$ and a is locally Lipschitz with respect to the second variable, then the solution u of problem (4.3.1) belongs to $H^{s+\kappa-\varepsilon}(\Omega)$ for all $0 < \varepsilon < s$, where $\kappa = \frac{1}{2}$ for $\frac{1}{2} < s < 1$ and $\kappa = s - \varepsilon$ for $0 < s \leq \frac{1}{2}$. Moreover, we have the bound*

$$\|u\|_{H^{s+\kappa-\varepsilon}(\Omega)} \lesssim C \varepsilon^{-\nu} \|q - a(\cdot, 0)\|_{L^2(\Omega)} \quad \forall \varepsilon \in (0, s),$$

where $\nu = \frac{1}{2}$ for $\frac{1}{2} < s < 1$ and $\nu = \frac{1}{2} + \nu_0$ for $0 < s \leq \frac{1}{2}$. Here, ν_0 and C denote positive constants that depend on Ω and d and Ω , d , and s , respectively.

Proof. The proof follows from a direct application of [17, Theorem 2.1 and inequality (2.6)] using the fact that a is locally Lipschitz with respect to the second variable so that $\|q - a(\cdot, u)\|_{L^2(\Omega)} \lesssim \|q - a(\cdot, 0)\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)} \lesssim \|q - a(\cdot, 0)\|_{L^2(\Omega)}$. \square

4.3.1 Finite element discretization

We now present a finite element approximation of problem (4.3.1) under the additional assumption that Ω is a Lipschitz *polytope*. Let $\{\mathcal{T}_h\}_{h>0}$ be a collection of conforming and quasi-uniform meshes \mathcal{T}_h made of closed simplices T , where $h = \max\{h_T : T \in \mathcal{T}_h\}$ and $h_T = \text{diam}(T)$. For each mesh \mathcal{T}_h , we introduce the following standard finite element space:

$$\mathbb{V}_h := \{v_h \in C(\bar{\Omega}) : v_h|_T \in \mathbb{P}_1(T) \ \forall T \in \mathcal{T}_h, v_h = 0 \text{ on } \partial\Omega\}. \quad (4.3.3)$$

The discrete approximation of (4.3.1) is as follows: Find $\mathbf{u}_h \in \mathbb{V}_h$ such that

$$\mathcal{A}(\mathbf{u}_h, v_h) + \int_{\Omega} a(x, \mathbf{u}_h) v_h dx = \int_{\Omega} q v_h dx \quad \forall v_h \in \mathbb{V}_h. \quad (4.3.4)$$

The existence of a discrete solution follows from Brouwer's fixed point theorem; uniqueness follows from the monotonicity of a . Moreover, we have $\|\mathbf{u}_h\|_s \lesssim \|q\|_{H^{-s}(\Omega)}$.

We now state a priori error estimates. For this purpose, we will assume that

$$|a(x, u) - a(x, v)| \leq |\phi(x)| |u - v| \quad (4.3.5)$$

for a.e. $x \in \Omega$ and $u, v \in \mathbb{R}$. The function ϕ belongs to $L^{\eta}(\Omega)$, where $\eta = d/2s$.

Theorem 4.3.2 (a priori error estimates). *Let $s \in (0, 1)$ and let $q \in L^r(\Omega)$ with $r > d/2s$. Let $u \in \tilde{H}^s(\Omega)$ be the solution to (4.3.1) and let $\mathbf{u}_h \in \mathbb{V}_h$ be its finite element approximation obtained as the solution to (4.3.4). If a satisfies (4.3.5), then we have*

$$\|u - \mathbf{u}_h\|_s \lesssim \|u - v_h\|_s \quad \forall v_h \in \mathbb{V}_h. \quad (4.3.6)$$

If, in addition, $q \in L^2(\Omega)$, a is locally Lipschitz with respect to the second variable, and $a(\cdot, 0) \in L^2(\Omega)$, then we have the error bound

$$\|u - \mathbf{u}_h\|_s \lesssim h^{\gamma} |\log h|^{\varphi} \|q - a(\cdot, 0)\|_{L^2(\Omega)}, \quad \gamma = \min\{s, \frac{1}{2}\}, \quad (4.3.7)$$

If, in addition, a satisfies the condition (4.3.5) with $\eta = d/s$, then we have

$$\|u - \mathbf{u}_h\|_{L^2(\Omega)} \lesssim h^{2\gamma} |\log h|^{2\varphi} \|q - a(\cdot, 0)\|_{L^2(\Omega)}, \quad \gamma = \min\{s, \frac{1}{2}\}. \quad (4.3.8)$$

Here, $\varphi = \nu$ if $s \neq \frac{1}{2}$, $\varphi = 1 + \nu$ if $s = \frac{1}{2}$, and $\nu \geq \frac{1}{2}$ is the constant in Theorem 4.3.1.

Proof. The proof of (4.3.6) can be found in [89, Theorem 5.2]. The error estimates (4.3.7) and (4.3.8)

can be found in [90, Theorem 5.1] and [90, Theorem 5.2]. \square

We conclude this section with the following convergence result.

Lemma 4.3.3 (convergence). *Let $s \in (0, 1)$ and let $u_h \in \mathbb{V}_h$ be the solution to*

$$\mathcal{A}(u_h, v_h) + \int_{\Omega} a(x, u_h)v_h dx = \int_{\Omega} q_h v_h dx \quad \forall v_h \in \mathbb{V}_h,$$

where $q_h \in L^r(\Omega)$ with $r > d/2s$. If a satisfies the condition (4.3.5), then we have $q_h \rightharpoonup q$ in $L^r(\Omega) \implies u_h \rightarrow u$ in $L^r(\Omega)$ as $h \rightarrow 0$. Here, $\mathfrak{r} \leq 2d/(d - 2s)$.

Proof. See [89, Proposition 5.3] for a proof. We note that in the statement of [89, Proposition 5.3] it is assumed that $\Omega \in C^2$. However, this assumption does not play any role and the same proof can be performed if Ω is a Lipschitz polytope. \square

4.4 The optimal control problem

In this section, we present the following weak formulation for the optimal control problem introduced in section 4.1: Find

$$\min\{J(u, q) : (u, q) \in \tilde{H}^s(\Omega) \times \mathbb{Q}_{ad}\} \quad (4.4.1)$$

subject to the *fractional semilinear*, and *elliptic* state equation

$$\mathcal{A}(u, v) + \int_{\Omega} a(x, u)v dx = \int_{\Omega} qv dx \quad \forall v \in \tilde{H}^s(\Omega). \quad (4.4.2)$$

Here, $a = a(x, u) : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a monotonically increasing in u Carathéodory function that satisfies (4.3.2) and $a(\cdot, 0) \in L^r(\Omega)$ with $r > d/2s$. As explained in section 4.3, problem (4.4.2) is well-posed under these assumptions on a . We therefore introduce the *control to state map* $\mathcal{S} : L^r(\Omega) \rightarrow \tilde{H}^s(\Omega) \cap L^\infty(\Omega)$ which, given a control q , associates to it the unique state u that solves (4.4.2).

4.4.1 Existence of an optimal solution

We begin this section by introducing the concept of global solution. We say that $\bar{q} \in \mathbb{Q}_{ad}$ is a global solution of (4.4.1)–(4.4.2) if $J(\mathcal{S}\bar{q}, \bar{q}) \leq J(\mathcal{S}q, q)$ for all $q \in \mathbb{Q}_{ad}$. The existence of an optimal solution $(\bar{u}, \bar{q}) \in \tilde{H}^s(\Omega) \times \mathbb{Q}_{ad}$ for our optimal control problem is as follows.

Theorem 4.4.1 (existence of an optimal solution). *Let $s \in (0, 1)$. Let $L = L(x, u) : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ be a Carathéodory function. Assume that, for every $\mathfrak{m} > 0$, there exists $\varphi_{\mathfrak{m}} \in L^r(\Omega)$ with $r > d/2s$ and*

$\sigma_m \in L^1(\Omega)$ such that

$$|a(x, u)| \leq \varphi_m(x), \quad |L(x, u)| \leq \sigma_m(x) \quad \text{a.e. } x \in \Omega, \quad u \in [-\mathbf{m}, \mathbf{m}]. \quad (4.4.3)$$

Thus, (4.4.1)–(4.4.2) admits at least one solution $(\bar{u}, \bar{q}) \in \tilde{H}^s(\Omega) \cap L^\infty(\Omega) \times \mathbb{Q}_{ad}$.

Proof. Let $\{(u_k, q_k)\}_{k \in \mathbb{N}}$ be a minimizing sequence, i.e., for $k \in \mathbb{N}$, $q_k \in \mathbb{Q}_{ad}$ and $u_k = \mathcal{S}q_k \in \tilde{H}^s(\Omega)$ are such that $J(u_k, q_k) \rightarrow j := \inf\{J(\mathcal{S}q, q) : q \in \mathbb{Q}_{ad}\}$ as $k \uparrow \infty$. The arguments in the proof of [89, Theorem 4.1] show that, up to a nonrelabeled subsequence, $q_k \xrightarrow{*} \bar{q}$ in $L^\infty(\Omega)$ and $u_k \rightarrow \bar{u}$ in $\tilde{H}^s(\Omega)$ as $k \uparrow \infty$, where $\bar{u} = \mathcal{S}\bar{q}$. On the other hand, the Lebesgue dominated convergence theorem combined with (4.4.3) and the fact that $u_k \rightarrow \bar{u}$ in $L^r(\Omega)$ for every $r < 2d/(d - 2s)$ show that $|\int_\Omega (L(x, u_k(x)) - L(x, \bar{u}))dx| \rightarrow 0$ as $k \uparrow \infty$. Since $\|\cdot\|_{L^1(\Omega)}$ and the square of $\|\cdot\|_{L^2(\Omega)}$ are continuous and convex in $L^1(\Omega)$ and $L^2(\Omega)$, respectively, we can arrive at $J(\bar{u}, \bar{q}) \leq j$. This completes the proof. \square

4.4.2 First-order necessary optimality conditions

In this section, we develop necessary first-order optimality conditions for (4.4.1)–(4.4.2). Since this problem is nonconvex, we discuss optimality conditions in the context of local solutions: We say that $\bar{q} \in \mathbb{Q}_{ad}$ is a local solution in $L^2(\Omega)$ for (4.4.1)–(4.4.2) if there exists $\varepsilon > 0$ such that

$$J(\mathcal{S}\bar{q}, \bar{q}) \leq J(\mathcal{S}q, q) \quad \forall q \in \mathbb{Q}_{ad} : \|q - \bar{q}\|_{L^2(\Omega)} \leq \varepsilon.$$

The element \bar{q} is called a strict local solution in $L^2(\Omega)$ for (4.4.1)–(4.4.2) if there exists $\varepsilon > 0$ such that $J(\mathcal{S}\bar{q}, \bar{q}) < J(\mathcal{S}q, q)$ for all $q \in \mathbb{Q}_{ad} \setminus \{\bar{q}\}$ such that $\|q - \bar{q}\|_{L^2(\Omega)} \leq \varepsilon$.

We now introduce $F : L^2(\Omega) \rightarrow \mathbb{R}$ and $j : L^1(\Omega) \rightarrow \mathbb{R}$ by

$$F(q) = \int_\Omega L(x, \mathcal{S}q)dx + \frac{\lambda}{2} \|q\|_{L^2(\Omega)}^2, \quad j(q) = \|q\|_{L^1(\Omega)}.$$

We also introduce the *reduced cost functional* $j : \mathbb{Q}_{ad} \rightarrow \mathbb{R}$ by $j(q) = F(q) + \mu j(q)$.

In what follows, we discuss differentiability properties for \mathcal{S} and F .

Proposition 4.4.1 (differentiability properties of \mathcal{S}). *Let $s \in (0, 1)$ and let $r > d/2s$. Assume that (A.1)–(A.3) hold. Then, the control to state map $\mathcal{S} : L^r(\Omega) \rightarrow \tilde{H}^s(\Omega) \cap L^\infty(\Omega)$ is of class C^2 . In addition, if $q, w \in L^r(\Omega)$, then $\phi = \mathcal{S}'(q)w \in \tilde{H}^s(\Omega) \cap L^\infty(\Omega)$ corresponds to the unique solution to the problem*

$$\mathcal{A}(\phi, v) + \int_\Omega \frac{\partial a}{\partial u}(x, u)\phi v dx = \int_\Omega w v dx \quad \forall v \in \tilde{H}^s(\Omega), \quad (4.4.4)$$

where $u = \mathcal{S}q$. If $w_1, w_2 \in L^r(\Omega)$, then $\psi = \mathcal{S}''(q)(w_1, w_2) \in \tilde{H}^s(\Omega) \cap L^\infty(\Omega)$ corresponds to the unique

solution to

$$\mathcal{A}(\psi, v) + \int_{\Omega} \frac{\partial a}{\partial u}(x, u) \psi v dx = - \int_{\Omega} \frac{\partial^2 a}{\partial u^2}(x, u) \phi_{w_1} \phi_{w_2} v dx \quad \forall v \in \tilde{H}^s(\Omega), \quad (4.4.5)$$

where $u = \mathcal{S}q$ and $\phi_{w_i} = \mathcal{S}'(q)w_i$, with $i \in \{1, 2\}$.

Proof. See [89, Theorem 4.3] for a proof. \square

To present the following result, we introduce the *adjoint state* p , which corresponds to the solution to the problem: Find $p \in \tilde{H}^s(\Omega) \cap L^\infty(\Omega)$ such that

$$\mathcal{A}(v, p) + \int_{\Omega} \frac{\partial a}{\partial u}(x, u) p v dx = \int_{\Omega} \frac{\partial L}{\partial u}(x, u) v dx \quad \forall v \in \tilde{H}^s(\Omega), \quad (4.4.6)$$

where $u = \mathcal{S}q \in \tilde{H}^s(\Omega) \cap L^\infty(\Omega)$. Since $\partial a / \partial u(x, u) \geq 0$ for a.e. $x \in \Omega$ and for all $u \in \mathbb{R}$ and $\partial L / \partial u(\cdot, u) \in L^r(\Omega)$ for every $\mathbf{m} > 0$ and $u \in [-\mathbf{m}, \mathbf{m}]$, the adjoint problem (4.4.6) is well-posed.

Proposition 4.4.2 (differentiability properties of F). *Let $s \in (0, 1)$ and let $r > d/2s$. Assume that (A.1)–(A.3) and (B.1)–(B.2) hold. Then, $F : L^2(\Omega) \cap L^r(\Omega) \rightarrow \mathbb{R}$ is of class C^2 . In addition, if $q, w \in L^2(\Omega) \cap L^r(\Omega)$, then*

$$F'(q)w = \int_{\Omega} (p + \lambda q) w dx, \quad (4.4.7)$$

where p solves (4.4.6). If $w_1, w_2 \in L^2(\Omega) \cap L^r(\Omega)$, then we have

$$F''(q)(w_1, w_2) = \int_{\Omega} \left(\frac{\partial^2 L}{\partial u^2}(x, u) \phi_{w_1} \phi_{w_2} + \lambda w_1 w_2 - p \frac{\partial^2 a}{\partial u^2}(x, u) \phi_{w_1} \phi_{w_2} \right) dx, \quad (4.4.8)$$

where $u = \mathcal{S}q$ and $\phi_{w_i} = \mathcal{S}'(q)w_i$, with $i \in \{1, 2\}$.

Proof. See [89, Proposition 4.5] for a proof. \square

The necessary first-order optimality conditions are as follows.

Theorem 4.4.2 (first-order optimality conditions). *If $\bar{q} \in \mathbb{Q}_{ad}$ is a local solution to (4.4.1)–(4.4.2), then there exists $\bar{\eta} \in \partial j(\bar{q})$ such that*

$$\int_{\Omega} (\bar{p} + \lambda \bar{q} + \mu \bar{\eta})(q - \bar{q}) dx \geq 0 \quad \forall q \in \mathbb{Q}_{ad}. \quad (4.4.9)$$

Here, \bar{p} denotes the solution to (4.4.6) where u is replaced by $\bar{u} = \mathcal{S}\bar{q}$.

Proof. Since \bar{q} is a local solution, there exists $\varepsilon > 0$ such that $j(\bar{q}) \leq j(q)$ for every $q \in \mathbb{Q}_{ad}$ such that $\|\bar{q} - q\|_{L^2(\Omega)} \leq \varepsilon$. Let $q \in \mathbb{Q}_{ad}$ and let $\rho \in (0, 1)$ be sufficiently small so that $q_\rho := \bar{q} + \rho(q - \bar{q}) =$

$\rho q + (1 - \rho)\bar{q} \in \mathbb{Q}_{ad}$ satisfies $\|\bar{q} - q_\rho\|_{L^2(\Omega)} \leq \varepsilon$. Thus

$$0 \leq j(q_\rho) - j(\bar{q}) = [F(q_\rho) - F(\bar{q})] + \mu[j(q_\rho) - j(\bar{q})]. \quad (4.4.10)$$

Note that $j(q_\rho) - j(\bar{q}) \leq \rho(j(q) - j(\bar{q}))$ because j is convex. Dividing (4.4.10) by ρ and taking the limit as $\rho \downarrow 0$ yield $0 \leq F'(\bar{q})(q - \bar{q}) + \mu(j(q) - j(\bar{q}))$ for every $q \in \mathbb{Q}_{ad}$. This inequality can be rewritten as follows: $\bar{q} \in \mathbb{Q}_{ad}$ is such that

$$\ell(\bar{q}) \leq \ell(q) \quad \forall q \in \mathbb{Q}_{ad}, \quad \ell : L^2(\Omega) \rightarrow \mathbb{R}, \quad \ell(q) := F'(\bar{q})q + \mu j(q),$$

i.e., \bar{q} is a global minimizer of ℓ over \mathbb{Q}_{ad} . We must thus have that $0 \in \partial(\ell + \mathbb{1}_{\mathbb{Q}_{ad}})(\bar{q}) = \partial\ell(\bar{q}) + \partial\mathbb{1}_{\mathbb{Q}_{ad}}(\bar{q})$ [96, page 134 and Proposition 4.5.1]. Here, $\mathbb{1}_{\mathbb{Q}_{ad}}$ corresponds to the indicator functional of \mathbb{Q}_{ad} ; see section 4.2.2. We now use that $F'(\bar{q}) = (\bar{p} + \lambda\bar{q})$ and let $\bar{\eta} \in \partial j(\bar{q})$ to arrive at $-(\bar{p} + \lambda\bar{q}) - \mu\bar{\eta} \in \partial\mathbb{1}_{\mathbb{Q}_{ad}}(\bar{q})$. This allows us to conclude. \square

Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}$ be such that $\mathbf{a} < \mathbf{b}$. We introduce the operator $\Pi_{[\mathbf{a}, \mathbf{b}]} : L^1(\Omega) \rightarrow \mathbb{Q}_{ad}$ by $\Pi_{[\mathbf{a}, \mathbf{b}]}(v) = \min\{\mathbf{b}, \max\{\mathbf{a}, v\}\}$ and present the following result.

Theorem 4.4.3 (projection formulas). *If $\bar{q}, \bar{u}, \bar{p}$, and $\bar{\eta}$ are as in the statement of Theorem 4.4.2, then*

$$\bar{q}(x) = \Pi_{[\alpha, \beta]}(-\lambda^{-1}(\bar{p}(x) + \mu\bar{\eta}(x))), \quad \bar{q}(x) = 0 \Leftrightarrow |\bar{p}(x)| \leq \mu, \quad (4.4.11)$$

$$\bar{\eta}(x) = \Pi_{[-1, 1]}(-\mu^{-1}\bar{p}(x)) \quad (4.4.12)$$

for a.e. $x \in \Omega$. In particular, the subgradient $\bar{\eta} \in \partial j(\bar{q})$ is uniquely determined and both \bar{q} and $\bar{\eta}$ belong to $\tilde{H}^s(\Omega) \cap L^\infty(\Omega)$.

Proof. The derivation of the projection formula for \bar{q} in (4.4.11) is standard in PDE-constrained optimization. The equivalence $\bar{q}(x) = 0 \Leftrightarrow |\bar{p}(x)| \leq \mu$ for a.e. $x \in \Omega$ can be found in [26, Corollary 3.2]. The projection formula for $\bar{\eta}$ in (4.4.12) can also be found in [26, Corollary 3.2]. This formula directly guarantees the uniqueness of $\bar{\eta}$. The desired regularity properties for \bar{q} and $\bar{\eta}$ follow from the projection formulas in (4.4.11) and (4.4.12), the fact that $\max\{0, \tau\} = (\tau + |\tau|)/2$ for all $\tau \in \mathbb{R}$, and [86, Theorem 1]. \square

4.4.3 Second-order optimality conditions

In this section, we assume that $d \in \{2, 3\}$ and $s > d/4$.

Let $\bar{q} \in \mathbb{Q}_{ad}$ be a local minimum and let $\bar{\eta} \in \partial j(\bar{q})$ be the corresponding subgradient. We define

$$C_{\bar{q}} := \{w \in L^2(\Omega) : w \text{ satisfies (4.4.13) and } F'(\bar{q})w + \mu j'(\bar{q}; w) = 0\},$$

where $F'(\bar{q})(\cdot)$ and $j'(\bar{q}, \cdot)$ are described in Proposition 4.4.2 and (4.2.2), respectively, and

$$w(x) \geq 0 \text{ if } \bar{q}(x) = \alpha, \quad w(x) \leq 0 \text{ if } \bar{q}(x) = \beta. \quad (4.4.13)$$

The set $C_{\bar{q}}$ is a closed and convex cone in $L^2(\Omega)$ [26, Proposition 3.4].

We formulate necessary second-order optimality conditions as follows.

Theorem 4.4.4 (second-order necessary optimality conditions). *If $\bar{q} \in \mathbb{Q}_{ad}$ is a local minimum of problem (4.4.1)–(4.4.2), then $F''(\bar{q})w^2 \geq 0$ for every $w \in C_{\bar{q}}$.*

Proof. The proof follows the same arguments as in [26, Theorem 3.7]. \square

To present the following result, for $\tau > 0$, we introduce the cone

$$C_{\bar{q}}^\tau := \{w \in L^2(\Omega) : w \text{ satisfies (4.4.13) and } F'(\bar{q})w + \mu j'(\bar{q}; w) \leq \tau \|w\|_{L^2(\Omega)}\}.$$

Theorem 4.4.5 (equivalence). *Let $\bar{q} \in \mathbb{Q}_{ad}$ be a local minimum and let $\bar{\eta} \in \partial j(\bar{q})$ be the corresponding subgradient such that (4.4.9) holds. Then, the following statements are equivalent:*

(i) $F''(\bar{q})w^2 > 0$ for all $w \in C_{\bar{q}} \setminus \{0\}$.

(ii) There exist $\tau, \delta > 0$ such that $F''(\bar{q})w^2 \geq \delta \|w\|_{L^2(\Omega)}^2$ for all $w \in C_{\bar{q}}^\tau$.

Proof. Since $C_{\bar{q}} \subset C_{\bar{q}}^\tau$ for every $\tau > 0$, it is immediate that (ii) implies (i).

We now prove that (i) implies (ii). To do so, we follow the arguments of the proof of [26, Theorem 3.8] and proceed by contradiction. Indeed, we assume the existence of a sequence $\{v_k\}_{k \in \mathbb{N}}$ such that

$$v_k \in C_{\bar{q}}^{1/k}, \quad F''(\bar{q})v_k^2 < k^{-1} \|v_k\|_{L^2(\Omega)}^2, \quad k \in \mathbb{N}.$$

Define $w_k := \|v_k\|_{L^2(\Omega)}^{-1} v_k$. Note that $w_k \in C_{\bar{q}}^{1/k}$ because $C_{\bar{q}}^{1/k}$ is a cone. Moreover,

$$\|w_k\|_{L^2(\Omega)} = 1, \quad F''(\bar{q})w_k^2 < k^{-1}, \quad k \in \mathbb{N}. \quad (4.4.14)$$

Since $\{w_k\}_{k \in \mathbb{N}}$ is uniformly bounded in $L^2(\Omega)$, we can extract a nonrelabeled subsequence such that $w_k \rightharpoonup w$ in $L^2(\Omega)$ as $k \uparrow \infty$. Note that w satisfies (4.4.13). Apply [26, Lemma 3.5] to derive that $F'(\bar{q})w + \mu j'(\bar{q}; w) \geq 0$. On the other hand, since $j'(\bar{q}; \cdot)$ is weakly lower semicontinuous in $L^2(\Omega)$ and $F'(\bar{q})w_k \rightarrow F'(\bar{q})w$ as $k \uparrow \infty$ we obtain

$$F'(\bar{q})w + \mu j'(\bar{q}; w) \leq \liminf_{k \uparrow \infty} F'(\bar{q})w_k + \mu j'(\bar{q}; w_k) \leq \liminf_{k \uparrow \infty} k^{-1} = 0.$$

To obtain the last inequality we used that $w_k \in C_{\bar{q}}^{1/k}$. This proves that $F'(\bar{q})w + \mu j'(\bar{q}; w) = 0$ and thus that $w \in C_{\bar{q}}$.

We now prove that $w \equiv 0$. Since (i) holds, it suffices to show that $F''(\bar{q})w^2 \leq 0$. To accomplish this task, we use the characterization (4.4.8) and write

$$F''(\bar{q})w_k^2 = \int_{\Omega} \left(\frac{\partial^2 L}{\partial u^2}(x, \bar{u})\phi_k^2 + \lambda w_k^2 - \bar{p} \frac{\partial^2 a}{\partial u^2}(x, \bar{u})\phi_k^2 \right) dx. \quad (4.4.15)$$

Here, $\phi_k = \mathcal{S}'(\bar{q})w_k \in \tilde{H}^s(\Omega) \cap L^\infty(\Omega)$ solves (4.4.4) with u and w replaced by $\bar{u} = \mathcal{S}(\bar{q})$ and w_k , respectively and $\bar{p} \in \tilde{H}^s(\Omega) \cap L^\infty(\Omega)$ solves (4.4.6) with u replaced by \bar{u} . Since $w_k \rightharpoonup w$ in $L^2(\Omega)$ as $k \uparrow \infty$, we deduce that $\phi_k \rightharpoonup \phi = \mathcal{S}'(\bar{q})w$ in $\tilde{H}^s(\Omega)$ as $k \uparrow \infty$. Note that ϕ solves (4.4.4) with u replaced by \bar{u} . This convergence result and the compact embedding of section 4.2.3 show that $\phi_k \rightarrow \phi$ in $L^r(\Omega)$ for every $r < 2d/(d-2s)$. Thus, we invoke the assumptions (A.3) and (B.2) and obtain

$$\begin{aligned} \left| \int_{\Omega} \frac{\partial^2 L}{\partial u^2}(x, \bar{u}) (\phi_k^2 - \phi^2) dx \right| &\leq \left\| \frac{\partial^2 L}{\partial u^2}(\cdot, \bar{u}) \right\|_{L^r(\Omega)} \|\phi_k - \phi\|_{L^t(\Omega)} \|\phi_k + \phi\|_{L^t(\Omega)}, \\ \left| \int_{\Omega} \frac{\partial^2 a}{\partial u^2}(x, \bar{u}) \bar{p} (\phi_k^2 - \phi^2) dx \right| &\leq \left\| \frac{\partial^2 a}{\partial u^2}(\cdot, \bar{u}) \right\|_{L^r(\Omega)} \|\bar{p}\|_{L^\infty(\Omega)} \|\phi_k - \phi\|_{L^t(\Omega)} \|\phi_k + \phi\|_{L^t(\Omega)}. \end{aligned}$$

Here $r^{-1} + 2t^{-1} = 1$. Note that, since $r > d/2s$, we have that $t < 2d/(d-2s)$.

Since the square of $\|\cdot\|_{L^2(\Omega)}$ is continuous and convex in $L^2(\Omega)$, the aforementioned convergence results based on (4.4.15) and (4.4.14) yield

$$F''(\bar{q})w^2 \leq \liminf_{k \rightarrow \infty} F''(\bar{q})w_k^2 \leq 0.$$

The condition (i), which reads: $F''(\bar{q})w^2 > 0$ for all $w \in C_{\bar{q}} \setminus \{0\}$, therefore implies that $w = 0$ and that $F''(\bar{q})w_k^2 \rightarrow 0$ as $k \uparrow \infty$.

After proving that $w = 0$, we conclude the proof by arriving at a contradiction. We begin by noting that $w_k \rightarrow 0$ in $L^2(\Omega)$ implies that $\phi_k \rightarrow 0$ in $L^r(\Omega)$ for every $r < 2d/(d-2s)$. We use the latter convergence result and $\|w_k\|_{L^2(\Omega)} = 1$ to obtain

$$F''(\bar{q})w_k^2 = \lambda + \int_{\Omega} \left(\frac{\partial^2 L}{\partial u^2}(x, \bar{u})\phi_k^2 - \bar{p} \frac{\partial^2 a}{\partial u^2}(x, \bar{u})\phi_k^2 \right) dx \rightarrow \lambda > 0, \quad k \uparrow \infty.$$

This is a contradiction because $F''(\bar{q})w_k^2 \rightarrow 0$ as $k \uparrow \infty$. This concludes the proof. \square

We conclude this section with second-order sufficient optimality conditions.

Theorem 4.4.6 (sufficient optimality conditions). *Let $(\bar{u}, \bar{p}, \bar{q}, \bar{\eta}) \in \tilde{H}^s(\Omega) \times \tilde{H}^s(\Omega) \times \mathbb{Q}_{ad} \times \partial j(\bar{q})$ satisfy the first-order optimality conditions (4.4.2), (4.4.6), and (4.4.9). If $F''(\bar{q})w^2 > 0$ for all $w \in C_{\bar{q}} \setminus \{0\}$, then there exist $\delta > 0$ and $\varepsilon > 0$ such that*

$$j(q) \geq j(\bar{q}) + \frac{\delta}{4} \|q - \bar{q}\|_{L^2(\Omega)}^2 \quad \forall q \in \mathbb{Q}_{ad} : \|q - \bar{q}\|_{L^2(\Omega)} \leq \varepsilon.$$

Proof. The proof follows the same arguments as in [26, Theorem 3.9]. \square

4.4.4 Regularity properties

Let \bar{q} be a local minimum of problem (4.4.1)–(4.4.2). In this section, we provide regularity results for all involved optimal control variables, i.e., $\bar{q}, \bar{u}, \bar{p}$, and $\bar{\eta}$. To accomplish this task, we assume that the conditions (A.1)–(A.3) and (B.1)–(B.2) hold together with the additional property (C.1).

To present the following result, we define

$$\Lambda(L, u) = \left\| \frac{\partial L}{\partial u}(\cdot, u) \right\|_{L^2(\Omega)}. \quad (4.4.16)$$

Theorem 4.4.7 (regularity properties). *Let \bar{q} be a local minimum of (4.4.1)–(4.4.2) and let \bar{u}, \bar{p} , and $\bar{\eta}$ be the associated optimal variables. Then, $\bar{u}, \bar{p}, \bar{q}, \bar{\eta} \in H^{s+\kappa-\varepsilon}(\Omega)$ for all $\varepsilon \in (0, s)$, where $\kappa = \frac{1}{2}$ for $\frac{1}{2} < s < 1$ and $\kappa = s - \varepsilon$ for $0 < s \leq \frac{1}{2}$. Moreover,*

$$\|\bar{u}\|_{H^{s+\kappa-\varepsilon}(\Omega)} \lesssim \varepsilon^{-\nu} \|\bar{q} - a(\cdot, 0)\|_{L^2(\Omega)}, \quad (4.4.17)$$

$$\|\bar{p}\|_{H^{s+\kappa-\varepsilon}(\Omega)} \lesssim \varepsilon^{-\nu} \Lambda(L, \bar{u}), \quad (4.4.18)$$

and

$$\|\bar{q}\|_{H^{s+\kappa-\varepsilon}(\Omega)} + \|\bar{\eta}\|_{H^{s+\kappa-\varepsilon}(\Omega)} \lesssim \varepsilon^{-\nu} [1 + \Lambda(L, \bar{u})], \quad (4.4.19)$$

where $\nu = \frac{1}{2}$ for $\frac{1}{2} < s < 1$ and $\nu = \frac{1}{2} + \nu_0$ for $0 < s \leq \frac{1}{2}$, with $\nu_0 = \nu_0(\Omega, d) > 0$. The hidden constant in all inequalities is independent of ε .

Proof. Since $\bar{q} \in \mathbb{Q}_{ad} \subset L^2(\Omega) \cap L^r(\Omega)$ and a is locally Lipschitz with respect to the second variable and satisfies (A.1), the desired regularity property for \bar{u} follows from Theorem 4.3.1. We recall that $\bar{u} \in L^\infty(\Omega)$ and that $\|\bar{u}\|_{L^\infty(\Omega)} \lesssim \|\bar{q} - a(\cdot, 0)\|_{L^r(\Omega)}$. We now focus on the optimal adjoint variable \bar{p} . We first note that since $\bar{u} \in L^\infty(\Omega)$ and, for every $\mathbf{m} > 0$, $|\partial L / \partial u(x, u)| \leq \psi_{\mathbf{m}}(x)$ for a.e. $x \in \Omega$ and $u \in [-\mathbf{m}, \mathbf{m}]$, with $\psi_{\mathbf{m}} \in L^r(\Omega)$, we can conclude that $\bar{p} \in L^\infty(\Omega)$ [89, Theorem 3.1]. With this

regularity result, we invoke the assumptions (A.3) and (C.1) to arrive at

$$\frac{\partial L}{\partial u}(\cdot, \bar{u}) - \frac{\partial a}{\partial u}(\cdot, \bar{u})\bar{p} \in L^2(\Omega).$$

Thus, an application of [17, Theorem 2.1 and inequality (2.6)] show that $\bar{p} \in H^{s+\kappa-\varepsilon}(\Omega)$ together with the estimate

$$\|\bar{p}\|_{H^{s+\kappa-\varepsilon}(\Omega)} \lesssim \varepsilon^{-\nu} (\Lambda(\cdot, \bar{u}) + \|\bar{p}\|_{L^2(\Omega)}) \lesssim \varepsilon^{-\nu} \Lambda(\cdot, \bar{u}).$$

We note that since $\bar{u} \in L^\infty(\Omega)$ and the assumption (A.3) holds, $|\partial a / \partial u(x, \bar{u})| \leq C_m$ for a.e. $x \in \Omega$. The desired regularity property for $\bar{\eta}$ and part of the estimate in (4.4.19) follow from the projection formula (4.4.12), the fact that $\max\{0, \tau\} = (\tau + |\tau|)/2$ for all $\tau \in \mathbb{R}$, and [86, Theorem 1], which applies because $s + \kappa - \varepsilon = s + \frac{1}{2} - \varepsilon < \frac{3}{2}$ if $\frac{1}{2} < s < 1$ and $s + \kappa - \varepsilon = 2s - 2\varepsilon \leq 1 - 2\varepsilon < \frac{3}{2}$ if $0 < s \leq \frac{1}{2}$. The desired regularity property for \bar{q} follows the same arguments. This concludes the proof. \square

4.5 A fully discrete scheme for the optimal control problem

Before we begin our analysis, we would like to mention that in this section we assume that Ω is a Lipschitz *polytope*. In the following, we propose and analyze the following fully discrete finite element approximation of our optimal control problem (4.4.1)–(4.4.2): Find $\min J(u_h, q_h)$ subject to the *discrete state equation*

$$\mathcal{A}(u_h, v_h) + \int_{\Omega} a(x, u_h)v_h dx = \int_{\Omega} q_h v_h dx \quad \forall v_h \in \mathbb{V}_h \quad (4.5.1)$$

and the *control constraints* $q_h \in \mathbb{Q}_{ad,h}$. Here, $\mathbb{Q}_{ad,h} := \mathbb{Q}_{ad} \cap \mathbb{Q}_h$, where $\mathbb{Q}_h := \{q_h \in L^\infty(\Omega) : q_h|_T \in \mathbb{P}_0(T) \forall T \in \mathcal{T}_h\}$. \mathbb{V}_h denotes the finite element space defined in (4.3.3).

The existence of at least one optimal solution is standard. To provide first-order optimality conditions, we introduce $F_h : L^2(\Omega) \rightarrow \mathbb{R}$ and $j_h : \mathbb{Q}_h \rightarrow \mathbb{R}$ to be such that

$$F_h(q) = \int_{\Omega} L(x, \mathcal{S}_h q) dx + \frac{\lambda}{2} \|q\|_{L^2(\Omega)}^2, \quad j_h(q_h) = \|q_h\|_{L^1(\Omega)},$$

where $\mathcal{S}_h : \mathbb{Q}_{ad,h} \rightarrow \mathbb{V}_h$ denotes the *discrete control to state map*.

first-order optimality conditions for the fully discrete scheme are as follows.

Theorem 4.5.1 (first-order optimality conditions). *If $\bar{q}_h \in \mathbb{Q}_{ad,h}$ is a local solution to the fully discrete scheme, then there exists $\bar{\eta}_h \in \partial j_h(\bar{q}_h)$ such that*

$$\int_{\Omega} (\bar{p}_h + \lambda \bar{q}_h + \mu \bar{\eta}_h)(q_h - \bar{q}_h) dx \geq 0 \quad \forall q_h \in \mathbb{Q}_{ad,h}. \quad (4.5.2)$$

Here, \bar{p}_h is the solution to the discrete adjoint equation: Find $\bar{p}_h \in \mathbb{V}_h$ such that

$$\mathcal{A}(v_h, \bar{p}_h) + \int_{\Omega} \frac{\partial a}{\partial u}(x, \bar{u}_h) \bar{p}_h v_h dx = \int_{\Omega} \frac{\partial L}{\partial u}(x, \bar{u}_h) v_h dx \quad \forall v_h \in \mathbb{V}_h, \quad (4.5.3)$$

where $\bar{u}_h = \mathcal{S}_h \bar{q}_h$.

Proof. The proof follows the same arguments as in Theorem 4.4.2. \square

Analogous to the continuous case, we obtain projection formulas, but now on a discrete level.

Theorem 4.5.2 (discrete projection formulas). *If $\bar{q}_h \in \mathbb{Q}_{ad,h}$ and $\bar{\eta}_h \in \partial j_h(\bar{q}_h)$ are as in the statement of Theorem 4.5.1, then for every $T \in \mathcal{T}_h$, we have the formulas*

$$\bar{q}_h|_T = \Pi_{[\alpha, \beta]} \left[-\frac{1}{\lambda} \left(\frac{1}{|T|} \int_T \bar{p}_h dx + \mu \bar{\eta}_h|_T \right) \right], \quad (4.5.4)$$

$$\bar{q}_h|_T = 0 \Leftrightarrow \frac{1}{|T|} \left| \int_T \bar{p}_h dx \right| \leq \mu, \quad (4.5.5)$$

$$\bar{\eta}_h|_T = \Pi_{[-1, 1]} \left(-\frac{1}{\mu|T|} \int_T \bar{p}_h dx \right). \quad (4.5.6)$$

In particular, the subgradient $\bar{\eta}_h \in \partial j_h(\bar{q}_h)$ is uniquely determined.

Proof. From the variational inequality (4.5.2) we have that, for every $q_T \in [\alpha, \beta]$,

$$\left(\int_T \bar{p}_h dx + [\lambda \bar{q}_h|_T + \mu \bar{\eta}_h|_T] |T| \right) (q_T - \bar{q}_h|_T) \geq 0.$$

The projection formula (4.5.4) can be derived from this inequality. The sparse property (4.5.5) and the projection formula (4.5.6) are derived in [26, equivalence (4.4b)] and [26, formula (4.4c)], respectively. \square

We conclude this section with the following error bounds for the discretization of the adjoint equation and the corresponding subdifferential variables by finite elements. These error bounds hold under the assumption that discrete solutions u_h of the problem (4.5.1) are uniformly bounded in $L^\infty(\Omega)$, i.e.,

$$\exists C > 0 : \quad \|u_h\|_{L^\infty(\Omega)} \leq C \quad \forall h > 0. \quad (4.5.7)$$

To present a proof of some of the aforementioned error bounds, we introduce $\mathcal{P}_h : L^2(\Omega) \rightarrow \mathbb{Q}_h$, the orthogonal projection operator on piecewise constant functions. As a final ingredient, we let $\{\bar{q}_h\}_{h>0} \subset \mathbb{Q}_{ad,h}$ be a sequence of local minima of the fully discrete optimal control problems such that $\|\bar{q} - \bar{q}_h\|_{L^2(\Omega)} \rightarrow 0$ as $h \rightarrow 0$, where \bar{q} corresponds to a local solution of (4.4.1)–(4.4.2); see Theorems 4.5.4 and 4.5.5 below.

We are now ready to present error bounds.

Theorem 4.5.3 (error bounds). *Let $\bar{\eta} \in \partial j(\bar{q})$ be as in Theorems 4.4.2 and 4.4.3, and let \bar{p} be the solution to (4.4.6) with u replaced by $\bar{u} = \mathcal{S}\bar{q}$. Let $\bar{\eta}_h \in \partial j_h(\bar{q}_h)$ be as in Theorems 4.5.1 and 4.5.2, and let \bar{p}_h be the solution to (4.5.3). Let us assume that (A.1)–(A.3), (B.1)–(B.2), (C.1), and (4.5.7) hold. Then, we have*

$$\|\bar{p} - \bar{p}_h\|_s \lesssim h^\gamma |\log h|^\varphi + \|\bar{q} - \bar{q}_h\|_{L^2(\Omega)}, \quad \gamma = \min\{s, \frac{1}{2}\} \quad (4.5.8)$$

$$\|\bar{p} - \bar{p}_h\|_{L^2(\Omega)} + \|\bar{\eta} - \bar{\eta}_h\|_{L^2(\Omega)} \lesssim h^{2\gamma} |\log h|^{2\varphi} + \|\bar{q} - \bar{q}_h\|_{L^2(\Omega)}, \quad (4.5.9)$$

where $\varphi = \nu$ if $s \neq \frac{1}{2}$, $\varphi = 1 + \nu$ if $s = \frac{1}{2}$, and $\nu \geq \frac{1}{2}$ is the constant in Theorem 4.3.1.

Proof. The error estimates for $\bar{p} - \bar{p}_h$ in $\tilde{H}^s(\Omega)$ and $L^2(\Omega)$ can be found in [90, Theorem 6.2, estimate (6.17)] and [90, Theorem 6.2, estimate (6.18)], respectively. It is worth noting that in the proof of [90, Theorem 6.2] it is assumed that $\partial L/\partial u$ is locally Lipschitz with respect to the second variable. This assumption can be removed at the expense of having the assumption on the second derivative in (C.1). It remains to prove the estimate for $\bar{\eta} - \bar{\eta}_h$ in $L^2(\Omega)$. To do this, we use the projection formulas for $\bar{\eta}$ and $\bar{\eta}_h$, which are given in (4.4.12) and (4.5.6), respectively, and arrive at

$$\|\bar{\eta} - \bar{\eta}_h\|_{L^2(\Omega)} \leq \mu^{-1} \|\bar{p} - \mathcal{P}_h \bar{p}_h\|_{L^2(\Omega)}.$$

A simple application of a triangle inequality thus results in $\|\bar{\eta} - \bar{\eta}_h\|_{L^2(\Omega)} \leq \mu^{-1} \|\bar{p} - \mathcal{P}_h \bar{p}\|_{L^2(\Omega)} + C\mu^{-1} \|\bar{p} - \bar{p}_h\|_{L^2(\Omega)}$ using the fact that \mathcal{P}_h is stable in $L^2(\Omega)$. The regularity property for \bar{p} derived in Theorem 4.4.7, namely, $\bar{p} \in H^{s+\kappa-\varepsilon}(\Omega)$, where κ and ε are the same as in the statement of Theorem 4.4.7, and a basic error estimate for the orthogonal projection \mathcal{P}_h yield

$$\|\bar{p} - \mathcal{P}_h \bar{p}\|_{L^2(\Omega)} \lesssim h^{2\gamma} |\log h|^\nu \Lambda(L, \bar{u}), \quad \gamma = \min\{s, \frac{1}{2}\}.$$

This and the derived error estimate for $\bar{p} - \bar{p}_h$ in $L^2(\Omega)$ allow us to complete the error estimate (4.5.9). This completes the proof. \square

4.5.1 Convergence of discretizations

We begin this section with a convergence result that guarantees that a sequence of discrete global solutions $\{\bar{q}_h\}_{h>0}$ contains subsequences that converge to global solutions of problem (4.4.1)–(4.4.2) as $h \rightarrow 0$.

Theorem 4.5.4 (convergence of discrete global solutions). *Let us assume that (A.1)–(A.3), (B.1)–(B.2), and (C.1) hold. Let $h > 0$ and let $\bar{q}_h \in \mathbb{Q}_{ad,h}$ be a global solution to the fully discrete control*

problem. Then, there exist nonrelabeled subsequences of $\{\bar{q}_h\}_{h>0}$ such that $\bar{q}_h \xrightarrow{*} \bar{q}$ in the weak* topology of $L^\infty(\Omega)$ as $h \rightarrow 0$ and \bar{q} corresponds to a global solution of the continuous control problem (4.4.1)–(4.4.2). Furthermore,

$$\lim_{h \rightarrow 0} \|\bar{q} - \bar{q}_h\|_{L^2(\Omega)} = 0, \quad \lim_{h \rightarrow 0} j_h(\bar{q}_h) = j(\bar{q}). \quad (4.5.10)$$

If, in addition, (4.5.7) holds, then

$$\|\bar{\eta} - \bar{\eta}_h\|_{L^2(\Omega)} \rightarrow 0 \quad (4.5.11)$$

as $h \rightarrow 0$.

Proof. Since $\{\bar{q}_h\}_{h>0} \subset \mathbb{Q}_{ad,h}$ is uniformly bounded in $L^\infty(\Omega)$, we can extract a nonrelabeled subsequence such that $\bar{q}_h \xrightarrow{*} \bar{q}$ in the weak* topology of $L^\infty(\Omega)$ as $h \rightarrow 0$. In the following, we prove that the limit \bar{q} is a global solution to (4.4.1)–(4.4.2).

Let $\bar{q} \in \mathbb{Q}_{ad}$ be a global solution to (4.4.1)–(4.4.2) and define $\mathbf{q}_h = \mathcal{P}_h \bar{q} \in \mathbb{Q}_h$. We note that according to the definition of \mathcal{P}_h , \mathbf{q}_h is such that $\mathbf{q}_h|_T = \int_T \bar{q}(x) dx / |T|$ for each $T \in \mathcal{T}_h$. This immediately guarantees that $\mathbf{q}_h \in \mathbb{Q}_{ad,h}$, because $\bar{q} \in \mathbb{Q}_{ad}$. We now take advantage of the fact that $\bar{q} \in H^{s+\kappa-\varepsilon}(\Omega)$, where κ and ε are as in the statement of Theorem 4.4.7, and a basic error estimate for \mathcal{P}_h to deduce that $\|\bar{q} - \mathbf{q}_h\|_{L^2(\Omega)} \rightarrow 0$ as $h \rightarrow 0$. We now use the global optimality of $\bar{q} \in \mathbb{Q}_{ad}$ in conjunction with the fact that $\bar{q} \in \mathbb{Q}_{ad}$, a convergence result based on Lemma 4.3.3 and the fact that $\bar{q}_h \xrightarrow{*} \bar{q}$ in the weak* topology of $L^\infty(\Omega)$ as $h \rightarrow 0$, the global discrete optimality of $\bar{q}_h \in \mathbb{Q}_{ad,h}$ combined with the fact that $\mathbf{q}_h \in \mathbb{Q}_{ad,h}$, and another convergence result based on Lemma 4.3.3 and the fact that $\|\bar{q} - \mathbf{q}_h\|_{L^2(\Omega)} \rightarrow 0$ as $h \rightarrow 0$ to obtain

$$j(\bar{q}) \leq j(\bar{q}) \leq \liminf_{h \rightarrow 0} j_h(\bar{q}_h) \leq \limsup_{h \rightarrow 0} j_h(\bar{q}_h) \leq \limsup_{h \rightarrow 0} j_h(\mathbf{q}_h) = j(\bar{q}).$$

As a result, we have obtained that $j(\bar{q}) \leq j(\bar{q}) \leq j(\bar{q})$, which guarantees the global optimality of \bar{q} . We also proved that $j_h(\bar{q}_h) \rightarrow j(\bar{q})$ as $h \rightarrow 0$.

We now prove that $\|\bar{q} - \bar{q}_h\|_{L^2(\Omega)} \rightarrow 0$ as $h \rightarrow 0$. Due to Lemma 4.3.3, we have that $\bar{u}_h = \mathcal{S}_h \bar{q}_h \rightarrow \bar{u} = \mathcal{S} \bar{q}$ in $L^\tau(\Omega)$ for every $\tau \leq 2d/(d-2s)$. From this follows $|\int_\Omega (L(x, \bar{u}_h) - L(x, \bar{u})) dx| \rightarrow 0$ as $h \rightarrow 0$. Since $j_h(\bar{q}_h) \rightarrow j(\bar{q})$ as $h \rightarrow 0$, we obtain

$$\frac{\lambda}{2} \|\bar{q}_h\|_{L^2(\Omega)}^2 + \mu \|\bar{q}_h\|_{L^1(\Omega)} \rightarrow \frac{\lambda}{2} \|\bar{q}\|_{L^2(\Omega)}^2 + \mu \|\bar{q}\|_{L^1(\Omega)}, \quad h \rightarrow 0. \quad (4.5.12)$$

Define $g = \text{sign}(\bar{q}) \in L^\infty(\Omega)$. Let $h > 0$. We note that, since $|\bar{q}_h| \geq \bar{q}_h$ in Ω , we have

$$\|\bar{q}_h\|_{L^1(\Omega)} - \int_{\Omega} \bar{q}_h g dx = \int_{\Omega_q^+} (|\bar{q}_h| - \bar{q}_h) dx + \int_{\Omega_q^-} (|\bar{q}_h| + \bar{q}_h) dx + \int_{\Omega_q^0} |\bar{q}_h| dx \geq 0.$$

This bound, the limit in (4.5.12), and the fact that $\bar{q}_h \xrightarrow{*} \bar{q}$ in the weak* topology of $L^\infty(\Omega)$ as $h \rightarrow 0$, allow us to obtain the following result:

$$\begin{aligned} 0 &\leq \lim_{h \rightarrow 0} \left[\frac{\lambda}{2} \|\bar{q}_h - \bar{q}\|_{L^2(\Omega)}^2 + \mu \left(\|\bar{q}_h\|_{L^1(\Omega)} - \int_{\Omega} \bar{q}_h g dx \right) \right] \\ &= \left(\frac{\lambda}{2} \|\bar{q}\|_{L^2(\Omega)}^2 + \mu \|\bar{q}\|_{L^1(\Omega)} \right) - \frac{\lambda}{2} \|\bar{q}\|_{L^2(\Omega)}^2 - \mu \|\bar{q}\|_{L^1(\Omega)} = 0. \end{aligned}$$

This immediately implies that $\|\bar{q}_h - \bar{q}\|_{L^2(\Omega)} \rightarrow 0$ as $h \rightarrow 0$.

The convergence of the term $\|\bar{\eta} - \bar{\eta}_h\|_{L^2(\Omega)}$ follows immediately from Theorem 4.5.3. This concludes the proof. \square

To present the next result, we introduce $B_\epsilon(\bar{q}) := \{q \in L^2(\Omega) : \|\bar{q} - q\|_{L^2(\Omega)} \leq \epsilon\}$.

Theorem 4.5.5 (convergence to a local solution). *Let the assumptions of Theorem 4.5.4 hold. If \bar{q} is a strict local minimum of the control problem (4.4.1)–(4.4.2), then there exist $h_\dagger > 0$ and a sequence $\{\bar{q}_h\}_{0 < h < h_\dagger}$ of local minima of the fully discrete control problems such that (4.5.10) and (4.5.11) hold.*

Proof. Since \bar{q} is a strict local minimum of (4.4.1)–(4.4.2), there exists $\epsilon > 0$ such that \bar{q} is the unique solution to the following problem: Find $\min\{j(q) : q \in \mathbb{Q}_{ad} \cap B_\epsilon(\bar{q})\}$.

We now introduce the following discrete problem for each $h > 0$: Find $\min\{j_h(q_h) : q_h \in \mathbb{Q}_{ad,h} \cap B_\epsilon(\bar{q})\}$.

We note that there exists $h_\star > 0$ so that for each $h \in (0, h_\star)$ the discrete function $\mathcal{P}_h \bar{q}$ belongs to $\mathbb{Q}_{ad,h} \cap B_\epsilon(\bar{q})$. Consequently, $\mathbb{Q}_{ad,h} \cap B_\epsilon(\bar{q})$ is not empty and the previously introduced discrete problem admits a solution.

Let $h \in (0, h_\star)$ and let $\bar{q}_h \in \mathbb{Q}_{ad,h} \cap B_\epsilon(\bar{q})$ be a global solution of the aforementioned discrete problem. The arguments elaborated in the proof of Theorem 4.5.4 show the existence of a nonrelabeled subsequence of $\{\bar{q}_h\}_{0 < h < h_\star}$ such that it converges strongly in $L^2(\Omega)$ as $h \rightarrow 0$ to a global solution of the following problem: Find $\min\{j(q) : q \in \mathbb{Q}_{ad} \cap B_\epsilon(\bar{q})\}$. As mentioned at the beginning of the proof, this problem admits a unique solution \bar{q} . Consequently, the whole sequence $\{\bar{q}_h\}_{0 < h < h_\star}$ must converge to \bar{q} in $L^2(\Omega)$ as $h \rightarrow 0$. As a result, there exists $h_\dagger \in (0, h_\star)$ such that the constraint $\bar{q}_h \in \mathbb{Q}_{ad,h} \cap B_\epsilon(\bar{q})$ is not active for $h \leq h_\dagger$, i.e., \bar{q}_h solves the fully discrete scheme. This concludes the proof. \square

4.5.2 Error estimates

This section is dedicated to the derivation of error estimates. For this purpose, we assume that $d \in \{2, 3\}$ and $s > d/2$ and we let $\{\bar{q}_h\}_{h>0} \subset \mathbb{Q}_{ad,h}$ be a sequence of local minima of the fully discrete optimal control problems such that $\|\bar{q} - \bar{q}_h\|_{L^2(\Omega)} \rightarrow 0$ as $h \rightarrow 0$, where \bar{q} corresponds to a local solution of (4.4.1)–(4.4.2); see Theorems 4.5.4 and 4.5.5. The main goal of this section is to obtain an error estimate for $\bar{q} - \bar{q}_h$ in $L^2(\Omega)$, namely

$$\|\bar{q} - \bar{q}_h\|_{L^2(\Omega)} \lesssim h^{2\gamma} |\log h|^{2\varphi} \quad \forall h \leq h_\square, \quad \gamma = \min\left\{s, \frac{1}{2}\right\}, \quad (4.5.13)$$

where $\varphi = \nu$ if $s \neq \frac{1}{2}$, $\varphi = 1 + \nu$ if $s = \frac{1}{2}$, $\nu \geq \frac{1}{2}$ is as in Theorem 4.3.1, and $h_\square > 0$ is the constant in Theorem 4.5.6 below.

The following result is helpful to obtain (4.5.13).

Theorem 4.5.6 (instrumental error estimate). *Let us assume that (A.1)–(A.3), (B.1)–(B.2), (C.1), and (4.5.7) hold. Let $\bar{q} \in \mathbb{Q}_{ad}$ satisfy the second-order optimality condition (i), or equivalently (ii) in Theorem 4.4.5. If (4.5.13) is false, then there exists $h_\square > 0$ such that*

$$\mathfrak{C} \|\bar{q} - \bar{q}_h\|_{L^2(\Omega)}^2 \leq [F'(\bar{q}_h) - F'(\bar{q})](\bar{q}_h - \bar{q}) \quad \forall h \in (0, h_\square], \quad (4.5.14)$$

where $\mathfrak{C} := 2^{-1}\delta$ and δ is the constant that appears in the item (ii) of Theorem 4.4.5.

Proof. In a first step, we invoke the C^2 regularity of F in $L^2(\Omega) \cap L^r(\Omega)$, with $r > d/2s$ (cf. Proposition 4.4.2), the $L^2(\Omega)$ -convergence of \bar{q}_h to \bar{q} as $h \rightarrow 0$, and the mean value theorem to conclude the existence of $\epsilon > 0$ and $h_\epsilon > 0$ such that

$$[F'(\bar{q}_h) - F'(\bar{q})](\bar{q}_h - \bar{q}) \geq F''(\bar{q})(\bar{q}_h - \bar{q})^2 - \frac{\delta}{2} \|\bar{q}_h - \bar{q}\|_{L^2(\Omega)}^2 \quad \forall h \leq h_\epsilon.$$

It therefore suffices to prove that $\bar{q}_h - \bar{q} \in C_{\bar{q}}^\tau$ for some $\tau > 0$ and for every $h \leq h_\tau$, where $h_\tau > 0$, to apply item (ii) of Theorem 4.4.5 and deduce (4.5.14) with $h_\square := \min\{h_\tau, h_\epsilon\}$. Therefore, the rest of the proof is devoted to proving that $\bar{q}_h - \bar{q} \in C_{\bar{q}}^\tau$.

Since (4.5.13) is false, there are sequences $\{h_\ell\}_{\ell=1}^\infty$ and $\{\bar{q}_{h_\ell}\}_{\ell=1}^\infty$ such that $h_\ell \rightarrow 0$ as $\ell \uparrow \infty$ and

$$(h_\ell^{2\gamma} |\log h_\ell|^{2\varphi})^{-1} \|\bar{q} - \bar{q}_{h_\ell}\|_{L^2(\Omega)} \rightarrow \infty$$

as $h_\ell \rightarrow 0$. In the following, we omit the subindex ℓ to simplify notation. For each $h > 0$, we define the function $v_h := (\bar{q}_h - \bar{q}) / \|\bar{q}_h - \bar{q}\|_{L^2(\Omega)}$. Since $\{v_h\}_{h>0}$ is uniformly bounded in $L^2(\Omega)$, there exists a nonrelabeled subsequence such that $v_h \rightharpoonup v$ in $L^2(\Omega)$ as $h \rightarrow 0$. We now prove the existence of $\tau > 0$

and $h_\tau > 0$ such that $v_h \in C_q^\tau$ for every $h \leq h_\tau$. The fact that every v_h satisfies the sign conditions (4.4.13) is trivial. It is therefore sufficient to prove that

$$F'(\bar{q})v_h + \mu j'(\bar{q}; v_h) \leq \tau \quad \forall h \leq h_\tau. \quad (4.5.15)$$

Since [26, Lemma 3.5] guarantees that $F'(\bar{q})v_h + \mu j'(\bar{q}; v_h) \geq 0$, which holds for every $h > 0$ because $v_h \in L^2(\Omega)$ satisfies (4.4.13), we will obtain (4.5.15) with the help of the limit $F'(\bar{q})v_h + \mu j'(\bar{q}; v_h) \rightarrow 0$ as $h \rightarrow 0$. This will be now the focus of the proof.

Let us first note that the arguments developed in the proof of [89, Theorem 7.4] based on the weak convergence $v_h \rightharpoonup v$ in $L^2(\Omega)$ as $h \rightarrow 0$ show that

$$\left\{ F'(\bar{q})v + \mu \int_{\Omega} \bar{\eta} v dx \right\} = \lim_{h \rightarrow 0} \left\{ F'(\bar{q}_h)v_h + \mu \int_{\Omega} \bar{\eta}_h v_h dx \right\} \leq 0. \quad (4.5.16)$$

This is the place where we use the assumption that (4.5.13) is false; see the proof of [89, Theorem 7.4, page 22] for details. On the other hand, $F'(\bar{q})v + \mu \int_{\Omega} \bar{\eta} v dx = \int_{\Omega} [\bar{p} + \lambda \bar{q} + \mu \bar{\eta}] v dx \geq 0$ as a consequence of the variational inequality (4.4.9) and the fact that v satisfies (4.4.13). The relation $F'(\bar{q})v + \mu \int_{\Omega} \bar{\eta} v dx = 0$ can be derived from this.

We now prove that $j'(\bar{q}; v_h) \rightarrow \int_{\Omega} \bar{\eta} v dx$ as $h \rightarrow 0$; recall that $j'(\bar{q}; \cdot)$ is defined in (4.2.2). To do this, we first note that $v_h \rightharpoonup v$ in $L^2(\Omega)$ as $h \rightarrow 0$ yields

$$\lim_{h \rightarrow 0} \left\{ \int_{\Omega_q^+} v_h dx - \int_{\Omega_q^-} v_h dx \right\} = \int_{\Omega_q^+} v dx - \int_{\Omega_q^-} v dx. \quad (4.5.17)$$

We now study $\int_{\Omega_q^0} |v_h| dx$, where $\Omega_q^0 = \{x \in \Omega : \bar{q}(x) = 0\}$. In view of (4.4.11), the set Ω_q^0 can be rewritten as $\Omega_q^0 = \{x \in \Omega : |\bar{p}(x)| \leq \mu\}$. We decompose this set as follows:

$$\begin{aligned} \Omega_q^0 &= \Omega_\mu^+ \cup \Omega_\mu^- \cup \Omega_\mu^{\text{less}}, & \Omega_\mu^+ &:= \{x \in \Omega : \bar{p}(x) = \mu\}, \\ \Omega_\mu^- &:= \{x \in \Omega : \bar{p}(x) = -\mu\}, & \Omega_\mu^{\text{less}} &:= \{x \in \Omega : |\bar{p}(x)| < \mu\}. \end{aligned}$$

With these sets at hand, we can write the integral $\int_{\Omega_q^0} |v_h| dx$ as follows:

$$\int_{\Omega_q^0} |v_h| dx = \int_{\Omega_\mu^{\text{less}}} |v_h| dx + \int_{\Omega_\mu^+} |v_h| dx + \int_{\Omega_\mu^-} |v_h| dx := \mathfrak{J}_h + \mathfrak{K}_h + \mathfrak{L}_h. \quad (4.5.18)$$

In the following, we proceed in three steps to examine the three preceding terms.

Step 1. We study the limit value of \mathfrak{J}_h as $h \rightarrow 0$. For this purpose, for each $h > 0$ and $T \in \mathcal{T}_h$, we

introduce the average $\bar{p}_{h,T} := \int_T \bar{p}_h dx / |T|$. We also define

$$\mathcal{T}_{1,h} := \{T \in \mathcal{T}_h : |\bar{p}_{h,T}| \leq \mu\}, \quad \Omega_{1,h} := \Omega_\mu^{\text{less}} \cap \mathcal{T}_{1,h}, \quad (4.5.19)$$

$$\mathcal{T}_{2,h} := \{T \in \mathcal{T}_h : |\bar{p}_{h,T}| > \mu\}, \quad \Omega_{2,h} := \Omega_\mu^{\text{less}} \cap \mathcal{T}_{2,h}. \quad (4.5.20)$$

In an abuse of notation, here and in what follows, by $\mathcal{T}_{1,h}$ and $\mathcal{T}_{2,h}$, we will indistinctively denote either these sets or the union of the triangles that comprise them. *Step 2.* We now study the limit value of \mathfrak{K}_h as $h \rightarrow 0$. To do this, we define

$$\mathcal{T}_{3,h} := \{T \in \mathcal{T}_h : \bar{p}_{h,T} < -\mu\}, \quad \Omega_{3,h} := \Omega_\mu^+ \cap \mathcal{T}_{3,h},$$

$$\mathcal{T}_{4,h} := \{T \in \mathcal{T}_h : \bar{p}_{h,T} > +\mu\}, \quad \Omega_{4,h} := \Omega_\mu^+ \cap \mathcal{T}_{4,h},$$

and $\Omega_{5,h} := \Omega_\mu^+ \cap \mathcal{T}_{1,h}$, where $\mathcal{T}_{1,h}$ is defined in (4.5.19). If we proceed as at the beginning of *Step 1*, we can deduce that $v_h(x) = 0$ for a.e. $x \in \Omega_{5,h}$. Consequently,

$$\mathfrak{K}_h = \int_{\Omega_\mu^+} |v_h| dx = \int_{\Omega_{3,h}} |v_h| dx + \int_{\Omega_{4,h}} |v_h| dx.$$

Let $T \in \mathcal{T}_{3,h}$. By definition, $\bar{p}_{h,T} < -\mu$. Recall that $\bar{p}(x) = \mu$ for a.e. $x \in \Omega_\mu^+$. Thus,

$$0 \leq 4\mu^2 |T \cap \Omega_\mu^+| = \int_{T \cap \Omega_\mu^+} (\mu - (-\mu))^2 dx < \int_T (\bar{p} - \mathcal{P}_h \bar{p}_h)^2 dx = \|\bar{p} - \mathcal{P}_h \bar{p}_h\|_{L^2(T)}^2.$$

Summing over all elements T in $\mathcal{T}_{3,h}$ we obtain $|\Omega_{3,h}| \rightarrow 0$ as $h \rightarrow 0$. To complete the proof of this step, we let $T \in \mathcal{T}_{4,h}$. Note that $\bar{p}_{h,T} > \mu$. Using the projection formulas (4.5.4) and (4.5.6), we can deduce that $\bar{q}_h|_T < 0$. This implies that $v_h(x) < 0$ for a.e. $x \in T \cap \Omega_\mu^+$. As a result, $|v_h(x)| = -v_h(x)$ for a.e. $x \in T \cap \Omega_\mu^+$. Thus,

$$\begin{aligned} \lim_{h \rightarrow 0} \mathfrak{K}_h &= - \lim_{h \rightarrow 0} \int_{\Omega_{4,h}} v_h dx = - \lim_{h \rightarrow 0} \int_{\Omega_{4,h}} v_h dx - \lim_{h \rightarrow 0} \int_{\Omega_{3,h}} v_h dx \\ &\quad - \lim_{h \rightarrow 0} \int_{\Omega_{5,h}} v_h dx = - \lim_{h \rightarrow 0} \int_{\Omega_\mu^+} v_h dx = - \int_{\Omega_\mu^+} v dx. \end{aligned} \quad (4.5.21)$$

To obtain the first relation, we used that $\int_{\Omega_{3,h}} |v_h| dx \rightarrow 0$ as $h \rightarrow 0$. This follows from $|\Omega_{3,h}| \rightarrow 0$ as $h \rightarrow 0$ and $\|v_h\|_{L^2(\Omega)} = 1$ for each $h > 0$. The second relation follows from the fact that $\int_{\Omega_{3,h}} v_h dx \rightarrow 0$ as $h \rightarrow 0$ and $\int_{\Omega_{5,h}} v_h dx = 0$ for each $h > 0$. The third relation is a consequence of $\Omega_\mu^+ = \Omega_{3,h} \cup \Omega_{4,h} \cup \Omega_{5,h}$. Finally, the last relation follows from the use of the weak convergence $v_h \rightharpoonup v$

in $L^2(\Omega)$ as $h \rightarrow 0$. We now note that for each $T \in \mathcal{T}_{1,h}$, $\bar{q}_h|_T = 0$ as a consequence of (4.5.5). Thus,

$$v_h(x)|_T = \frac{\bar{q}_h(x) - \bar{q}(x)}{\|\bar{q}_h - \bar{q}\|_{L^2(\Omega)}} \Big|_T = \frac{-\bar{q}(x)}{\|\bar{q}_h - \bar{q}\|_{L^2(\Omega)}} \Big|_T = 0$$

for a.e. $x \in \Omega_\mu^{\text{less}} \cap T$ and for each $T \in \mathcal{T}_{1,h}$ due to (4.4.11). As a result, we obtain

$$\int_{\Omega_{1,h}} |v_h| dx = \sum_{T \in \mathcal{T}_{1,h}} \int_{T \cap \Omega_\mu^{\text{less}}} |v_h| dx = 0,$$

which implies that

$$\mathfrak{J}_h = \int_{\Omega_\mu^{\text{less}}} |v_h| dx = \int_{\Omega_{2,h}} |v_h| dx.$$

We now bound $\int_{\Omega_{2,h}} |v_h| dx$. For this purpose we let $T \in \mathcal{T}_{2,h}$ and note that

$$0 < |\mu - |\bar{p}(x)|| < \|\bar{p}_{h,T}\| - |\bar{p}(x)|$$

for a.e. $x \in \Omega_\mu^{\text{less}} \cap T$. Integrating over $\Omega_\mu^{\text{less}} \cap T$ results in

$$0 \leq \int_{T \cap \Omega_\mu^{\text{less}}} |\mu - |\bar{p}||^2 dx < \int_T \|\mathcal{P}_h \bar{p}_h\| - |\bar{p}|^2 dx \leq \|\mathcal{P}_h \bar{p}_h - \bar{p}\|_{L^2(T)}^2.$$

Summing over all elements T in $\mathcal{T}_{2,h}$ and using the arguments from the proof of Theorem 4.5.3 which guarantee that $\|\mathcal{P}_h \bar{p}_h - \bar{p}\|_{L^2(\Omega)} \rightarrow 0$ as $h \rightarrow 0$, it follows that $|\Omega_{2,h}| \rightarrow 0$ as $h \rightarrow 0$. This implies that

$$\lim_{h \rightarrow 0} \mathfrak{J}_h = 0. \quad (4.5.22)$$

Step 2. We now study the limit value of \mathfrak{K}_h as $h \rightarrow 0$. To do this, we define

$$\begin{aligned} \mathcal{T}_{3,h} &:= \{T \in \mathcal{T}_h : \bar{p}_{h,T} < -\mu\}, & \Omega_{3,h} &:= \Omega_\mu^+ \cap \mathcal{T}_{3,h}, \\ \mathcal{T}_{4,h} &:= \{T \in \mathcal{T}_h : \bar{p}_{h,T} > +\mu\}, & \Omega_{4,h} &:= \Omega_\mu^+ \cap \mathcal{T}_{4,h}, \end{aligned}$$

and $\Omega_{5,h} := \Omega_\mu^+ \cap \mathcal{T}_{1,h}$, where $\mathcal{T}_{1,h}$ is defined in (4.5.19). If we proceed as at the beginning of *Step 1*, we can deduce that $v_h(x) = 0$ for a.e. $x \in \Omega_{5,h}$. Consequently,

$$\mathfrak{K}_h = \int_{\Omega_\mu^+} |v_h| dx = \int_{\Omega_{3,h}} |v_h| dx + \int_{\Omega_{4,h}} |v_h| dx.$$

Let $T \in \mathcal{T}_{3,h}$. By definition, $\bar{p}_{h,T} < -\mu$. Recall that $\bar{p}(x) = \mu$ for a.e. $x \in \Omega_\mu^+$. Thus,

$$0 \leq 4\mu^2 |T \cap \Omega_\mu^+| = \int_{T \cap \Omega_\mu^+} (\mu - (-\mu))^2 dx < \int_T (\bar{p} - \mathcal{P}_h \bar{p}_h)^2 dx = \|\bar{p} - \mathcal{P}_h \bar{p}_h\|_{L^2(T)}^2.$$

Summing over all elements T in $\mathcal{T}_{3,h}$ we obtain $|\Omega_{3,h}| \rightarrow 0$ as $h \rightarrow 0$. To complete the proof of this step, we let $T \in \mathcal{T}_{4,h}$. Note that $\bar{p}_{h,T} > \mu$. Using the projection formulas (4.5.4) and (4.5.6), we can deduce that $\bar{q}_h|_T < 0$. This implies that $v_h(x) < 0$ for a.e. $x \in T \cap \Omega_\mu^+$. As a result, $|v_h(x)| = -v_h(x)$ for a.e. $x \in T \cap \Omega_\mu^+$. Thus,

$$\begin{aligned} \lim_{h \rightarrow 0} \mathfrak{K}_h &= - \lim_{h \rightarrow 0} \int_{\Omega_{4,h}} v_h dx = - \lim_{h \rightarrow 0} \int_{\Omega_{4,h}} v_h dx - \lim_{h \rightarrow 0} \int_{\Omega_{3,h}} v_h dx \\ &\quad - \lim_{h \rightarrow 0} \int_{\Omega_{5,h}} v_h dx = - \lim_{h \rightarrow 0} \int_{\Omega_\mu^+} v_h dx = - \int_{\Omega_\mu^+} v dx. \end{aligned} \quad (4.5.23)$$

To obtain the first relation, we used that $\int_{\Omega_{3,h}} |v_h| dx \rightarrow 0$ as $h \rightarrow 0$. This follows from $|\Omega_{3,h}| \rightarrow 0$ as $h \rightarrow 0$ and $\|v_h\|_{L^2(\Omega)} = 1$ for each $h > 0$. The second relation follows from the fact that $\int_{\Omega_{3,h}} v_h dx \rightarrow 0$ as $h \rightarrow 0$ and $\int_{\Omega_{5,h}} v_h dx = 0$ for each $h > 0$. The third relation is a consequence of $\Omega_\mu^+ = \Omega_{3,h} \cup \Omega_{4,h} \cup \Omega_{5,h}$. Finally, the last relation follows from the use of the weak convergence $v_h \rightharpoonup v$ in $L^2(\Omega)$ as $h \rightarrow 0$.

Step 3. Using arguments similar to those in *Step 2* we arrive at

$$\lim_{h \rightarrow 0} \mathfrak{L}_h = \int_{\Omega_\mu^-} v dx. \quad (4.5.24)$$

Collecting (4.5.18), (4.5.22), (4.5.23), and (4.5.24) we conclude that

$$\lim_{h \rightarrow 0} \int_{\Omega_\mu^0} |v_h| dx = - \int_{\Omega_\mu^+} v dx + \int_{\Omega_\mu^-} v dx. \quad (4.5.25)$$

After we have proved (4.5.25) we use (4.5.17) and the fact that $\bar{\eta}(x) = 1$ for a.e. $x \in \Omega_{\bar{q}}^+$ and $x \in \Omega_\mu^-$ and that $\bar{\eta}(x) = -1$ for a.e. $x \in \Omega_{\bar{q}}^-$ and $x \in \Omega_\mu^+$ to deduce that

$$\begin{aligned} \lim_{h \rightarrow 0} j'(\bar{q}; v_h) &= \int_{\Omega_{\bar{q}}^+} v dx - \int_{\Omega_{\bar{q}}^-} v dx - \int_{\Omega_\mu^+} v dx + \int_{\Omega_\mu^-} v dx \\ &= \int_{\Omega_{\bar{q}}^+} \bar{\eta} v dx + \int_{\Omega_{\bar{q}}^-} \bar{\eta} v dx + \int_{\Omega_\mu^+} \bar{\eta} v dx + \int_{\Omega_\mu^-} \bar{\eta} v dx = \int_{\Omega} \bar{\eta} v dx. \end{aligned} \quad (4.5.26)$$

To obtain the last equality we have used that $\|v_h\|_{L^1(\Omega_\mu^{\text{less}})} \rightarrow 0$, which follows from (4.5.22), and that $v_h \rightharpoonup v$ in $L^2(\Omega)$ as $h \rightarrow 0$ to conclude that $v = 0$ a.e. in Ω_μ^{less} .

The limit (4.5.26) and the relation (4.5.16) allows us to conclude that

$$\lim_{h \rightarrow 0} (F'(\bar{q})v_h + \mu j'(\bar{q}; v_h)) = 0.$$

Since $F'(\bar{q})v_h + \mu j'(\bar{q}; v_h) \geq 0$ for each $h > 0$, we deduce the existence of $\tau > 0$ and $h_\tau > 0$ such that $F'(\bar{q})v_h + \mu j'(\bar{q}; v_h) \leq \tau$ for all $h \leq h_\tau$. This concludes the proof. \square

We now provide a proof for the main result of this section.

Theorem 4.5.7 (error bound for the approximation of an optimal control). *Let us assume that (A.1)–(A.3), (B.1)–(B.2), (C.1), and (4.5.7) hold. Let $\bar{q} \in \mathbb{Q}_{\text{ad}}$ satisfy the second-order optimality condition (i), or equivalently (ii) in Theorem 4.4.5. Then, there is $h_\square > 0$ so that the estimate (4.5.13) holds.*

Proof. We proceed by contradiction: If we assume that (4.5.13) is false, then there exists $h_\square > 0$ such that the estimate (4.5.14) of Theorem 4.5.6 holds for every $h \in (0, h_\square]$. Based on the instrumental error estimate (4.5.14), we use the continuous and discrete optimality conditions, (4.4.9) and (4.5.2), respectively, to obtain [26, ineq. (4.14)]

$$\begin{aligned} \mathfrak{C} \|\bar{q} - \bar{q}_h\|_{L^2(\Omega)}^2 &\leq [F'_h(\bar{q}_h) - F'(\bar{q}_h)](\bar{q} - \bar{q}_h) + [F'_h(\bar{q}_h) - F'(\bar{q})](\mathcal{P}_h \bar{q} - \bar{q}) \\ &\quad + \left[F'(\bar{q})(\mathcal{P}_h \bar{q} - \bar{q}) + \mu \int_{\Omega} \bar{\eta}(\mathcal{P}_h \bar{q} - \bar{q}) dx \right] + \mu \int_{\Omega} (\bar{\eta} - \bar{\eta}_h)(\bar{q}_h - \bar{q}) dx \\ &\quad + \mu \int_{\Omega} (\bar{\eta}_h - \bar{\eta})(\mathcal{P}_h \bar{q} - \bar{q}) dx := \mathbf{I}_h + \mathbf{II}_h + \mathbf{III}_h + \mathbf{IV}_h + \mathbf{V}_h, \quad \forall h \leq h_\square. \end{aligned}$$

In the following, we proceed in several steps and estimate each of the terms \mathbf{I}_h , \mathbf{II}_h , \mathbf{III}_h , \mathbf{IV}_h , and \mathbf{V}_h individually.

Step 1. We first control the term \mathbf{IV}_h . To do this, we first use that $\bar{\eta} \in \partial j(\bar{q})$ and the definition of the subgradient given in (4.2.1) to obtain

$$\int_{\Omega} \bar{\eta}(\bar{q}_h - \bar{q}) dx \leq \|\bar{q}_h\|_{L^1(\Omega)} - \|\bar{q}\|_{L^1(\Omega)}.$$

Since $\bar{\eta}_h \in \partial j_h(\bar{q}_h)$, the characterization in [26, eq. (4.3)] leads to the conclusion that

$$\int_{\Omega} \bar{\eta}_h(\bar{q} - \bar{q}_h) dx = \int_{\Omega} \bar{\eta}_h \bar{q} dx - \|\bar{q}_h\|_{L^1(\Omega)} \leq \|\bar{q}\|_{L^1(\Omega)} - \|\bar{q}_h\|_{L^1(\Omega)}.$$

Consequently, we can control the term \mathbf{IV}_h as follows:

$$\mathbf{IV}_h \leq \mu (\|\bar{q}_h\|_{L^1(\Omega)} - \|\bar{q}\|_{L^1(\Omega)} + \|\bar{q}\|_{L^1(\Omega)} - \|\bar{q}_h\|_{L^1(\Omega)}) = 0. \quad (4.5.27)$$

Step 2. We estimate \mathbf{III}_h . To do so, we use the characterization of $F'(\bar{q})$ described in (4.4.7) and standard properties for the orthogonal projection operator \mathcal{P}_h to obtain

$$\begin{aligned} \mathbf{III}_h &= \int_{\Omega} (\bar{p} + \lambda\bar{q} + \mu\bar{\eta} - \mathcal{P}_h(\bar{p} + \lambda\bar{q} + \mu\bar{\eta}))(\mathcal{P}_h\bar{q} - \bar{q})dx \\ &\leq (\|\bar{p} - \mathcal{P}_h\bar{p}\|_{L^2(\Omega)} + \mu\|\bar{\eta} - \mathcal{P}_h\bar{\eta}\|_{L^2(\Omega)}) \|\bar{q} - \mathcal{P}_h\bar{q}\|_{L^2(\Omega)}. \end{aligned} \quad (4.5.28)$$

Here, $\bar{p} \in \tilde{H}^s(\Omega) \cap L^\infty(\Omega)$ denotes the solution to (4.4.6) with u replaced by $\bar{u} = \mathcal{S}\bar{q}$. The control of the error $\|\bar{p} - \mathcal{P}_h\bar{p}\|_{L^2(\Omega)}$ follows from a standard error estimate for \mathcal{P}_h in conjunction with the regularity property $\bar{p} \in H^{s+\kappa-\varepsilon}(\Omega)$ for all $\varepsilon \in (0, s)$, where $\kappa = \frac{1}{2}$ for $\frac{1}{2} < s < 1$ and $\kappa = s - \varepsilon$ for $0 < s \leq \frac{1}{2}$ (see Theorem 4.4.7). In fact, we have

$$\|\bar{p} - \mathcal{P}_h\bar{p}\|_{L^2(\Omega)} \lesssim h^{2\gamma} |\log h|^\nu \Lambda(L, \bar{u}), \quad \gamma = \min\{s, \frac{1}{2}\}, \quad (4.5.29)$$

where $\nu = \frac{1}{2}$ for $\frac{1}{2} < s < 1$ and $\nu = \frac{1}{2} + \nu_0$ for $0 < s \leq \frac{1}{2}$, with $\nu_0 = \nu_0(\Omega, d) > 0$. The term $\Lambda(L, \bar{u})$ is defined in (4.4.16). Similarly, we have

$$\|\bar{q} - \mathcal{P}_h\bar{q}\|_{L^2(\Omega)} + \|\bar{\eta} - \mathcal{P}_h\bar{\eta}\|_{L^2(\Omega)} \lesssim h^{2\gamma} |\log h|^\nu [1 + \Lambda(L, \bar{u})]. \quad (4.5.30)$$

If we replace the estimates obtained in (4.5.29) and (4.5.30) into (4.5.28), we obtain

$$\mathbf{III}_h \lesssim h^{4\gamma} |\log h|^{2\nu} [1 + \Lambda(L, \bar{u})]^2. \quad (4.5.31)$$

Step 3. An estimate for the term \mathbf{V}_h follows easily from the definition of \mathcal{P}_h and the estimate (4.5.30) derived in the previous step:

$$\mathbf{V}_h = \mu \int_{\Omega} (\mathcal{P}_h\bar{\eta} - \bar{\eta})(\mathcal{P}_h\bar{q} - \bar{q})dx \lesssim h^{4\gamma} |\log h|^{2\nu} [1 + \Lambda(L, \bar{u})]^2. \quad (4.5.32)$$

Step 4. The aim of this step is to estimate the term \mathbf{I}_h . To accomplish this task, we introduce the variables $\hat{u} \in \tilde{H}^s(\Omega)$ and $\hat{p} \in \tilde{H}^s(\Omega)$, which solve, respectively,

$$\mathcal{A}(\hat{u}, v) + \int_{\Omega} a(x, \hat{u})vdx = \int_{\Omega} \bar{q}_h vdx \quad \forall v \in \tilde{H}^s(\Omega)$$

and

$$\mathcal{A}(\hat{p}, v) + \int_{\Omega} \frac{\partial a}{\partial u}(x, \hat{u})\hat{p}vdx = \int_{\Omega} \frac{\partial L}{\partial u}(x, \hat{u})vdx \quad \forall v \in \tilde{H}^s(\Omega).$$

With these variables, we can rewrite and estimate the term \mathbf{I}_h as follows:

$$\mathbf{I}_h = \int_{\Omega} [(\bar{p}_h + \lambda \bar{q}_h) - (\hat{p} + \lambda \bar{q}_h)] (\bar{q} - \bar{q}_h) dx \leq \frac{1}{\mathfrak{C}} \|\bar{p}_h - \hat{p}\|_{L^2(\Omega)}^2 + \frac{\mathfrak{C}}{4} \|\bar{q} - \bar{q}_h\|_{L^2(\Omega)}^2. \quad (4.5.33)$$

To obtain the last inequality, we used Young's inequality. Here, $\mathfrak{C} = 2^{-1}\delta$ is the constant that appears in Theorem 4.5.6.

The rest of this step is dedicated to bound the term $\|\bar{p}_h - \hat{p}\|_{L^2(\Omega)}$. For this purpose, we introduce \tilde{p} as the solution to: Find $\tilde{p} \in \tilde{H}^s(\Omega)$ such that

$$\mathcal{A}(\tilde{p}, v) + \int_{\Omega} \frac{\partial a}{\partial u}(x, \bar{u}_h) \tilde{p} v dx = \int_{\Omega} \frac{\partial L}{\partial u}(x, \bar{u}_h) v dx \quad \forall v \in \tilde{H}^s(\Omega).$$

We note that, in view of (4.5.7) and the assumptions on a and L all terms in this weak formulation are well-posed. With the variable \tilde{p} at hand, the triangle inequality in $L^2(\Omega)$ yields $\|\bar{p}_h - \hat{p}\|_{L^2(\Omega)} \leq \|\bar{p}_h - \tilde{p}\|_{L^2(\Omega)} + \|\tilde{p} - \hat{p}\|_{L^2(\Omega)}$. To bound $\|\bar{p}_h - \tilde{p}\|_{L^2(\Omega)}$, we use that \bar{p}_h corresponds to the finite element approximation of \tilde{p} within \mathbb{V}_h . Indeed, an application of a suitable modification of Theorem [90, Theorem 6.1] yields

$$\|\bar{p}_h - \tilde{p}\|_{L^2(\Omega)} \lesssim h^{2\gamma} |\log h|^{2\varphi} \Lambda(L, \bar{u}_h). \quad (4.5.34)$$

We note that the same arguments we used to derive the regularity results for \bar{p} in Theorem 4.4.7 also apply to \tilde{p} with a similar estimate. It remains to bound the term $\|\tilde{p} - \hat{p}\|_{L^2(\Omega)}$. To do this, we first note that $\tilde{p} - \hat{p}$ is such that

$$\begin{aligned} \tilde{p} - \hat{p} \in \tilde{H}^s(\Omega) : \quad \mathcal{A}(\tilde{p} - \hat{p}, v) + \int_{\Omega} \frac{\partial a}{\partial u}(x, \hat{u})(\tilde{p} - \hat{p}) v dx \\ = \int_{\Omega} \frac{\partial^2 L}{\partial u^2}(x, u_{\theta})(\bar{u}_h - \hat{u}) v dx + \int_{\Omega} \frac{\partial^2 a}{\partial u^2}(x, u_{\theta})(\hat{u} - \bar{u}_h) \tilde{p} v dx \end{aligned} \quad (4.5.35)$$

for all $v \in \tilde{H}^s(\Omega)$, where $u_{\theta} := \hat{u} + \theta_h(\bar{u}_h - \hat{u})$ and $u_{\vartheta} := \bar{u}_h + \vartheta_h(\hat{u} - \bar{u}_h)$ with $\theta_h, \vartheta_h \in (0, 1)$. Given the assumptions (C.1) and (4.5.7), the $L^\infty(\Omega)$ -regularity of \tilde{p} , and Hölder's inequality we have that all terms in (4.5.35) are well-defined. We now invoke a stability estimate for problem (4.5.35), assumptions (A.3) and (C.1), the $L^\infty(\Omega)$ -regularity of \tilde{p} , and the embedding $H^s(\Omega) \hookrightarrow L^\tau(\Omega)$, which holds for every $\tau \leq 2d/(d - 2s)$, to derive the estimates

$$\begin{aligned} \|\tilde{p} - \hat{p}\|_{L^2(\Omega)} &\lesssim \|\tilde{p} - \hat{p}\|_s \leq \left\| \left(\frac{\partial^2 L}{\partial u^2}(\cdot, u_{\theta}) - \frac{\partial^2 a}{\partial u^2}(\cdot, u_{\vartheta}) \tilde{p} \right) (\bar{u}_h - \hat{u}) \right\|_{H^{-s}(\Omega)} \\ &\lesssim \left(\left\| \frac{\partial^2 L}{\partial u^2}(\cdot, u_{\theta}) \right\|_{L^{\frac{d}{s}}(\Omega)} + \|\tilde{p}\|_{L^\infty(\Omega)} \right) \|\bar{u}_h - \hat{u}\|_{L^2(\Omega)} \\ &\lesssim h^{2\gamma} |\log h|^{2\varphi} \|\bar{q}_h - a(\cdot, 0)\|_{L^2(\Omega)}. \end{aligned} \quad (4.5.36)$$

To obtain the last estimate, we used the fact that \bar{u}_h corresponds to the finite element approximation of \hat{u} within \mathbb{V}_h and Theorem 4.3.2. If we replace the estimates (4.5.34) and (4.5.36) in (4.5.33), we obtain that

$$\mathbf{I}_h \leq Ch^{4\gamma} |\log h|^{4\varphi} (\|\bar{q}_h - a(\cdot, 0)\|_{L^2(\Omega)} + \Lambda(L, \bar{u}_h))^2 + \frac{\mathfrak{C}}{4} \|\bar{q} - \bar{q}_h\|_{L^2(\Omega)}^2. \quad (4.5.37)$$

Step 5. In this step, we bound \mathbf{II}_h . To do this, we use the definition of the orthogonal projection operator \mathcal{P}_h and the regularity properties of \bar{p} and \bar{q} to obtain

$$\begin{aligned} \mathbf{II}_h &= \int_{\Omega} ((\bar{p}_h + \lambda\bar{q}_h - (\bar{p} + \lambda\bar{q}))(\mathcal{P}_h\bar{q} - \bar{q})) dx = \int_{\Omega} (\bar{p}_h - \bar{p})(\mathcal{P}_h\bar{q} - \bar{q}) dx \\ &+ \lambda \|\mathcal{P}_h\bar{q} - \bar{q}\|_{L^2(\Omega)}^2 \leq \|\bar{p}_h - \bar{p}\|_{L^2(\Omega)} \|\mathcal{P}_h\bar{q} - \bar{q}\|_{L^2(\Omega)} + \lambda \|\mathcal{P}_h\bar{q} - \bar{q}\|_{L^2(\Omega)}^2. \end{aligned}$$

Use the estimates (4.5.30) and (4.5.9), as well as Young's inequality, to obtain

$$\mathbf{II}_h \leq Ch^{4\gamma} |\log h|^{4\varphi} [1 + \Lambda(L, \bar{u})]^2 + \frac{\mathfrak{C}}{4} \|\bar{q} - \bar{q}_h\|_{L^2(\Omega)}^2. \quad (4.5.38)$$

Step 6. Through the collection of (4.5.27), (4.5.31), (4.5.32), (4.5.37), and (4.5.38), we conclude that (4.5.13) holds, which is a contradiction. This concludes the proof. \square

As a corollary, we present the following estimate for $\bar{\eta} - \bar{\eta}_h$.

Corollary 4.5.8 (error bound for the approximation of an optimal subgradient). *In the framework of Theorem 4.5.7, we have the error bound*

$$\|\bar{\eta} - \bar{\eta}_h\|_{L^2(\Omega)} \lesssim h^{2\gamma} |\log h|^{2\varphi} \quad \forall h \leq h_{\square}, \quad \gamma = \min \left\{ s, \frac{1}{2} \right\},$$

where $\varphi = \nu$ if $s \neq \frac{1}{2}$, $\varphi = 1 + \nu$ if $s = \frac{1}{2}$, and $\nu \geq \frac{1}{2}$ is the constant in Theorem 4.3.1.

Proof. The error bound is an immediate consequence of Theorems 4.5.3 and 4.5.7. \square

4.6 A semidiscrete scheme for the optimal control problem

In the following, we propose a semidiscretization strategy based on the variational discretization approach [67]. Here, only the state space is discretized (the control space is not discretized). The semidiscrete approach is as follows: Find $\min J(u_h, \mathbf{q})$ subject to

$$\mathcal{A}(u_h, v_h) + \int_{\Omega} a(x, u_h) v_h dx = \int_{\Omega} \mathbf{q} v_h dx \quad \forall v_h \in \mathbb{V}_h, \quad (4.6.1)$$

and the control constraints $\mathbf{q} \in \mathbb{Q}_{ad}$. Standard arguments show that there is at least one optimal solution to this problem. Furthermore, as in Theorem 4.4.2, it can be proved that if $\bar{\mathbf{q}} \in \mathbb{Q}_{ad}$ denotes a local solution, then there exists $\bar{\eta} \in \partial j(\bar{\mathbf{q}})$ such that

$$\int_{\Omega} (\bar{p}_h + \lambda \bar{\mathbf{q}} + \mu \bar{\eta})(q - \bar{\mathbf{q}}) dx \geq 0 \quad \forall q \in \mathbb{Q}_{ad}. \quad (4.6.2)$$

Here, \bar{p}_h solves the problem (4.5.3), where \bar{u}_h corresponds to the solution of (4.6.1) with \mathbf{q} replaced by $\bar{\mathbf{q}}$. As in Theorem 4.4.3, the following projection formulas can be derived for every $x \in \Omega$:

$$\begin{aligned} \bar{\mathbf{q}}(x) &= \Pi_{[\alpha, \beta]} (-\lambda^{-1}(\bar{p}_h(x) + \mu \bar{\eta}(x))), & \bar{\mathbf{q}}(x) = 0 &\Leftrightarrow |\bar{p}_h(x)| \leq \mu, \\ \bar{\eta}(x) &= \Pi_{[-1, 1]} (-\mu^{-1} \bar{p}_h(x)). \end{aligned}$$

Since $\bar{\mathbf{q}}$ and $\bar{\eta}$ implicitly depend on h , we will use the notation $\bar{\mathbf{q}}_h$ and $\bar{\eta}_h$ in the following. Assuming that discrete solutions \bar{u}_h of (4.6.1) are uniformly bounded in $L^\infty(\Omega)$ and that (A.1)–(A.3), (B.1)–(B.2), and (C.1) hold, we can provide error bounds for the approximation error of the adjoint state and subdifferential variables. The error estimate for the latter is simpler than that in Theorem 4.5.3 because of the bound $\|\bar{\eta} - \bar{\eta}_h\|_{L^2(\Omega)} \leq \mu^{-1} \|\bar{p} - \bar{p}_h\|_{L^2(\Omega)}$. Furthermore, minor adjustments in the proofs of the Theorems 4.5.4 and 4.5.5 lead to the following convergence results.

- Let $h > 0$ and let $\bar{\mathbf{q}}_h \in \mathbb{Q}_{ad}$ be a global solution of the semidiscrete scheme. Then, there exist nonrelabelled subsequences of $\{\bar{\mathbf{q}}_h\}_{h>0}$ such that $\bar{\mathbf{q}}_h \xrightarrow{*} \bar{q}$ in $L^\infty(\Omega)$ as $h \rightarrow 0$ and \bar{q} corresponds to a global solution to (4.4.1)–(4.4.2). In addition, the convergence results (4.5.10) and (4.5.11) hold.
- If $\bar{q} \in \mathbb{Q}_{ad}$ is a strict local minimum of the control problem (4.4.1)–(4.4.2), then there exists a sequence of local minima $\{\bar{\mathbf{q}}_h\}_{0 < h < h_*}$ of the semidiscrete scheme such that (4.5.10) and (4.5.11) hold.

We now derive the error bound for the semidiscrete scheme given in (4.6.4). For this purpose, we assume that $d \in \{2, 3\}$ and $s > d/4$ and we let $\{\bar{\mathbf{q}}_h\} \subset \mathbb{Q}_{ad}$ be a sequence of local minima of such a scheme such that $\bar{\mathbf{q}}_h \rightarrow \bar{q}$ in $L^2(\Omega)$ as $h \rightarrow 0$, where $\bar{q} \in \mathbb{Q}_{ad}$ is a local solution to (4.4.1)–(4.4.2). In a first step, we provide an instrumental result that is analogous to Theorem 4.5.6.

Theorem 4.6.1 (instrumental error estimate). *Let us assume that (A.1)–(A.3), (B.1)–(B.2), (C.1) and (4.5.7) hold. Let $\bar{q} \in \mathbb{Q}_{ad}$ satisfy the second-order optimality condition (i), or equivalently (ii) in Theorem 4.4.5. Then, there exists $h_* > 0$ such that*

$$\mathfrak{C} \|\bar{q} - \bar{\mathbf{q}}_h\|_{L^2(\Omega)}^2 \leq [F'(\bar{\mathbf{q}}_h) - F'(\bar{q})](\bar{\mathbf{q}}_h - \bar{q}) \quad \forall h \in (0, h_*], \quad (4.6.3)$$

where $\mathfrak{C} = 2^{-1}\delta$ and δ is the constant that appears in the item (ii) of Theorem 4.4.5.

Proof. The proof follows analogous arguments as in the proof of Theorem 4.5.6. The main modifications are the redefinitions of the sets

$$\begin{aligned}\Omega_{1,h} &:= \{x \in \Omega_\mu^{\text{less}} : |\bar{p}_h(x)| \leq \mu\}, & \Omega_{2,h} &:= \{x \in \Omega_\mu^{\text{less}} : |\bar{p}_h(x)| > +\mu\}, \\ \Omega_{3,h} &:= \{x \in \Omega_\mu^+ : \bar{p}_h(x) < -\mu\}, & \Omega_{4,h} &:= \{x \in \Omega_\mu^+ : \bar{p}_h(x) > +\mu\},\end{aligned}$$

and $\Omega_{5,h} := \{x \in \Omega_\mu^+ : |\bar{p}_h(x)| \leq \mu\}$. For the sake of simplicity, we skip the details. \square

We now derive the main result of this section.

Theorem 4.6.2 (error bound for the approximation of an optimal control). *Let us assume that (A.1)–(A.3), (B.1)–(B.2), (C.1), and (4.5.7) hold. Let $\bar{q} \in \mathbb{Q}_{ad}$ satisfy the second-order optimality condition (i), or equivalently (ii) in Theorem 4.4.5. Then, there exists $h_\bullet > 0$ such that*

$$\|\bar{q} - \bar{q}_h\|_{L^2(\Omega)} \lesssim h^{2\gamma} |\log h|^{2\varphi} \quad \forall h \leq h_\bullet \quad \gamma = \min\left\{s, \frac{1}{2}\right\}, \quad (4.6.4)$$

where $\varphi = \nu$ if $s \neq \frac{1}{2}$, $\varphi = 1 + \nu$ if $s = \frac{1}{2}$, and $\nu \geq \frac{1}{2}$ is the constant in Theorem 4.3.1.

Proof. We proceed as in the proof of Theorem 4.5.7 and use the instrumental error bound (4.6.3) and the variational inequalities (4.4.9) and (4.6.2) with $q = \bar{q}_h$ and $q = \bar{q}$, respectively, to obtain (cf. [26, Theorem 5.1])

$$\mathfrak{C} \|\bar{q} - \bar{q}_h\|_{L^2(\Omega)}^2 \leq [F'_h(\bar{q}_h) - F'(\bar{q}_h)](\bar{q} - \bar{q}_h) + \mu \int_\Omega (\bar{\eta} - \bar{\eta}_h)(\bar{q}_h - \bar{q}) dx \quad \forall h \leq h_*.$$

We immediately notice that the first and second terms on the right-hand side of the previous inequality correspond to \mathbf{I}_h and \mathbf{IV}_h , respectively, from the proof of Theorem 4.5.7. These terms, \mathbf{I}_h and \mathbf{IV}_h , are estimated in (4.5.37) and (4.5.27), respectively. \square

4.7 Numerical examples

We present a numerical experiment that illustrates the performance of the fully and semidiscrete methods presented Sections 4.5 and 4.6, respectively, when used to approximate a solution of the control problem (4.4.1)–(4.4.2). A MATLAB implementation is used for the experiment, and the methods are solved using a semi-smooth Newton method.

The setting of the experiment is as follows: we set $d = 2$, $\Omega = B(0, 1)$, and $\lambda = 1$, where $B(0, 1)$ denotes the unit disc. We let $a(\cdot, u) = u^3$ and $L(\cdot, u) = (u - u_\Omega)^2/2$, where u_Ω is such that the exact

optimal state and the optimal adjoint state are

$$\bar{u}(x) = \bar{p}(x) = (2^{2s}\Gamma^2(1+s))^{-1}(1-|x|^2)_+^s, \quad t_+ = \max\{0, t\}. \quad (4.7.1)$$

We also consider $a = -1$, $b = 1$, and $s \in \{0.2, 0.4, 0.6, 0.8\}$. Note that we go beyond the theory presented and illustrate the performance of the methods for different values of $s \in (0, 1)$. Additionally, for $s \leq 0.5$, we set $\mu = 0.6$, and for $s > 0.5$, we choose $\mu = 0.25$.

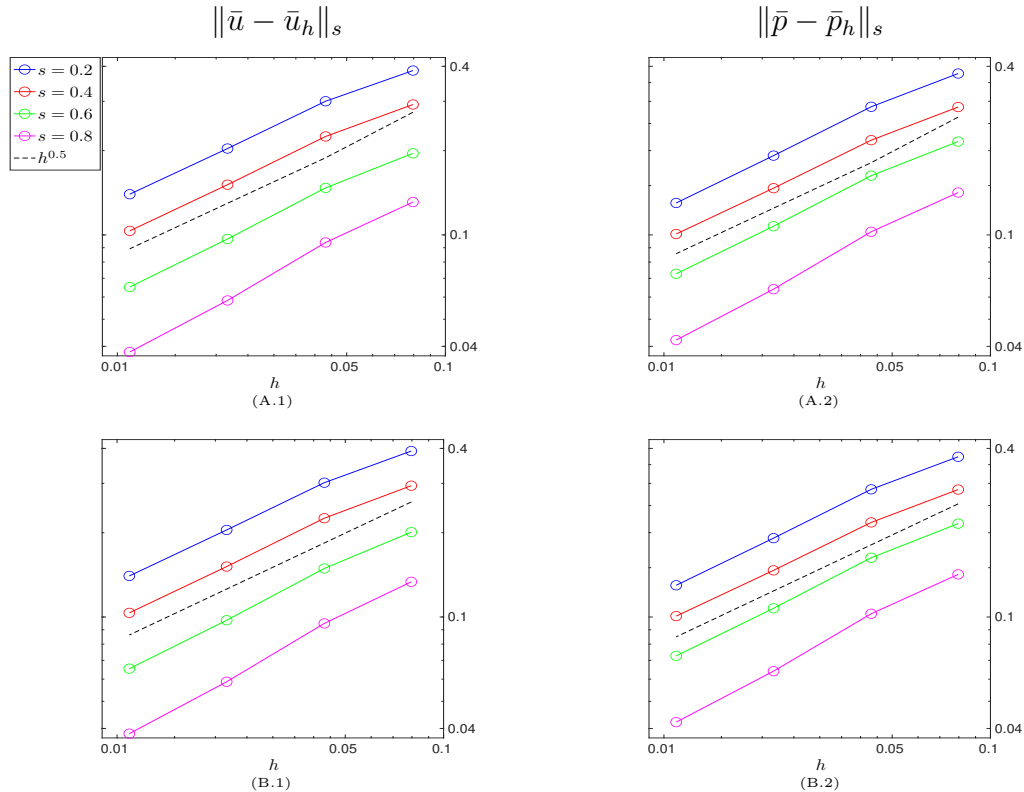


Figure 4.1: Experimental rates of convergence for $\|\bar{u} - \bar{u}_h\|_s$ and $\|\bar{p} - \bar{p}_h\|_s$ considering the fully discrete (A.1)–(A.2) and semidiscrete schemes (B.1)–(B.2) for $s \in \{0.2, 0.4, 0.6, 0.8\}$.

Figures 4.1, 4.2, and 4.3 show the results for the fully discrete and semidiscrete schemes. Figure 4.1 shows the experimental convergence rates for $\|\bar{u} - \bar{u}_h\|_s$ and $\|\bar{p} - \bar{p}_h\|_s$ for $s \in \{0.2, 0.4, 0.6, 0.8\}$. The experimental convergence rates for $\|\bar{u} - \bar{u}_h\|_{L^2(\Omega)}$ and $\|\bar{p} - \bar{p}_h\|_{L^2(\Omega)}$ are shown in Figure 4.2, while the results for $\|\bar{q} - \bar{q}_h\|_{L^2(\Omega)}$ and $\|\bar{\eta} - \bar{\eta}_h\|_{L^2(\Omega)}$ are shown in Figure 4.3. It can be observed that when $s \geq 0.5$ the experimental convergence rates for all involved approximation errors are in agreement with the error estimates obtained in Sections 4.5.2 and 4.6.

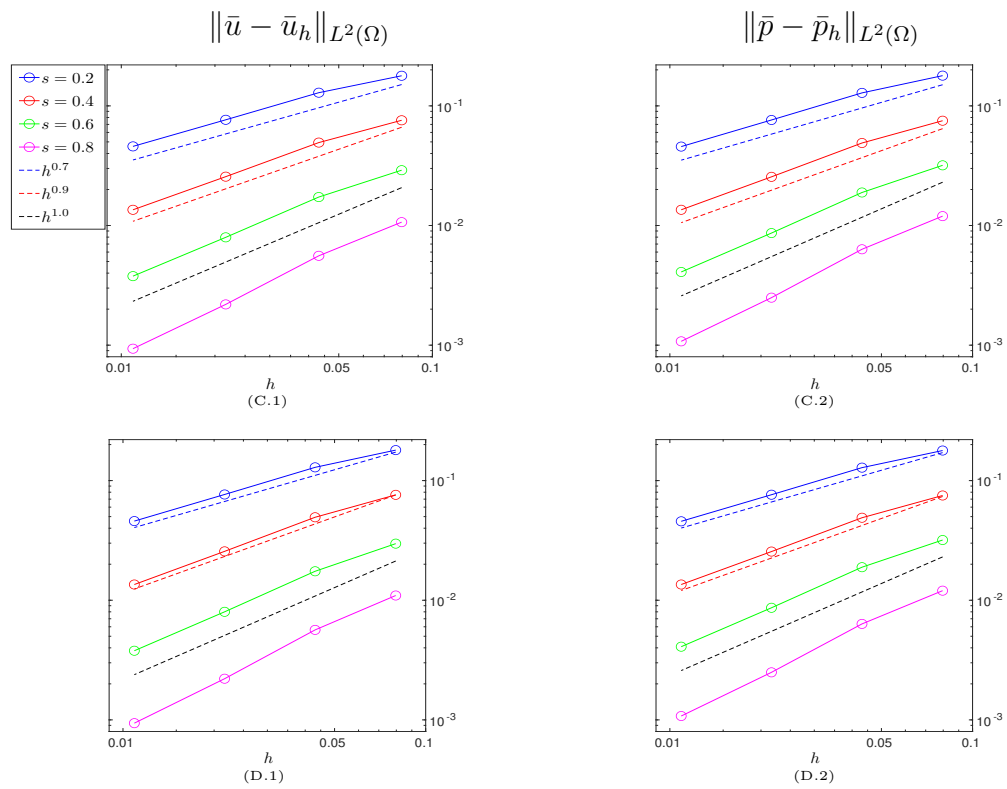


Figure 4.2: Experimental rates of convergence for $\|\bar{u} - \bar{u}_h\|_{L^2(\Omega)}$ and $\|\bar{p} - \bar{p}_h\|_{L^2(\Omega)}$ considering the fully discrete (C.1)–(C.2) and semidiscrete schemes (D.1)–(D.2) for $s \in \{0.2, 0.4, 0.6, 0.8\}$.

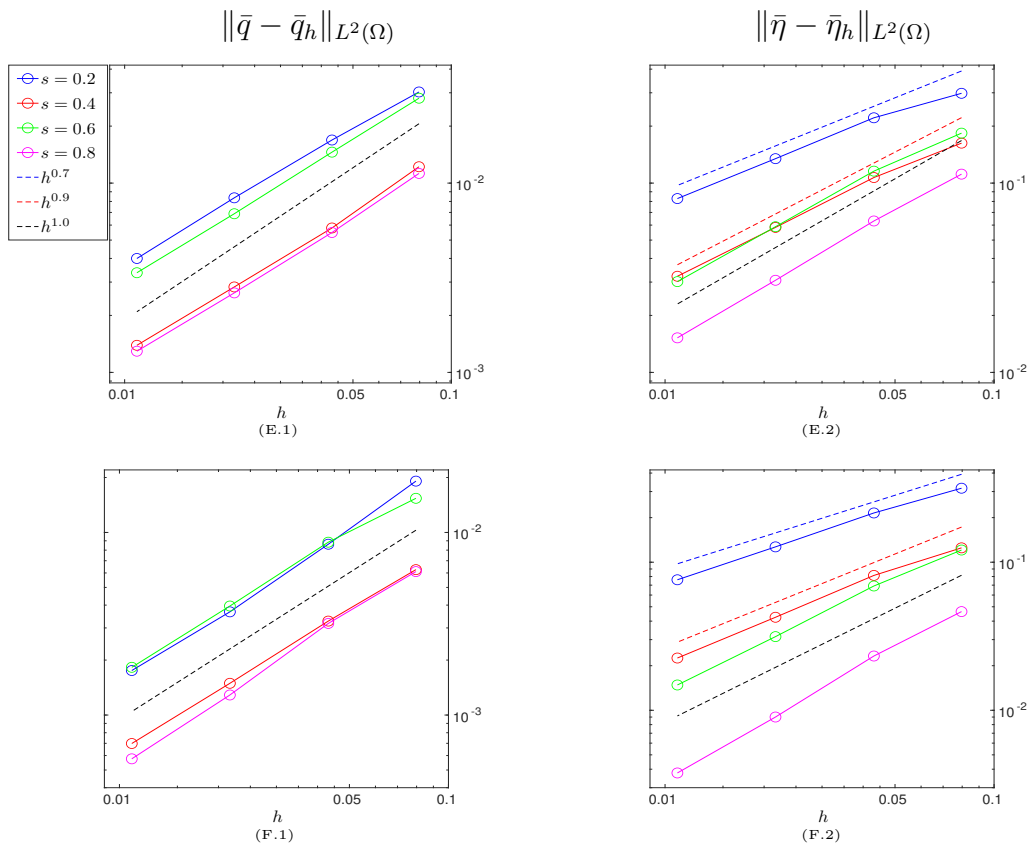


Figure 4.3: Experimental rates of convergence for $\|\bar{q} - \bar{q}_h\|_{L^2(\Omega)}$ and $\|\bar{\eta} - \bar{\eta}_h\|_{L^2(\Omega)}$ considering the fully discrete (E.1)–(E.2) and semidiscrete schemes (F.1)–(F.2) for $s \in \{0.2, 0.4, 0.6, 0.8\}$.

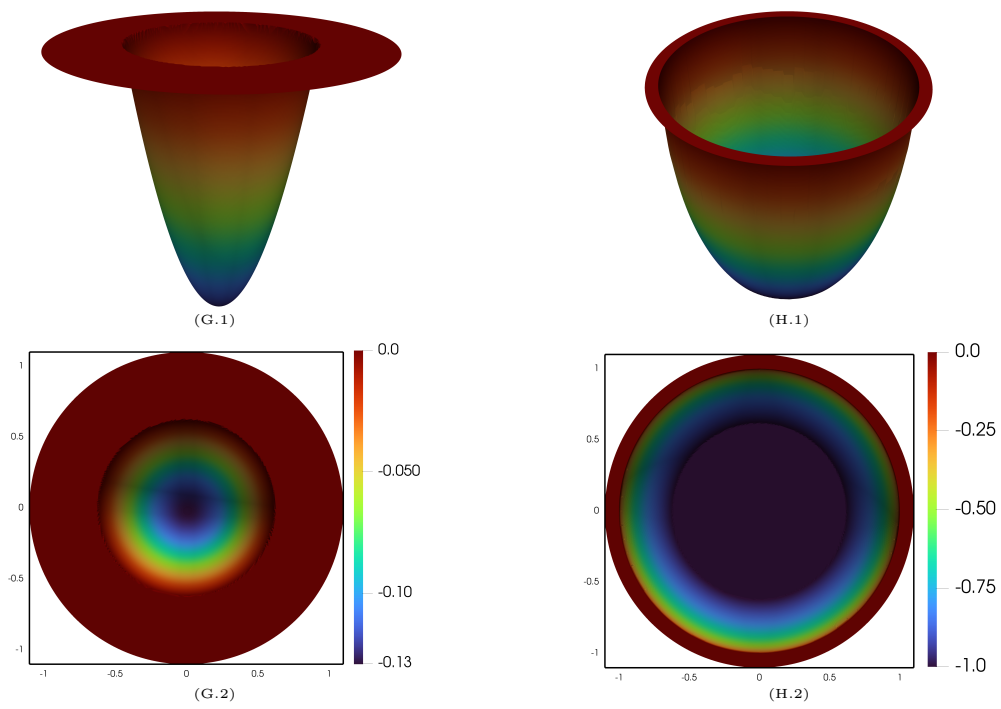


Figure 4.4: Finite element solutions \bar{q}_h (left) and $\bar{\eta}_h$ (right), obtained by the semidiscrete scheme with $s = 0.4$. The sparse behavior in the control variable \bar{q}_h is evident. In addition, a singular behavior can be observed for $\bar{\eta}_h$ near the boundary $\partial\Omega$.

Chapter 5

A nonlocal coupled system: analysis and discretization

5.1 Introduction

Let Ω_1 and Ω_2 be two open, connected, and bounded domains with Lipschitz boundary in \mathbb{R}^n ($n \geq 2$) such that $\bar{\Omega}_1 \cap \bar{\Omega}_2 = \emptyset$. Let $s_1, s_2 \in (0, 1)$. Let $J : \mathbb{R}^n \rightarrow \mathbb{R}$ be a nonnegative and symmetric measurable function such that $J \in L^1(\mathbb{R}^n)$ and $\text{supp}(J) \supset B_r(0)$, where $r > d$ and $d = \text{dist}(\Omega_1, \Omega_2)$. In this paper, we will analyze the following *nonlocal coupled system* and develop solution techniques for it: Given two functions $f_1 : \Omega_1 \rightarrow \mathbb{R}$ and $f_2 : \Omega_2 \rightarrow \mathbb{R}$, find $u := u_1\chi_{\Omega_1} + u_2\chi_{\Omega_2}$ such that

$$\begin{aligned} 2\mathcal{C}_1 \int_{\Omega_2^c} \frac{u_1(x) - u_1(y)}{|x - y|^{n+2s_1}} dy + \int_{\Omega_2} J(x - y) (u_1(x) - u_2(y)) dy &= f_1(x), \quad x \in \Omega_1, \\ 2\mathcal{C}_2 \int_{\Omega_1^c} \frac{u_2(x) - u_2(y)}{|x - y|^{n+2s_2}} dy + \int_{\Omega_1} J(x - y) (u_2(x) - u_1(y)) dy &= f_2(x), \quad x \in \Omega_2. \end{aligned} \tag{5.1.1}$$

Here, for $i \in \{1, 2\}$, $\mathcal{C}_i = c(n, s_i)/2$, where $c(n, s_i)$ denotes a positive constant, and χ_{Ω_i} corresponds to the characteristic function over the domain Ω_i .

Nonlocal models have attracted great interest in recent years, mainly due to the extensive list of applications including finance [24, 39], image science [23, 60], and peridynamics [48, 70]. Nonlocal models are able to describe phenomena that go beyond the scope of classical PDEs. These phenomena include the presence of discontinuities in the solution, as often observed in fractures or in continuum mechanics [47, 99], and the occurrence of anomalous diffusion effects, such as super- and subdiffusion in subsurface transport and turbulence [97, 101]. These models are usually governed by integro-differential equations, where the value of the corresponding nonlocal operator at a point depends on the behavior

of the solution over a finite or infinite region of the domain. An important example from the family of nonlocal operators is the *integral fractional Laplacian* $(-\Delta)^s$ ($0 < s < 1$), which corresponds to the infinitesimal generator of a stable Lévy process [106].

For many years, several authors have studied the coupling of local and nonlocal models; see [1, 21, 49, 61, 72]. As described in [42] and also in [1], this study is motivated by the fact that it is often the case that nonlocal effects are concentrated only in some parts of the domain, while the system can be accurately described by a PDE in the remaining parts. The goal of coupling local and nonlocal models is to combine a local equation (a PDE) with a nonlocal equation (an integral equation), assuming that the location of the local and nonlocal effects can be determined in advance. In this context, one of the challenges of a coupling strategy is to find a mathematically consistent formulation. An additional advantage is that the above-mentioned coupling provides a feasible way to circumvent the non-trivial task of specifying nonlocal boundary conditions.

In this paper, we are mainly interested in a model consisting of two particle systems located in different and disjoint domains. Each particle system follows its own law, which can be described as a random walk that allows long jumps. In addition, there is a suitable coupling law between these two systems that allows a flow of particles from one domain to the other domain, and this coupling law can also allow long jumps. This model gives rise to a system of two nonlocal equations associated with the restricted fractional Laplacian defined on different domains and a nonlocal coupling term.

The content of this work is organized as follows. In section 5.2 we introduce the notation, assumptions and functional framework that we will use in our work. In section 5.3, we derive a weak formulation of (5.1.1) by minimizing a suitable energy functional and characterizing the unique minimum as the solution of the corresponding Euler-Lagrange equations. With this weak formulation in hand, we provide a finite element discretization scheme in section 5.4 to approximate the solution. In section 5.5, we propose and analyze an alternating method for both the continuous and discrete problems, and provide results on the convergence of the solutions. Finally, in section 5.6, we present a numerical experiment that illustrates the performance of our developed theory.

5.2 Notation and preliminary remarks

Let us establish the notation and recall some facts that will be useful later.

5.2.1 Notation

In the course of this work, we let $n \geq 2$. If $\Omega \subset \mathbb{R}^n$ is an open and bounded domain, we denote by $\partial\Omega$ and Ω^c the boundary and the complement of Ω , respectively. Given $r > 0$ and $x \in \mathbb{R}^n$, we denote by $B_r(x)$ the (open) Euclidean ball of radius r centered at x . We set $B_r = B_r(0)$.

If \mathcal{X} and \mathcal{Y} are Banach function spaces, we write $\mathcal{X} \hookrightarrow \mathcal{Y}$ to denote that \mathcal{X} is continuously embedded in \mathcal{Y} . We denote by \mathcal{X}' and $\|\cdot\|_{\mathcal{X}}$ the dual and the norm of \mathcal{X} , respectively. We denote by $\langle \cdot, \cdot \rangle_{\mathcal{X}', \mathcal{X}}$ the duality pairing between \mathcal{X}' and \mathcal{X} and simply write $\langle \cdot, \cdot \rangle$ if the spaces \mathcal{X}' and \mathcal{X} are clear from the context. The relation $\mathbf{a} \lesssim \mathbf{b}$ indicates that $\mathbf{a} \leq C\mathbf{b}$, with a positive constant C that does not depend on \mathbf{a} , \mathbf{b} , or the discretization parameters, but may depend on s , n , and Ω . The value of C might change at each occurrence.

5.2.2 Assumptions

Let Ω_1 and Ω_2 be two open, connected, and bounded domains in \mathbb{R}^n . We assume that $\partial\Omega_1$ and $\partial\Omega_2$ are Lipschitz and that $\bar{\Omega}_1 \cap \bar{\Omega}_2 = \emptyset$. The distance between the sets Ω_1 and Ω_2 is denoted by [55, Section 0.6]

$$d := \text{dist}(\Omega_1, \Omega_2) := \inf\{|x - y| : x \in \Omega_1, y \in \Omega_2\}.$$

We note that d is a strictly positive constant because $\bar{\Omega}_1 \cap \bar{\Omega}_2 = \emptyset$.

On the other hand, given $\Omega \subset \mathbb{R}^n$, we denote the distance from a point $x \in \mathbb{R}^n$ to Ω by [55, Section 0.6],

$$\delta_{\Omega}(x) = \inf\{|x - t| : t \in \Omega\}. \quad (5.2.1)$$

In our paper, we will operate under the following assumptions on the kernel J :

- (J1) $J : \mathbb{R}^n \rightarrow \mathbb{R}$ is a non-negative and symmetric function, i.e., $J(z) \geq 0$ for a.e. $z \in \mathbb{R}^n$ and $J(z) = J(-z)$ for all $z \in \mathbb{R}^n$.
- (J2) $J \in L^1(\mathbb{R}^n)$.
- (J3) There exists $C > 0$ and $r > d$ such that $B_r \subset \text{supp}(J)$.

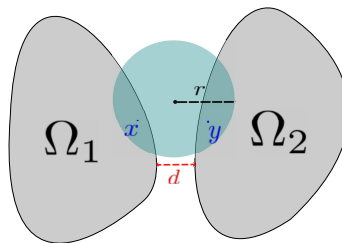


Figure 5.1: Nonlocal interactions between Ω_1 and Ω_2 because of the kernel J .

5.2.3 Convolution

Let $f \in L^1(\mathbb{R}^n)$ and let $g \in L^p(\mathbb{R}^n)$ with $1 \leq p \leq \infty$. We define the convolution of f with g as

$$(f \star g)(x) = \int_{\mathbb{R}^n} f(x-y)g(y)dy. \quad (5.2.2)$$

The following is a classical result [19, Theorem 4.15].

Lemma 5.2.1 (continuity). *If $f \in L^1(\mathbb{R}^n)$ and $g \in L^p(\mathbb{R}^n)$ with $1 \leq p \leq \infty$, then*

$$\|f \star g\|_{L^p(\mathbb{R}^n)} \leq \|f\|_{L^1(\mathbb{R}^n)} \|g\|_{L^p(\mathbb{R}^n)}. \quad (5.2.3)$$

5.2.4 Function spaces

Fractional Sobolev spaces provide a natural framework for analyzing problems with different definitions of the fractional Laplace operator. In \mathbb{R}^n , a family of such spaces can be defined based on the Fourier transform \mathcal{F} : For any $s \geq 0$, we define [102, Definition 15.7], [78, Chapter 1, Section 7]

$$H^s(\mathbb{R}^n) := \{v \in L^2(\mathbb{R}^n) : (1 + |\xi|^2)^{\frac{s}{2}} \mathcal{F}(v) \in L^2(\mathbb{R}^n)\},$$

endowed with the norm $\|v\|_{H^s(\mathbb{R}^n)} := \|(1 + |\xi|^2)^{\frac{s}{2}} \mathcal{F}(v)\|_{L^2(\mathbb{R}^n)}$.

Let Ω be an open and bounded domain with Lipschitz boundary. We define $\tilde{H}^s(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $H^s(\mathbb{R}^n)$ [81, page 77] and note that it can be equivalently characterized as the following space of zero-extension functions [81, Theorem 3.29]:

$$\tilde{H}^s(\Omega) = \{v|_\Omega : v \in H^s(\mathbb{R}^n), \text{ supp } v \subset \bar{\Omega}\}.$$

If $s \in (0, 1)$, we equip the space $\tilde{H}^s(\Omega)$ with the following inner product and the following semi-norm [81, page 75]:

$$(v, w)_{\tilde{H}^s(\Omega)} := \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(v(x) - v(y))(w(x) - w(y))}{|x - y|^{n+2s}} dx dy, \quad |v|_{\tilde{H}^s(\Omega)} := (v, v)_{\tilde{H}^s(\Omega)}^{\frac{1}{2}}.$$

Owing to the fractional Poincaré inequality [3, Proposition 2.4]

$$\|v\|_{L^2(\Omega)} \leq C|v|_{\tilde{H}^s(\Omega)} \quad \forall v \in \tilde{H}^s(\Omega), \quad C = C(\Omega, n, s), \quad (5.2.4)$$

we have that $|\cdot|_{\tilde{H}^s(\Omega)}$ is actually a norm in $\tilde{H}^s(\Omega)$ and that $\tilde{H}^s(\Omega)$ is Hilbert. We denote by $H^{-s}(\Omega)$ the dual space of the fractional Sobolev space $\tilde{H}^s(\Omega)$.

For $\mu \in (0, 1)$, we introduce the Slobodeckii–Gagliardo semi-norm [81, (3.18)]

$$|v|_{H^\mu(\Omega)} = \left(\int_{\Omega} \int_{\Omega} \frac{|v(x) - v(y)|^2}{|x - y|^{n+2\mu}} dx dy \right)^{\frac{1}{2}}. \quad (5.2.5)$$

Let $r \in \mathbb{N}_0$ and $\mu \in (0, 1)$ be such that $s = r + \mu$. We define [81, page 74]

$$H^s(\Omega) := \{v \in H^r(\Omega) : |\partial^\alpha v|_{H^\mu(\Omega)} < \infty \text{ for } |\alpha| = r\}$$

and endow this space with the following norm:

$$\|v\|_{H^s(\Omega)} := \left(\|v\|_{H^r(\Omega)}^2 + \sum_{|\alpha|=r} |\partial^\alpha v|_{H^\mu(\Omega)}^2 \right)^{\frac{1}{2}}.$$

We conclude this section with the following Sobolev embedding results.

Lemma 5.2.2 (embedding results). *Let $s \in (0, 1)$. If $\mathfrak{r} \in [1, 2n/(n - 2s)]$, then $H^s(\Omega) \hookrightarrow L^\mathfrak{r}(\Omega)$. If $\mathfrak{r} \in [1, 2n/(n - 2s))$, then $H^s(\Omega) \hookrightarrow L^\mathfrak{r}(\Omega)$ is compact.*

Proof. A proof of $H^s(\Omega) \hookrightarrow L^\mathfrak{r}(\Omega)$ can be found in [4, Theorem 7.34]. The fact that the embedding is compact for $\mathfrak{r} < 2n/(n - 2s)$ follows from [46, Corollary 7.2]. \square

5.3 The nonlocal coupled model

Let $s_1, s_2 \in (0, 1)$. We define the space

$$H := \tilde{H}^{s_1}(\Omega_1) \oplus \tilde{H}^{s_2}(\Omega_2) = \{u = u_1 + u_2 : u_1 \in \tilde{H}^{s_1}(\Omega_1), u_2 \in \tilde{H}^{s_2}(\Omega_2)\}.$$

We equip the space H with the following semi-norm:

$$|u|_H := \left(|u_1|_{\tilde{H}^{s_1}(\Omega_1)}^2 + |u_2|_{\tilde{H}^{s_2}(\Omega_2)}^2 \right)^{\frac{1}{2}}, \quad u = u_1 + u_2. \quad (5.3.1)$$

Owing to the Poincaré inequality (5.2.4), the semi-norm $|\cdot|_H$ is actually a norm in H .

5.3.1 The energy functionals

We introduce two energy functionals. First, we define $E : H \rightarrow \mathbb{R}_0^+$ by

$$E(u) := \frac{C_1}{2} \int_{\Omega_2^c} \int_{\Omega_2^c} \frac{(u_1(x) - u_1(y))^2}{|x - y|^{n+2s_1}} dy dx + \frac{C_2}{2} \int_{\Omega_1^c} \int_{\Omega_1^c} \frac{(u_2(x) - u_2(y))^2}{|x - y|^{n+2s_2}} dy dx + \frac{1}{2} \int_{\Omega_1} \int_{\Omega_2} J(x - y)(u_1(x) - u_2(y))^2 dy dx. \quad (5.3.2)$$

Since J belongs to $L^1(\mathbb{R}^n)$ and is a non-negative function, it is immediate that E is well-defined and non-negative over H . Second, given two functions $f_1 \in L^2(\Omega_1)$ and $f_2 \in L^2(\Omega_2)$, we define the energy functional $\mathcal{E} : H \rightarrow \mathbb{R}$ by

$$\mathcal{E}(u) := E(u) - \int_{\Omega_1} f_1 u_1 dx - \int_{\Omega_2} f_2 u_2 dx. \quad (5.3.3)$$

5.3.2 Properties of the energy functionals

In this section, we examine some properties satisfied by the energy functionals E and \mathcal{E} satisfy. We begin our analysis with an auxiliary bound for E .

Lemma 5.3.1 (A lower bound in L^2 for E). *There exists $\mathcal{C} > 0$ such that*

$$E(u) \geq \mathcal{C} \left(\|u_1\|_{L^2(\Omega_1)}^2 + \|u_2\|_{L^2(\Omega_2)}^2 \right) \quad \forall u \in H. \quad (5.3.4)$$

Proof. We proceed by contradiction and assume that there is $\{u_k\}_{k \in \mathbb{N}} \subset H$, so that for each $k \in \mathbb{N}$ we have

$$u_k = u_{1,k} + u_{2,k}, \quad \|u_{1,k}\|_{L^2(\Omega_1)}^2 + \|u_{2,k}\|_{L^2(\Omega_2)}^2 = 1, \quad E(u_k) \leq k^{-1}. \quad (5.3.5)$$

From (5.3.5), we can directly deduce that $E(u_k) \rightarrow 0$ as $k \uparrow \infty$. Without loss of generality, we can assume that $\{u_{1,k}\}_{k \in \mathbb{N}}$ is such that $u_{1,k} \neq 0$ in Ω_1 for every $k \in \mathbb{N}$.

As a first step, we note that the non-negativity of E in conjunction with its definition given in (5.3.2) allows us to conclude that

$$\int_{\Omega_2^c} \int_{\Omega_2^c} \frac{(u_{1,k}(x) - u_{1,k}(y))^2}{|x - y|^{n+2s_1}} dy dx \rightarrow 0 \implies \|u_{1,k}\|_{H^{s_1}(\Omega_1)}^2 \rightarrow 0 \quad (5.3.6)$$

as $k \uparrow \infty$. This convergence result in conjunction with the fact that $\|u_{1,k}\|_{L^2(\Omega_1)} \leq 1$ for all $k \in \mathbb{N}$ shows that the sequence $\{u_{1,k}\}_{k \in \mathbb{N}}$ is uniformly bounded in $H^{s_1}(\Omega_1)$. Similar arguments show that $\{u_{2,k}\}_{k \in \mathbb{N}}$ is uniformly bounded in $H^{s_2}(\Omega_2)$. We can thus extract nonrelabeled subsequences $\{u_{1,k}\}_{k \in \mathbb{N}}$

and $\{u_{2,k}\}_{k \in \mathbb{N}}$ such that

$$\begin{aligned} u_{1,k} &\rightharpoonup u_1 \text{ in } H^{s_1}(\Omega_1), & u_{1,k} &\rightarrow u_1 \text{ in } L^2(\Omega_1), & k &\uparrow \infty, \\ u_{2,k} &\rightharpoonup u_2 \text{ in } H^{s_2}(\Omega_2), & u_{2,k} &\rightarrow u_2 \text{ in } L^2(\Omega_2), & k &\uparrow \infty. \end{aligned} \quad (5.3.7)$$

We now note that (5.3.6) and a similar convergence argument for $\{u_{2,k}\}_{k \in \mathbb{N}}$ show that the previously extracted subsequences satisfy the strong convergence properties:

$$u_{1,k} \rightarrow u_1 \text{ in } H^{s_1}(\Omega_1), \quad u_{2,k} \rightarrow u_2 \text{ in } H^{s_2}(\Omega_2), \quad k \uparrow \infty \quad (5.3.8)$$

because weak convergence and convergence of norms imply strong convergence in $H^{s_1}(\Omega_1)$ and $H^{s_2}(\Omega_2)$. We again invoke (5.3.6) and a similar argument for $\{u_{2,k}\}_{k \in \mathbb{N}}$ to conclude that $u_1 = C_1$ and $u_2 = C_2$, where C_1 and C_2 are real constants. Note that here we have used that both sets Ω_1 and Ω_2 are connected.

On the other hand, since $E(u_k) \rightarrow 0$ as $k \uparrow \infty$ it also follows that

$$\int_{\Omega_1} \int_{\Omega_2} J(x-y)(u_{1,k}(x) - u_{2,k}(y))^2 dy dx \rightarrow 0, \quad k \uparrow \infty.$$

This convergence property in conjunction with the fact that $u_1 = C_1$ and $u_2 = C_2$, the non-negativity of J , and the property $\text{supp}(J) \supset B_r$, where $r > d = \text{dist}(\Omega_1, \Omega_2)$, allows us to conclude that

$$(C_1 - C_2)^2 \int_{\Omega_1} \int_{\Omega_2} J(x-y) dy dx = 0 \implies C_1 = C_2.$$

As a final step, we show that $C_1 = C_2 = 0$. To do this, we proceed as follows. On the one hand, we have that

$$\int_{\Omega_2^c} \int_{\Omega_2^c} \frac{(u_{1,k}(x) - u_{1,k}(y))^2}{|x-y|^{n+2s_1}} dy dx \rightarrow 0 \implies \int_{\Omega_1} \int_{\Omega_1^c \cap \Omega_2^c} \frac{u_{1,k}(x)^2}{|x-y|^{n+2s_1}} dy dx \rightarrow 0, \quad k \uparrow \infty. \quad (5.3.9)$$

On the other hand, defining $\mathcal{A} = \{z \in \Omega_2^c : \text{dist}(z, \Omega_2) \leq d/2\}$, we have

$$\int_{\Omega_1} \int_{\Omega_1^c \cap \Omega_2^c} \frac{u_{1,k}(x)^2}{|x-y|^{n+2s_1}} dy dx \geq \int_{\Omega_1} \int_{\mathcal{A}} \frac{u_{1,k}(x)^2}{|x-y|^{n+2s_1}} dy dx.$$

Define $h : \Omega_1 \rightarrow \mathbb{R}$ by $h(x) = \int_{\mathcal{A}} |x-y|^{-n-2s_1} dy$. We note that $h \in L^\infty(\Omega_1)$ and in particular that $h(x) \geq C_{\mathcal{A}} > 0$ for all $x \in \Omega_1$. As a result,

$$\int_{\Omega_1} \int_{\mathcal{A}} \frac{u_{1,k}(x)^2}{|x-y|^{n+2s_1}} dy dx = \int_{\Omega_1} u_{1,k}(x)^2 h(x) dx \geq C_{\mathcal{A}} \int_{\Omega_1} u_{1,k}(x)^2 dx$$

for every $k \in \mathbb{N}$. If we take the limit as $k \uparrow \infty$ in the previous inequality and use the strong convergence of $\{u_{1,k}\}_{k \in \mathbb{N}}$ in $L^2(\Omega_1)$ as $k \uparrow \infty$ (5.3.7), we arrive at

$$C_{\mathcal{A}} \int_{\Omega_1} u_1(x)^2 dx \leq 0,$$

where we have also used (5.3.9). This shows that $C_{\mathcal{A}}|\Omega_1|C^2 = 0$, which implies that $C = 0$. As a result, $C_1 = C_2 = 0$, and then $u_{1,k} \rightarrow 0$ in $L^2(\Omega_1)$ and $u_{2,k} \rightarrow 0$ in $L^2(\Omega_2)$ as $k \uparrow \infty$. This is a contradiction with (5.3.5) and concludes the proof. \square

Lemma 5.3.2 (\mathcal{E} is convex and continuous). *The energy functional \mathcal{E} defined in (5.3.3) is convex and continuous on H .*

Proof. The fact that \mathcal{E} is convex on H is clear. Let us now prove that \mathcal{E} is continuous on H . To do so, let $\{u_k\}_{k \in \mathbb{N}} \subset H$ be such that

$$u_k = u_{1,k} + u_{2,k}, \quad u_{1,k} \rightarrow u_1 \text{ in } \tilde{H}^{s_1}(\Omega_1), \quad u_{2,k} \rightarrow u_2 \text{ in } \tilde{H}^{s_2}(\Omega_2), \quad k \uparrow \infty. \quad (5.3.10)$$

In the following, we prove that $\mathcal{E}(u_k) \rightarrow \mathcal{E}(u)$ as $k \uparrow \infty$. From (5.3.10) we immediately deduce that

$$\int_{\Omega_1} f_1 u_{1,k} dx \rightarrow \int_{\Omega_1} f_1 u_1 dx, \quad \int_{\Omega_2} f_2 u_{2,k} dx \rightarrow \int_{\Omega_2} f_2 u_2 dx, \quad k \uparrow \infty.$$

It remains to prove that $E(u_k) \rightarrow E(u)$ as $k \uparrow \infty$. To do this, we begin with a simple application of the triangle inequality and obtain

$$\begin{aligned} \left| |u_{1,k}|_{H^{s_1}(\Omega_2^c)} - |u_1|_{H^{s_1}(\Omega_2^c)} \right| &\leq |u_{1,k} - u_1|_{H^{s_1}(\Omega_2^c)} \leq |u_{1,k} - u_1|_{H^{s_1}(\mathbb{R}^n)} \rightarrow 0, \\ \left| |u_{2,k}|_{H^{s_2}(\Omega_1^c)} - |u_2|_{H^{s_2}(\Omega_1^c)} \right| &\leq |u_{2,k} - u_2|_{H^{s_2}(\Omega_1^c)} \leq |u_{2,k} - u_2|_{H^{s_2}(\mathbb{R}^n)} \rightarrow 0 \end{aligned}$$

as $k \uparrow \infty$. In other words, we have

$$\begin{aligned} \int_{\Omega_2^c} \int_{\Omega_2^c} \frac{|u_{1,k}(x) - u_{1,k}(y)|^2}{|x - y|^{n+2s_1}} dy dx &\rightarrow \int_{\Omega_2^c} \int_{\Omega_2^c} \frac{|u_1(x) - u_1(y)|^2}{|x - y|^{n+2s_1}} dy dx, \\ \int_{\Omega_1^c} \int_{\Omega_1^c} \frac{|u_{2,k}(x) - u_{2,k}(y)|^2}{|x - y|^{n+2s_2}} dy dx &\rightarrow \int_{\Omega_1^c} \int_{\Omega_1^c} \frac{|u_2(x) - u_2(y)|^2}{|x - y|^{n+2s_2}} dy dx \end{aligned} \quad (5.3.11)$$

as $k \uparrow \infty$. It remains to prove the convergence of the nonlocal term that represents the coupling. In a

first step we write

$$\begin{aligned} \int_{\Omega_1} \int_{\Omega_2} J(x-y)(u_{1,k}(x) - u_{2,k}(y))^2 dy dx &= \int_{\Omega_1} \int_{\Omega_2} J(x-y)u_{1,k}(x)^2 dy dx \\ &\quad - 2 \int_{\Omega_1} \int_{\Omega_2} J(x-y)u_{1,k}(x)u_{2,k}(y) dy dx + \int_{\Omega_1} \int_{\Omega_2} J(x-y)u_{2,k}(y)^2 dy dx := \text{I}_k + \text{II}_k + \text{III}_k. \end{aligned}$$

Let us analyze I_k . For this purpose, we define $g : \Omega_1 \rightarrow \mathbb{R}$ by $g(x) := \int_{\Omega_2} J(x-y)dy$ for a.e. $x \in \Omega_1$ and note that $g \in L^\infty(\Omega_1)$. In fact, for a.e. $x \in \Omega_1$, we have

$$|g(x)| \leq \int_{\mathbb{R}^n} |J(x-y)|dy = \|J\|_{L^1(\mathbb{R}^n)}.$$

We now note that the convergence property $u_{1,k} \rightarrow u_1$ in $L^2(\Omega_1)$ as $k \uparrow \infty$ guarantees that $\{u_{1,k}\}_{k \in \mathbb{N}}$ is uniformly bounded in $L^2(\Omega_1)$ so that there exists $\mathcal{M} > 0$ such that $\|u_{1,k} + u_1\|_{L^2(\Omega_1)} \leq \mathcal{M}$ for every $k \in \mathbb{N}$. This and $g \in L^\infty(\Omega_1)$ allow us to obtain

$$\begin{aligned} \left| \int_{\Omega_1} g(x)(u_{1,k}^2(x) - u_1^2(x))dx \right| &\leq \|g\|_{L^\infty(\Omega_1)} \int_{\Omega_1} |u_{1,k}^2(x) - u_1^2(x)|dx \\ &\leq \mathcal{M} \|g\|_{L^\infty(\Omega_1)} \left(\int_{\Omega_1} |u_{1,k}(x) - u_1(x)|^2 dx \right)^{\frac{1}{2}} \rightarrow 0, \quad k \uparrow \infty. \end{aligned}$$

As a result,

$$\text{I}_k = \int_{\Omega_1} u_{1,k}(x)^2 g(x) dx \rightarrow \int_{\Omega_1} \int_{\Omega_2} J(x-y)u_1(x)^2 dy dx, \quad k \uparrow \infty. \quad (5.3.12)$$

Similar arguments combined with Fubini's Theorem show that

$$\text{III}_k \rightarrow \int_{\Omega_1} \int_{\Omega_2} J(x-y)u_2(y)^2 dy dx, \quad k \uparrow \infty. \quad (5.3.13)$$

It only remains to analyze II_k . In the following, we prove that

$$\text{II}_k \rightarrow \text{II} := -2 \int_{\Omega_1} \int_{\Omega_2} J(x-y)u_1(x)u_2(y) dy dx, \quad k \uparrow \infty. \quad (5.3.14)$$

To this end, for each $k \in \mathbb{N}$, we define $\vartheta_k : \mathbb{R}^n \rightarrow \mathbb{R}$ by $\vartheta_k(x) = J * (u_{2,k}\chi_{\Omega_2})(x)$. We also define $\vartheta : \mathbb{R}^n \rightarrow \mathbb{R}$ by $\vartheta(x) = J * (u_2\chi_{\Omega_2})(x)$. We bound the difference $\vartheta - \vartheta_k$ in $L^2(\Omega_1)$ with the help of the continuity property of Lemma 5.2.1:

$$\|\vartheta - \vartheta_k\|_{L^2(\Omega_1)} \leq \|J\|_{L^1(\mathbb{R}^n)} \|u_2 - u_{2,k}\|_{L^2(\Omega_2)} \rightarrow 0, \quad k \uparrow \infty.$$

In particular, we have that $\{\vartheta_k\}_{k \in \mathbb{N}}$ is uniformly bounded in $L^2(\Omega_1)$. Consequently,

$$\begin{aligned} |\Pi_k - \Pi| &\leq 2 \int_{\Omega_1} |u_{1,k}(x) - u_1(x)| |\vartheta_k(x)| dx + 2 \int_{\Omega_1} |u_1(x)| |\vartheta_k(x) - \vartheta(x)| dx \\ &\leq 2 \|u_{1,k} - u_1\|_{L^2(\Omega_1)} \|\vartheta_k\|_{L^2(\Omega_1)} + 2 \|u_1\|_{L^2(\Omega_1)} \|\vartheta_k - \vartheta\|_{L^2(\Omega_1)} \rightarrow 0, \quad k \uparrow \infty. \end{aligned}$$

Consequently, $\Pi_k \rightarrow \Pi$ as $k \uparrow \infty$, as we intended to show.

The collection of the convergence properties (5.3.11)–(5.3.14) allows us to conclude that $E(u_k) \rightarrow E(u)$ as $k \uparrow \infty$. This concludes the proof. \square

5.3.3 Existence of a minimizer

We are now ready to provide one of the most important results of this section.

Theorem 5.3.3 (existence of a unique minimizer). *Let $s_1, s_2 \in (0, 1)$, let $f_1 \in L^2(\Omega_1)$, and let $f_2 \in L^2(\Omega_2)$. Then, there exists a unique minimizer u of \mathcal{E} in H .*

Proof. We proceed according to the direct method of the calculus of variations [36, Theorem 5.51]. To apply such a method, we need to verify that \mathcal{E} is proper, convex, lower semicontinuous, and coercive in H . Since \mathcal{E} is well-defined over H , it is trivial that \mathcal{E} is proper in H . The convexity and continuity of \mathcal{E} are a consequence of the results in Lemma 5.3.2. It remains to prove that \mathcal{E} is coercive in H .

Let $u = u_1 + u_2 \in H$, where $u_1 \in \tilde{H}^{s_1}(\Omega_1)$ and $u_2 \in \tilde{H}^{s_2}(\Omega_2)$. To prove the coercivity of \mathcal{E} , we will first prove that there exists a constant $C > 0$ such that

$$\begin{aligned} &\frac{C_1}{2} \int_{\Omega_1^c} \int_{\Omega_2^c} \frac{|u_1(x) - u_1(y)|^2}{|x - y|^{n+2s_1}} dy dx + \frac{C_2}{2} \int_{\Omega_1^c} \int_{\Omega_1^c} \frac{|u_2(x) - u_2(y)|^2}{|x - y|^{n+2s_2}} dy dx \\ &\geq \frac{C_1}{2} |u_1|_{\tilde{H}^{s_1}(\Omega)}^2 + \frac{C_2}{2} |u_2|_{\tilde{H}^{s_2}(\Omega)}^2 - C \left(\|u_1\|_{L^2(\Omega_1)}^2 + \|u_2\|_{L^2(\Omega_2)}^2 \right). \end{aligned} \quad (5.3.15)$$

As a first step in the derivation of (5.3.15), we use Fubini's theorem and the fact that $u_1 = 0$ in Ω_2 to obtain the identity

$$|u_1|_{\tilde{H}^{s_1}(\Omega_1)}^2 = \int_{\Omega_2^c} \int_{\Omega_2^c} \frac{|u_1(x) - u_1(y)|^2}{|x - y|^{n+2s_1}} dy dx + 2 \int_{\Omega_2^c} \int_{\Omega_2} \frac{|u_1(x) - u_1(y)|^2}{|x - y|^{n+2s_1}} dy dx.$$

Let us now control the second second term on the right-hand side of the previous expression. To this end, we first note that if $x \in \Omega_1$ and $y \in \Omega_2$, then $|x - y| \geq d = \text{dist}(\Omega_1, \Omega_2)$. We now further exploit that $u_1 = 0$ in Ω_2 and write

$$\int_{\Omega_2^c} \int_{\Omega_2} \frac{|u_1(x) - u_1(y)|^2}{|x - y|^{n+2s_1}} dy dx = \int_{\Omega_1} \int_{\Omega_2} \frac{|u_1(x)|^2}{|x - y|^{n+2s_1}} dy dx \leq |\Omega_2| d^{-n-2s_1} \|u_1\|_{L^2(\Omega_1)}^2.$$

As a result, we can obtain the following bound

$$|u_1|_{\dot{H}^{s_1}(\Omega_1)}^2 - 2|\Omega_2|d^{-n-2s_1}\|u_1\|_{L^2(\Omega_1)}^2 \leq \int_{\Omega_2^c} \int_{\Omega_2^c} \frac{|u_1(x) - u_1(y)|^2}{|x - y|^{n+2s_1}} dy dx. \quad (5.3.16)$$

Analogously, the following estimate can be derived

$$|u_2|_{\dot{H}^{s_2}(\Omega_2)}^2 - 2|\Omega_1|d^{-n-2s_2}\|u_2\|_{L^2(\Omega_2)}^2 \leq \int_{\Omega_1^c} \int_{\Omega_1^c} \frac{|u_2(x) - u_2(y)|^2}{|x - y|^{n+2s_2}} dy dx. \quad (5.3.17)$$

If we add the inequalities in (5.3.16) and (5.3.17), we obtain the desired estimate (5.3.15). With the bound (5.3.15) at hand, we use the non-negativity property of the kernel J to deduce that

$$E(u) \geq \frac{\mathcal{C}_1}{2}|u_1|_{\dot{H}^{s_1}(\Omega)}^2 + \frac{\mathcal{C}_2}{2}|u_2|_{\dot{H}^{s_2}(\Omega)}^2 - C \left(\|u_1\|_{L^2(\Omega_1)}^2 + \|u_2\|_{L^2(\Omega_2)}^2 \right).$$

We are now in a position to apply the estimate of Lemma 5.3.1 to conclude that

$$\mathfrak{C}E(u) \geq \frac{\mathcal{C}_1}{2}|u_1|_{\dot{H}^{s_1}(\Omega)}^2 + \frac{\mathcal{C}_2}{2}|u_2|_{\dot{H}^{s_2}(\Omega)}^2, \quad (5.3.18)$$

where $\mathfrak{C} = 1 + \mathcal{C}^{-1}C$. If we combine this bound with Hölder's inequality and Young's inequality, we can conclude that

$$\begin{aligned} \mathfrak{C}\mathcal{E}(u) &\geq \frac{\mathcal{C}_1}{2}|u_1|_{\dot{H}^{s_1}(\Omega)}^2 + \frac{\mathcal{C}_2}{2}|u_2|_{\dot{H}^{s_2}(\Omega)}^2 \\ &\quad - \mathfrak{C} \left(\epsilon_1 \|u_1\|_{L^2(\Omega_1)}^2 + \epsilon_2 \|u_2\|_{L^2(\Omega_2)}^2 + \frac{1}{4\epsilon_1} \|f_1\|_{L^2(\Omega_1)}^2 + \frac{1}{4\epsilon_2} \|f_2\|_{L^2(\Omega_2)}^2 \right), \end{aligned}$$

where $\epsilon_1, \epsilon_2 > 0$. We now apply the Poincaré inequality (5.2.4) and choose ϵ_1 and ϵ_2 sufficiently small to obtain

$$\mathcal{E}(u) \gtrsim |u_1|_{\dot{H}^{s_1}(\Omega)}^2 + |u_2|_{\dot{H}^{s_2}(\Omega)}^2 - \|f_1\|_{L^2(\Omega_1)}^2 - \|f_2\|_{L^2(\Omega_2)}^2.$$

We can thus conclude that \mathcal{E} is coercive in H , i.e., $\mathcal{E}(u) \rightarrow \infty$ as $|u|_H \rightarrow \infty$.

After we have proved that \mathcal{E} is coercive in H , the existence of a minimizer follows from the direct method of the calculus of variations. The uniqueness follows from the strict convexity of the functional \mathcal{E} . \square

5.3.4 The Euler–Lagrange equations

Let u be the minimizer of \mathcal{E} in H . Recall that the existence and uniqueness of such a minimizer is guaranteed by the results of Theorem 5.3.3. In the following, we derive the corresponding nonlocal coupled problem that solves u . Since u is a minimizer of \mathcal{E} in H , then for every function v in H , we

have that

$$\left. \frac{\partial}{\partial t} \mathcal{E}(u + tv) \right|_{t=0} = 0.$$

Elementary and standard calculations thus show that u solves the weak formulation:

$$u \in H : \quad \mathcal{A}(u, v) = \int_{\Omega_1} f_1(x)v_1(x)dx + \int_{\Omega_2} f_2(x)v_2(x)dx \quad \forall v \in H. \quad (5.3.19)$$

Here, $\mathcal{A} : H \times H \rightarrow \mathbb{R}$ is defined as

$$\begin{aligned} \mathcal{A}(u, v) = & \mathcal{C}_1 \int_{\Omega_2^c} \int_{\Omega_2^c} \frac{(u_1(x) - u_1(y))(v_1(x) - v_1(y))}{|x - y|^{n+2s_1}} dy dx \\ & + \mathcal{C}_2 \int_{\Omega_1^c} \int_{\Omega_1^c} \frac{(u_2(x) - u_2(y))(v_2(x) - v_2(y))}{|x - y|^{n+2s_2}} dy dx \\ & + \int_{\Omega_1} \int_{\Omega_2} J(x - y)(u_1(x) - u_2(y))(v_1(x) - v_2(y)) dy dx. \end{aligned} \quad (5.3.20)$$

From the weak formulation (5.3.19), we now obtain a nonlocal coupled system as follows: First, we consider $v \in H$ such that $v_2 = 0$ in (5.3.19) to arrive at

$$\begin{aligned} \mathcal{C}_1 \int_{\Omega_2^c} \int_{\Omega_2^c} \frac{(u_1(x) - u_1(y))(v_1(x) - v_1(y))}{|x - y|^{n+2s_1}} dy dx \\ + \int_{\Omega_1} \int_{\Omega_2} J(x - y)(u_1(x) - u_2(y))v_1(x) dy dx = \int_{\Omega_1} f_1(x)v_1(x) dx \end{aligned} \quad (5.3.21)$$

for all $v_1 \in \tilde{H}^{s_1}(\Omega_1)$. Second, we consider $v \in H$ such that $v_1 = 0$ in (5.3.19) to obtain

$$\begin{aligned} \mathcal{C}_2 \int_{\Omega_1^c} \int_{\Omega_1^c} \frac{(u_2(x) - u_2(y))(v_2(x) - v_2(y))}{|x - y|^{n+2s_2}} dy dx \\ + \int_{\Omega_1} \int_{\Omega_2} J(x - y)(u_2(y) - u_1(x))v_2(y) dy dx = \int_{\Omega_2} f_2(x)v_2(x) dx \end{aligned} \quad (5.3.22)$$

for all $v_2 \in \tilde{H}^{s_2}(\Omega_2)$.

Remark 5.3.4 (equivalence and weak formulation). The following comments are now appropriate.

1. The system (5.3.21)–(5.3.22) corresponds to a weak formulation of (2.2.2).
2. By construction, if $u = u_1 + u_2 \in H$ solves (5.3.19), then u_1 and u_2 solve the nonlocal coupled problem (5.3.21)–(5.3.22). On the other hand, since $H = \tilde{H}^{s_1}(\Omega_1) \oplus \tilde{H}^{s_2}(\Omega_2)$, (5.3.19) can be obtained by adding (5.3.21) and (5.3.22). Consequently, the problem (5.3.19) and the system (5.3.21)–(5.3.22) are equivalent.

Since it will be useful later, in the next lemma we present the most important properties that the form \mathcal{A} satisfies.

Lemma 5.3.5 (properties of \mathcal{A}). *The form \mathcal{A} is bilinear, continuous, and coercive in the space H .*

Proof. We divide the proof in three steps.

Step 1. The fact that \mathcal{A} is bilinear is trivial.

Step 2. We now prove the boundedness of \mathcal{A} in $H \times H$. To do this, we let $u, v \in H$ and first note that

$$\mathcal{C}_1 \int_{\Omega_2^c} \int_{\Omega_2^c} \frac{(u_1(x) - u_1(y))(v_1(x) - v_1(y))}{|x - y|^{n+2s_1}} dy dx \leq \mathcal{C}_1 |u_1|_{\tilde{H}^{s_1}(\Omega_1)} |v_1|_{\tilde{H}^{s_1}(\Omega_1)}, \quad (5.3.23)$$

$$\mathcal{C}_2 \int_{\Omega_1^c} \int_{\Omega_1^c} \frac{(u_2(x) - u_2(y))(v_2(x) - v_2(y))}{|x - y|^{n+2s_1}} dy dx \leq \mathcal{C}_2 |u_2|_{\tilde{H}^{s_2}(\Omega_2)} |v_2|_{\tilde{H}^{s_2}(\Omega_2)}. \quad (5.3.24)$$

We now analyze the nonlocal term that represents the coupling. For this purpose, we first bound it as

$$\begin{aligned} & \int_{\Omega_1} \int_{\Omega_2} J(x - y)(u_1(x) - u_2(y))(v_1(x) - v_2(y)) dy dx \\ & \leq \left[\int_{\Omega_1 \times \Omega_2} J(x - y) |u_1(x) - u_2(y)|^2 dy dx \right]^{\frac{1}{2}} \left[\int_{\Omega_1 \times \Omega_2} J(x - y) |v_1(x) - v_2(y)|^2 dy dx \right]^{\frac{1}{2}}. \end{aligned}$$

Let us now write the square of the first term on the left-hand side of the previous bound as follows:

$$\begin{aligned} & \int_{\Omega_1} \int_{\Omega_2} J(x - y)(u_1(x) - u_2(y))^2 dy dx = \int_{\Omega_1} \int_{\Omega_2} J(x - y) u_1^2(x) dy dx \\ & - 2 \int_{\Omega_1} \int_{\Omega_2} J(x - y) u_1(x) u_2(y) dy dx + \int_{\Omega_1} \int_{\Omega_2} J(x - y) u_2^2(y) dy dx := \mathbf{I}_1 - 2\mathbf{I}_2 + \mathbf{I}_3. \end{aligned} \quad (5.3.25)$$

The control of \mathbf{I}_1 , \mathbf{I}_2 , \mathbf{I}_3 follows from the arguments presented in the proof of Lemma 5.3.2. In the following, we briefly present the arguments. First, we have the bounds

$$\mathbf{I}_1 \leq \|J\|_{L^1(\mathbb{R}^n)} \|u_1\|_{L^2(\Omega_1)}^2, \quad \mathbf{I}_3 \leq \|J\|_{L^1(\mathbb{R}^n)} \|u_2\|_{L^2(\Omega_2)}^2. \quad (5.3.26)$$

Second, using the definition of convolution, the Cauchy–Schwartz inequality, the estimate of Lemma 5.2.1, and Young’s inequality, we arrive at

$$\begin{aligned} \mathbf{I}_2 &= \int_{\Omega_1} u_1(x) (J \star u_2 \chi_{\Omega_2})(x) dx \leq \|u_1\|_{L^2(\Omega_1)} \|J \star u_2 \chi_{\Omega_2}\|_{L^2(\mathbb{R}^n)} \\ &\leq \|u_1\|_{L^2(\Omega_1)} \|J\|_{L^1(\mathbb{R}^n)} \|u_2\|_{L^2(\Omega_2)} \leq \frac{\|J\|_{L^1(\mathbb{R}^n)}}{2} \left[\|u_1\|_{L^2(\Omega_1)}^2 + \|u_2\|_{L^2(\Omega_2)}^2 \right]. \end{aligned} \quad (5.3.27)$$

A collection of the bounds (5.3.23), (5.3.24), (5.3.26), and (5.3.27) yield the continuity of \mathcal{A} :

$$\mathcal{A}(u, v) \lesssim |u|_H |v|_H \quad \forall u, v \in H. \quad (5.3.28)$$

Step 3. Finally, we prove that the bilinear form \mathcal{A} is coercive in $H \times H$. Given $u \in H$, we immediately notice that from the definitions of E and \mathcal{A} given in (5.3.2) and (5.3.20), respectively, we deduce that $\mathcal{A}(u, u) = 2E(u)$. We thus use the bound (5.3.18) to obtain

$$\mathcal{A}(u, u) \gtrsim |u_1|_{\tilde{H}^{s_1}(\Omega)}^2 + |u_2|_{\tilde{H}^{s_2}(\Omega)}^2. \quad (5.3.29)$$

We have thus proved that \mathcal{A} is bilinear, bounded, and coercive in H . This concludes the proof. \square

Remark 5.3.6 (inner product). Since \mathcal{A} is symmetric in $H \times H$, the results of Lemma 5.3.5 immediately guarantee that \mathcal{A} defines an inner product in $H \times H$.

We conclude this section with the following stability bound.

Theorem 5.3.7 (stability bound). *The solution $u = u_1 + u_2 \in \tilde{H}^{s_1}(\Omega_1) \oplus \tilde{H}^{s_2}(\Omega_2)$ of (5.3.19), or equivalently of the coupled system (5.3.21) and (5.3.22), satisfies the following stability bound*

$$|u_1|_{\tilde{H}^{s_1}(\Omega_1)} + |u_2|_{\tilde{H}^{s_2}(\Omega_2)} \lesssim \|f_1\|_{L^2(\Omega_1)} + \|f_2\|_{L^2(\Omega_2)}, \quad (5.3.30)$$

with a hidden constant that is independent of u , f_1 , and f_2 .

Proof. Set $v = u \in H$ in (5.3.19) to obtain $\mathcal{A}(u, u) = (f_1, u_1)_{L^2(\Omega_1)} + (f_2, u_2)_{L^2(\Omega_2)}$. The coercivity property of the bilinear form \mathcal{A} stated in (5.3.29) combined with standard inequalities and the Poincaré inequality (5.2.4) yields the desired stability estimate (5.3.30). This concludes the proof. \square

5.3.5 Regularity properties

We now explore regularity results for the solution of (5.3.19).

Theorem 5.3.8 (Sobolev regularity). *Let $s_1, s_2 \in (0, 1)$, let $f_1 \in L^2(\Omega_1)$, and let $f_2 \in L^2(\Omega_2)$. Let $u = u_1 + u_2 \in H$ be the unique solution to (5.3.19). Let $i \in \{1, 2\}$. Then, $u_i \in H^{s_i + \kappa_i - \varepsilon_i}(\Omega_i)$ for all $0 < \varepsilon_i < s_i$, where $\kappa_i = \frac{1}{2}$ for $\frac{1}{2} < s_i < 1$ and $\kappa_i = s_i - \varepsilon_i$ for $0 < s_i \leq \frac{1}{2}$. In addition, we have the bound*

$$\|u_i\|_{H^{s_i + \kappa_i - \varepsilon_i}(\Omega_i)} \leq C_i \varepsilon_i^{-\nu_i} \|f_i\|_{L^2(\Omega_i)}, \quad \forall \varepsilon_i \in (0, s_i),$$

where $\nu_i = \frac{1}{2}$ for $\frac{1}{2} < s_i < 1$ and $\nu_i = \frac{1}{2} + \nu_{0,i}$ for $0 < s_i \leq \frac{1}{2}$. Here, $\nu_{0,i}$ and C_i denote positive constants that depend on Ω_i and n and Ω_i , n , and s_i , respectively.

Proof. We develop a proof for u_1 ; the argument for u_2 is similar. From the relation (5.3.21) we can conclude that the function u_1 corresponds to the weak solution of the following problem:

$$u_1 \in \tilde{H}^{s_1}(\Omega_1) : \quad (-\Delta)^{s_1} u_1 = u_1(2\mathcal{C}_1 h - g) + \vartheta + f_1 \text{ in } \Omega_1, \quad (5.3.31)$$

supplemented with the volumetric boundary condition $u_1 = 0$ in Ω_1^c . Here, the functions $h, g, \vartheta : \Omega_1 \rightarrow \mathbb{R}$ are defined as follows:

$$h(x) = \int_{\Omega_2} |x - y|^{-n-2s_1} dy, \quad g(x) = \int_{\Omega_2} J(x - y) dy, \quad \vartheta(x) = \int_{\Omega_2} J(x - y) u_2(y) dy.$$

Since $\text{dist}(\Omega_1, \Omega_2) = d > 0$, it is immediate that the function h belongs to $L^\infty(\Omega_1)$. On the other hand, since $J \in L^1(\mathbb{R}^n)$, the function $g \in L^\infty(\Omega_1)$. From this we deduce that $u_1(2\mathcal{C}_1 h - g) \in L^2(\Omega_1)$. An application of the continuity bound of Lemma 5.2.1 shows that $\vartheta \in L^2(\Omega_1)$. All these results reveal that the forcing term of problem (5.3.31) belongs to $L^2(\Omega_1)$ so that the desired regularity follows from a direct application of [17, Theorem 2.1]. This concludes the proof. \square

5.4 A finite element discretization

Under the additional assumption that Ω_1 and Ω_2 are both Lipschitz polytopes, we now introduce a finite element solution technique to approximate the solution of the coupled system (5.3.21)–(5.3.22). We start with some terminology and describe the construction of the underlying finite element spaces. For this purpose, we first introduce the families

$$\{\mathcal{T}_{1,h}\}_{h>0}, \quad \{\mathcal{T}_{2,\mathfrak{h}}\}_{\mathfrak{h}>0},$$

of conforming and quasi-uniform meshes of $\bar{\Omega}_1$ and $\bar{\Omega}_2$, respectively, made of closed simplices T . Here, $h := \max\{h_T : T \in \mathcal{T}_{1,h}\}$ and $\mathfrak{h} := \max\{h_T : T \in \mathcal{T}_{2,\mathfrak{h}}\}$ denote the mesh sizes of $\mathcal{T}_{1,h}$ and $\mathcal{T}_{2,\mathfrak{h}}$, respectively, and $h_T = \text{diam}(T)$.

Given the meshes $\mathcal{T}_{1,h}$ and $\mathcal{T}_{2,\mathfrak{h}}$, we introduce the finite element spaces

$$\begin{aligned} \mathbb{V}_{1,h} &:= \{v_h \in C(\bar{\Omega}_1) : v_h|_T \in \mathbb{P}_1(T) \forall T \in \mathcal{T}_{1,h}, v_h = 0 \text{ on } \partial\Omega_1\}. \\ \mathbb{V}_{2,\mathfrak{h}} &:= \{v_h \in C(\bar{\Omega}_2) : v_h|_T \in \mathbb{P}_1(T) \forall T \in \mathcal{T}_{2,\mathfrak{h}}, v_h = 0 \text{ on } \partial\Omega_2\}. \end{aligned} \tag{5.4.1}$$

Remark 5.4.1 (homogeneous Dirichlet boundary conditions). We note that we enforce a classical homogeneous Dirichlet boundary condition on $\partial\Omega_1$ ($\partial\Omega_2$) and that discrete functions in $\mathbb{V}_{1,h}$ ($\mathbb{V}_{2,\mathfrak{h}}$) can be trivially extended to Ω_1^c (Ω_2^c) by zero.

In view of the comments in Remark 5.4.1, $\mathbb{V}_{1,h} \subset \tilde{H}^{s_1}(\Omega_1)$ and $\mathbb{V}_{2,\mathfrak{h}} \subset \tilde{H}^{s_2}(\Omega_2)$ for every $s_1, s_2 \in (0, 1)$.

5.4.1 The discrete scheme

We propose the following finite element approximation of the system (5.3.21)–(5.3.22): Find $u_{1,h} \in \mathbb{V}_{1,h}$ and $u_{2,h} \in \mathbb{V}_{2,h}$ such that

$$\begin{aligned} \mathcal{C}_1 \int_{\Omega_2^c} \int_{\Omega_2^c} \frac{(u_{1,h}(x) - u_{1,h}(y))(v_{1,h}(x) - v_{1,h}(y))}{|x - y|^{n+2s_1}} dy dx \\ + \int_{\Omega_1} \int_{\Omega_2} J(x - y)(u_{1,h}(x) - u_{2,h}(y))v_{1,h}(x) dy dx = \int_{\Omega_1} f_1(x)v_{1,h}(x) dx \end{aligned} \quad (5.4.2)$$

for all $v_{1,h} \in \mathbb{V}_{1,h}$, and

$$\begin{aligned} \mathcal{C}_2 \int_{\Omega_1^c} \int_{\Omega_1^c} \frac{(u_{2,h}(x) - u_{2,h}(y))(v_{2,h}(x) - v_{2,h}(y))}{|x - y|^{n+2s_2}} dy dx \\ + \int_{\Omega_1} \int_{\Omega_2} J(x - y)(u_{2,h}(y) - u_{1,h}(x))v_{2,h}(y) dy dx = \int_{\Omega_2} f_2(x)v_{2,h}(x) dx \end{aligned} \quad (5.4.3)$$

for all $v_{2,h} \in \mathbb{V}_{2,h}$.

Define $\mathbb{H}_{h,h} := \mathbb{V}_{1,h} \oplus \mathbb{V}_{2,h}$. If we add the discrete equations (5.4.2) and (5.4.3) and exploit the structure of the discrete space $\mathbb{H}_{h,h}$, we obtain the following equivalent weak formulation:

$$u_{h,h} \in \mathbb{H}_{h,h} : \quad \mathcal{A}(u_{h,h}, v_{h,h}) = \int_{\Omega_1} f_1(x)v_{1,h}(x) dx + \int_{\Omega_2} f_2(x)v_{2,h}(x) dx \quad (5.4.4)$$

for all $v_{h,h} \in \mathbb{H}_{h,h}$.

We have the following result for the discrete problem introduced earlier.

Theorem 5.4.2 (existence and uniqueness). *The discrete problem (5.4.4), or equivalently the discrete coupled system (5.4.2) and (5.4.3), admits a unique solution $u_{h,h} = u_{1,h} + u_{2,h} \in \mathbb{H}_{h,h} = \mathbb{V}_{1,h} \oplus \mathbb{V}_{2,h}$. In addition, we have the following stability bound:*

$$|u_{1,h}|_{\tilde{H}^{s_1}(\Omega_1)} + |u_{2,h}|_{\tilde{H}^{s_2}(\Omega_2)} \lesssim \|f_1\|_{L^2(\Omega_1)} + \|f_2\|_{L^2(\Omega_2)}, \quad (5.4.5)$$

where the hidden constant is independent of $u_{1,h}$, $u_{2,h}$, f_1 , f_2 , and the discretization parameters h and h .

Proof. Since \mathcal{A} is bilinear, continuous, and coercive in $H \times H$ and the finite-dimensional space $\mathbb{H}_{h,h} \subset H$, the existence and uniqueness of a solution $u_{h,h}$ follows directly from an application of the Lax-Milgram lemma. The desired stability estimate (5.4.5) is obtained by substituting $v_{1,h} = u_{1,h}$ and $v_{2,h} = u_{2,h}$ into (5.4.4) and using the coercivity of the bilinear form \mathcal{A} from (5.3.29) as well as standard inequalities. This concludes the proof. \square

5.4.2 A priori error bounds

We begin this section with the following result.

Lemma 5.4.3 (Galerkin orthogonality). *Let $u \in H$ and let $u_{h,\mathfrak{h}} \in \mathbb{H}_{h,\mathfrak{h}}$ be the solutions of problems (5.3.19) and (5.4.4), respectively. Then,*

$$\mathcal{A}(u - u_{h,\mathfrak{h}}, v_{h,\mathfrak{h}}) = 0 \quad \forall v_{h,\mathfrak{h}} \in \mathbb{H}_{h,\mathfrak{h}}. \quad (5.4.6)$$

Proof. Since $\mathbb{H}_{h,\mathfrak{h}} \subset H$, we are allowed to set $v_{h,\mathfrak{h}} = v_{1,h} + v_{2,\mathfrak{h}} \in \mathbb{H}_{h,\mathfrak{h}}$ as a test function in problem (5.3.19) and obtain

$$\mathcal{A}(u, v_{h,\mathfrak{h}}) = \int_{\Omega_1} f_1(x) v_{1,h}(x) dx + \int_{\Omega_2} f_2(x) v_{2,\mathfrak{h}}(x) dx \quad \forall v_{h,\mathfrak{h}} \in \mathbb{H}_{h,\mathfrak{h}}.$$

If we subtract the discrete equation (5.4.4) from the previous relation, we obtain the desired Galerkin orthogonality property. \square

We now derive the following quasi-best approximation result.

Lemma 5.4.4 (Cea's lemma). *Let $u = u_1 + u_2 \in H$ and let $u_{h,\mathfrak{h}} = u_{1,h} + u_{2,\mathfrak{h}} \in \mathbb{H}_{h,\mathfrak{h}}$ be the solutions of problems (5.3.19) and (5.4.4), respectively. Then,*

$$|u_1 - u_{1,h}|_{\tilde{H}^{s_1}(\Omega_1)}^2 + |u_2 - u_{2,\mathfrak{h}}|_{\tilde{H}^{s_2}(\Omega_2)}^2 \lesssim \min_{v_{h,\mathfrak{h}} \in \mathbb{H}_{h,\mathfrak{h}}} \left(|u_1 - v_{1,h}|_{\tilde{H}^{s_1}(\Omega_1)}^2 + |u_2 - v_{2,\mathfrak{h}}|_{\tilde{H}^{s_2}(\Omega_2)}^2 \right). \quad (5.4.7)$$

Proof. The proof is standard. We present a brief proof for the sake of completeness. Relying on the coercivity and continuity of the bilinear form \mathcal{A} derived in Lemma 5.3.5 and the Galerkin orthogonality property (5.4.6), it follows that

$$\begin{aligned} |u - u_{h,\mathfrak{h}}|_H^2 &\lesssim \mathcal{A}(u - u_{h,\mathfrak{h}}, u - u_{h,\mathfrak{h}}) \\ &= \mathcal{A}(u - u_{h,\mathfrak{h}}, u - v_{h,\mathfrak{h}}) \lesssim |u - u_{h,\mathfrak{h}}|_H |u - v_{h,\mathfrak{h}}|_H \quad \forall v_{h,\mathfrak{h}} \in \mathbb{H}_{h,\mathfrak{h}}. \end{aligned}$$

This concludes the proof. \square

We are now in a position to state and prove the following a priori error bound.

Theorem 5.4.5 (a priori error bound). *Let $s_1, s_2 \in (0, 1)$, let $f_1 \in L^2(\Omega_1)$, and let $f_2 \in L^2(\Omega_2)$. Let $u = u_1 + u_2 \in H$ and let $u_{h,\mathfrak{h}} = u_{1,h} + u_{2,\mathfrak{h}} \in \mathbb{H}_{h,\mathfrak{h}}$ be the solutions of problems (5.3.19) and (5.4.4), respectively. Then, we have the following error bound*

$$|u_1 - u_{1,h}|_{\tilde{H}^{s_1}(\Omega_1)} + |u_2 - u_{2,\mathfrak{h}}|_{\tilde{H}^{s_2}(\Omega_2)} \lesssim h^{\gamma_1} |\log h|^{\varphi_1} \|f_1\|_{L^2(\Omega_1)} + \mathfrak{h}^{\gamma_2} |\log \mathfrak{h}|^{\varphi_2} \|f_2\|_{L^2(\Omega_2)}. \quad (5.4.8)$$

Here, for $i \in \{1, 2\}$, $\gamma_i = \min\{s_i, \frac{1}{2}\}$, $\varphi_i = \nu_i$ if $s_i \neq \frac{1}{2}$, $\varphi_i = 1 + \nu_i$ if $s_i = \frac{1}{2}$, and $\nu_i \geq \frac{1}{2}$ is the constant in Theorem 5.3.8.

Proof. With the quasi-best approximation estimate (5.4.7) in hand, we can consider $v_{h,\mathfrak{h}} = \Pi_h u_1 + \Pi_{\mathfrak{h}} u_2$, where Π_h and $\Pi_{\mathfrak{h}}$ denote appropriate quasi-interpolation operators [17] to obtain the bound

$$|u_1 - u_{1,h}|_{\tilde{H}^{s_1}(\Omega_1)}^2 + |u_2 - u_{2,\mathfrak{h}}|_{\tilde{H}^{s_2}(\Omega_2)}^2 \lesssim |u_1 - \Pi_h u_1|_{\tilde{H}^{s_1}(\Omega_1)}^2 + |u_2 - \Pi_{\mathfrak{h}} u_2|_{\tilde{H}^{s_2}(\Omega_2)}^2.$$

The desired estimate thus follows from the arguments given in the proof of [17, Theorem 3.5] in combination with the regularity results of Theorem 5.3.8. This concludes the proof. \square

5.5 Alternating schemes

Inspired by the method proposed by Schwarz in [98], we propose and analyze in this section the continuous alternating **Algorithm 1**, and prove that it converges to the solution of the coupled problem (5.3.21)–(5.3.22). Even though we already know that the coupled problem (5.3.21)–(5.3.22) has a unique solution (cf. Theorem 5.3.3), the analysis of this algorithm forms the basis for proposing and analyzing the fully discrete alternating **Algorithm 2**, which is an iterative algorithm that converges to the solution of the finite element approximation (5.4.2)–(5.4.3).

5.5.1 The continuous alternating scheme

We begin this section with the introduction of the continuous alternating scheme mentioned above (**Algorithm 1**).

Algorithm 1: The continuous alternating scheme

1 Input: $\Omega_1, \Omega_2 \subset \mathbb{R}^n$, $s_1, s_2 \in (0, 1)$, $f_1 \in L^2(\Omega_1)$, $f_2 \in L^2(\Omega_2)$, $J \in L^1(\mathbb{R}^n)$, and $u_2^0 \in \tilde{H}^{s_2}(\Omega_2)$.

2 Step 0: Define $u^0 = 0 + u_2^0 \in H$.

3 For $i = 1$ **until convergence do**

1: Find the solution $u_1^{2i-1} \in \tilde{H}^{s_1}(\Omega_1)$ of (5.3.21) with u_2 replaced by u_2^{2i-2} .

2: Define $u_2^{2i-1} = u_2^{2i-2} \in \tilde{H}^{s_2}(\Omega_2)$ and $u^{2i-1} = u_1^{2i-1} + u_2^{2i-1} \in H$.

3: Find the solution $u_2^{2i} \in \tilde{H}^{s_2}(\Omega_2)$ of (5.3.22) with u_1 replaced by u_1^{2i-1} .

4: Define $u_1^{2i} = u_1^{2i-1} \in \tilde{H}^{s_1}(\Omega_1)$ and $u^{2i} = u_1^{2i} + u_2^{2i} \in H$.

5.5.1.1 Solution operators

In order to present an analysis for **Algorithm 1**, we introduce the following suitable solution operators. Given forcing terms $f_1 \in L^2(\Omega_1)$ and $f_2 \in L^2(\Omega_2)$, we define

$$\mathcal{L}_{f_1} : L^2(\Omega_2) \rightarrow \tilde{H}^{s_1}(\Omega_1), \quad w_2 \mapsto \mathbf{u}_1 = \mathcal{L}_{f_1}(w_2), \quad (5.5.1)$$

$$\mathcal{L}_{f_2} : L^2(\Omega_1) \rightarrow \tilde{H}^{s_2}(\Omega_2), \quad w_1 \mapsto \mathbf{u}_2 = \mathcal{L}_{f_2}(w_1). \quad (5.5.2)$$

Here, \mathbf{u}_1 corresponds to the solution of (5.3.21), where u_2 is replaced by w_2 , and \mathbf{u}_2 corresponds to the solution of (5.3.22), where u_1 is replaced by w_1 . Let $i \in \{1, 2\}$. In the case that $f_i \equiv 0$, we simply write \mathcal{L}_i . We note that \mathcal{L}_i is a linear map.

We now prove that the operators \mathcal{L}_{f_1} and \mathcal{L}_{f_2} are well-defined and continuous.

Lemma 5.5.1 (\mathcal{L}_{f_1} and \mathcal{L}_{f_2}). *The operators \mathcal{L}_{f_1} and \mathcal{L}_{f_2} , defined in (5.5.1) and (5.5.2), respectively, are well-defined and satisfy the following bounds:*

$$|\mathcal{L}_{f_1}(w_2)|_{\tilde{H}^{s_1}(\Omega_1)} \lesssim \|f_1\|_{L^2(\Omega_1)} + \|J\|_{L^1(\mathbb{R}^n)} \|w_2\|_{L^2(\Omega_2)}, \quad (5.5.3)$$

$$|\mathcal{L}_{f_2}(w_1)|_{\tilde{H}^{s_2}(\Omega_2)} \lesssim \|f_2\|_{L^2(\Omega_2)} + \|J\|_{L^1(\mathbb{R}^n)} \|w_1\|_{L^2(\Omega_1)}. \quad (5.5.4)$$

Proof. We analyze the operator \mathcal{L}_{f_1} ; the analysis for the map \mathcal{L}_{f_2} is similar.

Let $w_2 \in L^2(\Omega_2)$. Define $\mathfrak{E}_1 : \tilde{H}^{s_1}(\Omega_1) \rightarrow \mathbb{R}_0^+$ and $\mathcal{E}_1 : \tilde{H}^{s_1}(\Omega_1) \rightarrow \mathbb{R}$ as follows:

$$\mathfrak{E}_1(\mathbf{u}_1) := \frac{\mathcal{C}_1}{2} \int_{\Omega_2^c} \int_{\Omega_2^c} \frac{(\mathbf{u}_1(x) - \mathbf{u}_1(y))^2}{|x - y|^{n+2s_1}} dy dx + \frac{1}{2} \int_{\Omega_1} \int_{\Omega_2} J(x - y) \mathbf{u}_1^2(x) dy dx, \quad (5.5.5)$$

$$\mathcal{E}_1(\mathbf{u}_1) := \mathfrak{E}_1(\mathbf{u}_1) - \int_{\Omega_1} f_1(x) \mathbf{u}_1(x) dx - \int_{\Omega_1} \int_{\Omega_2} J(x - y) w_2(y) \mathbf{u}_1(x) dy dx. \quad (5.5.6)$$

We first note that $\mathfrak{E}_1(\mathbf{u}_1) \geq \mathcal{C} \|\mathbf{u}_1\|_{L^2(\Omega_1)}^2$ for all $\mathbf{u}_1 \in \tilde{H}^{s_1}(\Omega_1)$. This result follows from an adaptation of the arguments elaborated in the proof of Lemma 5.3.1. With this bound in hand, we follow the arguments in the proof of Theorem 5.3.3 and use (5.3.16), the non-negativity property of the kernel J , and the continuity property of Lemma 5.2.1 to derive the following coercivity properties:

$$\mathfrak{E}_1(\mathbf{u}_1) \gtrsim |\mathbf{u}_1|_{\tilde{H}^{s_1}(\Omega_1)}^2 \quad \forall \mathbf{u}_1 \in \tilde{H}^{s_1}(\Omega_1), \quad (5.5.7)$$

$$\mathcal{E}_1(\mathbf{u}_1) \gtrsim |\mathbf{u}_1|_{\tilde{H}^{s_1}(\Omega_1)}^2 - \|f_1\|_{L^2(\Omega_1)}^2 - \|J\|_{L^1(\mathbb{R}^n)}^2 \|w_2\|_{L^2(\Omega_2)}^2 \quad \forall \mathbf{u}_1 \in \tilde{H}^{s_1}(\Omega_1). \quad (5.5.8)$$

Let us also note that \mathcal{E}_1 is convex and continuous in $\tilde{H}^{s_1}(\Omega_1)$. The fact that \mathcal{E}_1 is convex is clear and the fact that \mathcal{E}_1 is continuous follows from the arguments elaborated in the proof of Lemma 5.3.2. With all these properties in hand, the direct method of the calculus of variations allows us to deduce

the existence of a minimizer \mathbf{u}_1 of \mathcal{E}_1 in $\tilde{H}^{s_1}(\Omega_1)$. The strict convexity of \mathcal{E}_1 guarantees the uniqueness of \mathbf{u}_1 . Finally, as in section 5.3.4, it can be shown that \mathbf{u}_1 is the unique solution of the problem

$$\begin{aligned} \mathcal{C}_1 \int_{\Omega_2^c} \int_{\Omega_2^c} \frac{(\mathbf{u}_1(x) - \mathbf{u}_1(y))(v(x) - v(y))}{|x - y|^{n+2s_1}} dy dx \\ + \int_{\Omega_1} \int_{\Omega_2} J(x - y) \mathbf{u}_1(x) v(x) dy dx = \int_{\Omega_1} f_1(x) v(x) dx + \int_{\Omega_1} \int_{\Omega_2} J(x - y) w_2(y) v(x) dy dx \end{aligned} \quad (5.5.9)$$

for all $v \in \tilde{H}^{s_1}(\Omega_1)$, i.e., $\mathcal{L}_{f_1}(w_2) = \mathbf{u}_1$. This shows that \mathcal{L}_{f_1} is well-defined. We now set $v = \mathbf{u}_1$ in (5.5.9) and use the coercivity bound (5.5.7) and the Poincaré inequality (5.2.4) to obtain an stability bound for \mathbf{u}_1 that implies the desired one for \mathcal{L}_{f_1} . \square

Having defined the operators \mathcal{L}_{f_1} and \mathcal{L}_{f_2} , we can rewrite the continuous alternating scheme as follows: Given $u_2^0 \in \tilde{H}^{s_2}(\Omega_2)$, we define $u^0 = 0 + u_2^0 \in H$ and compute, for $i = 1$ until convergence,

$$\begin{aligned} u_1^{2i-1} &= \mathcal{L}_{f_1}(u_2^{2i-2}), & u_2^{2i-1} &= u_2^{2i-2}, & u^{2i-1} &= u_1^{2i-1} + u_2^{2i-1} \in H, \\ u_2^{2i} &= \mathcal{L}_{f_2}(u_1^{2i-1}), & u_1^{2i} &= u_1^{2i-1}, & u^{2i} &= u_1^{2i} + u_2^{2i} \in H. \end{aligned} \quad (5.5.10)$$

5.5.1.2 Definition of convergence and equivalence

We say that **Algorithm 1** converges if the sequence $\{u^i\}_{i \geq 0} \subset H$, where $u^i = u_1^i + u_2^i$, converges to the solution $u = u_1 + u_2$ of problem (5.3.21)–(5.3.22) in the following sense:

$$u^i \rightarrow u \text{ in } H, \quad i \uparrow \infty.$$

Remark 5.5.2 (equivalence). Let $f_1 \in L^2(\Omega_1)$ and let $f_2 \in L^2(\Omega_2)$. Let $u = u_1 + u_2$ be the unique solution of the coupled system (5.3.21)–(5.3.22). Let us consider the sequence $\{u^i\}_{i \geq 0}$ generated by the **Algorithm 1** as described in (5.5.10) with initial datum u_2^0 . Let us note that this sequence satisfies the following property: Given the initial datum $u_2 - u_2^0$, the sequence $\{u - u^i\}_{i \geq 0}$ verifies for every $i \geq 1$

$$\begin{aligned} u_1 - u_1^{2i-1} &= \mathcal{L}_{f_1}(u_2) - \mathcal{L}_{f_1}(u_2^{2i-2}) = \mathcal{L}_1(u_2 - u_2^{2i-2}), \\ u_2 - u_2^{2i-1} &= u_2 - u_2^{2i-2}, \\ u_2 - u_2^{2i} &= \mathcal{L}_{f_2}(u_1) - \mathcal{L}_{f_2}(u_1^{2i-1}) = \mathcal{L}_2(u_1 - u_1^{2i-1}), \\ u_1 - u_1^{2i} &= u_1 - u_1^{2i-1}. \end{aligned} \quad (5.5.11)$$

We recall that \mathcal{L}_i is defined as in (5.5.1)–(5.5.2) with $f_i \equiv 0$. Let us now similarly consider the sequence $\{u - u^i\}_{i \geq 0}$, which satisfies (5.5.11). It is clear that $\{u^i\}_{i \geq 0}$ verifies (5.5.10).

Given this equivalence, we will from now on assume that $f_1 \equiv f_2 \equiv 0$ and show the convergence of the

following iterative method: Given $u_2^0 \in \tilde{H}^{s_2}(\Omega_2)$, define $\mathbf{u}^0 = 0 + u_2^0 \in H$ and compute, for $i = 1$ until convergence,

$$\begin{aligned} \mathbf{u}_1^{2i-1} &= \mathcal{L}_1(\mathbf{u}_2^{2i-2}), & \mathbf{u}_2^{2i-1} &= \mathbf{u}_2^{2i-2}, & \mathbf{u}^{2i-1} &= \mathbf{u}_1^{2i-1} + \mathbf{u}_2^{2i-1} \in H, \\ \mathbf{u}_2^{2i} &= \mathcal{L}_2(\mathbf{u}_1^{2i-1}), & \mathbf{u}_1^{2i} &= \mathbf{u}_1^{2i-1}, & \mathbf{u}^{2i} &= \mathbf{u}_1^{2i} + \mathbf{u}_2^{2i} \in H. \end{aligned} \quad (5.5.12)$$

5.5.1.3 Convergence analysis

We define

$$V_1 = \{u = u_1 + u_2 \in H : u_1 = \mathcal{L}_1(u_2), u_2 \in \tilde{H}^{s_2}(\Omega_2)\}, \quad (5.5.13)$$

$$V_2 = \{u = u_1 + u_2 \in H : u_2 = \mathcal{L}_2(u_1), u_1 \in \tilde{H}^{s_1}(\Omega_1)\}. \quad (5.5.14)$$

Since \mathcal{L}_1 and \mathcal{L}_2 are both linear, we deduce that V_1 and V_2 are both subspaces of H .

Lemma 5.5.3 ($H = V_1 \oplus V_2$). *It holds that $H = V_1 \oplus V_2$.*

Proof. We divide the proof into two steps.

Step 1. $V_1 \cap V_2 = \{0\}$. Let $u = u_1 + u_2 \in V_1 \cap V_2$. From the definition of V_1 and V_2 we deduce that $u_1 = \mathcal{L}_1(u_2) \in \tilde{H}^{s_1}(\Omega_1)$ and $u_2 = \mathcal{L}_2(u_1) \in \tilde{H}^{s_2}(\Omega_2)$. We now use the definition of the operators \mathcal{L}_1 and \mathcal{L}_2 from (5.5.1) and (5.5.2), respectively (with $f_1 \equiv f_2 \equiv 0$), to conclude that $u = u_1 + u_2 \in H$ solves the coupled problem (5.3.21)–(5.3.22) with $f_1 \equiv f_2 \equiv 0$. Since this problem is well-posed, we immediately deduce that $u_1 \equiv u_2 \equiv 0$ and then that $u \equiv 0$.

Step 2. $H = V_1 + V_2$. Let $u = u_1 + u_2 \in H$. We need to prove that there exist $v \in V_1$ and $w \in V_2$ such that $u = v + w$. By the definition of the subspaces V_1 and V_2 , we have that $v = v_1 + v_2$, where $v_1 = \mathcal{L}_1(v_2)$, and that $w = w_1 + w_2$, where $w_2 = \mathcal{L}_2(w_1)$. In view of this fact, in the following we prove the existence of v_2 and w_1 , so that $u_1 = \mathcal{L}_1(v_2) + w_1$ and $u_2 = v_2 + \mathcal{L}_2(w_1)$.

Step 2.1 Existence of v_2 . Applying the linear operator \mathcal{L}_2 to the relation $u_1 = \mathcal{L}_1(v_2) + w_1$ gives $\mathcal{L}_2(u_1) = \mathcal{L}_2(\mathcal{L}_1(v_2)) + \mathcal{L}_2(w_1)$. We use this relation and $u_2 = v_2 + \mathcal{L}_2(w_1)$ to obtain $\mathcal{L}_2(u_1) = \mathcal{L}_2(\mathcal{L}_1(v_2)) + u_2 - v_2$, which can be rewritten as follows:

$$u_2 - \mathcal{L}_2(u_1) = v_2 - \mathcal{L}_2(\mathcal{L}_1(v_2)) = (I - \mathcal{L}_2 \circ \mathcal{L}_1)v_2. \quad (5.5.15)$$

We now note that the operator $\mathcal{L}_2 \circ \mathcal{L}_1 : L^2(\Omega_2) \rightarrow \tilde{H}^{s_2}(\Omega_2)$. Given the compact embedding $\tilde{H}^{s_2}(\Omega_2) \hookrightarrow L^2(\Omega_2)$, we can consider $\mathcal{L}_2 \circ \mathcal{L}_1 : L^2(\Omega_2) \rightarrow L^2(\Omega_2)$, which is thus a linear and compact operator.

With this setting in hand, let us now show that there exists a unique v_2 that verifies (5.5.15). To accomplish this task, we will rely on the Fredholm alternative and prove that $\text{Ker}(I - \mathcal{L}_2 \circ \mathcal{L}_1) = \{0\}$ to conclude that $(I - \mathcal{L}_2 \circ \mathcal{L}_1)$ is a bijection. Let $\mathbf{v}_2 \in L^2(\Omega_2)$ be such that $(I - \mathcal{L}_2 \circ \mathcal{L}_1)\mathbf{v}_2 = 0$.

This implies that $(\mathcal{L}_2 \circ \mathcal{L}_1)(\mathbf{v}_2) = \mathbf{v}_2$. Using the definition of the linear maps \mathcal{L}_1 and \mathcal{L}_2 , we can show that $\mathcal{L}_1(\mathbf{v}_2) + \mathbf{v}_2 = \mathcal{L}_1(\mathbf{v}_2) + \mathcal{L}_2(\mathcal{L}_1(\mathbf{v}_2)) \in H$ solves the coupled problem (5.3.21)–(5.3.22) with $f_1 \equiv f_2 \equiv 0$. Since this problem is well-posed, we can thus conclude that $\mathbf{v}_2 = 0$. It follows that that $I - \mathcal{L}_2 \circ \mathcal{L}_1 : L^2(\Omega_2) \rightarrow L^2(\Omega_2)$ is a bijection and thus that there exists a unique v_2 that verifies (5.5.15).

Step 2.2 Existence of w_1 . With v_2 in hand, we consider $w_1 \in \tilde{H}^{s_1}(\Omega_1)$ as

$$w_1 = u_1 - \mathcal{L}_1(v_2). \quad (5.5.16)$$

Step 2.3. $u = v + w$. From the relations (5.5.15) and (5.5.16) we obtain

$$u_1 = w_1 + \mathcal{L}_1(v_2), \quad u_2 = v_2 + \mathcal{L}_2(w_1),$$

as we intended to show. □

To continue our analysis, we define the linear operators

$$\mathcal{P}_1 : H \rightarrow H, \quad u = u_1 + u_2 \mapsto \mathcal{P}_1(u) = \mathcal{L}_1(u_2) + u_2, \quad (5.5.17)$$

$$\mathcal{P}_2 : H \rightarrow H, \quad u = u_1 + u_2 \mapsto \mathcal{P}_2(u) = u_1 + \mathcal{L}_2(u_1). \quad (5.5.18)$$

We note that \mathcal{P}_1 and \mathcal{P}_2 are linear. In the following, we analyze orthogonality properties of these operators with respect to the bilinear form \mathcal{A} , which induces an inner product in $H \times H$; see Remark 5.3.6. For this purpose, we define

$$\langle u, v \rangle_{\mathcal{A}} := \mathcal{A}(u, v), \quad |v|_{\mathcal{A}} := \langle v, v \rangle_{\mathcal{A}}^{\frac{1}{2}} \quad \forall u, v \in H. \quad (5.5.19)$$

Lemma 5.5.4 (orthogonal projection). *Let $i \in \{1, 2\}$. The map \mathcal{P}_i is an orthogonal projection from H onto V_i with respect to the inner product defined in (5.5.19).*

Proof. We analyze the operator $\mathcal{P}_1 : H \rightarrow V_1$; the analysis for \mathcal{P}_2 is similar.

The fact that \mathcal{P}_1 is a projection is trivial. In fact, $\mathcal{P}_1(\mathcal{P}_1(u)) = \mathcal{P}_1(\mathcal{L}_1(u_2) + u_2) = \mathcal{L}_1(u_2) + u_2 = \mathcal{P}_1(u)$ for all $u = u_1 + u_2 \in H$. We now prove that $\mathcal{P}_1 : H \rightarrow V_1$ is an orthogonal map with respect to the inner product $\langle \cdot, \cdot \rangle_{\mathcal{A}}$ defined in (5.5.19). Given $u = u_1 + u_2 \in H$, we prove below that

$$\langle u - \mathcal{P}_1(u), v \rangle_{\mathcal{A}} = \langle (u_1 - \mathcal{L}_1(u_2)) + 0, v \rangle_{\mathcal{A}} = 0 \quad \forall v = \mathcal{L}_1(v_2) + v_2 \in V_1. \quad (5.5.20)$$

Denote $U_1 = u_1 - \mathcal{L}_1(u_2) \in \tilde{H}^{s_1}(\Omega_1)$. From the definition of the inner product $\langle \cdot, \cdot \rangle_{\mathcal{A}}$ and the bilinear

form $\mathcal{A}(\cdot, \cdot)$, it follows that

$$\begin{aligned} \langle u - \mathcal{P}_1(u), v \rangle_{\mathcal{A}} &= \mathcal{C}_1 \int_{\Omega_2^s} \int_{\Omega_2^s} \frac{(U_1(x) - U_1(y))(\mathcal{L}_1(v_2)(x) - \mathcal{L}_1(v_2)(y))}{|x - y|^{n+2s_1}} dy dx \\ &\quad + \int_{\Omega_1} \int_{\Omega_2} J(x - y) U_1(x) (\mathcal{L}_1(v_2)(x) - v_2(y)) dy dx. \end{aligned}$$

On the other hand, let us note that $\mathcal{L}_1(v_2) \in \tilde{H}^{s_1}(\Omega_1)$ solves the following problem:

$$\begin{aligned} \mathcal{C}_1 \int_{\Omega_2^s} \int_{\Omega_2^s} \frac{(\mathcal{L}_1(v_2)(x) - \mathcal{L}_1(v_2)(y))(z_1(x) - z_1(y))}{|x - y|^{n+2s_1}} dy dx \\ + \int_{\Omega_1} \int_{\Omega_2} J(x - y) ((\mathcal{L}_1(v_2)(x) - v_2(y)) z_1(x)) dy dx = 0 \quad \forall z_1 \in \tilde{H}^{s_1}(\Omega_1). \end{aligned}$$

If we set $z_1 = U_1$ in the previous weak formulation, we obtain (5.5.20), as we intended to show. This shows that $\mathcal{P}_1 : H \rightarrow V_1$ is an orthogonal map with respect to $\langle \cdot, \cdot \rangle_{\mathcal{A}}$. \square

To present the main result of this section, we note that the elements of the sequence $\{u^i\}_{i \geq 0}$ defined in (5.5.12) can be rewritten in terms of the orthogonal projections \mathcal{P}_1 and \mathcal{P}_2 , defined in (5.5.17) and (5.5.18), respectively, as follows: $u^0 = 0 + u_2^0$ and for every $i \geq 1$,

$$u^{2i-1} = \mathcal{L}_1(u_2^{2i-2}) + u_2^{2i-2} = \mathcal{P}_1(u^{2i-2}), \quad (5.5.21)$$

$$u^{2i} = u_1^{2i-1} + \mathcal{L}_2(u_1^{2i-1}) = \mathcal{P}_2(u^{2i-1}). \quad (5.5.22)$$

Theorem 5.5.5 (convergence of **Algorithm 1**). *Given $f_1 \in L^2(\Omega_1)$ and $f_2 \in L^2(\Omega_2)$, the sequence $\{u^i\}_{i \geq 0}$ generated by the continuous alternating method described in (5.5.10) converges to the unique minimizer $u = u_1 + u_2 \in H$ of the energy \mathcal{E} defined in (5.3.3), which is characterized as the unique solution of the system (5.3.21)–(5.3.22):*

$$u_1^i \rightarrow u_1 \text{ in } \tilde{H}^{s_1}(\Omega_1), \quad u_2^i \rightarrow u_2 \text{ in } \tilde{H}^{s_2}(\Omega_2), \quad i \uparrow \infty. \quad (5.5.23)$$

Moreover, the method is geometrically convergent: there exists $\kappa \in (0, 1)$ such that

$$|u_1 - u_1^i|_{\tilde{H}^{s_1}(\Omega_1)} + |u_2 - u_2^i|_{\tilde{H}^{s_2}(\Omega_2)} \lesssim \kappa^i \quad \forall i \in \mathbb{N}. \quad (5.5.24)$$

Proof. Given the equivalence presented in Remark 5.5.2, we assume that $f_1 \equiv 0$ and $f_2 \equiv 0$ and analyze the convergence of the sequence $\{u^i\}_{i \geq 0}$ described in (5.5.12). Let us note that within this setting, i.e., $f_1 \equiv 0$ and $f_2 \equiv 0$, the unique solution of the coupled problem (5.3.21)–(5.3.22) is $u \equiv 0 + 0$.

Given the previously derived results (Lemma 5.5.3 and Lemma 5.5.4) we are able to apply [79, Theorem

I.1] and obtain convergence. In fact, \mathcal{P}_i , for $i \in \{1, 2\}$, is an orthogonal projection from H onto V_i with respect to the inner product $\langle \cdot, \cdot \rangle_{\mathcal{A}}$ (cf. Lemma 5.5.4) and $H = V_1^\top \oplus V_2^\top$ (cf. Lemma 5.5.3 combined with [19, Corollary 2.15]). We can thus obtain that

$$\mathbf{u}^i \rightarrow 0 \text{ in } (H, |\cdot|_{\mathcal{A}}) \text{ as } i \uparrow \infty, \quad |\mathbf{u}^{i+1}|_{\mathcal{A}} \leq \kappa^i |\mathbf{u}^0|_{\mathcal{A}} \quad \forall i \geq 0.$$

From this we can deduce that $\mathbf{u}_1^i \rightarrow 0$ in $\tilde{H}^{s_1}(\Omega_1)$ and that $\mathbf{u}_2^i \rightarrow 0$ in $\tilde{H}^{s_2}(\Omega_2)$ as $i \uparrow \infty$. Let us now use the coercivity property (5.3.29) of the bilinear form \mathcal{A} to obtain

$$|\mathbf{u}_1^{i+1}|_{\tilde{H}^{s_1}(\Omega_1)}^2 + |\mathbf{u}_2^{i+1}|_{\tilde{H}^{s_2}(\Omega_2)}^2 \lesssim |\mathbf{u}^{i+1}|_{\mathcal{A}}^2 \lesssim \kappa^{2i} \quad \forall i \geq 0.$$

This directly implies that $|\mathbf{u}_1^i|_{\tilde{H}^{s_1}(\Omega_1)} \lesssim \kappa^i$ and $|\mathbf{u}_2^i|_{\tilde{H}^{s_2}(\Omega_2)} \lesssim \kappa^i$ for all $i \geq 0$. \square

5.5.2 The discrete alternating scheme

In this section, we present the discrete alternating **Algorithm 2**, which is an iterative method for solving the finite element discretization (5.4.2)–(5.4.3).

Algorithm 2: The discrete alternating scheme

- 1 **Input:** $\Omega_1, \Omega_2 \subset \mathbb{R}^n$, $s_1, s_2 \in (0, 1)$, $f_1 \in L^2(\Omega_1)$, $f_2 \in L^2(\Omega_2)$, $J \in L^1(\mathbb{R}^n)$, $\mathcal{T}_{1,h}$, $\mathcal{T}_{2,h}$, and $u_{2,h}^0 \in \mathbb{V}_{2,h}$.
 - 2 **Step 0:** Define $u_{h,h}^0 = 0 + u_{2,h}^0 \in H_{h,h}$.
 - 3 **For** $i = 1$ **until convergence do**
 - 1: Find the solution $u_{1,h}^{2i-1} \in \mathbb{V}_{1,h}$ of (5.4.2) with $u_{2,h}$ replaced by $u_{2,h}^{2i-2}$.
 - 2: Define $u_{2,h}^{2i-1} = u_{2,h}^{2i-2} \in \mathbb{V}_{2,h}$ and $u_{h,h}^{2i-1} = u_{1,h}^{2i-1} + u_{2,h}^{2i-1} \in \mathbb{H}_{h,h}$.
 - 3: Find the solution $u_{2,h}^{2i} \in \mathbb{V}_{2,h}$ of (5.4.3) with $u_{1,h}$ replaced by $u_{1,h}^{2i-1}$.
 - 4: Define $u_{1,h}^{2i} = u_{1,h}^{2i-1} \in \mathbb{V}_{1,h}$ and $u_{h,h}^{2i} = u_{1,h}^{2i} + u_{2,h}^{2i} \in \mathbb{H}_{h,h}$.
-

According to the arguments developed in section 5.5.1, the following results emerge.

Theorem 5.5.6 (convergence of **Algorithm 2**). *Given $f_1 \in L^2(\Omega_1)$ and $f_2 \in L^2(\Omega_2)$, the sequence $\{u_{h,h}^i\}_{i \geq 0}$ generated by the discrete alternating method described in **Algorithm 2** converges to the unique solution of the discrete coupled system (5.4.2)–(5.4.3):*

$$u_{1,h}^i \rightarrow u_{1,h} \text{ in } \mathbb{V}_{1,h}, \quad u_{2,h}^i \rightarrow u_{2,h} \text{ in } \mathbb{V}_{2,h}, \quad i \uparrow \infty.$$

Moreover, the method is geometrically convergent: there exists $\kappa \in (0, 1)$ such that

$$|u_{1,h} - u_{1,h}^i|_{\tilde{H}^{s_1}(\Omega_1)} + |u_{2,h} - u_{2,h}^i|_{\tilde{H}^{s_2}(\Omega_2)} \lesssim \kappa^i \quad \forall i \in \mathbb{N}.$$

5.6 Numerical experiment

In this section, we present a numerical experiment that illustrates the performance of the a priori error bounds derived in Section 5.4.2. The experiment was performed with a MATLAB code that implements a modified version of the fractional Laplacian problem from [2] in combination with the alternating method described in Algorithm 2. Since the domains are separated, the integrals associated with the interactions governed by the kernel J are approximated by Gaussian quadratures.

The setting of this experiment is as follows: we set $d = 2$ and consider

$$x_1 = -x_2 = (1, 1), \quad \Omega_1 = B_1(x_1), \quad \Omega_2 = B_1(x_2), \quad s_1, s_2 \in \{0.2, 0.4, 0.6, 0.8\}.$$

We also consider the kernel $J = \chi_{B_6(0)} |B_6(0)|^{-1}$, where $\chi_{B_6(0)}$ denotes the characteristic function supported at $B_6(0)$.

The exact solutions \bar{u}_1 and \bar{u}_2 are given by

$$\begin{aligned} \bar{u}_1(x) &= (2^{2s_1} \Gamma^2(1 + s_1))^{-1} (1 - |x - x_1|_+^{2s_1}), \\ \bar{u}_2(x) &= (2^{2s_2} \Gamma^2(1 + s_2))^{-1} (1 - |x - x_2|_+^{2s_2}), \quad t_+ = \max\{0, t\}. \end{aligned}$$

For this example, the source terms f_1 and f_2 are computed as:

$$f_1(x) = 1 - \bar{u}_1(x)(2\mathcal{C}_1 h_1(x) - g_1) - \vartheta_1, \quad f_2(x) = 1 - \bar{u}_2(x)(2\mathcal{C}_2 h_2(x) - g_2) - \vartheta_2,$$

where

$$\begin{aligned} h_1(x) &= \int_{\Omega_2} |x - y|^{-2-2s_1} dy, & h_2(x) &= \int_{\Omega_1} |x - y|^{-2-2s_2} dy, \\ \vartheta_1(x) &= \frac{1}{|B_6(0)|} \int_{\Omega_2} \bar{u}_2(y) dy, & \vartheta_2(x) &= \frac{1}{|B_6(0)|} \int_{\Omega_1} \bar{u}_1(y) dy, \end{aligned}$$

and $g_1 = g_2 = 1/36$.

In this experiment, we consider one mesh parameter for Ω_1 and Ω_2 , that is: $h = \mathfrak{h}$. Finally, we define $e_{i,h} := u_i - u_{i,h}$, with $i \in \{1, 2\}$.

In Table 5.1 we present the errors $|e_{1,h}|_{H^{s_1}(\Omega_1)}$ and $|e_{2,h}|_{H^{s_2}(\Omega_2)}$ as well as the experimental convergence rates for different combinations of $s_1, s_2 \in \{0.2, 0.4, 0.6, 0.8\}$ and for different mesh sizes h . We observe

(s_1, s_2)	$ e_{i,h} $	mesh size h	0.8	0.4	0.2	0.1	0.05	Rate
(0.2, 0.4)	$ e_{1,h} _{H^{s_1}(\Omega_1)}$		0.8875	0.7409	0.5413	0.3764	0.2745	0.4884
	$ e_{2,h} _{H^{s_2}(\Omega_2)}$		0.7555	0.5809	0.4018	0.2609	0.1678	0.5775
(0.2, 0.6)	$ e_{1,h} _{H^{s_1}(\Omega_1)}$		0.8878	0.7431	0.5474	0.3883	0.2926	0.4682
	$ e_{2,h} _{H^{s_2}(\Omega_2)}$		0.6122	0.4266	0.2741	0.1623	0.0824	0.6971
(0.2, 0.8)	$ e_{1,h} _{H^{s_1}(\Omega_1)}$		0.8882	0.7451	0.5523	0.3971	0.3053	0.4275
	$ e_{2,h} _{H^{s_2}(\Omega_2)}$		0.4796	0.2931	0.1614	0.0671	0.0543	0.7855
(0.4, 0.6)	$ e_{1,h} _{H^{s_1}(\Omega_1)}$		0.7559	0.5854	0.4146	0.2867	0.2101	0.4902
	$ e_{2,h} _{H^{s_2}(\Omega_2)}$		0.6125	0.4288	0.2809	0.1774	0.1121	0.6623
(0.4, 0.8)	$ e_{1,h} _{H^{s_1}(\Omega_1)}$		0.7563	0.5873	0.4192	0.2950	0.2224	0.4524
	$ e_{2,h} _{H^{s_2}(\Omega_2)}$		0.4798	0.2952	0.1687	0.0879	0.0252	1.024
(0.6, 0.8)	$ e_{1,h} _{H^{s_1}(\Omega_1)}$		0.6123	0.4327	0.2911	0.1970	0.1441	0.5308
	$ e_{2,h} _{H^{s_2}(\Omega_2)}$		0.4801	0.2970	0.1746	0.1018	0.05962	0.7563

Table 5.1: Experimental rates of convergence for $|e_{1,h}|_{H^{s_1}(\Omega_1)}$ and $|e_{2,h}|_{H^{s_2}(\Omega_2)}$ for $s_1, s_2 \in \{0.2, 0.4, 0.6, 0.8\}$. The expected convergence rate is 0.5.

that the experimental convergence rates agree with the theoretical result provided in Section 5.4.2. It is noteworthy that the asymptotic convergence rate is reached faster for the solution corresponding to the smaller fractional parameter.

Chapter 6

Conclusions

We have provided a complete study of three optimal control problems and one nonlocal coupled system. We have also analyzed finite element discretization schemes for each problem and provide convergence and a priori error estimates results.

In Chapter 2, we have employed the branches solution framework general and elaborated results, particularly the existence of optimal solutions and optimality conditions. The corresponding optimal control problem have result in a finite dimensional instance. As for the problems investigated in Chapters 3 and 4, we required more sophisticated tools to derive second order optimality conditions and a priori error estimates. All of this put in evidence that the treatment to investigate optimal control problems is far from being a standard research. Finally, in Chapter 5, we have investigated a nonlocal coupled system, providing existence, uniqueness and regularity of solution properties, together with a complete analysis of the discretization scheme. We have also studied an alternating method to approximate the continuous and discrete solutions, and derived convergence properties. This work constitute a first contribution in the study of coupled behaviours under purely nonlocal phenomena. In terms of future work, this thesis provide significant tools to develop an important number of research. In the following we adress few of them.

- Numerical analysis (a priori and a posteriori) of the optimal control problem for the Navier-Stokes equations with singular sources.
- Numerical analysis for optimal control problems for nonlocal coupled systems.
- Coupled systems involving fluid mechanics and nonlocal diffusion models.

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