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ON BRANNAN'S COEFFICIENT CONJECTURE AND APPLICATIONS

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Abstract. D. Brannan's conjecture says that for $0 < \alpha, \beta \leq 1$, $|x| = 1$, and $n \in \mathbb{N}$ one has $|A_{2n-1}(\alpha, \beta, x)| \leq |A_{2n-1}(\alpha, \beta, 1)|$, where

$$\frac{(1+xz)^\alpha}{(1-z)^\beta} = \sum_{n=0}^{\infty} A_n(\alpha, \beta, x) z^n.$$

We prove this for the case $\alpha = \beta$, and also prove a differentiated version of the Brannan conjecture. This has applications to estimates for Gegenbauer polynomials and also to coefficient estimates for univalent functions in the unit disk that are 'starlike with respect to a boundary point'. The latter application has previously been conjectured by H. Silverman and E. Silvia. The proofs make use of various properties of the Gauss hypergeometric function.

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1. Introduction. Let

$$\frac{(1+xz)^\alpha}{(1-z)^\beta} = \sum_{n=0}^{\infty} A_n(\alpha, \beta, x) z^n, \quad \alpha, \beta > 0. \tag{1.1}$$

In the context of coefficient problems for functions with bounded boundary rotation the question arose for which combinations of $\alpha, \beta, n > 0$ the relation

$$|A_n(\alpha, \beta, x)| \leq |A_n(\alpha, \beta, 1)|, \quad |x| = 1, \tag{1.2}$$

will hold. Special cases have been discussed in various papers. After the verification of (1.2) for $n \leq 13$ (assuming $\alpha = \beta \geq 1$) by D. Brannan, J. G. Clunie & W. E. Kirwan [**5**] the case $\alpha \geq 1$ (with $\beta = 1$, $n \in \mathbb{N}$) were established by D. Aharonov and S. Friedland [**2**], and these imply (1.2) for $\alpha, \beta \geq 1$, $n \in \mathbb{N}$. Later Brannan [**4**] showed that the

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situation changes for values of $\alpha < 1$ and/or $\beta < 1$. He proved that for each α, with $0 < \alpha < 1$, there exists $n_\alpha \in \mathbb{N}$ such that

$$\max_{|x|=1} |A_{2n}(\alpha, 1, x)| > A_{2n}(\alpha, 1, 1), \quad n > n_\alpha. \tag{1.3}$$

Based on a theorem for $n = 3$ and numerical data for larger n Brannan then made the following conjecture.

Conjecture 1. (Brannan [**4**]). The relation (1.2) holds for $0 < \alpha, \beta \le 1$ and odd n.

This conjecture, for the special case $\beta = 1$, has been established by Brannan [**4**] ($n = 3$), J. G. Milcetich [**7**] ($n = 5$) and recently by R. W. Barnard, K. Pearce and W. Wheeler [**3**] ($n = 7$). For the cases $\alpha = \beta$ it has been verified by H. Silverman and E. Silvia [**8**] ($n = 3$; for the context of their work compare Section 3) and very recently by R. Geisler [**6**] for $n \le 33$, making heavy use of computer algebra.

Our main result here is the following theorem.

Theorem 1. *The Brannan conjecture holds for the cases* $\alpha = \beta =: \lambda \in (0, 1)$*, i.e.*

$$|A_{2n-1}(\lambda, \lambda, x)| \le |A_{2n-1}(\lambda, \lambda, 1)|, \quad |x| = 1, \ n \in \mathbb{N}. \tag{1.4}$$

Another question, related to the subject of the Brannan conjecture, arises for the coefficients of

$$\frac{1}{(1 + xz)^\alpha (1 - z)^\beta} = \sum_{n=0}^{\infty} A_n(-\alpha, \beta, x) z^n.$$

It is clear that for $\alpha, \beta > 0$ and $n \in \mathbb{N}$ we always have

$$\max_{|x|=1} |A_n(-\alpha, \beta, x)| = |A_n(-\alpha, \beta, -1)|,$$

so it seems that nothing interesting is to be discovered here. However, the situation changes if we look at the quantities

$$\max_{|x|=1} q(x)^\gamma |A_n(-\alpha, \beta, x)| \tag{1.5}$$

for some positive γ and

$$q(x) := \left| \frac{1 + x}{2} \right|.$$

Now we find something similar to the assertion in the Brannan conjecture.

Theorem 2. *For* $0 < \alpha, \beta$ *with* $\alpha + \beta \le 2$ *and* $n \in \mathbb{N}$ *we have*

$$q(x)^{\frac{\alpha+\beta}{2}} |A_{2n}(-\alpha, \beta, x)| \le |A_{2n}(-\alpha, \beta, 1)|, \quad |x| = 1. \tag{1.6}$$

Note that $A_{2n-1}(-\alpha, \alpha, 1) = 0$ for $\alpha > 0$, so that nothing like (1.6) can be expected for odd indices in general.

Writing $A'_n(\alpha, \beta, x) := \frac{\partial}{\partial x} A_n(\alpha, \beta, x)$ we find that

$$(1 - \alpha) A_n(-\alpha, \beta, x) = A'_{n+1}(1 - \alpha, \beta, x).$$

Looking at Theorem 2 in this way, it actually turns out to be a differentiated version of the inequality in Brannan's conjecture, with slightly different restrictions on α, β.

COROLLARY 1. *For* $-1 < \alpha < 1$ *and* $0 < \beta \leq 1 + \alpha$ *we have*

$$q(x)^{\frac{1-\alpha+\beta}{2}} |A'_{2n-1}(\alpha, \beta, x)| \leq |A'_{2n-1}(\alpha, \beta, 1)|, \quad |x| = 1, \ n \in \mathbb{N}. \tag{1.7}$$

Theorem 2, for $\alpha = \beta$, has an interesting interpretation in terms of the Gegenbauer orthogonal polynomial system. These polynomials $C_n^{(\alpha)}(t)$, for $\alpha > 0$, are given through a generating function (see [**1**, 22.9.3]):

$$\frac{1}{(1 - 2tz + z^2)^\alpha} = \sum_{n=0}^{\infty} C_n^{(\alpha)}(t) z^n.$$

This implies that

$$A_n(-\alpha, \alpha, -e^{2i\varphi}) = e^{in\varphi} C_n^{(\alpha)}(\cos\varphi),$$

and Theorem 2 gives

COROLLARY 2. *For* $0 < \alpha \leq 1, \ n \in \mathbb{N}$*, we have*

$$(1 - t^2)^{\frac{\alpha}{2}} \left| C_{2n}^{(\alpha)}(t) \right| \leq \left| C_{2n}^{(\alpha)}(0) \right|, \quad t \in [-1, 1]. \tag{1.8}$$

It is easily checked that this is not generally true for $\alpha > 1$, which also proves the sharpness of the bound $\alpha + \beta \leq 2$ in Theorem 2.

Corollary 2 is an improvement over the known estimate [**1**, 22.14.3]

$$(1 - t^2)^{\frac{\alpha}{2}} \left| C_m^{(\alpha)}(t) \right| < \frac{(m/2)^{\alpha-1}}{\Gamma(\alpha)}, \quad t \in [-1, 1], \ \alpha \in (0, 1),$$

for even m.

Using the hypergeometric function

$${}_2F_1(a, b, c, z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n,$$

we find that

$$A_n(\alpha, \beta, x) = \frac{(\beta)_n}{n!} {}_2F_1(-n, -\alpha, 1 - \beta - n, -x), \tag{1.9}$$

where $(a)_n := a(a + 1) \ldots (a + n - 1)$ is the Pochhammer symbol. It turns out that both Theorems 1 and 2 are contained in the following result concerning ${}_2F_1$.

THEOREM 3. *Let* $0 < \alpha, \beta$ *with* $\alpha + \beta \leq 2$ *and* $n \in \mathbb{N}$. *Then*

$$q(x)^{\frac{\alpha+\beta}{2}} |{}_2F_1(-2n, \alpha, \alpha + \beta, 1 + x)| \leq {}_2F_1(-2n, \alpha, \alpha + \beta, 2), \quad |x| = 1. \tag{1.10}$$

The proofs of these results are given in the next section. Section 3 contains a discussion and partial proof of a conjecture of H. Silverman and E. Silvia [**8**], which is closely related to the Brannan conjecture and has an interesting application to the coefficient problem for univalent functions f in the unit disk $\mathbb{D}$, with $f(\mathbb{D})$ 'starlike with respect to a boundary point'.

2. Proofs of Theorems 1–3.

Proof of Theorem 3. For $\alpha > 0$, $\beta > 0$, $|x| = 1$ we have (compare [**1**, 15.3.1])

$$ {}_2F_1(-2n, \alpha, \alpha+\beta, 1+x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}(1-t(1+x))^{2n} dt, $$

so that we have to estimate

$$ R(x) := \left| (1+x)^{\frac{\alpha+\beta}{2}} \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}(1-t(1+x))^{2n} dt \right|. \tag{2.1} $$

Using $\tau := 1 - t(1+x)$ and Cauchy's theorem for the triangle with the vertices $1, 0, -x$ we find that

$$ \begin{aligned} R(x) &= \left| \int_1^{-x} \left(\frac{1-\tau}{\sqrt{1+x}} \right)^{\alpha-1} \left(\frac{x+\tau}{\sqrt{1+x}} \right)^{\beta-1} \tau^{2n} d\tau \right| \\ &= \left| -\int_0^1 + \int_0^{-x} \right| \\ &= \left| \int_0^1 \left(S(x,\tau) + (-1)^{2n} x^{2n+1} T(x,\tau) \right) \tau^{2n} d\tau \right|, \end{aligned} $$

with

$$ \begin{aligned} S(x,\tau) : &= \left(\frac{1-\tau}{\sqrt{1+x}} \right)^{\alpha-1} \left(\frac{x+\tau}{\sqrt{1+x}} \right)^{\beta-1}, \\ T(x,\tau) : &= \left(\frac{1+x\tau}{\sqrt{1+x}} \right)^{\alpha-1} \left(\frac{x(1-\tau)}{\sqrt{1+x}} \right)^{\beta-1}. \end{aligned} $$

To prove $R(x) \le R(1)$, it suffices to show

$$ |S(x,\tau)| + |T(x,\tau)| \le S(1,\tau) + T(1,\tau), \quad |x| = 1,\ 0 \le \tau \le 1. $$

For τ fixed this is equivalent to

$$ U(x) := \frac{(1-\tau)^\gamma |x+\tau|^\gamma}{|1+x|^\gamma} \left[q(x) + \frac{1}{q(x)} \right] \le U(1), $$

where

$$ \gamma := \frac{\alpha+\beta-2}{2}, \quad q(x) := \left| \frac{1-\tau}{x+\tau} \right|^{(\alpha-\beta)/2}. $$

Since $q(x) + 1/q(x) \le q(1) + 1/q(1)$ and $\gamma \le 0$ it remains to show that

$$ \frac{|x+\tau|}{|1+x|} \ge \frac{1+\tau}{2}, \quad |x| = 1, $$

which is easily verified. □

Proof of Theorem 1. Using

$$\left(\frac{1+xz}{1-z}\right)^{\lambda} = \left(1+\frac{(1+x)z}{1-z}\right)^{\lambda} = \sum_{k=0}^{\infty}\binom{\lambda}{k}(1+x)^k\left(\frac{z}{1-z}\right)^k$$

we find

$$A_n(\lambda,\lambda,x) = \sum_{k=1}^{n}\binom{n-1}{k-1}\binom{\lambda}{k}(1+x)^k, \quad n\geq 1,$$

(compare also Todorov [**9**]) which can be written as

$$A_n(\lambda,\lambda,x) = \lambda\,(1+x)\,{}_2F_1(1-n,1-\lambda,2,1+x), \quad n\geq 1.$$

The assertion follows from Theorem 3 using $\alpha := 1-\lambda$, $\beta := 1+\lambda$ and with n replaced by $n-1$ (which, by assumption, is even). □

Proof of Theorem 2. Using (1.9) and the general formula [**1**, 15.3.6]

$$\begin{aligned}{}_2F_1(A,B,C,Z) = {} & \frac{\Gamma(C)\Gamma(C-A-B)}{\Gamma(C-A)\Gamma(C-B)}\,{}_2F_1(A,B,A+B+1-C,1-Z)\\ & + \frac{\Gamma(C)\Gamma(A+B-C)}{\Gamma(A)\Gamma(B)}(1-Z)^{C-A-B}\\ & \times {}_2F_1(C-A,C-B,1+C-A-B,1-Z),\end{aligned}$$

valid for $|\arg(1-Z)|<\pi$, we obtain

$$A_n(-\alpha,\beta,x) = \frac{(\alpha+\beta)_n}{n!}\,{}_2F_1(-n,\alpha,\alpha+\beta,1+x).$$

The conclusion follows from Theorem 3. □

3. The Silverman-Silvia conjecture and functions starlike with respect to a boundary point. In connection with their work on univalent functions f in the unit disk $\mathbb{D}$, with $f(\mathbb{D})$ 'starlike with respect to a boundary point' H. Silverman and E. Silvia [**8**] independently proposed a conjecture which is an extended version of the special case $\alpha=\beta$ of the Brannan conjecture 1. For odd n (and $\alpha=\beta$) the two conjectures actually coincide (and have been settled in Theorem 1 above). The remaining part of their conjecture is as follows.

CONJECTURE 2. (Silverman & Silvia). Let $\alpha=\beta=:\lambda$. For $n\in\mathbb{N}$ even there exists a unique $\lambda_n\in(0,1)$ such that

$$|A_n(\lambda,\lambda,x)|\leq|A_n(\lambda,\lambda,1)|, \quad |x|=1,$$

holds for $\lambda_n\leq\lambda\leq 1$ but fails to hold for $0<\lambda<\lambda_n$.

Conjecture 2 has been verified by Silverman and Silvia [**8**] for $n=2$ ($\lambda_2=1/\sqrt{2}$, see also Brannan [**4**]), and by P. Todorov [**9**] for $n=4$ ($\lambda_4=0.74\ldots$, a solution of a bi-cubic equation).

We have the following partial solution for Conjecture 2.

THEOREM 4. *For $n \in \mathbb{N}$ even there exist numbers $0 < \lambda_n^* \le \lambda_n^{**} < 1$ such that for $\lambda \ge \lambda_n^{**}$ we have*

$$|A_n(\lambda, \lambda, x)| \le A_n(\lambda, \lambda, 1), \quad |x| = 1, \tag{3.1}$$

and for $0 < \lambda < \lambda_n^*$

$$\max_{|x|=1} |A_n(\lambda, \lambda, x)| > A_n(\lambda, \lambda, 1). \tag{3.2}$$

Conjecture 2 claims that $\lambda_n^* = \lambda_n^{**}$ which we cannot prove yet. However, there is strong evidence that the equation

$$\frac{\partial^2}{\partial \varphi^2} \left|A_n(\lambda, \lambda, e^{i\varphi})\right|^2\Big|_{\varphi=0} = 0$$

has exactly one solution $\lambda \in (0, 1)$ for even n, and that this solution equals both, λ_n^* and λ_n^{**}, and therefore represents λ_n. This is true, at least, for the solved cases $n = 2, 4$.

The work of Silverman and Silvia [**8**] shows that Theorems 1 and 4 are the required final bits of information for the proof of the following theorem. For the background we refer to their paper and omit further details.

THEOREM 5. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be univalent in $\mathbb{D}$ such that*
(i) $f(0) = 1$, $\lim_{r\to 1} f(r) = 0$,
(ii) *There exist $\gamma, \lambda \in \mathbb{R}$ with $0 < \lambda \le 2$ such that $|\arg(e^{i\gamma} f(z))| < \frac{1}{2}\lambda\pi,\ z \in \mathbb{D}$,*
(iii) *$f(\mathbb{D})$ is starlike with respect to the boundary point 0.*
Then

$$|a_{2n-1}| \le A_{2n-1}(\lambda, \lambda, 1), \quad n \in \mathbb{N}. \tag{3.3}$$

Furthermore, if $\lambda \ge \lambda_n^{**}$, we have

$$|a_{2n}| \le A_{2n}(\lambda, \lambda, 1).$$

All these bounds are sharp and attained for $f(z) = ((1 + z)/(1 - z))^\lambda$.

Proof of Theorem 4. Let $n \in \mathbb{N}$. We define

$$a_n(\lambda, x) := \frac{-\Gamma(1 + \lambda)\Gamma(1 - \lambda)}{\lambda} A_n(\lambda, \lambda, x),$$

and, by the same manipulations as in the proof of Theorem 3, we find that

$$a_n(\lambda, x) = \int_0^1 \left[\left(\frac{1-\tau}{x+\tau}\right)^\lambda \tau^{n-1} - (-x)^n \left(\frac{\overline{x}+\tau}{1-\tau}\right)^\lambda \tau^{n-1} \right] d\tau.$$

For $\frac{1}{2} \le \lambda \le 1$ and $x = e^{i\varphi}$ we write $a_n(\lambda, x) = I_1(\varphi) + I_2(\varphi) + I_3(\varphi)$, where

$$I_1(\varphi) = \int_0^1 \left(\frac{1-\tau}{x+\tau}\right)^\lambda \tau^{n-1}\, d\tau,$$

$$I_2(\varphi) = -(-x)^n \int_0^{1/2} \left(\frac{\overline{x}+\tau}{1-\tau}\right)^\lambda \tau^{n-1}\, d\tau,$$

$$I_3(\varphi) = -(-x)^n \int_{1/2}^1 \left(\frac{\overline{x}+\tau}{1-\tau}\right)^\lambda \tau^{n-1}\, d\tau.$$

Then

$$|I_1(\varphi)| \le \int_0^1 \left|\frac{1-\tau}{x+\tau}\right|^\lambda \tau^{n-1}\,d\tau \le 1,$$

$$|I_2(\varphi)| \le \int_0^{1/2} \left|\frac{\overline{x}+\tau}{1-\tau}\right|^\lambda \tau^{n-1}\,d\tau \le 2.$$

Now let $\delta > 0$ and assume that $\delta \le |\varphi| \le \pi$. Then

$$|I_3(\varphi)| \le \left(\max_{\{\psi,\lambda,\tau\}\in G_\delta} \left|\frac{e^{i\psi}+\tau}{1+\tau}\right|^\lambda\right) \int_{1/2}^1 \left(\frac{1+\tau}{1-\tau}\right)^\lambda \tau^{n-1}\,d\tau \le m(\delta) I_3(0),$$

where $G_\delta = \{\{\psi,\lambda,\tau\} : \delta \le |\psi| \le \pi, \frac{1}{2} \le \lambda \le 1, \frac{1}{2} \le \tau \le 1\}$ and $m(\delta) < 1$. Note that

$$I_3(0) \ge 2^{1-n} \int_{1/2}^1 (1-\tau)^{-\lambda} d\tau \;\to\; \infty, \quad (\lambda \to 1),$$

so that

$$\left|\frac{a_n(\lambda,x)}{a_n(\lambda,1)}\right| \le \frac{3+m(\delta)I_3(0)}{|I_3(0)-3|} = m(\delta) + o(1), \quad (\lambda \to 1).$$

The same conclusion holds for $a_n(\lambda, x)$ replaced by $A_n(\lambda,\lambda,x)$, and implies that

$$\limsup_{\lambda\to 1} \sup_{\delta\le|\varphi|\le\pi} \left|\frac{A_n(\lambda,\lambda,e^{i\varphi})}{A_n(\lambda,\lambda,1)}\right| < 1. \tag{3.4}$$

To complete our proof we need to estimate $|A_n(\lambda,\lambda,e^{i\varphi})|$ in the neighborhood of $\varphi = 0$. A simple calculation yields

$$A_n(\lambda,\lambda,x) = \sum_{k=0}^n \binom{\lambda}{k}\binom{\lambda+n-k-1}{n-k} x^k =: \sum_{k=0}^n b_k(\lambda) x^k,$$

and expanding $h(\lambda,\varphi) := |A_n(\lambda,\lambda,e^{i\varphi})|^2$ at $\varphi = 0$ gives

$$h(\lambda,\varphi) = h(\lambda,0) - \varphi^2 \frac{1}{2} \sum_{k=0}^n \sum_{j=0}^n (j-k)^2 b_k(\lambda) b_j(\lambda) + M(\varphi,\lambda)\varphi^3,$$

where $M(\varphi,\lambda)$ is uniformly bounded by a constant M in some set $[\mu_1, 1] \times [-\delta, \delta]$ with $\mu_1 < 1, \delta > 0$. One easily finds that

$$\frac{1}{2} \sum_{k=0}^n \sum_{j=0}^n (j-k)^2 b_k(1) b_j(1) = 1,$$

so that, by continuity, we have

$$w_n(\lambda) := \frac{1}{2} \sum_{k=0}^n \sum_{j=0}^n (j-k)^2 b_k(\lambda) b_j(\lambda) > \frac{1}{2},$$

for all $\mu_2 \leq \lambda \leq 1$, for some $\mu_2 < 1$ Hence, for $|\varphi|$ sufficiently small and all λ in some interval $(\mu_3, 1]$ we have

$$\left|\frac{A_n(\lambda, \lambda, e^{i\varphi})}{A_n(\lambda, \lambda, 1)}\right|^2 \leq 1 - \varphi^2 \frac{\frac{1}{2} - M\varphi}{|A_n(\lambda, \lambda, 1)|^2} \leq 1.$$

Together with (3.4), with a sufficiently small δ, this proves the existence of $\lambda_n^{**} < 1$ with

$$|A_n(\lambda, \lambda, x)| \leq |A_n(\lambda, \lambda, 1)|, \quad |x| = 1, \ \lambda \geq \lambda_n^{**},$$

for every $n \in \mathbb{N}$.

Furthermore, one readily verifies that $w_n(\lambda) = (-1)^{n+1}\lambda^2 + O(\lambda^3)$, so that $h(\lambda, \varphi) > h(\lambda, 0)$ holds for small values of $\lambda > 0$ and φ (n even, fixed). This proves the existence of λ_n^* as asserted. The proof of Theorem 4 is complete. □

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