UNIVERSIDAD TÉCNICA FEDERICO SANTA MARÍA

DEPARTAMENTO DE MATEMÁTICA

An evolution problem associated with the Infinity Fractional Laplacian

Tesis presentada por

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como requisito parcial para optar al grado académico de Magíster en Ciencias Mención Matemática y al título profesional de Ingeniero Civil Matemático

Director de tesis

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Yo, **Nicolás Alejandro Muñoz León**, presento este trabajo titulado, **"An evolution problem associated with the Infinity Fractional Laplacian"**, en cumplimiento parcial de los requisitos para el título de Ingeniería Civil Matemática y el grado de Magíster en Ciencias Mención Matemática de la Universidad Técnica Federico Santa María.

Comisión evaluadora

Alexander Quaas Berger
Universidad Técnica Federico Santa María ____________________

Dedicado a Fresia Gomez.

"*Este canto vem de longe,*
A distância não sei dizer,
Salve, salve a toda a gente,
Que vive e deixa viver,
Aqui vai o nosso abraço,
Com o som e o saber,
Tirando de nossas mentes,
As palavras pra dizer,
A música segura o mundo,
Enquanto a gente viver,
É a maior fonte sem fim,
De alegria e prazer,
Toquem, cantem, minha gente,
Até o dia amanhecer."

Hermeto Pascoal

Resumen

El *Infinito Laplaciano Fraccionario*, denotado Δ_∞^s, es un operador integro-diferencial fuertemente no lineal derivado a partir del estudio de una versión no local del juego *Tug-of-War*. Este operador es el objeto principal de este trabajo. En el contexto de las soluciones viscosas, estudiamos existencia, unicidad y regularidad de las soluciones de un problema parabolico asociado a este operador. Dado el comportamiento discontinuo de $\Delta_\infty^s\phi(x)$ en puntos donde $\nabla\phi(x) = 0$, un resultado de estabilidad es probado. Sin embargo, es necesario asegurar que nuestro candidato a solución, dado por el Método de Perron, tiene gradiente no nulo para poder aplicar el resultado. Esto se realiza en dos dominios, a saber, una banda infinita y un anillo, estos dominios comparten la propiedad que sus complementos estan compuestos de dos componentes disconexas, por lo que podemos considerar la misma condición externa en ambos escenarios y establecer tasas de crecimiento y decrecimiento que inducen una monotonicidad estricta de la solución.

Abstract

The *Infinity fractional Laplacian*, denoted Δ_∞^s, is a fully non-linear, degenerate integro-differential operator derived from the study of a non-local *Tug-of-War* game. This operator is the main subject of this work. In the frame of viscosity solutions, we study existence, uniqueness and regularity of solutions to a parabolic problem associated with the operator. Given the discontinuous behavior of $\Delta_\infty^s \phi(x)$ at points where $\nabla\phi(x) = 0$, a stability result is proved. However, it is necessary to ensure that our candidate solution, given by Perron's Method, has non-zero gradient in order to apply it. This is done on two different domains, namely, an infinite strip and a annular domain, these domains share the property that their complement is made up of two disjoint components, so we are able to consider the same data in both settings and establish growth and decay rates that induce strict monotonicity on our solution.

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Chapter 1

Introducción

Estudiaremos un problema en evolución asociado al *Infinito Laplaciano Fraccionario.* Introducido el 2011 por C. Bjorland, L. Caffarelli, y A. Figalli ([9]), el *Infinito Laplaciano Fraccionario* es un operador degenerado integro-diferencial, fuertemente no lineal, derivado como el operador presente en la ecuación que modela una versión no local del juego *Tug-of-War.* Dado $s \in (\frac{1}{2}, 1)$, podemos escribir este operador de la siguiente manera:

$$\Delta_\infty^s \phi(x) := \sup_{y \in S^{N-1}} \inf_{z \in S^{N-1}} \int_0^\infty \frac{\phi(x + \eta y) + \phi(x - \eta z) - 2\phi(x)}{\eta^{1+2s}} \, d\eta. \tag{1.1}$$

Tug-of-War es un juego de dos jugadores con suma cero, introducido por Y. Peres, O. Schramm, S. Sheffield y D. Wilson en su trabajo [23], donde se muestra que el valor del juego (cuando cierto parámetro tiende a cero) es una solución para la ecuación estándar del *Infinito Laplaciano*, es decir, resuelve la siguiente ecuación,

$$\begin{cases} \Delta_\infty u(x) & = 0 \qquad \text{si } x \in \Omega, \\ \quad u(x) & = g(x) \quad \text{si } x \in \partial\Omega. \end{cases}$$

La versión local del *Tug-of-War* se describe a continuación: Consideremos un dominio acotado $\Omega \subset \mathbb{R}^N$ y un $\varepsilon > 0$ fijo. Inicialmente, una ficha se encuentra en un punto $x_0 \in \Omega$. Los dos jugadores, Jugador 1 y Jugador 2, juegan de acuerdo a las siguientes reglas: en el primer turno se tira una moneda equilibrada, y el jugador que gana la tirada de la moneda mueve la ficha a cualquier punto $x_1 \in B_\varepsilon(x_0)$ de su elección. A partir del siguiente turno, el juego continua de la misma manera a partir del punto x_1. Cuando la ficha sale del dominio Ω, digamos en el $\tau-$ésimo turno, es decir, $x_\tau \in \Omega^c$, el juego termina. Dada una función de pago final $g : \Omega^c \to \mathbb{R}$, Jugador 2 le paga a Jugador 1 la cantidad dada por $g(x_\tau)$, es decir, Jugador 1 gana $g(x_\tau)$ y Jugador 2 gana $-g(x_\tau)$.

En la versión no local del juego, descrito por primera vez en [9], al comienzo de cada turno, en lugar de tirar una moneda, cada jugador elige una dirección $y, z \in S^{N-1}$, y un proceso estocástico determina la dirección y cuanto se mueve la ficha en esa dirección.

En este trabajo no proveemos una interpretación de la ecuación parabólica no local que estudiamos desde el punto de vista de la teoría de juegos. Sin embargo, la versión local del juego con dependencia del tiempo ha sido estudiada. En esta versión, el juego se juega en un cilindro parabólico, es decir, $\Omega_T = \Omega \times (0, T)$, donde $\Omega \subset \mathbb{R}^N$ y $T > 0$ esta fijo. Aquí, la cantidad de turnos es limitada. El juego comienza en algún $0 < t_0 < T$, y cada turno la ficha se mueve de la misma manera que en la primera versión, con la adición de que el tiempo se reduce en $\varepsilon^2/2$, i.e, $t_k = t_{k-1} - \varepsilon^2/2$. Podemos pensar que la ficha se mueve en Ω_T, y el juego termina cuando la ficha sale del cilindro parabólico. Para conocer mas detalles al respecto, referimos a [10], [17] y las referencias allí citadas.

En [9], los autores estudian dos problemas estacionarios con el *Infinito Laplaciano Fraccionario*, a saber, un problema de Dirichlet, y un problema de obstáculo doble. El problema de Dirichlet es planteado como sigue: Dado $\Omega \subset \mathbb{R}^N$ y una función $g : \Omega^c \to \mathbb{R}$, se considera,

$$\begin{cases} \Delta_\infty^s u(x) & = 0 \qquad \text{si } x \in \Omega, \\ \quad u(x) & = g(x) \quad \text{si } x \in \Omega^c. \end{cases} \tag{1.2}$$

Motivados por el estudio de (1.2), aquí estudiamos el siguiente problema: Dado un dominio $\Omega \subset \mathbb{R}^N$, $T > 0$ fijo, una condición inicial $u_0 : \mathbb{R}^N \to \mathbb{R}$, y una función, o dato, $g : \mathbb{R}^{N+1} \to \mathbb{R}$, consideramos,

$$\begin{cases} \partial_t u(x,t) - \Delta_\infty^s u(x,t) & = & 0 & \text{si } (x,t) \in \Omega \times (0,T), \\ u(x,t) & = & g(x,t) & \text{si } (x,t) \in \Omega^c \times (0,T), \\ u(x,0) & = & u_0(x) & \text{si } x \in \mathbb{R}^N. \end{cases} \tag{$\mathcal{P}$}$$

Dado que trataremos con un operador integro-diferencial fuertemente no lineal que no esta en forma de divergencia, trabajaremos en el marco de las *soluciones viscosas*. Ademas, la expresión (1.1) no es la definición original del operador dada en [9], sin embargo, es una forma equivalente de representar el operador cuando ϕ es suave. La definición original del operador es la siguiente:

Definición 1.1. Dado $s \in (\frac{1}{2}, 1)$, el *Infinito Laplaciano Fraccionario* $\Delta_\infty^s : C^{1,1}(x) \cap BC(\mathbb{R}^N) \to \mathbb{R}$ en un punto $x \in \mathbb{R}^N$ viene dado por:

- Si $\nabla\phi(x) \neq 0$ entonces,

$$\Delta_\infty^s \phi(x) = \int_0^\infty \frac{\phi(x+\eta v) + \phi(x-\eta v) - 2\phi(x)}{\eta^{1+2s}}\, d\eta.$$

- Si $\nabla\phi(x) = 0$ entonces,

$$\Delta_\infty^s \phi(x) = \sup_{y \in S^{N-1}} \int_0^\infty \frac{\phi(x+\eta y) - \phi(x)}{\eta^{1+2s}}\, d\eta + \inf_{z \in S^{N-1}} \int_0^\infty \frac{\phi(x-\eta z) - \phi(x)}{\eta^{1+2s}}\, d\eta$$

donde $v \in S^{N-1}$ es la dirección de $\nabla\phi(x)$, es decir, $v = \frac{\nabla\phi(x)}{|\nabla\phi(x)|}$.

La expresión (1.1) coincide con la de la Definición 1.1 cuando ϕ es suave. Si ϕ no es suave a priori, $\Delta_\infty^s \phi(x)$ puede no estar bien definido. Por lo que la noción de *solución viscosa*, con la regularidad apropiada, nos permite darle sentido operador, y asegurar que esta bien definido. La definición de solución viscosa que usaremos viene dada en la Definición 3.6. Para mas detalles sobre la noción de soluciones viscosas, referimos al apéndice A, o a [15] para una introducción al tema.

En el capítulo 3 probaremos que si $\phi \in C^{1,1}(x) \cap BC(\mathbb{R}^N)$, entonces $\Delta_\infty^s \phi(x)$ esta bien definido, y daremos un bosquejo de la demostración de la equivalencia entre ambas expresiones para Δ_∞^s.

Tal como en el problema de Dirichlet estudiado en [9], la mayor dificultad cuando trabajamos con Δ_∞^s es probar la existencia de una solución. Esto se debe al comportamiento discontinuo del operador en puntos donde $\nabla\phi(x) = 0$. Por ejemplo ([9]), consideremos la función $p : \mathbb{R}^2 \to \mathbb{R}$ dada por $p(x,y) = (1 - 2x^2 - y^2) \vee 0$, y notemos que

$$\lim_{h\to 0} \Delta_\infty^s p(h,0) \neq \lim_{h \to 0} \Delta_\infty^s p(0,h) \neq \Delta_\infty^s p(0,0).$$

Por lo tanto $\Delta_\infty^s \phi$ puede ser discontinuo incluso si ϕ es una función suave, y en particular, Δ_∞^s es inestable bajo limites uniformes en puntos donde la función límite tiene gradiente nulo, precisamente, si $\phi_n \to \phi_0$ uniformemente y $\nabla\phi_0(x_0) = 0$, entonces, en general,

$$\lim_{n\to\infty} \Delta_\infty^s \phi_n(x_0) \neq \Delta_\infty^s \phi_0(x_0).$$

Por esto, siguiendo el enfoque de [9], consideramos dominios donde es posible probar que nuestro candidato a solución de $\mathcal{P}$ tiene gradiente no nulo en estos dominios. Para probar la existencia de una solución empleamos el *Método de Perron*. Este método consiste en definir un candidato a solución del problema como el supremo puntual de una familia de subsoluciones, precisamente, definimos,

$$U(x,t) = \sup\{u(x,t) : u \in \mathcal{F}\} \tag{1.3}$$

donde $\mathcal{F}$ es una familia de subsoluciones viscosas. El motivo de la importancia de que U tenga gradiente no nulo, es que las subsoluciones y supersoluciones de $\mathcal{P}$ son estables bajo limites uniformes en puntos donde sus gradientes son no nulos. La idea es: si $\{u_n\}_{n\in\mathbb{N}}$ es una sucesión de subsoluciones (o supersoluciones) tal que cada función en la sucesión tiene gradiente no nulo, y estas convergen uniformemente a $\tilde{u}$, entonces $\tilde{u}$ también es una subsolución con gradiente no nulo. Por lo que, una vez probado que U tiene gradiente no nulo, podemos tomar ventaja de este resultado de estabilidad.

Como ya mencionamos, consideraremos escenarios en los que sea posible probar que U tiene gradiente no nulo en el interior del dominio. El **primer escenario** que consideramos (Capítulo 4) es uno donde el dominio Ω esta definido como una banda infinita, ortogonal al primer eje canónico, con dato 0 a un lado de la banda, y 1 en el otro. Mediante el uso de barreras, es posible probar ciertas tasa de crecimiento y decrecimiento de U, que inducen cierta monotonía de U dentro de Ω. El dominio en este escenario es igual al considerado en [9].

El **segundo escenario** (Capítulo 5) es un anillo, precisamente, consideramos $\Omega = B_R \setminus \overline{B_r}$ con $0 < r < R$, con el mismo dato que en el primer escenario. En este caso, establecemos tasas de crecimiento y decrecimiento similares a las del primer escenario que inducen monotonía radial. Esto es posible gracias a la similitud entre ambos dominios y el dato que consideramos. A pesar de que la regularidad de la frontera del dominio deje algo de ambigüedad con respecto a la dirección del gradiente de U, logramos probar que esta es monótona en la dirección radial.

La similitud mencionada en el parrafo anterior, es el hecho que ambos dominios abiertos estan "entre" las dos componentes disconexas de sus respectivos complementos. Tecnicamente hablando, esto significa que ambos dominios satisfacen,

$$\mathbb{R}^N = \Omega^{c,-} \cup \Omega^{c,+} \cup \Omega \quad \text{y} \quad \Omega^{c,-} \cap \Omega^{c,+} = \emptyset,$$

donde $\Omega^{c,-}$ y $\Omega^{c,+}$ son las componentes disconexas de Ω^c. Dado que ambos dominios satisfacen lo anterior, podemos considerar el mismo dato en ambos escenarios como $g \equiv 0$ en $\Omega^{c,-}$ y $g \equiv 1$ en $\Omega^{c,+}$, y así tener una idea acerca de la dirección de crecimiento de U, y por lo tanto, de la monotonía de U. Esta similitud entre ambos dominios es el motivo de elegir un anillo como nuevo dominio de estudio.

En ambos escenarios probaremos resultados de existencia, regularidad y unicidad. Siguiendo el enfoque de [9], la idea principal para probar la existencia de soluciones, es construir subsoluciones y supersoluciones apropiadas y usarlas como barreras para probar las tasas de crecimiento y decrecimiento de U. Estas estimaciones inducen una monotonicidad uniforme estricta en el sentido de la Definición 4.4, lo que implica que U tiene gradiente no nulo. Luego, mediante la construcción de una función *bump* y el resultado de estabilidad, demostramos que U es una solución. Los resultados de unicidad vienen de principios de comparación. Para el segundo escenario, establecemos un principio de comparación para dominios acotados, mientras que para el primer escenario necesitamos la monotonicidad de subsoluciones y supersoluciones junto a una cota en particular para el dominio Ω.

Como se señala en [9], es natural considerar g en el espacio de las funciones Hölder continuas con exponente $2s - 1$, esto da una manera de probar un resultado de regularidad para subsoluciones de (2.2). Adaptamos la demostración de este resultado para obtener regularidad para subsoluciones de $\mathcal{P}$ en la variable espacial. Ademas, usaremos la función $|x - x_0|^\alpha$, con $\alpha > 0$ y $x \neq x_0$, como barrera para obtener un nuevo resultado de regularidad en ambas variables.

Trabajos relacionados

Desde ell trabajo de G. Aronsson [3, 2, 4] en la década de 1960, la ecuación del *Infinito Laplaciano* a sido estudiada de forma extensiva. El tema mas estudiado en aquellos tiempos fueron los minimizadores absolutos. De acuerdo a Aronsson, un minimizador absoluto en un dominio $\Omega \subset \mathbb{R}^N$ es una función continua a valores reales que posee la constante de Lipschitz mas pequeña posible en cada conjunto abierto compactamente contenido en Ω. En [18], un trabajo fundamental de R. Jensen, la equivalencia entre los minimizadores absolutos y las soluciones viscosas de la ecuación homogenea del Infinito Laplaciano fue probado. Ademas de proveer una demostración de la unicidad de los minimizadores absolutos.

Desde entonces, muchas personas han contribuido a la teoría de los minimizadores absolutos. también llamados *funciones infinito armónicas*. Para nombrar a algunos trabajos importantes dentro de esta teoría, nos referimos a [14, 12, 13] de Crandall, Evans y Gariepy, [21] de Lindqvist y Manfredi, [19] de Juutinen, y [7] de Barron, Jensen y Wang.

En [26] el siguiente problema en evolución ese estudiado,

$$\begin{cases} \partial_t u(x,t) &= \Delta_\infty^s u(x,t), \quad x \in \mathbb{R}^N, t > 0, \\ u(x,0) &= u_0(x), \quad x \in \mathbb{R}^N, \end{cases} \tag{1.4}$$

donde los autores desarrollan la existencia de soluciones viscosas basándose en aproximaciones con esquemas monótonos. Esta viene siendo la principal diferencia en la teoría de existencia entre [26] y este trabajo. Ademas, estudian el comportamiento asintótico de las soluciones construidas, y obtienen un principio de Harnack global. Por otro lado, como (2.4) es planteado en $\mathbb{R}^N$, [26] no posee un resultado de unicidad de una solución viscosa, esto es consistente con nuestro enfoque de escoger ciertos escenarios específicos en los que podemos probar unicidad.

En [20] se estudia una versión del operador $p-$Laplaciano, introducido en [8]. Para $p \geq 2$, $s \in (1/2, 1)$ y una función acotada u de regularidad $C^{1,1}(x)$ con $\nabla u(x) \neq 0$, se define:

$$\Delta_p^s u(x) := C_{N,p,s} \int_{T_p^{0,\infty}\left(\frac{\nabla u(x)}{|\nabla u(x)|}\right)} \frac{u(x+z) + u(x-z)2u(x)}{|z|^{N+2s}} dz, \tag{1.5}$$

donde $C_{N,p,s}$ es una constante especifica, y la integración ocurre en un cono infinito $T_p^{0,\infty}\left(\frac{\nabla u(x)}{|\nabla u(x)|}\right) \subset \mathbb{R}^N$ cuya directriz es la dirección del vector $\frac{\nabla u(x)}{|\nabla u(x)|}$, y cuyo angulo de apertura α depende de N, p. Cuando $p = 2$ este operador es consistente con la formula usual para $-(-\Delta)^s u(x)$, y cuando $p \to \infty$ se tiene $\alpha \to 0$ y el cono se reduce a una recta, lo que resulta consistente con la definición de $\Delta_\infty^s u(x)$.

Vale a pena mencionar que Δ_∞^s como se presenta aquí, no es la única versión no local de Δ_∞. Un *Infinito Laplaciano Fraccionario* variacional fue estudiado en [11], este no se ajusta al enfoque de la teoría de juegos, que es la principal motivación en [9].

Notación

Fijemos algo de notación que usaremos a lo largo del texto.

Dado $x_0 \in \mathbb{R}^N$, $t_0 \in \mathbb{R}$ y $r > 0$, denotamos a la bola abierta de radio r centrada en $(x_0, t_0) \in \mathbb{R}^{N+1}$ como $B_r(x_0, t_0) \subset \mathbb{R}^{N+1}$. Similarmente, $B_r(x_0) \subset \mathbb{R}^N$ y $B_r(t_0) \subset \mathbb{R}$ denotan las bolas abiertas de radio r centradas en x_0 y t_0 respectivamente. Ademas, B_r denota la bola abierta de radio $r > 0$ centrada en $0 \in \mathbb{R}^N$.

Denotamos al $i-$ésimo componente de un vector $x \in \mathbb{R}^N$ por x_i, y e_i denota al $i-$ésimo vector canónico. S^{N-1} denota la esfera unitaria $(N-1)-$dimensional contenida en $\mathbb{R}^N$.

La función distancia sera denotada por d. Precisamente, si $x \in \mathbb{R}^N$ y $V, W \subset \mathbb{R}^N$, consideramos,

$$d(x, V) = \inf_{y \in V} |x - y|, \quad \text{y} \quad d(V, W) = \inf_{\substack{y \in V \\ z \in W}} |y - z|.$$

Ademas, la función $d(\cdot, \partial\Omega)$ se extiende por cero a $\mathbb{R}^N$, i.e,

$$d(x, \partial\Omega) = \begin{cases} \operatorname{dist}(x, \partial\Omega) & \text{si } x \in \overline{\Omega} \\ 0 & \text{si } x \in \mathbb{R}^N \setminus \overline{\Omega}. \end{cases} \tag{1.6}$$

Dadas dos (o mas) funciones f y g, $f \vee g$ denota el máximo puntual entre ambas funciones, i.e $(f \vee g)(x) = \max\{f(x), g(x)\}$. Similarmente, $f \wedge g$ denotara al mínimo. Ademas, $f^+ := \max\{f, 0\}$ y $f^- := \min\{f, 0\}$.

Usaremos ∇ para denotar el gradiente de una función con respecto a la variable espacial. Cuando escribimos $P.V$ frente a una integral, significa que la estamos evaluando en el sentido del valor principal de Cauchy.

$BC(\mathbb{R}^N)$ denota al conjunto de funciones continuas acotadas en $\mathbb{R}^N$. Consideramos el conjunto de funciones Hölder continuas, cuya definición viene dada por:

Definición 1.2. Decimos que una función $f : \Omega \to \mathbb{R}$ es *Hölder continua con exponente* γ en Ω en

$$[f]_{C^{0,\gamma}} := \sup_{x,y\in\Omega} \frac{|f(x) - f(y)|}{|x - y|^\gamma} < \infty.$$

En cuyo caso escribimos $f \in C^{0,\gamma}(\Omega)$. Ademas, si

$$\lim_{\delta\to 0} \sup_{|x-y|\leq\delta} \frac{|f(x) - f(y)|}{|x - y|^\gamma} = 0,$$

escribimos $f \in C_0^{0,\gamma}(\Omega)$.

Chapter 2

Introduction

The main goal of this work is to study an evolution problem associated with the *Infinity Fractional Laplacian*. Introduced in 2011 by C. Bjorland, L. Caffarelli, and A. Figalli ([9]), the *Infinity Fractional Laplacian* is a fully non-linear,degenerate integro-differential operator, derived as the operator involved in the equation that models a non-local version of a *Tug-of-War* game. Given $s \in (\frac{1}{2}, 1)$, this operator can be written as follows,

$$\Delta_\infty^s \phi(x) := \sup_{y \in S^{N-1}} \inf_{z \in S^{N-1}} \int_0^\infty \frac{\phi(x+\eta y) + \phi(x - \eta z) - 2\phi(x)}{\eta^{1+2s}}\, d\eta. \tag{2.1}$$

Tug-of-War is a zero-sum two-player game, originally introduced by Y. Peres, O. Schramm, S. Sheffield and D. Wilson in [23], where it is shown that the value function of the game (as a certain parameter approaches cero) solves the standard *Infinity Laplace* equation, that is,

$$\begin{cases} \Delta_\infty u(x) &= 0 \quad \text{if } x \in \Omega, \\ u(x) &= g(x) \quad \text{if } x \in \partial\Omega. \end{cases}$$

The local version of the game goes as follows: Consider a bounded domain $\Omega \subset \mathbb{R}^N$ and a fixed $\varepsilon > 0$. At the start of the game a token is placed at a point $x_0 \in \Omega$. On each turn, the players flip a fair coin and the winner gets to move the token to another point $x_1 \in B_\varepsilon(x_0)$. Once the token leaves Ω, let's say on the $\tau-$th turn, i.e, $x_\tau \in \Omega^c$, the game ends. A function $g : \Omega^c \to \mathbb{R}$ is defined outside of Ω called *payoff function*, and Player 2 pays Player 1 the amount given by $g(x_\tau)$, that is, Player 1 gains $g(x_\tau)$ and Player 2 gains $-g(x_\tau)$.

In the non-local version of the game, described for the first time in [9], at the beginning of each turn, instead of flipping a coin, each player chooses a direction $y, z \in S^{N-1}$ and a stochastic process determines the direction and how much the token moves in that direction.

We don't provide a game theoretical interpretation of the non-local parabolic equation that we study in this work. However, the local version of the time dependent game has been studied. In that version, the game is played in a parabolic cylinder, that is, $\Omega_T = \Omega \times (0, T)$, where $\Omega \in \mathbb{R}^N$ and $T > 0$ is fixed. Here, the amount of turns is limited. The game begins at some $0 < t_0 < T$, and each turn the token is moved in the same way as the first version, with the addition that the clock steps $\varepsilon^2/2$ backwards, $t_k = t_{k-1} - \varepsilon^2/2$. We can think the token moves inside Ω_T, and the game ends when the token leaves the parabolic cylinder. For more details about this, we refer to [10], [17] and the references therein.

In [9], the authors study two stationary problems involving the *Infinity Fractional Laplacian*, namely, a Dirichlet problem, and a double-obstacle problem. The Dirichlet problem can be stated as follows: Given $\Omega \subset \mathbb{R}^N$ and data $g : \Omega^c \to \mathbb{R}$, consider,

$$\begin{cases} \Delta_\infty^s u(x) &= 0 \quad \text{if } x \in \Omega, \\ u(x) &= g(x) \quad \text{if } x \in \Omega^c. \end{cases} \tag{2.2}$$

Motivated by the study of (2.2), we study the following problem: Given an open domain $\Omega \subset \mathbb{R}^N$, fixed $T > 0$, initial condition $u_0 : \mathbb{R}^N \to \mathbb{R}$, and data $g : \mathbb{R}^{N+1} \to \mathbb{R}$, we consider

$$\begin{cases} \partial_t u(x,t) - \Delta_\infty^s u(x,t) &= 0 & \text{if } (x,t) \in \Omega \times (0,T), \\ u(x,t) &= g(x,t) & \text{if } (x,t) \in \Omega^c \times (0,T), \\ u(x,0) &= u_0(x) & \text{if } x \in \mathbb{R}^N. \end{cases} \tag{$\mathcal{P}$}$$

Given that we have a fully non-linear integro-differential operator that is not in divergence form, we will work in the frame of ***viscosity solutions***. Moreover, expression (2.1) is not the original definition of the operator given in [9], regardless, it is an equivalent way to represent the operator when ϕ is smooth enough. The original definition of the operator is the following:

Definition 2.1. For $s \in (\frac{1}{2}, 1)$, the *Infinity fractional Laplacian* $\Delta_\infty^s : C^{1,1}(x) \cap BC(\mathbb{R}^N) \to \mathbb{R}$ at a point $x \in \mathbb{R}^N$ is given by:

- If $\nabla\phi(x) \neq 0$ then,

$$\Delta_\infty^s \phi(x) = \int_0^\infty \frac{\phi(x+\eta v) + \phi(x-\eta v) - 2\phi(x)}{\eta^{1+2s}}\, d\eta.$$

- If $\nabla\phi(x) = 0$ then,

$$\Delta_\infty^s \phi(x) = \sup_{y \in S^{N-1}} \int_0^\infty \frac{\phi(x+\eta y) - \phi(x)}{\eta^{1+2s}}\, d\eta + \inf_{z \in S^{N-1}} \int_0^\infty \frac{\phi(x-\eta z) - \phi(x)}{\eta^{1+2s}}\, d\eta$$

where $v \in S^{N-1}$ is the direction of $\nabla\phi(x)$, that is, $v = \frac{\nabla\phi(x)}{|\nabla\phi(x)|}$.

Expression (2.1) coincides with the one in Definition 2.1 when ϕ is smooth. If ϕ is not smooth enough a priori, $\Delta_\infty^s \phi(x)$ can be not well defined, so the notion of ***viscosity solutions*** together with the appropiate regularity for test functions, allows us to make sense of operator and ensure that it is well defined. The definition of viscosity solution that we will use is given in Definition 3.6. For more details about viscosity solutions, we refer to appendix A, and for a full introduction to the topic, we refer to [15].

In Chapter 3 we prove that for $\phi \in C^{1,1}(x) \cap BC(\mathbb{R}^N)$, $\Delta_\infty^s \phi(x)$ is indeed well defined, and give a sketch of the proof of the equivalence between the two expressions for Δ_∞^s. The definition of $C^{1,1}(x)$ is given in Definition 3.1.

Just like in the Dirichlet problem studied in [9], the biggest difficulty when working with Δ_∞^s is proving the existence of a solution. This is because of the discontinuous behavior of the operator at points where $\nabla\phi(x) = 0$. For example ([9]), consider the function $p : \mathbb{R}^2 \to \mathbb{R}$ given by $p(x,y) = (1 - 2x^2 - y^2) \vee 0$, and note that

$$\lim_{h \to 0} \Delta_\infty^s p(h,0) \neq \lim_{h \to 0} \Delta_\infty^s p(0,h) \neq \Delta_\infty^s p(0,0).$$

So $\Delta_\infty^s \phi$ can be discontinuous even if ϕ is a smooth function, and in particular, Δ_∞^s is unstable under uniform limits on points where the limit function has zero gradient, precisely, if $\phi_n \to \phi_0$ uniformly and $\nabla\phi_0(x_0) = 0$, then, in general

$$\lim_{n \to \infty} \Delta_\infty^s \phi_n(x_0) \neq \Delta_\infty^s \phi_0(x_0).$$

This is why, following the approach of [9], we consider domains where it is possible to prove that our candidate solution to $\mathcal{P}$ has a non-zero gradient on these domains. To prove the existence of a solution we will employ *Perron's Method*. This method consists in defining a candidate solution to our problem as the supremum of a familiy of subsolutions, precisely, we will define,

$$U(x,t) = \sup\{u(x,t) : u \in \mathcal{F}\} \tag{2.3}$$

where $\mathcal{F}$ is a family of viscosity subsolutions. The reason it is important that U has non-zero gradient, is because subsolutions and supersolutions of $\mathcal{P}$ are stable under uniform limits at points where their gradients are non-zero, roughly, if $\{u_n\}_{n\in\mathbb{N}}$ is a sequence of subsolutions (or supersolutions) such that each function in the sequence has non-zero gradient, and they converge uniformly to $\tilde{u}$, then $\tilde{u}$ is also a subsolution that has non-zero gradient. So once we prove that U has non-zero gradient, we can take advantage of this stability result.

As previously mentioned, we will consider settings in which it is possible to prove that U has non-zero gradient in the interior of the domain. The **first setting** that we will consider (Chapter 4) is one where our domain Ω is defined as an infinite strip orthogonal to the e_1 axis, with data 0 on one side of the strip, and 1 on the other side. Through the use of barriers, it is possible to prove certain growth and decay rates of U that induce monotonicity of U on Ω. The domain considered in this setting is the same as the one considered in [9].

The **second setting** (Chapter 5) is an annular one, precisely, we consider $\Omega = B_R\backslash\overline{B_r}$ with $0 < r < R$, and with the same data as the first setting. In this case, we can establish similar growth and decay rates for U, that induce the same monotonicity. Although the regularity of the boundary in this case introduces some ambiguity concerning the direction of the gradient of U, we are able to prove that it is monotone along the radial direction.

The similarity mentioned in the last paragraph, is the fact that both open domains are "in between" the two disjont components of their respective complements. Technically speaking, both domains satisfy

$$\mathbb{R}^N = \Omega^{c,-} \cup \Omega^{c,+} \cup \Omega \quad \text{and} \quad \Omega^{c,-} \cap \Omega^{c,+} = \emptyset,$$

where $\Omega^{c,-}$ and $\Omega^{c,+}$ are the two disjont components of Ω^c. Since this holds for both domains, we can consider the same data on both settings, $g \equiv 0$ on $\Omega^{c,-}$ and $g \equiv 1$ on $\Omega c, +$, which gives us a clear idea of the growth direction of U, and in turn, of the monotonicity of U. This similarity between the domains is the main reason we chose to study the annular domain.

In both of these settings we will prove existence, regularity and uniqueness results. Following the approach in [9], the main idea to prove the existence of a solution on both settings, is to construct appropriate sub and super solutions to use as barriers to prove growth and decay rates of U. These rates induce a strict uniform monotonicity in the sense of Definition 4.4, which in turn implies that U has non-zero gradient. Then, through a standard construction of a bump function and the stability result, it is proved that U is a solution. The uniqueness results follow from comparison principles. For the second setting we establish a comparison principle for bounded domains, while for the first setting we will need monotonicity of the sub and super solutions together with a particular bound on Ω.

As pointed out in [9], it is natural to consider the data g in the space of Hölder continuous functions with exponent $2s-1$, this gives a way to prove a regularity result for subsolutions of (2.2). We adapt the proof of this result to get a regularity result in the spatial variable for subsolutions of $\mathcal{P}$. Moreover, we use the function $|x-x_0|^\alpha$, with $\alpha > 0$ and $x \neq x_0$, as a barrier to get a new regularity result in both the spatial and time variables.

Related Literature

Since G. Aronsson's work [3, 2, 4] in the 1960's, the *Infinity Laplace* equation has been under extensive study. The main subject studied in these earlier times was the concept of absolute minimizers. According to Aronsson, an absolute minimizer in a domain $\Omega \subset \mathbb{R}^N$ is a continuous real-valued function which has the least possible Lipschitz constant in every open set whose closure is compactly contained in Ω. In R. Jensen's fundamental work [18], equivalence of the absolute minimizers and viscosity solutions of the homogeneous Infinity Laplace equation was established and an original proof of the uniqueness of absolute minimizers was provided.

Since then, many people have contributed to the theory of absolute minimizers which are also called infinity harmonic functions. To mention a few of such contributors which are of course far from a complete list, we refer to [14, 12, 13] by Crandall, Evans and Gariepy, [21] by Lindqvist and Manfredi, [19] by Juutinen, and [7] by Barron, Jensen and Wang which help to complete the theory of absolute minimizers.

In [26] the following evolution problem is studied,

$$\left\{\begin{array}{lll} \partial_t u(x,t) & = \Delta_\infty^s u(x,t), & x \in \mathbb{R}^N, t>0, \\ u(x,0) & = u_0(x), & x \in \mathbb{R}^N, \end{array}\right. \tag{2.4}$$

where the authors develop an existence theory of suitable viscosity solutions based on approximation with monotone schemes, this being the key difference between [26] and this text. Moreover, they study the asymptotic behavior of the constructed solutions, and obtain a global Harnack type principle. Also, since (2.4) is posed in the whole $\mathbb{R}^N$, [26] lacks a uniqueness of viscosity solution result, this is consistent with our approach of choosing certain specific settings in which we can prove uniqueness.

In [20] a version of the $p-$Laplace operator is studied, which was introduced in [8]. For $p \geq 2$, $s \in (1/2,1)$, and for a given bounded u that is of regularity $C^{1,1}(x)$ with $\nabla u(x) \neq 0$, one defines:

$$\Delta_p^s u(x) := C_{N,p,s} \int_{T_p^{0,\infty}\left(\frac{\nabla u(x)}{|\nabla u(x)|}\right)} \frac{u(x+z)+u(x-z)2u(x)}{|z|^{N+2s}} dz, \tag{2.5}$$

where $C_{N,p,s}$ is a specific constant, and the integration happens on an infinite cone $T_p^{0,\infty}\left(\frac{\nabla u(x)}{|\nabla u(x)|}\right) \subset \mathbb{R}^N$ whose centerline is aligned with the vector $\frac{\nabla u(x)}{|\nabla u(x)|}$ and whose aperture angle α depends on N,p. When $p=2$ this operator is consistent with the usual formula for $-(-\Delta)^s u(x)$, and when $p \to \infty$ one has $\alpha \to 0$ and the cone reduces to a line, which is consistent with the definition of $\Delta_\infty^s u(x)$ given earlier.

It is also worth mentioning that Δ_∞^s as treated here, is not the only nonlocal counterpart of Δ_∞. A variational *Infinity Fractional Laplacian* was studied in [11], which is not suited for a game theoretical approach, which was the main motivation in [9].

Notation

Let us fix some notation that we will use throughout.

Given $x_0 \in \mathbb{R}^N$, $t_0 \in \mathbb{R}$ and $r>0$, we denote the open ball of radius r centered at $(x_0,t_0) \in \mathbb{R}^{N+1}$ as $B_r(x_0,t_0) \subset \mathbb{R}^{N+1}$. Similarly, $B_r(x_0) \subset \mathbb{R}^N$ and $B_r(t_0) \subset \mathbb{R}$ denote the open balls of radius r centered at x_0 and t_0 respectively. Also, B_r denotes the open ball of radius $r >$ centered at $0 \in \mathbb{R}^N$.

We denote the $i-$th component of a vector $x \in \mathbb{R}^N$ by x_i, and e_i will denote the $i-$th canonical vector. S^{N-1} will denote the $(N-1)-$dimensional unit sphere contained in $\mathbb{R}^N$.

The distance function will be denoted by d. Precisely, if $x \in \mathbb{R}^N$ and $V,W \subset \mathbb{R}^N$, we consider,

$$d(x,V) = \inf_{y \in V} |x-y|, \quad \text{and} \quad d(V,W) = \inf_{\substack{y \in V \\ z \in W}} |y-z|.$$

In addition, the function $d(\cdot,\partial\Omega)$ is extended by zero to the whole $\mathbb{R}^N$, i.e,

$$d(x,\partial\Omega) = \left\{\begin{array}{ll} \operatorname{dist}(x,\partial\Omega) & \text{if } x \in \overline{\Omega} \\ 0 & \text{if } x \in \mathbb{R}^N \setminus \overline{\Omega}. \end{array}\right. \tag{2.6}$$

Given two (or more) functions f and g, $f \vee g$ denotes the pointwise maximum of the two functions, i.e $(f \vee g)(x) = \max\{f(x),g(x)\}$. Similarly, $f \wedge g$ denotes the minimum. Also, $f^+ := \max\{f,0\}$ and $f^- := \min\{f,0\}$.

We use ∇ to denote the gradient of a function only with respect to the spatial variable. When we write $P.V$ in front of an integral, it means that we are evaluating it in the sense of the Cauchy Principal Value.

The set $BC(\mathbb{R}^N)$ denotes the set of bounded continuous functions on $\mathbb{R}^N$. We consider the set of Hölder continuous functions,

Definition 2.2. A function $f : \Omega \to \mathbb{R}$ is said to be *Hölder continuous with exponent* γ on Ω if

$$[f]_{C^{0,\gamma}} := \sup_{x,y\in\Omega} \frac{|f(x)-f(y)|}{|x-y|^{\gamma}} < \infty.$$

In which case we denote $f \in C^{0,\gamma}(\Omega)$. Also, if

$$\lim_{\delta\to 0} \sup_{|x-y|\leq\delta} \frac{|f(x)-f(y)|}{|x-y|^{\gamma}} = 0,$$

we denote $f \in C_0^{0,\gamma}(\Omega)$.

Chapter 3

The parabolic problem in an open domain

In this chapter we study $\mathcal{P}$ on an arbitrary open domain $\Omega \subset \mathbb{R}^N$. We lack a existence result in this setting. However, here we are able to prove a comparison principle, regularity results, and the stability Theorem mentioned beforehand. Most of these results, and the ideas therein, will be useful in chapters 4 and 5.

3.1 Preliminaries

To prove that $\Delta_\infty^s \phi(x)$ is well defined at a point $x \in \mathbb{R}^N$, and that (2.1) coincides with the expression given in Definition 2.1, we need the following sense of regularity,

Definition 3.1. A function ϕ is in $C^{1,1}(x_0)$, or equivalently is "$C^{1,1}$ at a point x_0" if there exists a vector $p \in \mathbb{R}^N$ and constants $M, \eta_0 > 0$ such that

$$|\phi(x_0 + x) - \phi(x_0) - p \cdot x| \leq M|x|^2$$

for $|x| < \eta_0$. We denote $\nabla\phi(x_0) := p$.

Remark 3.2. If $u : \mathbb{R}^N \to \mathbb{R}$ is a continuous function such that at every point there is a concave (resp. convex) paraboloid touching it from below (resp. above), informally we say that u is $C^{1,1}$ from below (resp. above).

Remark 3.3. Given an open set $\Omega \subset \mathbb{R}^N$, we say that $\phi \in C^{1,1}(\Omega)$ if $\phi \in C^{1,1}(x)$ for all $x \in \Omega$.

The following result is part of Lemma 2.1 in [25]. For the sake of completeness we provide the proof that $\Delta_\infty^s \phi$ is well defined.

Lemma 3.4. *Let $x \in \mathbb{R}^N$ and $\phi \in C^{1,1}(x) \cap BC(\mathbb{R}^N)$. Then $\Delta_\infty^s \phi(x)$ is finite and well defined.*

Proof. To ease notation, we define,

$$L_\phi(x, \eta y, \eta z) := \phi(x + \eta y) + \phi(x - \eta z) - 2\phi(x).$$

Let $y, z \in S^{N-1}$. Observe that, since $\phi \in C^{1,1}(x)$,

$$|\phi(x + \eta y) + \phi(x - \eta z) - 2\phi(x) - \eta\langle p, y - z\rangle| \leq 2M\eta^2, \tag{3.1}$$

with $0<\eta<\eta_0$, where $M,\eta_0>0$ and $p\in\mathbb{R}^N$ are as in Definition 3.1. Then, we obtain,

$$\begin{aligned}
&\left|\int_0^\infty \frac{L_\phi(x,\eta y,\eta z)}{\eta^{1+2s}}d\eta-\int_0^{\eta_0}\frac{\eta\langle p,y-z\rangle}{\eta^{1+2s}}d\eta\right|\\
&\qquad=\left|\int_0^{\eta_0}\frac{L_\phi(x,\eta y,\eta z)}{\eta^{1+2s}}d\eta+\int_{\eta_0}^\infty\frac{L_\phi(x,\eta y,\eta z)}{\eta^{1+2s}}d\eta-\int_0^{\eta_0}\frac{\eta\langle p,y-z\rangle}{\eta^{1+2s}}d\eta\right|\\
&\qquad=\left|\int_0^{\eta_0}\frac{L_\phi(x,\eta y,\eta z)-\eta\langle p,y-z\rangle}{\eta^{1+2s}}d\eta+\int_{\eta_0}^\infty\frac{L_\phi(x,\eta y,\eta z)}{\eta^{1+2s}}d\eta\right|\\
&\qquad\le\left|\int_0^{\eta_0}\frac{2M}{\eta^{2s-1}}d\eta+\int_{\eta_0}^\infty\frac{4||\phi||_\infty}{\eta^{1+2s}}d\eta\right|\\
&\qquad<\infty
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\sup_{y\in S^{N-1}}\inf_{z\in S^{N-1}}\langle p,y-z\rangle&=\sup_{y\in S^{N-1}}\inf_{z\in S^{N-1}}\langle p,y\rangle-\langle p,z\rangle\\
&=\sup_{y\in S^{N-1}}\langle p,y\rangle-\sup_{z\in S^{N-1}}\langle p,z\rangle\\
&=0
\end{aligned}$$

which implies,

$$\sup_{y\in S^{N-1}}\inf_{z\in S^{N-1}}\int_0^{\eta_0}\frac{\eta\langle p,y-z\rangle}{\eta^{1+2s}}d\eta=\sup_{y\in S^{N-1}}\inf_{z\in S^{N-1}}\langle p,y-z\rangle\cdot\int_0^{\eta_0}\frac{1}{\eta^{2s}}d\eta=0.$$

This yields,

$$\begin{aligned}
|\Delta_\infty^s\phi(x)|&=\left|\sup_{y\in S^{N-1}}\inf_{z\in S^{N-1}}\int_0^\infty\frac{L_\phi(x,\eta y,\eta z)}{\eta^{1+2s}}d\eta-\sup_{y\in S^{N-1}}\inf_{z\in S^{N-1}}\int_0^{\eta_0}\frac{\eta\langle p,y-z\rangle}{\eta^{1+2s}}d\eta\right|\\
&\le\sup_{y\in S^{N-1}}\sup_{z\in S^{N-1}}\left|\int_0^\infty\frac{L_\phi(x,\eta y,\eta z)}{\eta^{1+2s}}d\eta-\int_0^{\eta_0}\frac{\eta\langle p,y-z\rangle}{\eta^{1+2s}}d\eta\right|\\
&<\infty,
\end{aligned}$$

which concludes the proof. ☐

For an explicit bound on $\Delta_\infty^s\phi(x)$, we refer to Lemma 2.1 in [25].

The following result (Proposition 2.2 in [25]) gives the equivalence between both the definitions of the operator that we presented in the introduction. We include a sketch of the proof for completeness.

Lemma 3.5. *For $s\in(1/2,1)$, the operator $\Delta_\infty^s: C^{1,1}(x)\cap BC(\mathbb{R}^N)\to\mathbb{R}$ at a point $x\in\mathbb{R}^N$ is given by:*

$$\Delta_\infty^s\phi(x):=\sup_{y\in S^{N-1}}\inf_{z\in S^{N-1}}\int_0^\infty\frac{\phi(x+\eta y)+\phi(x-\eta z)-2\phi(x)}{\eta^{1+2s}}\,d\eta \tag{3.2}$$

Sketch of the Proof. When $\nabla\phi(x)\neq0$, it can be shown that the supremum and infimum of (2.1) is actually taken at $v:=\frac{\nabla\phi(x)}{|\nabla\phi(x)|}$. To see this, assume that that the supremum in (2.1) is taken at y, and let us argue that $y=v$. Indeed, by splitting the integral and using the definitions of $C^{1,1}$ and the infimum,

$$\Delta_\infty^s\phi(x)\le\int_0^\infty\frac{(\phi(x+\eta y)+\phi(x-\eta v)-2\phi(x))}{\eta^{1+2s}}d\eta\le\langle\nabla\phi(x),(y-v)\rangle\cdot\int_0^{\eta_0}\frac{\eta}{\eta^{2s+1}}d\eta+C,$$

where C is a finite constant. Using the bounds from the proof of Lemma 3.4, we get that the right hand side must be finite so, since the integral diverges, we must have $y=v$. A similar argument holds for the infimum. The equivalence when $\nabla\phi(x)=0$ follows from the fact that the integrals in this case are well-defined and can be combined to get (2.1). ☐

3.2 Viscosity solutions

To make sense of the operator when working with functions that are not necessarily smooth, we work with the notion of viscosity solutions of $\mathcal{P}$. As mentioned previously, $\Delta_\infty^s u(x)$ could be not well defined if $u \notin C^{1,1}$, so in this case we use the standard idea of test functions for viscosity solutions, replacing u locally with a $C^{1,1}$ function which touches it from below or above.

Definition 3.6. A function $u : \mathbb{R}^{N+1} \to \mathbb{R}$ such that $u \in USC(\overline{\Omega}\times[0,T])$ (resp. $u \in LSC(\overline{\Omega}\times[0,T])$), is said to be a ***viscosity subsolution*** (resp. ***supersolution***) of $\mathcal{P}$ if:

- $u \leq u_0$ (resp. $u \geq u_0$) on $\mathbb{R}^N \times \{0\}$,
- $u \leq g$ (resp. $u \geq g$) on $\Omega^c \times (0,T]$,

and, for every $(x_0,t_0) \in \Omega \times (0,T)$, everytime that all of the following holds: $\phi : \mathbb{R}^{N+1} \to \mathbb{R}$ is such that;

- $\phi \in C^1((t_0-r,t_0+r)); C^{1,1}(x_0) \cap C(\overline{B_r(x_0)})$ for $r > 0$,
- $\phi(x_0,t_0) = u(x_0,t_0)$,
- $\phi(x,t) > u(x,t)$ (resp. $\phi(x,t) < u(x,t)$) for all $(x,t) \in B_r(x_0,t_0) \setminus \{(x_0,t_0)\}$,

we have $\partial_t\tilde{u}(x_0,t_0) - \Delta_\infty^s\tilde{u}(x_0,t_0) \leq 0$ on $\Omega\times(0,T]$ (resp. $\partial_t\tilde{u}(x_0,t_0) - \Delta_\infty^s\tilde{u}(x_0,t_0) \geq 0$ on $\Omega\times(0,T]$), where

$$\tilde{u}(x,t) := \begin{cases} \phi(x,t), & \text{if } \ (x,t) \in B_r(x_0,t_0), \\ u(x,t), & \text{if } \ (x,t) \notin B_r(x_0,t_0). \end{cases} \tag{3.3}$$

Throughout the document, we refer to these functions ϕ as *test functions*, considering the same regularity expressed in the definition.

In this last definition we say a test function ϕ "touches u from above (resp. below) at (x_0,t_0)". Also, we say that a function $u : \mathbb{R}^n \times (0,T) \to \mathbb{R}$ is a "***subsolution (resp. supersolution) at non-zero gradient points***" if it satisfies the subsolution (resp. supersolution) condition of Definition 3.6 only when $\nabla\phi(x,t) \neq 0$. So, if u is a subsolution (resp. supersolution) then, in particular, it will be a subsolution (resp. supersolution) on non-zero gradient points. If a function is a subsolution and a supersolution of $\mathcal{P}$ at the same time, we say that it is a viscosity solution of $\mathcal{P}$.

From now on we drop the term "viscosity" and we will only write "solution" when referring to a viscosity solution. It is worth mentioning that the notion of solution given by Definition 3.6, is the only one that we will use throughout the whole document.

3.3 Comparison Principle

First we showcase a technical result from [9] that helps us prove the comparison principle on points where the gradient is zero.

Lemma 3.7. *Let $u, w \in C^{1,1}(x_0) \cap BC(\mathbb{R}^N)$ such that $\nabla u(x_0) = \nabla w(x_0) = 0$. Then,*

$$\begin{aligned} 2\inf_{z\in S^{N-1}} \int_0^\infty & \frac{[u-w](x_0-\eta z) - [u-w](x_0)}{\eta^{1+2s}} d\eta \\ & \leq \Delta_\infty^s u(x_0) - \Delta_\infty^s w(x_0) \\ & \leq 2\sup_{y\in S^{N-1}} \int_0^\infty \frac{[u-w](x_0+\eta y) - [u-w](x_0)}{\eta^{1+2s}} d\eta. \end{aligned}$$

Proof. Define

$$L(u,y,x_0) = \int_0^\infty \frac{u(x_0+\eta y) - u(x_0)}{\eta^{1+2s}} d\eta$$

Then,

$$\Delta_\infty^s u(x_0) = \sup_{y\in S^{N-1}} L(u,y,x_0) + \inf_{y\in S^{N-1}} L(u,y,x_0)$$

For all $\delta > 0$ there exists $\hat{y}, \bar{y} \in S^{N-1}$ such that

$$\sup_{y \in S^{N-1}} L(u, y, x_0) - L(u, \hat{y}, x_0) < \delta, \qquad \sup_{y \in S^{N-1}} L(w, y, x_0) - L(w, \bar{y}, x_0) < \delta$$

This implies,

$$L(u, \bar{y}, x_0) \leq \sup_{y \in S^{N-1}} L(u, y, x_0) < \delta + L(u, \hat{y}, x_0)$$
$$-L(w, \bar{y}, x_0) - \delta < -\sup_{y \in S^{N-1}} L(w, y, x_0) \leq -L(w, \hat{y}, x_0)$$

Putting everything together and using the linearity of L we get,

$$\begin{aligned} &\inf_{z \in S^{N-1}} L(u - w, -z, x_0) - \delta \\ &\quad < \sup_{y \in S^{N-1}} L(u, y, x_0) - \sup_{y \in S^{N-1}} L(w, y, x_0) \\ &\quad < \sup_{y \in S^{N-1}} L(u - w, y, x_0) + \delta \end{aligned}$$

An analogous argument holds for the infimum in the definition of Δ_∞^s, and the proof is complete by putting both cases together and taking $\delta \to 0$. ☐

Now we move on to proving a comparison principle on compact sets. We will suppose our functions grow less than $|x|^{2s}$ at infinity, so that the integral defining Δ_∞^s converges at infinity.

Theorem 3.8. *Let $\Omega \subset \mathbb{R}^N$ be a bounded set. Let $u, w : \mathbb{R}^{N+1} \to \mathbb{R}$, be two functions such that:*

- *u and w are subsolution and supersolution, respectively, of $\mathcal{P}$.*
- *u and w are continuous on $\Omega \times [0, T]$ and $\Omega^c \times [0, T]$.*
- *u and w grow less than $|x|^{2s}$ at infinity.*

Then, $u \leq w$ on $\Omega \times (0, T)$.

Proof. In this proof we use the inf-convolution and sup-convolution, for more details about these functions and useful results, we refer to Appendix B.

Arguing by contradiction, let's suppose that there exists $(x_0, t_0) \in \Omega \times (0, T)$ such that $u(x_0, t_0) > w(x_0, t_0)$. Fix $c > 0$ such that $u(x_0, t_0) - w(x_0, t_0) > c > 0$, and replace u and w for $u^{\varepsilon,\kappa}$ and $w_{\varepsilon,\kappa}$, the sup-convolution and inf-convolution respectively. Since

$$u \leq u^{\varepsilon,\kappa} \text{ and } w \geq w_{\varepsilon,\kappa} \qquad \text{on } \mathbb{R}^N \times [0, T],$$

we have $u^{\varepsilon,\kappa}(x_0, t_0) - w_{\varepsilon,\kappa}(x_0, t_0) > c$ for all $\varepsilon, \kappa > 0$. Also, since u and w are sub and super solution, we have $u \leq w$ on $(\Omega^c \times (0, T]) \cup (\mathbb{R}^N \times \{0\})$, this implies

$$(u^{\varepsilon,\kappa} - w_{\varepsilon.\kappa}) \vee 0 \to 0$$

locally uniformly on $(\Omega^c \times (0, T]) \cup (\mathbb{R}^N \times \{0\})$ as $\varepsilon, \kappa \to 0$. Then, the continuous function $u^{\varepsilon,\kappa} - w_{\varepsilon.\kappa}$ reaches its maximum on $\overline{\Omega} \times [0, T]$ at a point $(\overline{x}, \overline{t}) \in \Omega \times (0, T]$. Fix $\delta := (u^{\varepsilon,\kappa} - w_{\varepsilon,\kappa})(\overline{x}, \overline{t})$, since $u^{\varepsilon,\kappa}(\cdot, \overline{t})$ is $C^{1,1}$ from below, $(w_{\varepsilon,\kappa} + \delta)(\cdot, \overline{t})$ is $C^{1,1}$ from above and $(w_{\varepsilon,\kappa} + \delta)(\cdot, \overline{t})$ touches $u^{\varepsilon,\kappa}(\cdot, \overline{t})$ from below at $\overline{x}$, we have that both functions are in $C^{1,1}(\overline{x})$ and $\nabla u^{\varepsilon,\kappa}(\overline{x}, \overline{t}) = \nabla(w_{\varepsilon,\kappa} + \delta)(\overline{x}, \overline{t}) = \nabla w_{\varepsilon,\kappa}(\overline{x}, \overline{t})$. Then, we can evaluate $\Delta_\infty^s u^{\varepsilon,\kappa}(\overline{x}, \overline{t})$ and $\Delta_\infty^s(w_{\varepsilon,\kappa} + \delta)(\overline{x}, \overline{t})$ directly without appealing to test functions.

Thanks to Lemma B.3, $u^{\varepsilon,\kappa}$ and $w_{\varepsilon,\kappa}$ are sub and super solutions of $\mathcal{P}$, therefore

$$\partial_t u^{\varepsilon,\kappa}(\overline{x}, \overline{t}) - \Delta_\infty^s u^{\varepsilon,\kappa}(\overline{x}, \overline{t}) \leq 0$$

and

$$\partial_t(w_{\varepsilon,\kappa} + \delta)(\overline{x}, \overline{t}) - \Delta_\infty^s(w_{\varepsilon,\kappa} + \delta)(\overline{x}, \overline{t}) \geq 0,$$

Now, we study two cases:

Case I: Suppose that $\nabla u^{\varepsilon,\kappa}(\overline{x},\overline{t}) = \nabla w_{\varepsilon,\kappa}(\overline{x},\overline{t}) \neq 0$ and let $v \in S^{N-1}$ be the direction of this vector, then,

$$\partial_t(u^{\varepsilon,\kappa} - w_{\varepsilon,\kappa} - \delta)(\overline{x},\overline{t}) + \Delta^s_\infty(w_{\varepsilon,\kappa} + \delta)(\overline{x},\overline{t}) - \Delta^s_\infty u^{\varepsilon,\kappa}(\overline{x},\overline{t}) \leq 0$$

Note that, if $\overline{t} \neq T$, we have $\partial_t(u^{\varepsilon,\kappa} - w_{\varepsilon,\kappa} - \delta)(\overline{x},\overline{t}) = 0$, and if $\overline{t} = T$ then $\partial_t(u^{\varepsilon,\kappa} - w_{\varepsilon,\kappa} - \delta)(\overline{x},\overline{t}) \geq 0$, both cases are due to the fact that $(u^{\varepsilon,\kappa} - w_{\varepsilon,\kappa})$ reaches its maximum at $(\overline{x},\overline{t})$. It follows that,

$$\begin{aligned} 0 &\geq \Delta^s_\infty(w_{\varepsilon,\kappa} + \delta)(\overline{x},\overline{t}) - \Delta^s_\infty u^{\varepsilon,\kappa}(\overline{x},\overline{t}) \\ &= \int_0^\infty \frac{[w_{\varepsilon,\kappa} + \delta - u^{\varepsilon,\kappa}]\left(\overline{x} + \eta v,\overline{t}\right) + [w_{\varepsilon,\kappa} + \delta - u^{\varepsilon,\kappa}]\left(\overline{x} - \eta v,\overline{t}\right) - 2[w_{\varepsilon,\kappa} + \delta - u^{\varepsilon,\kappa}]\left(\overline{x},\overline{t}\right)}{|\eta|^{1+2s}} d\eta \end{aligned}$$

Since $[w_{\varepsilon,\kappa} + \delta - u^{\varepsilon,\kappa}](\overline{x},\overline{t}) = 0$ and $[w_{\varepsilon,\kappa} + \delta - u^{\varepsilon,\kappa}] \geq 0$ on $\mathbb{R}^N \times (0,T)$, we have $[w_{\varepsilon,\kappa} + \delta - u^{\varepsilon,\kappa}] \leq 0$ along the line $\{\overline{x} + \eta v, \overline{t}\}_{\eta \in \mathbb{R}}$. Then, taking $\varepsilon, \kappa \to 0$ we get $w + \delta - u \leq 0$, this implies $w < u$ on the line, which contradicts assumption $u \leq w$ on $\Omega^c \times (0,T)$, since, taking η large enough, we get $w(\cdot,\overline{t}) < u(\cdot,\overline{t})$ at some point outside Ω.

Case II: Suppose $\nabla u^{\varepsilon,\kappa}(\overline{x},\overline{t}) = \nabla w_{\varepsilon,\kappa}(\overline{x},\overline{t}) = 0$. We proceed in a similar way as the previous case, but now we make use of Lemma 3.7. Again, we have $\partial_t(u^{\varepsilon,\kappa} - w_{\varepsilon,\kappa} - \delta)(\overline{x},\overline{t}) \geq 0$, and

$$0 \geq \Delta^s_\infty(w_{\varepsilon,\kappa} + \delta)(\overline{x},\overline{t}) - \Delta^s_\infty u^{\varepsilon,\kappa}(\overline{x},\overline{t})$$

By Lemma 3.7, it follows,

$$0 \geq 2 \inf_{z \in S^{N-1}} \int_0^\infty \frac{[w_{\varepsilon,\kappa} + \delta - u^{\varepsilon,\kappa}]\left(\overline{x} - \eta z,\overline{t}\right) - [w_{\varepsilon,\kappa} + \delta - u^{\varepsilon,\kappa}]\left(\overline{x},\overline{t}\right)}{|\eta|^{1+2s}} d\eta.$$

Since $[w_{\varepsilon,\kappa} + \delta - u^{\varepsilon,\kappa}](\overline{x},\overline{t}) = 0$ and $[w_{\varepsilon,\kappa} + \delta - u^{\varepsilon,\kappa}] \geq 0$ on $\mathbb{R}^N \times (0,T)$, we have $[w_\varepsilon + \delta - u^\varepsilon] \leq 0$ on the line $\{\overline{x} + \eta z, \overline{t}\}_{\eta \in \mathbb{R}}$ for some direction $z \in S^{N-1}$. Then, we can conclude in the same way we did in the previous case. ☐

Remark 3.9. In any case that we study going forward, the functions will always grow at infinity at most as $|x|^{2s-1}$, so the last assumption from the Theorem will always be satisfied.

3.4 Regularity

We will show that solutions of $\mathcal{P}$ are Hölder continuous with exponent $2s-1$ on the spatial variable, and Lipschitz continuous on the time variable. The regularity on the spatial variable comes from the fact that

$$\Delta^s_\infty\left(A|x - x_0|^{2s-1} + B\right) = 0$$

for $x \neq x_0$ and any $A, B \in \mathbb{R}$, so we can use these cusps as barriers, this is the content of the following Lemma.

Lemma 3.10. *Let $A, B, \alpha \in \mathbb{R}$ and $x \in \mathbb{R}^N \setminus \{0\}$. Define*

$$\mathcal{C}(x) := A|x|^\alpha + B.$$

Then

$$\Delta^s_\infty \mathcal{C}(x) = A|x|^{\alpha-2s} F(\alpha),$$

where,

$$F(\alpha) = \int_0^\infty \frac{|1+t|^\alpha + |1-t|^\alpha - 2}{t^{1+2s}} dt$$

is a convex function in $C^\infty(0,\infty)$ such that $F(0) = F(2s-1) = 0$, $F(\alpha) < 0$ if $\alpha \in (0, 2s-1)$ and $F(\alpha) > 0$ if $\alpha > 2s-1$.

Proof. The first part of this proof follows ideas from the proof of Lemma 4 in [6].

If $x \neq 0$, $\mathcal{C}(x)$ is smooth enough to evaluate $\Delta^s_\infty \mathcal{C}(x)$ without appealing to test functions. Note that,

$$v := \frac{\nabla \mathcal{C}(x)}{|\nabla \mathcal{C}(x)|} = \frac{x}{|x|} \neq 0.$$

Then,

$$\begin{aligned}
\Delta^s_\infty \mathcal{C}(x) &= \int_0^\infty \frac{\mathcal{C}(x+\eta v) + \mathcal{C}(x-\eta v) - 2\mathcal{C}(x)}{\eta^{1+2s}} d\eta \\
&= \int_0^\infty \frac{A\left|x + \eta \frac{x}{|x|}\right|^\alpha + B + A\left|x - \eta\frac{x}{|x|}\right|^\alpha + B - 2A|x|^\alpha - 2B}{\eta^{1+2s}} d\eta \\
&= A\int_0^\infty \frac{|x+\eta\frac{x}{|x|}|^\alpha + |x-\eta\frac{x}{|x|}|^\alpha - 2|x|^\alpha}{\eta^{1+2s}} d\eta
\end{aligned}$$

By doing the change of variable $t = \frac{\eta}{|x|}$ we get

$$\begin{aligned}
\Delta^s_\infty \mathcal{C}(x) &= A\int_0^\infty \frac{|x+tx|^\alpha + |x-tx|^\alpha - 2|x|^\alpha}{(t|x|)^{1+2s}} |x| dt \\
&= A|x|^{\alpha-2s} \int_0^\infty \frac{|1+t|^\alpha + |1-t|^\alpha - 2}{t^{1+2s}} dt \\
&= A|x|^{\alpha-2s} F(\alpha)
\end{aligned}$$

Note that for all $k \in \mathbb{N}$, our candidate for the $k-$th derivative of $F^{(k)}(\alpha)$ is given by

$$\int_0^\infty \frac{|1+t|^k(\log(|1+t|))^\alpha + |1-t|^k(\log(|1-t|))^\alpha}{t^{1+2s}} dt$$

This integral converges for each $k \geq 1$. Indeed, by the Taylor expansion for $t \sim 0$ we can deduce that $|1+t|^k(\log(|1+t|))^\alpha + |1-t|^k(\log(|1-t|))^\alpha = O(t^2)$, so $F \in C^\infty(0,\infty)$. To see that F is convex, note that

$$F''(\alpha) = \int_0^\infty \frac{|1+t|^2(\log(|1+t|))^\alpha + |1-t|^2(\log(|1-t|))^\alpha}{t^{1+2s}} dt > 0.$$

Now, for the second part, we follow the proof of Lemma B.1 in [5]. Clearly $F(0) = 0$. To prove $F(2s-1) = 0$, we have,

$$\int_0^\infty \frac{|1+t|^\alpha + |1-t|^\alpha - 2}{|t|^{1+2s}} dt = \underbrace{P.V \int_{-1}^\infty \frac{|1+t|^\alpha - 1}{|t|^{1+2s}} dt}_{I} + \underbrace{\int_1^\infty \frac{|1-t|^\alpha - 1}{|t|^{1+2s}} dt}_{II}$$

First we compute I. By doing the change of variable $1 + t = e^z$ we get

$$\begin{aligned}
I &= P.V \int_{-\infty}^\infty \frac{e^{z\alpha} - 1}{|e^z - 1|^{1+2s}} e^z dz \\
&= P.V \int_{-\infty}^\infty \frac{e^{\frac{z\alpha}{2}}\left(e^{\frac{z\alpha}{2}} - e^{\frac{-z\alpha}{2}}\right)}{e^{\frac{z}{2}(1+2s)}|e^{\frac{z}{2}} - e^{\frac{-z}{2}}|^{1+2s}} e^z dz \\
&= P.V \int_{-\infty}^\infty \frac{2\sinh\left(\frac{z\alpha}{2}\right)}{\left|2\sinh\left(\frac{z}{2}\right)\right|^{1+2s}} e^{\frac{z}{2}(\alpha-1-2s+2)} dz
\end{aligned}$$

Note that, when $\alpha = 2s-1$ the integrand in I is an odd function, so, in this case $I = 0$. Now we compute II. In a similar way, we do the change of variable $t-1 = e^z$ to get

$$\begin{aligned} II &= \int_{-\infty}^{\infty} \frac{e^{z\alpha}-1}{|e^z+1|^{1+2s}} e^z dz \\ &= \int_{-\infty}^{\infty} \frac{e^{\frac{z\alpha}{2}}\left(e^{\frac{z\alpha}{2}} - e^{\frac{-z\alpha}{2}}\right)}{e^{\frac{z}{2}(1+2s)}|e^{\frac{z}{2}} + e^{\frac{-z}{2}}|^{1+2s}} e^z dz \\ &= \int_{-\infty}^{\infty} \frac{2\sinh\left(\frac{z\alpha}{2}\right)}{\left|2\cosh\left(\frac{z}{2}\right)\right|^{1+2s}} e^{\frac{z}{2}(\alpha-1-2s+2)} dz \end{aligned}$$

Again, when $\alpha = 2s-1$ the integrand in II is an odd function, so $II = 0$. We conclude $F(2s-1) = 0$. Finally, because of the strict convexity we have $F(\alpha) < 0$ for $\alpha \in (0, 2s-1)$ and $F(\alpha) > 0$ for $\alpha > 2s-1$. ☐

Taking $g(\cdot, t) \in C^{0,2s-1}(\mathbb{R}^N)$, and comparing a subsolution with a cusp $\mathcal{C}$, we get obtain a regularity result in the spatial variable. The idea of the proof is: If u is a subsolution, we can choose $A, B \in \mathbb{R}$ conveniently such that the cusp $\mathcal{C}$ centered on x_0, can only touch u on (x_0, t_0), then thanks to the assumptions on u, we conclude the result. The following result and its proof follow Theorem 3.7 in [9].

Theorem 3.11. *Assume $g(\cdot, t) \in C^{0,2s-1}(\mathbb{R}^N)$ for all $t \in (0, T)$. Let $\Omega \subset \mathbb{R}^N$ be an open set, and u be a subsolution of $\mathcal{P}$, with $u = g$ on $\Omega^c \times (0, T)$. Suppose that there exists constants $A > 0$ and $C_0 < \infty$, such that*

$$\sup_{z\in\Omega^c} \left\{ g(z,t) - A\,|x-z|^{2s-1} \right\} \le u(x,t) \le C_0, \qquad \forall (x,t) \in \Omega \times (0,T). \tag{3.4}$$

Then,

$$|u(x,t) - u(y,t)| \le A|x-y|^{2s-1} \qquad \forall x, y \in \mathbb{R}^N, t \in (0,T), \tag{3.5}$$

that is, $u(\cdot, t) \in C^{0,2s-1}(\mathbb{R}^N)$ and

$$[u(\cdot,t)]_{C^{0,2s-1}(\mathbb{R}^N)} \le A.$$

for all $t \in (0, T)$.

Proof. Fix $(x_0, t_0) \in \Omega \times (0, T)$. we will show that, for each $x \in \mathbb{R}^N$,

$$u(x,t_0) \le u(x_0,t_0) + A\,|x-x_0|^{2s-1}. \tag{3.6}$$

Fix $\varepsilon > 0$ and define $\mathcal{C}(x) = B + (A+\varepsilon)\,|x-x_0|^{2s-1}$, where $B \in \mathbb{R}$ will be chosen conveniently. Thanks to (3.4), if we take $B = u(x_0, t_0)$, then $\mathcal{C}(\cdot) > g(\cdot, t_0)$ on Ω^c. This implies that, if first we take $B = C_0$, so $\mathcal{C}(x) \ge u(x, t_0)$ on $\mathbb{R}^N$, and then lower B until $\mathcal{C}(x)$ touches $u(x, t_0)$ from above at a point $\bar{x}$, then $\bar{x} \in \Omega$.

We claim that $\bar{x} = x_0$. Assume by contradiction that $\bar{x} \neq x_0$, in this case $\mathcal{C}$ is smooth enough on a neighborhood of $\bar{x}$ so we can use it as a test function that touches u from above and define $\tilde{u}$ as in (3.3).

Since u is a subsolution we have

$$\partial_t \tilde{u}(\bar{x}, t_0) - \Delta^s_\infty \tilde{u}(\bar{x}, t_0) \le 0.$$

Then, as $\mathcal{C} \ge u(\cdot, t_0)$ on $\mathbb{R}^N$, for any $r \in (0, |\bar{x} - x_0|)$ we have

$$\begin{aligned}
0 \geq \partial_t \tilde{u}(\bar{x}, t_0) - \Delta^s_\infty \tilde{u}(\bar{x}, t_0) =&\partial_t \mathcal{C}(\bar{x}) - \int_0^r \frac{\mathcal{C}(\bar{x}+\eta v) + \mathcal{C}(\bar{x}-\eta v) - 2\mathcal{C}(\bar{x})}{\eta^{1+2s}} d\eta \\
&- \int_r^\infty \frac{u(\bar{x}+\eta v, t_0) + u(\bar{x}-\eta v, t_0) - 2u(\bar{x}, t_0)}{\eta^{1+2s}} d\eta \\
\geq& 0 - \int_0^r \frac{\mathcal{C}(\bar{x}+\eta v) + \mathcal{C}(\bar{x}-\eta v) - 2\mathcal{C}(\bar{x})}{\eta^{1+2s}} d\eta \\
&- \int_r^\infty \frac{\mathcal{C}(\bar{x}+\eta v) + \mathcal{C}(\bar{x}-\eta v) - 2\mathcal{C}(\bar{x})}{\eta^{1+2s}} d\eta \\
=& - \Delta^s_\infty \mathcal{C}(\bar{x}, t_0) \\
=& 0
\end{aligned} \tag{3.7}$$

where $v \in S^{N-1}$ denotes the direction of $\nabla \mathcal{C}(\bar{x}) \neq 0$. This inequality implies,

$$\int_r^\infty \frac{u(\bar{x}+\eta v, t_0) + u(\bar{x}-\eta v, t_0) - 2u(\bar{x}, t_0)}{\eta^{1+2s}} d\eta = \int_r^\infty \frac{\mathcal{C}(\bar{x}+\eta v) + \mathcal{C}(\bar{x}-\eta v) - 2\mathcal{C}(\bar{x})}{\eta^{1+2s}} d\eta$$

Using the fact that $\mathcal{C}(x) \geq u(x, t_0)$ with equality on $\bar{x}$, we conclude $u(x, t_0) = \mathcal{C}(x)$ on the set $\{\bar{x} + \eta v\}_{\eta \geq r}$, which contradicts $u = g$ on $\Omega^c \times \{t_0\}$. Then $\bar{x} = x_0$, which implies,

$$u(x, t_0) \leq \mathcal{C}(x) = u(x_0, t_0) + (A + \varepsilon) |x - x_0|^{2s-1} \quad \forall x \in \mathbb{R}^N.$$

Hence, taking $\varepsilon \to 0$ we get (3.6). On the other hand, if we fix $x \in \Omega$ and change the roles of x and x_0 in the proof we get (3.5), and thanks to the arbitrariness of t_0 we get the result. ☐

Now we move on to prove the regularity of solutions of $\mathcal{P}$ on the time variable under some extra assumptions on u_0 and g, when Ω is a bounded set. For this, we use the fact that $\mathcal{P}$ is invariant under translations in t and the Comparison Principle. Our approach is based on the regularity proofs presented in [1] and [22].

Lemma 3.12. *Assume the following:*

- $u_0 \in C^{1,1}(\Omega) \cap BC(\mathbb{R}^N)$, *and* $g(\cdot, t) \in C^{0,2s-1}(\mathbb{R}^N)$,
- $u_0(x) = g(x, 0)$ *on* $\mathbb{R}^N$,
- $|g(x,t) - g(x,t')| \leq ||\partial_t g||_\infty |t - t'|$ *for all* $x \in \mathbb{R}^N$ *and* $t, t' \in (0, T)$.

Let $\Omega \subset \mathbb{R}^N$ *be a bounded set, and* u *a solution of* $\mathcal{P}$, *then there exists* $\lambda > 0$, *that depends on* u_0 *and* g *such that*

$$|u(x,t) - u_0(x)| \leq \lambda t$$

for all $(x, t) \in \mathbb{R}^N \times (0, T)$.

Proof. Define the function $w^+(x,t) = u_0(x) + \lambda t$, where $\lambda > 0$ will be chosen conveniently. We have $w^+(x, 0) = u_0(x)$, and

$$\begin{aligned}
w^+(x,t) &= u_0(x) + \lambda t \\
&= g(x, 0) + \lambda t \\
&\geq g(x,t) - t||\partial_t g||_\infty + \lambda t \\
&= g(x,t) + t(\lambda - ||\partial_t g||_\infty) \\
&\geq g(x,t),
\end{aligned}$$

for $x \in \Omega^c$, if $\lambda \geq ||\partial_t g||_\infty$.

On the other hand, because of the regularity of u_0, we can evaluate $\Delta^s_\infty u_0$ directly without appealing to test functions, then, for $(x,t) \in \Omega \times (0,T)$ we have $\nabla u_0(x) = \nabla w^+(x,t)$ and

$$\begin{aligned} \partial_t w^+(x,t) - \Delta^s_\infty w^+(x,t) &= \lambda - \Delta^s_\infty u_0(x) \\ &\geq \lambda - ||\Delta^s_\infty u_0(x)||_\infty \\ &\geq 0 \end{aligned}$$

if $\lambda \geq ||\Delta^s_\infty u_0(x)||_\infty$. Hence, if $\lambda = \max\{||\Delta^s_\infty u_0(x)||_\infty, ||\partial_t g||_\infty\}$, w^+ is a supersolution of $\mathcal{P}$. In a similar way, the function $w^-(x,t) = u_0(x) - \lambda t$ is a subsolution if $\lambda = \max\{||\Delta^s_\infty u_0(x)||_\infty, ||\partial_t g||_\infty\}$. Then, using the Comparison principle we get

$$w^- \leq u \leq w^+,$$

which implies,

$$|u(x,t) - u_0(x)| \leq \lambda t.$$

□

Theorem 3.13. *Under the same assumptions of Lemma 3.12, let $\Omega \subset \mathbb{R}^N$ be a bounded set, and u a solution of $\mathcal{P}$, then*

$$|u(x,t) - u(x,s)| \leq \lambda |t-s|$$

for all $x \in \mathbb{R}^N$ and $t, s \in (0,T)$, with λ as in Lemma 3.12.

Proof. Fix $h \in (-T,T)$ and define $v(x,t) := u(x,t-h)$. Define the interval $(t_1,t_2) := (0,T) \cap (h, T+h)$ and consider the problem

$$\begin{cases} \partial_t w(x,t) - \Delta^s_\infty w(x,t) &= 0 & \text{if } (x,t) \in \Omega \times (t_1,t_2]; \\ w(x,t) &= g(x,t) & \text{if } (x,t) \in \Omega^c \times (t_1,t_2], \\ w(x,t_1) &= u(x,t_1) & \text{if } x \in \mathbb{R}^N. \end{cases} \tag{$\mathcal{P}_h$}$$

Note that $t_1 = \max\{0,h\}$ and $t_2 = \min\{T, T+h\}$. By Lemma 3.12 we have

$$|u(x,h) - u_0(x)| \leq \lambda |h|$$

if $h \geq 0$, and

$$|u(x,-h) - u_0(x)| \leq \lambda |h|$$

if $h \leq 0$. Thus,

$$|v(x,t_1) - u(x,t_1)| \leq \lambda |h| \tag{3.8}$$

We claim that $w^+(x,t) := v(x,t) + \lambda|h|$ is a supersolution of $\mathcal{P}_h$ in the sense of Definition 3.6. Let's check if it satisfies the conditions,

- By (3.8) we have $w^+(x,t_1) \geq u(x,t_1)$
- If $(x,t) \in \Omega^c \times (t_1,t_2]$, in particular $(x,t) \in \Omega^c \times (0,T)$, so

  $$\begin{aligned} v(x,t) + \lambda|h| &= u(x,t-h) + \lambda|h| \\ &= g(x,t-h) + \lambda|h| \\ &\geq g(x,t) - \lambda|h| + \lambda|h| \\ &= g(x,t) \end{aligned}$$

- Note that, by its definition, v is a solution of $\mathcal{P}$ on $\Omega \times (h, T+h)$, therefore, for $(x,t) \in \Omega \times (t_1,t_2] \subset \Omega \times (h, T+h]$ we have

  $$\partial_t(v(x,t) + \lambda|h|) - \Delta^s_\infty(v(x,t) + \lambda|h|) = 0$$

  in the viscosity sense.

Hence, w^+ is a supersolution of $\mathcal{P}_h$. Analogously, the function $w^-(x,t) = v(x,t) - \lambda|h|$ is a subsolution of $\mathcal{P}_h$. Then, by the comparison principle we get

$$w^- \leq u \leq w^+$$

which implies

$$|u(x,t) - u(x,t-h)| \leq \lambda|h|$$

for all $(x,t) \in \Omega \times (t_1,t_2]$. By the arbitrariness of h, we conclude the result. □

3.5 Stability

Now we move on to proving that the set of subsolutions and supersolutions of $\mathcal{P}$ at non-zero gradient points is closed. Later we will be able to apply the following results to help prove the existence of solutions to $\mathcal{P}$ on the settings mentioned in the introduction.

First, we prove a result that allows us to "combine" subsolutions, or supersolutions, to construct new ones.

Lemma 3.14. *The maximum (resp. minimum) of two subsolutions (resp. supersolutions) of $\mathcal{P}$ is a subsolution (resp. supersolution) of $\mathcal{P}$.*

Proof. Let u_1 and u_2 be two subsolutions of $\mathcal{P}$. We will check that $w = u_1 \vee u_2$ is a subsolution.

Let $(x_0,t_0) \in \Omega \times (0,T)$. If ϕ is a test function that touches w from above on $B_r(x_0,t_0)$, then

$$\phi(x_0,t_0) = u_1(x_0,t_0) \qquad \text{or} \qquad \phi(x_0,t_0) = u_2(x_0,t_0)$$

Let's assume without loss of generality that the first equality is true. Observe that Δ_∞^s is monotone increasing, i.e,

$$-\Delta_\infty^s \tilde{w}(x_0,t_0) \leq -\Delta_\infty^s \tilde{u}_1(x_0,t_0) \leq 0 \tag{3.9}$$

Indeed, let $y,z \in S^{N-1}$ be arbitrary, then,

$$\begin{aligned}&\int_0^r \frac{\phi(x_0+\eta y,t_0)+\phi(x_0-\eta z,t_0)-2\phi(x_0,t_0)}{\eta^{1+2s}}d\eta + \int_r^\infty \frac{u_1(x_0+\eta y,t_0)+u_1(x_0-\eta z,t_0)-2u_1(x_0,t_0)}{\eta^{1+2s}}d\eta \\ &\leq \int_0^r \frac{\phi(x_0+\eta y,t_0)+\phi(x_0-\eta z,t_0)-2\phi(x_0,t_0)}{\eta^{1+2s}}d\eta + \int_r^\infty \frac{w(x_0+\eta y,t_0)+w(x_0-\eta z,t_0)-2w(x_0,t_0)}{\eta^{1+2s}}d\eta\end{aligned}$$

since $\phi(x_0,t_0) = u_1(x_0,t_0) = w(x_0,t_0)$ and $w \geq u_1$. Then, as y,z are arbitrary and u_1 is a subsolution, we get (3.9). Furthermore, by the definition of $\tilde{w}$ and $\tilde{u}_1$,

$$\partial_t \tilde{w}(x_0,t_0) = \partial_t \tilde{u}_1(x_0,t_0) = \partial_t \phi(x_0,t_0).$$

This implies,

$$\partial_t \tilde{w}(x_0,t_0) - \Delta_\infty^s \tilde{w}(x_0,t_0) \leq \partial_t \tilde{u}_1(x_0,t_0) - \Delta_\infty^s \tilde{u}_1(x_0,t_0) \leq 0$$

On the other hand, since $u_1,u_2 \leq g$ on $\Omega^c \times (0,T)$ and $u_1,u_2 \leq u_0$ on $\mathbb{R}^N \times \{0\}$, we have $w \leq g$ on $\Omega^c \times (0,T)$ and $w \leq u_0$ on $\mathbb{R}^N \times \{0\}$. Hence, w is a subsolution. The proof for supersolutions is analogous. □

The following result, in a certain sense, motivates the choice of the domains in which we are able to prove that the Perron solution, defined as in (2.3), has a non-vanishing gradient. This is because it allows us to conclude that it is a subsolution by constructing a sequence of subsolutions.

Theorem 3.15. *Let $\Omega \subset \mathbb{R}^N$, $T > 0$ and $\{u_n\}_{n\in\mathbb{N}}$ be a sequence of subsolutions on non-zero gradient points of $\mathcal{P}$. Assume that,*

- *u_n converges uniformly to a function U, and*
- *there exists $\alpha < 2s, C > 0$, such that $|u_n(x,t)| \leq C(1+|x|)^\alpha$ for all n and for all $t \in (0,T)$.*

Then U is a subsolution on non-zero gradient points.

Proof. Let $(x_0,t_0)\in\Omega\times(0,T)$, and let ϕ be a test function that touches U from above at (x_0,t_0), such that $\nabla\phi(x_0,t_0)\neq 0$ and $\phi>U$ on $B_r(x_0,t_0)\setminus\{(x_0,t_0)\}$. Since u_n converges locally uniformly to U, for n large enough there exists $\delta_n\in\mathbb{R}$ such that $\phi+\delta_n$ touches u_n from above on some point $(x_n,t_n)\in B_r(x_0,t_0)$. Define $r_n=r-|(x_n,t_n)-(x_0,t_0)|$. Note that $(x_n,t_n)\to(x_0,t_0)$ as $n\to\infty$, so $r_n\to r$ as $n\to\infty$. Define

$$\tilde{u}_n(x,t)=\begin{cases}\phi(x,t)+\delta_n & \text{if } |(x,t)-(x_n,t_n)|<r_n,\\ u_n(x,t) & \text{if } |(x,t)-(x_n,t_n)|\geq r_n.\end{cases}$$

Since $\nabla\phi(x_0,t_0)\neq 0$, for n sufficiently large, we can ensure that $\nabla\phi(x_n,t_n)\neq 0$ thanks to the regularity of ϕ. Then, since u_n is a subsolution, we have $\partial_t\tilde{u}_n(x_n,t_n)-\Delta^s_\infty\tilde{u}_n(x_n,t_n)\leq 0$. Let $v_0,v_n\in S^{N-1}$ be the directions of the vectors $\nabla\phi(x_0,t_0)$ and $\nabla\phi(x_n,t_n)$ respectively.

We have

$$\begin{aligned}0\geq \partial_t\phi(x_n,t_n)-\int_0^{r_n}&\frac{\phi(x_n+\eta v_n,t_n)+\phi(x_n-\eta v_n,t_n)-2\phi(x_n,t_n)}{\eta^{2s+1}}d\eta\\ &+\int_{r_n}^{\infty}\frac{u_n(x_n+\eta v_n,t_n)+u_n(x_n-\eta v_n,t_n)-2u_n(x_n,t_n)}{\eta^{2s+1}}d\eta\end{aligned}$$

Thanks to the regularity of ϕ, the integrand in the first integral on the right hand side is bounded by the integrable function $M\eta^{1-2s}$, and because of the assumption on the growth of u_n, the integrand in the second integral is bounded by a integrable function independent of n. Furthermore, $v_n\to v_0$ as $x_n\to x_0$. Finally, because of the regularity of ϕ on t we have $\partial_t\phi(x_n,t_n)\to\partial_t\phi(x_0,t_0)$. Hence, by the convergence of u_n, and applying the Monotone Convergence Theorem, by taking $n\to\infty$ we get,

$$\begin{aligned}0\geq&\partial_t\phi(x_0,t_0)-\int_0^{r}\frac{\phi(x_0+\eta v_0,t_0)+\phi(x_0-\eta v_0,t_0)-2\phi(x_0,t_0)}{\eta^{2s+1}}d\eta\\ &+\int_{r}^{\infty}\frac{U(x_0+\eta v_0,t_0)+U(x_0-\eta v_0,t_0)-2U(x_0,t_0)}{\eta^{2s+1}}d\eta\\ =&\partial_t\tilde{U}(x_0,t_0)-\Delta^s_\infty\tilde{U}(x_0,t_0),\end{aligned}$$

where $\tilde{U}$ is defined as in (3.3). Finally, by the uniform convergence of u_n to U, $u_n\leq g$ on $\Omega^c\times(0,T)$ and $u_n(x,0)\leq u_0(x)$ on $\mathbb{R}^N\times\{0\}$ for all $n\in\mathbb{N}$, we have that $U\leq g$ on $\Omega^c\times(0,T)$ and $U(x,0)\leq u_0(x)$ on $\mathbb{R}^N\times\{0\}$. Therefore U is a subsolution. ☐

Remark 3.16. Observe that, in this Theorem we can take $\{u_n\}_{n\in\mathbb{N}}$ to be a sequence of subsolutions, since in particular, they will be subsolutions on non zero gradient points, but in either case we can only conclude that U is a subsolution on non zero gradient points.

Chapter 4

The problem on an Infinite Strip

In this chapter we consider Ω as a infinite strip, orthogonal to the e_1 canonical vector. This set Ω is defined as follows: Consider two functions $\Gamma_1, \Gamma_2 : \mathbb{R}^{N-1} \longrightarrow \mathbb{R}$ that define the boundaries,

$$\partial\Omega^- = \{(\Gamma_1(\hat{x}), \hat{x}_1, ..., \hat{x}_{n-1}) : \hat{x} \in \mathbb{R}^{N-1}\}$$
$$\partial\Omega^+ = \{(\Gamma_2(\hat{x}), \hat{x}_1, ..., \hat{x}_{n-1}) : \hat{x} \in \mathbb{R}^{N-1}\}.$$

These will be the left and right boundaries of Ω respectively. We assume that Γ_1 and Γ_2 are uniformly $C^{1,1}$ and are uniformly separate, precisely:

- There exists constants $M > m > 0$ such that for all $\hat{x} \in \mathbb{R}^{N-1}$, we have

$$0 \leq \Gamma_1(\hat{x}) \leq \Gamma_2(\hat{x}) \leq M, \text{ and} \tag{4.1}$$
$$m \leq \Gamma_2(\hat{x}) - \Gamma_1(\hat{x}). \tag{4.2}$$

  We denote $m_0 := d(\partial\Omega^-, \partial\Omega^+)$, i.e, m_0 is the smallest distance between $\partial\Omega^-$ and $\partial\Omega^+$.

- There exists a constant $C_1 > 0$ such that

$$\sup_{\hat{x} \in \mathbb{R}^{N-1}} |\partial_k \Gamma_i(\hat{x})| + |\partial_l \partial_k \Gamma_i(\hat{x})| \leq C_1 \tag{4.3}$$

  for each $i = 1, 2$ and $k, l = 1, ...n - 1$.

Remark 4.1. The fact that Γ_1 and Γ_2 are uniformly $C^{1,1}$ gives us the existence of neighborhoods $\mathcal{N}$ and $\mathcal{M}$ of $\partial\Omega^-$ and $\partial\Omega^+$ respectively, such that:

(a) for $\gamma, \kappa > 0$ small enough, $x + \gamma e_1 \in \mathcal{N}$ for all $x \in \partial\Omega^-$ and $x - \kappa e_1 \in \mathcal{M}$ for all $x \in \partial\Omega^+$;

(b) for $x \in \mathcal{N}$, the line with direction $\nabla d(x, \partial\Omega^-)$ that goes through x intersects $\mathcal{N}$ orthogonally;

(c) for $x \in \mathcal{M}$, the line with direction $\nabla d(x, \partial\Omega^+)$ that goes through x intersects $\mathcal{M}$ orthogonally.

Let's fix some notation:

- Because of (4.3), Γ_1 and Γ_2 are Lipschitz functions with the same Lipschitz constant, we denote this constant by L.
- Define $\theta = \text{arccotan}(L) \in (0, \frac{\pi}{2})$ and consider the cone

$$C^+ = \left\{ x \in \mathbb{R}^N \setminus \{0\} : \cos(\theta) < \frac{x \cdot e_1}{|x|} \leq 1 \right\},$$

  and we denote $C^- = -C^+$. Also, we denote $C_\theta := \cos(\theta) > 0$.

- We denote the two components of Ω^c as,

$$\Omega^{c,-} = \{x \in \mathbb{R}^N : x_1 \leq \Gamma_1(x_2, ..., x_n)\},$$
$$\Omega^{c,+} = \{x \in \mathbb{R}^N : x_1 \geq \Gamma_2(x_2, ..., x_n)\}.$$

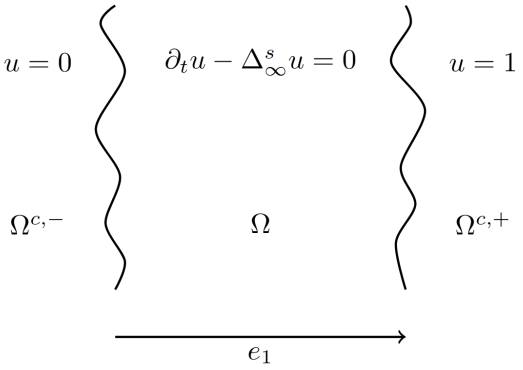

FIGURE 4.1: Parabolic problem in the Strip.

- Given $\kappa > 0$, define

$$\partial\Omega^-_\kappa = \{x + \kappa e_1 : x \in \partial\Omega^-\} \qquad \text{and} \qquad \partial\Omega^+_\kappa = \{x - \kappa e_1 : x \in \partial\Omega^+\},$$

and,

$$\Omega^-_\kappa = \{x + \gamma e_1 : x \in \partial\Omega^-, 0 < \gamma < \kappa\} \qquad \text{and} \qquad \Omega^+_\kappa = \{x - \gamma e_1 : x \in \partial\Omega^+, 0 < \gamma < \kappa\}.$$

In this context, we consider a function g such that

$$g(x) = \left\{ \begin{array}{ll} 0 & \text{if } x \in \Omega^{c,-}, \\ 1 & \text{if } x \in \Omega^{c,+}, \end{array} \right.$$

Consider the problem,

$$\left\{ \begin{array}{rcll} \partial_t u(x,t) - \Delta^s_\infty u(x,t) & = & 0 & \text{if } (x,t) \in \Omega \times (0,T), \\ u(x,t) & = & g(x) & \text{if } (x,t) \in \Omega^c \times (0,T), \\ u(x,0) & = & u_0(x) & \text{if } x \in \mathbb{R}^N. \end{array} \right. \tag{$\mathcal{P}_\mathcal{S}$}$$

For the initial condition, we assume the following:

(i) $u_0 \in C^{1,1}(\Omega) \cap BC(\mathbb{R}^N)$, $0 \le u_0 \le 1$ and it is compatible with the exterior condition of the problem, i.e, $u_0(x) = g(x)$ for all $x \in \Omega^c$.

(ii) u_0 is uniformly monotone on C^+, precisely, there exists $\beta_0 > 0$ such that for all $x \in \mathbb{R}^N$,

$$u_0(x) + \beta_0 h^{1+s} \le u_0(x + hy) \qquad \forall h > 0$$

for all $y \in C^+$.

(iii) There exists $\varepsilon_0 > 0$ and $\kappa_0 > 0$ such that:
  - $\Omega^-_{\kappa_0} \subset \mathcal{N}$ and $\varepsilon_0 d^s(x, \partial\Omega^-) \le u_0(x)$ for $x \in \Omega^-_{\kappa_0}$;
  - $\Omega^+_{\kappa_0} \subset \mathcal{M}$ and $1 - \varepsilon_0 d^s(x, \partial\Omega^+) \ge u_0(x)$ for $x \in \Omega^+_{\kappa_0}$.

(iv) There exists $\alpha_0 \in (0, 2s-1)$ such that $u_0 \in C^{0,\alpha_0}(\mathbb{R}^N)$ with constant $m_0^{-\alpha_0}$, that is

$$|u_0(x) - u_0(y)| \le m_0^{-\alpha_0}|x - y|^{\alpha_0} \qquad \forall x, y \in \mathbb{R}^N.$$

Remark 4.2. Note that assumption (iv) implies that,

$$\sup_{x_0 \in \partial\Omega^+} \left(1 - m_0^{-\alpha_0}|x - x_0|^{\alpha_0}\right) \le u_0(x) \tag{4.4}$$

for all $x \in \Omega$.

Assumption (iii) allows us to prove the existence of subsolutions that we can use as barriers. The necessity of this assumption comes from the fact that we need to prove that subsolutions are below the initial condition. We could ask for the initial condition to be below (resp. above) $1 - d^s(x, \partial\Omega^+)$ (resp. $d^s(x, \partial\Omega^-)$) globally in Ω, but it is sufficient to have these bounds on the neighborhoods $\mathcal{N}$ and $\mathcal{M}$, this is because we can take advantage of the monotonicity of u_0 (assumption (ii)) to prove the bounds needed inside Ω.

On the other hand, in (iv) we take $\alpha_0 \in (0, 2s-1)$ because we require the existence of a subsolution that is stricter than the one given by the function $|x|^{2s-1}$. This stricter subsolution (in the sense that $F(\alpha_0) < F(2s-1) = 0$ with F as in Lemma 3.10) will be used to contruct a barrier when proving a regularity result that involves both the time spatial variables. In addition, we assume that m_0 is such that

$$-F(\alpha_0)m_0^{-2s} > ||\Delta_\infty^s u_0||_\infty, \tag{4.5}$$

Recall that, $F(\alpha_0) < 0$ for $\alpha_0 \in (0, 2s-1)$, so $-F(\alpha_0)m_0^{-2s} > 0$. This assumption on m_0 will be useful when working with said barrier.

Following Perron's method, we will show that the supremum of a family of subsolutions $\mathcal{F}$ of $\mathcal{P_S}$ is the only solution. Precisely, we define the set $\mathcal{F}$ as,

$$u \in \mathcal{F} \Leftrightarrow u \text{ is a subsolution of } \mathcal{P_S} \text{ and } u(x,t) \leq 1 \text{ on } \Omega \times [0,T).$$

Observe that $u \equiv 0 \in \mathcal{F}$, hence the set $\mathcal{F}$ is not empty. Furthermore, each element in $\mathcal{F}$ is bounded above by 1 on all $\mathbb{R}^N \times (0,T)$. So, we can define

$$U(x,t) = \sup\{u(x,t) : u \in \mathcal{F}\}, \qquad \forall (x,t) \in \mathbb{R}^N \times [0,T). \tag{4.6}$$

as our candidate for a unique solution of $\mathcal{P_S}$.

Given that $u \equiv 0 \in \mathcal{F}$, and $u \leq 1$ on $\mathbb{R}^N \times (0,T)$ for all $u \in \mathcal{F}$, we have $U \leq 1$ on $\mathbb{R}^N \times (0,T)$. On the other hand, the indicator function $\mathbb{1}_{\Omega^{c,+}}$ is also in $\mathcal{F}$, which implies,

$$U = 0 \quad \text{on } \Omega^{c,-} \times (0,T) \qquad \text{and} \qquad U = 1 \quad \text{on } \Omega^{c,+} \times (0,T)$$

We have:

$$\begin{aligned} &0 \leq U \leq 1 \text{ on } \Omega \times (0,T) \\ &U = 0 \text{ on } \Omega^{c,-} \times (0,T) \\ &U = 1 \text{ on } \Omega^{c,+} \times (0,T) \end{aligned} \tag{4.7}$$

Furthermore, since $u \in \mathcal{F}$ satisfies $u(x,0) \leq u_0(x)$, this implies $U(x,0) \leq u_0(x)$ on Ω. Thus, U is well defined on all $\mathbb{R}^N \times [0,T)$.

Remark 4.3. From (4.7) we get $U(x,t) = g(x)$ on $\Omega^c \times (0,T)$. Moreover, $w^-(x,t) = u_0(x) - \lambda t$ is in $\mathcal{F}$ for $\lambda > 0$ sufficiently large. Indeed, for $(x,t) \in \Omega^c \times (0,T)$ we have,

$$w^-(x,t) = u_0(x) - \lambda t = g(x) - \lambda t \geq g(x).$$

On the other hand, for $t = 0$,

$$w^-(x,0) = u_0(x),$$

and for $(x,t) \in \Omega \times (0,T)$,

$$\begin{aligned} \partial_t w^-(x,t) - \Delta_\infty^s w^-(x,t) &= -\lambda - \Delta_\infty^s u_0(x) \\ &\leq -\lambda + ||\Delta_\infty^s u_0||_\infty \\ &\leq 0, \end{aligned}$$

for $\lambda \geq ||\Delta_\infty^s u_0||_\infty$. In addition, by (i), $w^- \leq 1$. Hence $w^- \in \mathcal{F}$. Then, since $w^-(x,0) = u_0(x)$ on $\mathbb{R}^N$, we have

$$U(x,0) = u_0(x)$$

for all $x \in \mathbb{R}^N$. Hence, it only remains to show that U satisfies the equation.

The remainder of this chapter is dedicated to proving that U is the only solution to $\mathcal{P}_{\mathcal{S}}$. The approach of this proof is based on finding sub and super solutions that establish a certain growth and decay of U, this estimates will imply a uniform monotonicity in the following sense.

Definition 4.4 (Uniform Monotonicity)**.** We say that a function $u : \Omega \to \mathbb{R}$ is uniformly monotone in the direction $y \in \mathbb{R}^N$ with exponent $\alpha > 0$ and constant $\beta > 0$ if the following statement holds: For all $x \in \Omega$ there exists $h_0 > 0$ such that

$$u(x) + \beta h^\alpha \leq u(x + hy) \qquad \forall\ 0 \leq h \leq h_0.$$

As we will see below, this monotonicity will imply that the test functions that touch U from above or below have non zero gradient, so we will be able to use the stability result, Theorem 3.15, to prove that U is a subsolution of $\mathcal{P}_{\mathcal{S}}$.

4.1 Monotonicity in a cone

First, we establish a result that has implications on the direction of the gradient of the test functions that touch U from above or below.

Proposition 4.5. *U is non-decreasing in C^+, that is, $U(x,t) \leq U(x+y,t)$ for all $(x,t) \in \mathbb{R}^N \times (0,T)$ and $y \in C^+$*

Proof. Note that if $x_0 \in \partial\Omega^-$, thanks to the $L-$Lipschitz regularity of the boundary, the cone $x_0 + C^+$ doesn't intersect $\partial\Omega^-$ except at x_0, i.e, $x_0 + C^+ \cap \partial\Omega^- = \{x_0\}$. Because of this and (4.7), it's enough to prove the result for $(x,t) \in \Omega \times (0,T)$.

Fix $y \in C^+$ and $(x_0,t_0) \in \Omega \times (0,T)$. Let $u^\delta \in \mathcal{F}$ be a sequence of subsolutions such that $U(x_0,t_0) \leq u^\delta(x_0,t_0) + \delta$ with $\delta \to 0$. Note that, since $0 \in \mathcal{F}$, we can assume that for all $\delta > 0$ we have $u^\delta \geq 0$, since we can take the maximum between 0 and u^δ, and by Lemma 3.14, this function will be in $\mathcal{F}$. Furthermore, since raising the value of u^δ outside of Ω increases the value of $\Delta^s_\infty u^\delta$ on Ω, we can assume that these subsolutions satisfy

$$u^\delta(x,t) = \left\{ \begin{array}{ll} 1, & \text{if } x \in \Omega^{c,+} \times (0,T), \\ 0, & \text{if } x \in \Omega^{c,-} \times (0,T). \end{array} \right. \tag{4.8}$$

We claim that the function

$$u^\delta_y(x,t) = \left\{ \begin{array}{rl} u^\delta(x-y,t), & \text{if } (x,t) \in \overline{\Omega} \times [0,T), \\ 1, & \text{if } (x,t) \in \Omega^{c,+} \times [0,T), \\ 0, & \text{if } (x,t) \in \Omega^{c,-} \times [0,T) \end{array} \right. \tag{4.9}$$

is a subsolution of $\mathcal{P}_{\mathcal{S}}$. Observe that

$$u^\delta(x-y,t) \leq u^\delta(x,t) \tag{4.10}$$

for all $(x,t) \in \Omega^c \times (0,T)$. Indeed, if $x \in \Omega^{c,+}$ we have $u^\delta(x,t) = 1 \geq u^\delta(x-y,t)$, and if $x \in \Omega^{c,-}$ we have $x - y \in \Omega^{c,-}$, thus $0 = u^\delta(x-y,t) = u^\delta(x,t)$. Then, by (4.10),

$$u^\delta_y(\cdot,t) \geq u^\delta(\cdot - y, t) \tag{4.11}$$

and they coincide on $\Omega \times (0,T)$.

Let φ be a test function that touches u^δ_y from above on $B_r(x_0,t_0)$ for some $r > 0$, that is,

$$\varphi(x_0,t_0) = u^\delta_y(x_0,t_0) \quad \text{ and } \quad \varphi > u^\delta_y \text{ on } B_r(x_0,t_0) \setminus \{(x_0,t_0)\}.$$

Suppose that $x_0 - y \in \Omega$, then,

$$\psi(x,t) := \varphi(x+y,t)$$

is a test function that touches u^δ from above on $B_r(x_0 - y, t_0)$. Define $\tilde{u}^\delta_y$ and $\overline{u}^\delta$ as in (3.3) with φ and ψ respectively. Since u^δ is a subsolution,

$$\partial_t \overline{u}^\delta(x_0 - y, t_0) - \Delta^s_\infty \overline{u}^\delta(x_0 - y, t_0) \leq 0$$

Furthermore,

$$\partial_t \overline{u}^\delta(x_0 - y, t_0) = \partial_t \psi(x_0 - y, t_0) = \partial_t \varphi(x_0, t_0) = \partial_t \tilde{u}^\delta_y(x_0, t_0) \tag{4.12}$$

As $\psi(x, t) = \varphi(x + y, t)$ on $B_r(x_0 - y, t_0)$, we have

$$\begin{aligned}\int_0^r & \frac{\varphi(x_0 + \eta v, t_0) + \varphi(x_0 - \eta z, t_0) - 2\varphi(x_0, t_0)}{\eta^{1+2s}} d\eta \\ &= \int_0^r \frac{\psi(x_0 - y + \eta v, t_0) + \psi(x_0 - y - \eta z, t_0) - 2\psi(x_0 - y, t_0)}{\eta^{1+2s}} d\eta\end{aligned}$$

for all $v, z \in S^{N-1}$, this combined with (4.11) gives,

$$\Delta^s_\infty \tilde{u}^\delta_y(x_0, t_0) \geq \Delta^s_\infty \overline{u}^\delta(x_0 - y, t_0)$$

Now, using (4.12),

$$\partial_t \tilde{u}^\delta_y(x_0, t_0) - \Delta^s_\infty \tilde{u}^\delta_y(x_0, t_0) \leq \partial_t \overline{u}^\delta(x_0 - y, t_0) - \Delta^s_\infty \overline{u}^\delta(x_0 - y, t_0) \leq 0.$$

If $x_0 - y \in \Omega^c$, $u^\delta_y(x_0, t_0) = u^\delta(x_0 - y, t_0) = 0$ and $\partial_t \varphi(x_0, t_0) = 0$. In this case we have,

$$\Delta^s_\infty \tilde{u}^\delta_y(x_0, t_0) = \sup_{v \in S^{N-1}} \int_0^\infty \frac{\tilde{u}^\delta_y(x_0 + \eta v, t_0)}{\eta^{1+2s}} d\eta + \inf_{z \in S^{N-1}} \int_0^\infty \frac{\tilde{u}^\delta_y(x_0 - \eta z, t_0)}{\eta^{1+2s}} d\eta.$$

because u^δ_y has null gradient. Since $u^\delta \geq 0$, we have $\Delta^s_\infty \tilde{u}^\delta_y(x_0, t_0) \geq 0$, which implies

$$\partial_t \tilde{u}^\delta_y(x_0, t_0) - \Delta^s_\infty \tilde{u}^\delta_y(x_0, t_0) \leq 0.$$

On the other hand, since u^δ is a subsolution, because of the monotonicity of u_0 on C^+ (assumption (ii)), we have

$$u^\delta_y(x, 0) = u^\delta(x - y, 0) \leq u_0(x - y) \leq u_0(x), \qquad \forall x \in \mathbb{R}^N.$$

Therefore u^δ_y is a subsolution, that is, it is below U on $\mathbb{R}^N \times [0, T)$. This gives,

$$U(x_0, t_0) \leq u^\delta(x_0, t_0) + \delta = u^\delta_y(x_0 + y, t_0) + \delta \leq U(x_0 + y, t_0) + \delta$$

thanks to the fact that if $x_0 + y \in \Omega^{c,+}$ we have $u^\delta_y(x_0 + y, t_0) = 1$, and by (4.7), $U(x_0 + y, t_0) = 1$. Finally, we conclude by taking $\delta \to 0$. □

The previous result has implications on the possible values that the gradient of a test function that touches it from above or below can take: if ϕ touches U from above on $(x_0, t_0) \in \Omega \times (0, T)$, then $\nabla \phi(x_0, t_0) = 0$ or $\cos(\frac{\pi}{2} - \theta) \leq v \cdot e_1 \leq 1$, where $v \in S^{N-1}$ is the direction of $\nabla \phi(x_0, t_0)$. Indeed, let $y \in S^{N-1} \cap C^+$. we have,

$$\begin{aligned}\nabla \phi(x_0, t_0) \cdot y &= \lim_{h \to 0} \frac{\phi(x_0 - hy, t_0) - \phi(x_0, t_0)}{h} \\ &\geq \limsup_{h \to 0} \frac{U(x_0 - hy, t_0) - U(x_0, t_0)}{h} \geq 0.\end{aligned}$$

This implies that the angle between $\nabla \phi(x_0, t_0)$ and y is less than $\frac{\pi}{2}$, and since y is arbitrary on C^+, the angle between $\nabla \phi(x_0, t_0)$ and e_1 is less than $\frac{\pi}{2} - \theta$. A similar argument holds for test functions that touch U from below.

4.2 Barriers

Now we construct appropriate barriers to estimate the growth and decay of U on Ω. Precisely, we will prove that there exists a constant $\varepsilon > 0$ such that,

$$\varepsilon d^s(x, \partial\Omega^-) \leq U \leq 1 - \varepsilon d^s(x, \partial\Omega^+) \tag{4.13}$$

This estimate of U is suggested by the fact that the function $\mathbb{R} \ni \eta \mapsto (\eta^+)^s$ is a solution of the one dimensional $s-$fractional Laplacian on $\mathbb{R}^+$, that is,

$$\Delta^s(\eta^+)^s = 0 \text{ on } (0, \infty). \tag{4.14}$$

We aim to show the existence of a constant $\varepsilon > 0$ such that the function,

$$g_\varepsilon(x) = \begin{cases} \varepsilon d^s(x, \partial\Omega^-) & \text{if } x \in \overline{\Omega}, \\ 1 & \text{if } x \in \Omega^{c,+}, \\ 0 & \text{if } x \in \Omega^{c,-}. \end{cases}$$

is a subsolution near $\partial\Omega^-$. This should hold because, by (4.14) we should have

$$\Delta^s_\infty \left(d^s(\cdot, \partial\Omega^-)\right)(x_0) = 0,$$

for all x_0 near $\partial\Omega^-$, since in the neighborhood $\mathcal{N}$ the function $d^s(\cdot, \partial\Omega^-)$ in the direction of its gradient behaves like $\eta \mapsto (\eta^+)^s$ in one dimension. This is thanks to $\nabla d(\cdot, \partial\Omega^-)$ being orthogonal to the boundary $\partial\Omega^-$ in this neighborhood.

A notable fact is that we can't prove that the function g_ε is a subsolution on the whole Ω. This is because of the regularity of the boundary: since it is not smooth enough a priori, we can't ensure that $\nabla d(\cdot, \partial\Omega^-)$ is orthogonal to the boundary $\partial\Omega^-$ on the whole interior of Ω, therefore we can't directly apply the result mentioned in the previous paragraph on the whole domain. So, we have to restrict the proof to the neighborhood $\mathcal{N}$.

To get the desired bound on the rest of Ω, it is necessary to construct additional barriers that are above g_ε outside $\mathcal{N}$. Specifically, we will prove that U grows like $d^{2s}(x, \partial\Omega^-)$ while moving away from $\partial\Omega^-$, and decays like $1 - d^{2s}(x, \partial\Omega^+)$ while moving away from $\partial\Omega^+$, both estimates inside the whole Ω. We will prove this by constructing a subsolution and supersolution made as envelopes of paraboloids that will bound these functions by above and below respectively. These constructions are the content of the following Lemmas.

First, we need to estimate the values of the operator when applied to one of these paraboloids.

Lemma 4.6. *For all* $s \in (\frac{1}{2}, 1)$ *there exists a constant* C_s *such that*

$$\Delta^s_\infty[-A(|x - x_0|^2 - r_0^2) \vee 0] \geq -C_s A r_0^{2-2s}$$

for all $x \neq x_0$ *and* $r_0 > 0$.

Proof. Define $\nu(x) = -A(|x - x_0|^2 - r_0^2)$, we have

$$v := \frac{\nabla\nu(x)}{|\nabla\nu(x)|} = \frac{-(x - x_0)}{|x - x_0|} \neq 0$$

Then,

$$\Delta^s_\infty \nu(x) = \int_0^\infty \frac{-A(|(x + \eta v) - x_0|^2 - r_0^2) - A(|(x - \eta v) - x_0|^2 - r_0^2) + 2A(|x - x_0|^2 - r_0^2)}{\eta^{1+2s}} d\eta.$$

By doing the change of variable $\sigma = \frac{\eta}{|x - x_0|}$ we get

$$\Delta^s_\infty \nu(x) = -A|x - x_0|^{2-2s} \int_0^\infty \frac{|1 + \sigma|^2 + |1 - \sigma|^2 - 2}{\sigma^{1+2s}} d\sigma = -A|x - x_0|^{2-2s} C_s < \infty.$$

Where,

$$C_s = \int_0^\infty \frac{|1+\sigma|^2 + |1-\sigma|^2 - 2}{\sigma^{1+2s}} d\sigma,$$

and $C_s = F(2) > 0$, with F as in Lemma 3.10. Therefore, if $|x - x_0| < r_0$, we have

$$\Delta_\infty^s \nu(x) = -A|x - x_0|^{2-2s} C_s \geq -AC_s r_0^{2-2s},$$

which concludes the result. ☐

Lemma 4.7. *There exists a constant* $c_0 > 0$*, that depends on* s *and the geometry of the problem such that, for any set* $S \subseteq \mathbb{R}^N$ *and constants* $\ell_0 \in (0, M]$*,* $A > 0$ *satisfying,*

(H.1) $AC_\theta^{2-2s} \ell_0^{2-2s} \leq c_0$*,*

(H.2) $AC_\theta^2 \ell_0^2 \leq \varepsilon_0 d^s(\partial\Omega^-_{(1-C_\theta)\ell_0}, \partial\Omega^-)$ *if* $(1 - C_\theta)\ell_0 \leq \kappa_0$*,*

(H.3) $AC_\theta^2 \ell_0^2 \leq \varepsilon_0 d^s(\partial\Omega^-_{\kappa_0}, \partial\Omega^-)$ *if* $(1 - C_\theta)\ell_0 > \kappa_0$*,*

the function

$$\mathbb{P}^+_{S,\ell_0}(x) = \begin{cases} P^+_{S,\ell_0}(x), & \text{if } x \in \overline{\Omega} \\ 1, & \text{if } x \in \Omega^{c,+} \\ 0, & \text{if } x \in \Omega^{c,-} \end{cases}$$

where,

$$P^+_{S,\ell_0}(x) = \sup_{x_0 \in \partial\Omega^-_{\ell_0} \cap S} \left(\sup_{x' - x_0 \in C^+} \{-A(|x - x'|^2 - C_\theta^2 \ell_0^2) \vee 0\} \right)$$

is a subsolution of $\mathcal{P}_S$*. Moreover,*

$$\mathbb{P}^-_{S,\ell_0}(x) = \begin{cases} P^-_{S,\ell_0}(x), & \text{if } x \in \overline{\Omega} \\ 1, & \text{if } x \in \Omega^{c,+} \\ 0, & \text{if } x \in \Omega^{c,-} \end{cases}$$

where,

$$P^-_{S,\ell_0}(x) = \inf_{x_0 \in \partial\Omega^+_{\ell_0} \cap S} \left(\inf_{x' - x_0 \in C^-} \{1 + A(|x - x'|^2 - C_\theta^2 \ell_0^2) \wedge 1\} \right)$$

is a supersolution of $\mathcal{P}_S$*. Where* ε_0 *and* κ_0 *are as in assumption (iii).*

This approach of constructing envelopes of paraboloids comes from following that of [9]. The idea behind the construction of $\mathbb{P}^+_{S,\ell_0}$ is to figure out how far we can raise a paraboloid near the boundary while still getting a subsolution. This works by using the boundary data inside $\Omega^{c,+}$ to balance out the concave shape of the paraboloid when we compute the operator. To make use of the values inside $\Omega^{c,+}$, we need to make sure that the gradient of our test function always stays inside C^+. That's why we build the barrier as a supremum of these paraboloids within the monotonicity cone, since the gradient of $\mathbb{P}^+_{S,\ell_0}$ always points into C^+. Finally, having a general set S lets us create more flexible barriers of this kind, which can be used either locally or globally, depending on the situation.

Something important to note in this context, compared to the elliptic case, is that we must adapt to the requirement imposed by the initial condition in order for a function to be a subsolution (resp. supersolution). That is, we must ensure that $\mathbb{P}^+_{S,\ell_0}$ and $\mathbb{P}^-_{S,\ell_0}$ lie below and above, respectively, the initial condition. This is the reason we consider assumption (iii), which allows us to bound $\mathbb{P}^+_{S,\ell_0}$ and $\mathbb{P}^-_{S,\ell_0}$ in the neighborhoods $\mathcal{N}$ and $\mathcal{M}$, respectively. Then, using the monotonicity of these functions and the assumptions (H.2) and (H.3), we can establish the bounds with respect to the initial condition to ensure that the subsolution (resp. supersolution) condition is satisfied.

Proof of Lemma 4.7. If $x \in \Omega$ is such that $\mathbb{P}^+_{S,\ell_0}(x) = 0$, then $\partial_t \mathbb{P}^+_{S,\ell_0}(x) - \Delta_\infty^s \mathbb{P}^+_{S,\ell_0}(x) \leq 0$ since $\partial_t \mathbb{P}^+_{S,\ell_0} = 0$ and $\mathbb{P}^+_{S,\ell_0} \geq 0$, which implies $\Delta_\infty^s \mathbb{P}^+_{S,\ell_0} \geq 0$ in the viscosity sense. So it would be enough to verify $\Delta_\infty^s \mathbb{P}^+_{S,\ell_0}(x) \geq 0$ when $\mathbb{P}^+_{S,\ell_0}(x) > 0$.

We estimate the value of $\Delta_\infty^s \mathbb{P}^+_{S,\ell_0}(x)$ by computing lower bounds for the positive and negative contributions to the operator. Fix $r_0 = C_\theta \ell_0$. For the positive part we estimate by bounding from below the contribution from $\Omega^{c,+}$. Since $\nabla \mathbb{P}^+_{S,\ell_0} \in C^+$, the worst case (when the contribution is as small as possible) happens when

$$C_\theta = v \cdot e_1 = \frac{\nabla \mathbb{P}^+_{S,\ell_0}(x)}{|\nabla \mathbb{P}^+_{S,\ell_0}(x)|} \cdot e_1.$$

Then, since $d(x, \partial\Omega^+) \leq M$, the contribution from $\Omega^{c,+}$ is at least

$$\int_{\frac{M}{C_\theta}}^{\infty} \frac{1 - 2[-A(|x - x_0|^2 - r_0^2)]}{\eta^{1+2s}} d\eta \geq \int_{\frac{M}{C_\theta}}^{\infty} \frac{1 - 2Ar_0^2}{\eta^{1+2s}} d\eta = \frac{1 - 2Ar_0^2}{2s} \left(\frac{C_\theta}{M}\right)^{2s} \tag{4.15}$$

For the negative part we use Lemma 4.6,

$$\inf_{x' \in B_{r_0}(x_0)} \{\Delta_\infty^s [-A(|x - x_0|^2 - r_0^2) \vee 0](x')\} \geq -C_s A r_0^{2-2s}.$$

Putting everything together,

$$\Delta_\infty^s \mathbb{P}^+_{S,\ell_0}(x) \geq \frac{1 - 2Ar_0^2}{2s} \left(\frac{C_\theta}{M}\right)^{2s} - C_s A r_0^{2-2s}$$

then, there exists a constant $c_0 > 0$ that depends on s and the geometry of Ω such that $\Delta_\infty^s \mathbb{P}^+_{S,\ell_0}(x) \geq 0$ if $Ar_0^{2-2s} = AC_\theta^{2-2s} \ell_0^{2-2s} \leq c_0$.

Now we're going to prove that, for any set $S \subset \mathbb{R}^N$ and for all $\ell_0 \in (0, M]$, we have $\mathbb{P}^+_{S,\ell_0}(x) \leq u_0(x)$ for all $x \in \mathbb{R}^N$. Since $\mathbb{P}^+_{\mathbb{R}^N,\ell_0} \geq \mathbb{P}^+_{S,\ell_0}$ for any $S \subset \mathbb{R}^N$ and $\mathbb{P}^+_{\mathbb{R}^N,\ell_0} = g$ on Ω^c, it is enough to prove that for any $\ell_0 \in (0, M]$ one has,

$$\mathbb{P}^+_{\mathbb{R}^N,\ell_0} \leq u_0 \tag{4.16}$$

on Ω. For this purpose, we consider ε_0 and κ_0 as in (iii) and define the function

$$\mathcal{D}(x) := \begin{cases} \varepsilon_0 d^s(x, \partial\Omega^-) & \text{if } x \in \Omega^-_{\kappa_0}, \\ \varepsilon_0 d^s(\partial\Omega^-_{\kappa_0}, \partial\Omega^-) & \text{if } x \in \Omega \setminus \Omega^-_{\kappa_0}, \\ g(x) & \text{if } x \in \Omega^c. \end{cases}$$

By (iii), we have $u_0(x) \geq \varepsilon_0 d^s(x, \partial\Omega^-)$ on $\Omega^-_{\kappa_0}$, and by (ii) we get

$$u_0 \geq \varepsilon_0 d^s(\partial\Omega^-_{\kappa_0}, \partial\Omega^-) \qquad \text{on } \Omega \setminus \Omega^-_{\kappa_0}$$

Moreover, by (i), $u_0 = g$ on Ω^c, so $u_0 \geq \mathcal{D}$ on $\mathbb{R}^N$. Therefore, if we prove $\mathbb{P}^+_{\mathbb{R}^N,\ell_0} \leq \mathcal{D}$ on Ω, we conclude the result.

First, note that if $x \in \Omega$ is such that $\mathbb{P}^+_{\mathbb{R}^N,\ell_0}(x) > 0$ one has $x \in \Omega \setminus \Omega^-_{(1-C_\theta)\ell_0}$, and this implies

$$d(x, \partial\Omega^-) \geq d(\partial\Omega^-_{(1-C_\theta)\ell_0}, \partial\Omega^-) \tag{4.17}$$

If $x \in \Omega$ is such that $\mathbb{P}^+_{\mathbb{R}^N,\ell_0}(x) = 0$ automatically $\mathbb{P}^+_{\mathbb{R}^N,\ell_0}(x) \leq u_0(x)$, because (ii) and (iii) imply $u_0 > 0$ on Ω. Now, given $\ell_0 \in (0, M]$ we analyze two cases:

- If $(1 - C_\theta)\ell_0 \leq \kappa_0$ we have

  $$\Omega^-_{(1-C_\theta)\ell_0} \subseteq \Omega^-_{\kappa_0}$$

  then, for $x \in \Omega$ such that $\mathbb{P}^+_{\mathbb{R}^N,\ell_0}(x) > 0$ there exists two possibilities,

  - $x \in \Omega^-_{\kappa_0} \setminus \Omega^-_{(1-C_\theta)\ell_0}$, or
  - $x \in \Omega \setminus \Omega^-_{\kappa_0}$.

  In the first of this two subcases we use (H.2) and (4.17) to get

  $$\mathbb{P}^+_{\mathbb{R}^N,\ell_0}(x) \leq AC_\theta^2 \ell_0^2 \leq \varepsilon_0 d^s(\partial\Omega^-_{(1-C_\theta)\ell_0}, \partial\Omega^-) \leq \varepsilon_0 d^s(x, \partial\Omega^-)$$

then $\mathbb{P}^+_{\mathbb{R}^N,\ell_0} \leq \mathcal{D}$ on $\Omega^-_{\kappa_0}$. In the second subcase we use

$$d(\partial\Omega^-_{(1-C_\theta)\ell_0}, \partial\Omega^-) \leq d(\partial\Omega^-_{\kappa_0}, \partial\Omega^-)$$

to get

$$\mathbb{P}^+_{\mathbb{R}^N,\ell_0}(x) \leq AC_\theta^2\ell_0^2 \leq \varepsilon_0 d^s(\partial\Omega^-_{(1-C_\theta)\ell_0}, \partial\Omega^-) \leq \varepsilon_0 d^s(\partial\Omega^-_{\kappa_0}, \partial\Omega^-)$$

so we conclude $\mathbb{P}^+_{\mathbb{R}^N,\ell_0} \leq \mathcal{D}$ on Ω in this case.

- If $(1 - C_\theta)\ell_0 > \kappa_0$, for $x \in \Omega$ such that $\mathbb{P}^+_{\mathbb{R}^N,\ell_0}(x) > 0$, by 4.17 we have $x \in \Omega \setminus \Omega^-_{\kappa_0}$, then, by (H.3),

$$\mathbb{P}^+_{\mathbb{R}^N,\ell_0}(x) \leq AC_\theta^2\ell_0^2 \leq \varepsilon_0 d^s(\partial\Omega^-_{\kappa_0}, \partial\Omega^-)$$

which implies

$$\mathbb{P}^+_{\mathbb{R}^N,\ell_0}(x) \leq \mathcal{D}$$

on Ω.

Finally, we can conclude $\mathbb{P}^+_{\mathbb{R}^N,\ell_0}(x) \leq u_0$ on $\mathbb{R}^N$, thus $\mathbb{P}^+_{S,\ell_0}$ is a subsolution. To prove that $\mathbb{P}^-_{S,\ell_0}$ is a supersolution follows an analogous argument. ☐

Lemma 4.8. *There exists* $C > 0$ *such that*

$$Cd^{2s}(x, \partial\Omega^-) \leq U(x,t) \leq 1 - Cd^{2s}(x, \partial\Omega^+)$$

for all $(x,t) \in \Omega \times (0,T)$.

Proof. For any $x_0 \in \Omega$ there exists $\ell_0 \in (0, M]$ such that $x_0 \in \partial\Omega^-_{\ell_0}$. Clearly $\ell_0 \geq d(x_0, \partial\Omega^-)$, so it is enough to prove $U(x_0,t) \geq C\ell_0^{2s}$, for some constant $C > 0$ that doesn't depend on x_0, where $t \in (0,T)$. By Lemma 4.7, $\mathbb{P}^+_{S,\ell_0}$ with $S = \{x_0\}$ is a subsolution of $\mathcal{P}_{\mathcal{S}}$, so $U(x,t) \geq \mathbb{P}^+_{S,\ell_0}(x)$ for all $(x,t) \in \mathbb{R}^N \times (0,T)$. In particular, $U(x_0,t) \geq \mathbb{P}^+_{S,\ell_0}(x_0) = AC_\theta^2\ell_0^2$, so, by choosing smaller c_0 if necessary, and taking $A = c_0 C_\theta^{2s-2}\ell_0^{2-2s}$ we get

$$U(x_0,t) \geq \mathbb{P}^+_{S,\ell_0}(x_0) = c_0 C_\theta^{2s}\ell_0^{2s}$$

where the constant $C = c_0 C_\theta^{2s}$ doesn't depend on x_0 as we wanted.

Similarly, $\mathbb{P}^-_{S,\ell_0}$ is a supersolution of $\mathcal{P}_{\mathcal{S}}$ that is equal to 1 for all $x \in \Omega$ outside of the compact set $(x_0 + C^-) \cap \Omega$. So, using the Comparison Theorem 3.8, we conclude $u \leq \mathbb{P}^-_{S,\ell_0}$ for all $u \in \mathcal{F}$, then, taking the supremum over $u \in \mathcal{F}$, we get $U \leq \mathbb{P}^-_{S,\ell_0}$ and conclude in the same way that we did in the previuos case. ☐

Now we have the necessary barriers to prove (4.13). The following result and its proof are analogous to Lemma 4.7 in [9], with some straightforward changes concerning the initial condition and the partial derivative. We include the proof in detail for completeness. The idea of the proof is the following:

Thanks to the previous Lemmas, we can ensure that (4.13) holds in $\Omega \setminus (\mathcal{N} \cup \mathcal{M})$ as we will see in the proof. Therefore, for the lower bound in (4.13), it is going to be enough to analyze what happens when we compute $\Delta^s_\infty g_\varepsilon(x_0)$ for $x_0 \in \mathcal{N}$. This integral differs from $\Delta^s_\infty\left(d^s(\cdot, \partial\Omega^-)\right)(x_0)$ in two terms:

- If $x \in \Omega^{c,+}$ and $\varepsilon d^s(x, \partial\Omega^-) \leq 1$, then there is a "gain" on the integral, since $g_\varepsilon(x) = 1 \geq \varepsilon d^s(x, \partial\Omega^-)$.
- If $x \in \Omega^{c,+}$ and $\varepsilon d^s(x, \partial\Omega^-) \geq 1$, then there is a "loss", since $g_\varepsilon(x) = 1 \leq \varepsilon d^s(x, \partial\Omega^-)$

The goal is then to prove that the gain compensates for the loss when ε is small enough.

Lemma 4.9. *There exists a constant* $\varepsilon > 0$ *such that* $\varepsilon d^s(x, \partial\Omega^-) \leq U(x,t) \leq 1 - \varepsilon d^s(x, \partial\Omega^+)$ *for all* $(x,t) \in \Omega \times (0,T)$.

Proof. Recall from Remark 4.1 that the $C^{1,1}$ regularity of the boundary implies the existence of the neighborhood $\mathcal{N}$ of $\partial\Omega^-$ such that:

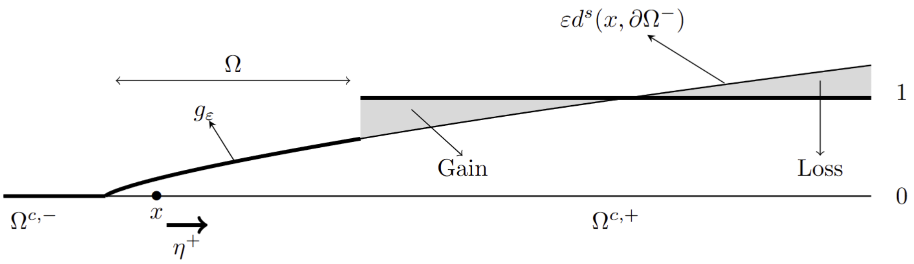

FIGURE 4.2: Gain and Loss from the proof of Lemma 4.9.

($\mathcal{H}$.1) for $\ell_0 > 0$ small enough, $x + 2\ell_0 e_1 \in \mathcal{N}$ for all $x \in \partial\Omega^-$;

($\mathcal{H}$.2) for any $x \in \mathcal{N}$ the line with direction $\nabla d^s(x,\partial\Omega^-)$ that goes through x intersects $\partial\Omega^-$ orthogonally.

Fix $\ell_0 > 0$ small, such that ($\mathcal{H}$.1) holds, and choose $A > 0$ such that (H.1)-(H.3) hold, thus $\mathbb{P}^+_{S,\ell_0}$ with $S = \mathbb{R}^N$ is a subsolution of $\mathcal{P}_{\mathcal{S}}$.

Define

$$g_\varepsilon(x) = \begin{cases} \varepsilon d^s(x,\partial\Omega^-) & \text{if } x \in \overline{\Omega}, \\ 1 & \text{if } x \in \Omega^{c,+}, \\ 0 & \text{if } x \in \Omega^{c,-}. \end{cases}$$

Taking $\varepsilon > 0$ small enough we can ensure $g_\varepsilon(x) < \mathbb{P}^+_{S,\ell_0}(x)$ for all $x \in \Omega \setminus \mathcal{N}$. Then, the result is proven if we verify that the function $w(x) = g_\varepsilon(x) \vee \mathbb{P}^+_{S,\ell_0}(x)$ is a subsolution.

Let $(x,t) \in \Omega \times (0,T)$ and ϕ be a test function that touches w from above at (x,t). Clearly, $\partial_t\phi(x,t) = 0$. If $x \in \Omega \setminus \mathcal{N}$ we have $\mathbb{P}^+_{S,\ell_0}(x) > g_\varepsilon(x)$, then $\Delta^s_\infty \tilde{w}(x) \geq \Delta^s_\infty \tilde{\mathbb{P}}^+_{S,\ell_0}(x) \geq 0$ (this is argued in the same way that we did in the proof of Proposition 4.5), where $\tilde{w}$ and $\tilde{\mathbb{P}}^+_{S,\ell_0}$ are defined as in (3.3) using ϕ. Hence, since $\Delta^s_\infty \tilde{w}(x) \geq \Delta^s_\infty \tilde{g}_\varepsilon(x)$ when $x \in \mathcal{N}$ and $g_\varepsilon(x) > \mathbb{P}^+_{S,\ell_0}(x)$, it remains to prove $\Delta^s_\infty \tilde{g}_\varepsilon(x) \geq 0$ in this case.

The regularity of the boundary $\partial\Omega^-$ allows to evaluate $\Delta^s_\infty g_\varepsilon(x)$ directly without appealing to test functions when $x \in \mathcal{N}$. Let $v \in S^{N-1}$ be the direction of $\nabla d(x,\partial\Omega^-)$. By ($\mathcal{H}$.2), if $x \in \mathcal{N}$, the line $\{x + \eta v\}_{\eta\in\mathbb{R}}$ intersects $\partial\Omega^-$ orthogonally. Let $\tau > 0$ (resp. $\tau' > 0$) be the distance between x and $\partial\Omega^-$ (resp. $\partial\Omega^+$) along this line. Replacing $\mathcal{N}$ for a smaller neighborhood if necessary, we can assume $\tau < \tau'$. Since $v \in C^+$, the maximum value that τ' can take is M/C_θ. Moreover, we suppose that ε is small enough to satisfy $\varepsilon^{-1/s} > 2M/C_\theta$. Using the fact that the function $\eta \mapsto (\eta^+)^s$ is a solution of $\Delta^s(\eta^+)^s$ on $\mathbb{R}^+$ (i.e, for all $\eta > 0$, $\int_{-\infty}^{\infty} \frac{((\eta+\tau)^+)^s - (\eta^+)^s}{|\tau|^{1+2s}} d\tau = 0$ in the principal value sense) we have

$$\Delta^s_\infty g_\varepsilon(x) \geq \int_{\tau'}^{\varepsilon^{-\frac{1}{s}}-\tau} \frac{1-\varepsilon(\tau+\eta)^s}{\eta^{1+2s}} d\eta - \int_{\varepsilon^{-\frac{1}{s}}-\tau}^{\infty} \frac{\varepsilon(\tau+\eta)^s - 1}{\eta^{1+2s}} d\eta.$$

The first integral in the right hand side represent the "gain", where $\varepsilon(\tau+\eta)^s \leq 1$, meanwhile the second integral represent the "loss" where $\varepsilon(\tau+\eta)^s \geq 1$. Now we check that the right hand side is positive if ε is small enough. For any $\eta > \tau' > \tau$ one has $\frac{\tau+\eta}{\eta} \leq 2$. So, taking $\varepsilon < C^s_\theta/(8M^s)$ and recalling that $\tau' < M/C_\theta < \varepsilon^{-1/s}/2$ we get

$$\begin{aligned}\int_{\tau'}^{\varepsilon^{-\frac{1}{s}}-\tau} \frac{1-\varepsilon(\tau+\eta)^s}{\eta^{1+2s}} d\eta &\geq \int_{\tau'}^{\varepsilon^{-\frac{1}{s}}-\tau} \left(\frac{1}{\eta^{1+2s}} - \frac{\varepsilon 2^s}{\eta^{1+s}}\right) d\eta \\ &\geq \frac{1}{s}\frac{1}{(\tau')^s}\left(\frac{1}{2}\frac{1}{(\tau')^s} - \varepsilon 2^s\right) - \frac{1}{2s}\frac{1}{\left(\varepsilon^{-\frac{1}{s}} - \tau\right)^{2s}} \\ &\geq \frac{1}{4s}\frac{C^{2s}_\theta}{M^{2s}} - \frac{2^{2s-1}}{s}\varepsilon^2\end{aligned}$$

Moreover,

$$\int_{\varepsilon^{-\frac{1}{s}}-\tau}^{\infty}\frac{\varepsilon(\tau+\eta)^s-1}{\eta^{1+2s}}d\eta \leq \int_{\varepsilon^{-\frac{1}{s}}-\tau}^{\infty}\left(\frac{2^s\varepsilon}{\eta^{1+s}}-\frac{1}{\eta^{1+2s}}\right)d\eta$$
$$\leq \frac{1}{s}\frac{\varepsilon 2^s}{\left(\varepsilon^{-\frac{1}{s}}-\tau\right)^s} \leq \frac{2^{2s}}{s}\varepsilon^2$$

Then, putting everything together

$$\Delta_\infty^s g_\varepsilon(x) \geq \frac{1}{4s}\frac{C_\theta^{2s}}{M^{2s}} - \frac{2^{2s}+2^{2s-1}}{s}\varepsilon^2.$$

From this we conclude that for ε small enough (how small depending only on s and the geometry of the problem) we get $\Delta_\infty^s g_\varepsilon(x) \geq 0$.

On the other hand, by assumption (iii), we can choose ε small enough such that $\varepsilon < \varepsilon_0$, this way we can ensure that $w(x) < u_0(x)$ on Ω. Thus, w is a subsolution of $\mathcal{P}_\mathcal{S}$, which implies $U \geq w$ and establishes the growth estimate from below.

To deduce the decay of U away from $\partial\Omega^+$, one can use similar techniques from the ones previous to construct a supersolution with the desired decay and then apply the Comparison Principle (Theorem 3.8), arguing essentially as in the last paragraph of the proof of Lemma 4.8. The key difference is that the comparison needs to be done in a bounded set. So, instead of defining the equivalent of g_ε, that would require a comparison principle in a non bounded set, one has to construct barriers like

$$g^\varepsilon(x) = \begin{cases} 1-\varepsilon d^s(x,\partial\mathscr{P}) & \text{if } x \in \overline{\Omega}\cap\mathscr{P}, \\ 1 & \text{if } x \in \Omega^{c,+}\cup(\Omega\backslash\mathscr{P}), \\ 0 & \text{if } x \in \Omega^{c,-}, \end{cases}$$

where $\mathscr{P}$ is a "parabolic set" that touches $\partial\Omega^+$ from the left. Precisely, at any point $\hat{x} \in \mathbb{R}^{N-1}$, the regularity of the boundary implies the existence of a paraboloid $p : \mathbb{R}^{N-1} \to \mathbb{R}$, with a uniform opening, that touches Γ_2 from below at $(\Gamma_2(\hat{x}), \hat{x})$. Define $\mathscr{P} = \left\{x \in \mathbb{R}^N : x_1 \leq p\left(x_2, \ldots, x_n\right)\right\}$, and choose S as a open ball centered at $(\Gamma_2(\hat{x}), \hat{x})$ that contains $\Omega\cap\mathscr{P}$. This way one constructs a family of supersolutions $g^\varepsilon \wedge \mathbb{P}^-_{S,\ell_0}$ (depending on the point $(\Gamma_2(\hat{x}), \hat{x})$) that are equal to 1 outside of a compact subset of Ω, so that one can apply the Comparison Principle on compact sets. Given that $\hat{x}$ is arbitrary, this proves the desired decay. □

4.3 Existence and regularity

Now we have the necessary barriers to prove the uniform monotonicity of U.

The approach to proving the uniform monotonicity follows a similar principle as the one seen in the proof of Proposition 4.5, that is, we approximate U at an arbitrary point and then prove that a certain convenient function is a subsolution (resp. supersolution) and conclude by using the definition of U and taking $\delta \to 0$ in the approximation.

Lemma 4.10. *There exists $\beta > 0$ such that $U(\cdot, t)$ is uniformly monotone in the direction e_1 with $\alpha = s+1$. More precisely, there exists $h_0 > 0$ such that, for all $x \in \Omega$,*

$$U(x,t) + \beta h^{1+s} \leq U(x+e_1 h, t) \qquad h < \min\{d(x.\partial\Omega), h_0\}$$

for all $t \in (0,T)$.

Proof. Fix $(x_0, t_0) \in \Omega \times (0,T)$, and consider a sequence $u^\delta \in \mathcal{F}$ of subsolutions of $\mathcal{P}_\mathcal{S}$ such that $U(x_0,t_0) \leq u^\delta(x_0,t_0) + \delta$. Replacing u^δ for u^δ_y, with u^δ_y is as in the proof of Proposition 4.5, we can assume that u^δ is monotone on C^+, $u^\delta = 1$ on $\Omega^{c,+}$ and $u^\delta = 0$ on $\Omega^{c,-}$. Let $u \in \mathcal{F}$ be such that

$$\varepsilon d^s(x,\partial\Omega^-) \leq u(x,t) \leq 1-\varepsilon d^s(x,\partial\Omega^+) \tag{4.18}$$

for ε small enough, and u monotone increasing in C^+. By taking the maximum between u and u^δ we can assume that u^δ satisfies 4.18.

Define, for all $\beta > 0$ and $h < d(x_0, \partial\Omega)$,

$$u_h^\delta(x,t) = \begin{cases} u^\delta(x,t), & \text{if } (x,t) \in \Omega \times (0,T) \text{ and } d(x, \partial\Omega^-) \leq h \\ u^\delta(x,t) \vee (u^\delta(x-he_1,t) + \beta h^{1+s}), & \text{if } (x,t) \in \Omega \times (0,T) \text{ and } d(x, \partial\Omega^-) > h \\ 1, & \text{if } (x,t) \in \Omega^{c,+} \times (0,T), \\ 0, & \text{if } (x,t) \in \Omega^{c,-} \times (0,T). \end{cases}$$

we will prove that there exists a universal constant $\beta > 0$ such that, for small h, u_h^δ is a subsolution of $\mathcal{P}_\mathcal{S}$. Once we establish that u_h^δ is a subsolution, it follows that $U(x,t) \geq u_h^\delta(x,t)$, hence if $d\left(x_0, \partial\Omega\right) > h$,

$$U\left(x_0, t_0\right) \leq u^\delta\left(x_0, t_0\right) + \delta \leq u_h^\delta\left(x_0 + he_1, t_0\right) - \beta h^{1+s} + \delta \leq U\left(x_0 + he_1, t_0\right) - \beta h^{1+s} + \delta.$$

Since $\delta > 0$ is arbitrary, this would conclude the proof.

Let ϕ be a test function that touches u_h^δ from above at $(x,t) \in \Omega \times (0,T)$. If ϕ touches u^δ from above, i.e, $u^\delta(x,t) \geq u^\delta\left(x - he_1, t\right) + \beta h^{1+s}$ then $\partial_t \tilde{u}_h^\delta(x,t) - \Delta_\infty^s \tilde{u}_h^\delta(x,t) \leq \partial_t \tilde{u}^\delta(x,t) - \Delta_\infty^s \tilde{u}^\delta(x,t) \leq 0$, since $u_h^\delta \geq u^\delta$ and $\partial_t \phi(x,t) = \partial_t u^\delta(x,t) = \partial_t \tilde{u}_h^\delta(x,t)$ in this case, where $\tilde{u}_h^\delta$ and $\tilde{u}^\delta$ are defined as in (3.3). Subsequently, it remains to analyze the case $u^\delta\left(x - he_1, t\right) + \beta h^{1+s} > u^\delta(x,t)$. To show that $\partial_t \tilde{u}_h^\delta(x,t) - \Delta_\infty^s \tilde{u}_h^\delta(x,t) \leq 0$ in this case, we use

$$\partial_t\left(\tilde{u}^\delta\left(x - he_1, t\right) + \beta h^{1+s}\right) - \Delta_\infty^s\left(\tilde{u}^\delta\left(x - he_1, t\right) + \beta h^{1+s}\right) \leq 0, \tag{4.19}$$

and

$$\partial_t\left(\tilde{u}^\delta\left(x - he_1, t\right) + \beta h^{1+s}\right) = \partial_t \tilde{u}_h^\delta(x,t) \tag{4.20}$$

and then compute the difference between the opearor applied to $u^\delta\left(\cdot - he_1, \cdot\right) + \beta h^{1+s}$ and the operator applied to u_h^δ, that is, we will prove that $\Delta_\infty^s \tilde{u}_h^\delta(x,t) - \Delta_\infty^s \tilde{u}^\delta\left(x - he_1, t\right) + \beta h^{1+s} \geq 0$.

By (4.18), there is a positive contribution, or “'gain”, induced by the growth (and decay) near the boundaries, and a negative contribution, or “loss”, that comes from changing the value of $u^\delta\left(\cdot - he_1, t\right) + \beta h^{1+s}$ on Ω^c. Consequently, as in the proof of Lemma 4.9, the goal becomes proving that the gain is greater than the loss for small β and h.

Suppose that ϕ touches $u^\delta(\cdot - he_1, \cdot) + \beta h^{1+s}$ from above at (x,t). Note that $\phi(\cdot,\cdot) - \beta h^{1+s}$ touches u^δ from above at $(x - he_1, t)$. Then, because of the monotonicity of u^δ in C^+, we have $\nabla\phi(x,t) = 0$ or $C_\theta \leq v \cdot e_1$, where v is the direction of $\nabla\phi(x,t)$. Let's consider the case where $\nabla\phi(x) \neq 0$ first. Let $\ell > 0$ be the distance between x and $\{z : d\left(z, \partial\Omega^-\right) \leq h\}$ along the line that goes through x with direction $\nabla\phi(x,t)$, and let $\ell' > 0$ be the distance between x and $\partial\Omega^+$ along this same line. Then,

$$\ell + h, \ell' + h < \frac{M}{C_\theta}. \tag{4.21}$$

Observe that, along this line with direction v, we have the following:

- For $\eta \in (-\ell - h, \ell)$ one has $u_h^\delta(x + \eta v) = u^\delta(x + \eta v)$ and for $\eta \in (\ell', \ell' + h)$ one has $u_h^\delta(x + \eta v, t) \geq u^\delta(x - he_1 + \eta v, t)$. In both these cases, we use (4.18) to estimate the gain.
- To estimate the loss that come from changing the value on $\Omega^{c,-}$ we consider the worst case, in which if $\eta \in (-\infty, -\ell - h)$, then $u_h^\delta(x + \eta v) = 0$. This is the “worst” case, as the smaller the value of $|\eta|$ for which $x + \eta v$ exits Ω, the larger the integral representing the loss becomes.
- Similarly to the previous point, to estimate the loss that comes from the change on $\Omega^{c,+}$, we consider $\eta \in (\ell' + h, \infty)$. Here, we choose this interval instead of (ℓ', ∞) because; for $\eta > \ell'$ it holds $x + \eta v \in \Omega^{c,+}$, but, the function $u^\delta(x - he_1, t) + \beta h^{1+s}$ is equal to $1 + \beta h^{1+s}$ when $x - he_1 + \eta v \in \Omega^{c,+}$, which corresponds to the change that the loss generates, and in the worst case, this holds when $\eta > \ell' + h$.

Considering all the previous points, we estimate

$$\begin{aligned}\partial_t \tilde{u}_h^\delta - \Delta_\infty^s \tilde{u}_h^\delta(x,t) &\geq \partial_t \tilde{u}_h^\delta - \Delta_\infty^s \tilde{u}_h^\delta(x,t) - \left(\partial_t \tilde{u}^\delta\left(x - e_1 h, t\right) - \Delta_\infty^s \tilde{u}^\delta\left(x - e_1 h, t\right)\right)\\ &\geq \int_{-\ell-h}^{-\ell} \frac{\left[\varepsilon(\ell+h+\eta)^s - \beta h^{1+s}\right]}{|\eta|^{1+2s}} d\eta - \int_{-\infty}^{-\ell-h} \frac{\beta h^{1+s}}{|\eta|^{1+2s}} d\eta\\ &\quad + \int_{\ell'}^{\ell'+h} \frac{\left[\varepsilon\left(\ell'+h-\eta\right)^s - \beta h^{1+s}\right]}{|\eta|^{1+2s}} d\eta - \int_{\ell'+h}^{\infty} \frac{\beta h^{1+s}}{|\eta|^{1+2s}} d\eta.\end{aligned}$$

The third and fourth terms represent the gain that comes from the growth near the boundary, while the second and fifth term represent the loss that comes from the change on Ω^c, everything deduced from the previous points. We need to prove

$$\frac{1}{2s} \frac{\beta h^{1+s}}{(\ell+h)^{2s}} = \int_{-\infty}^{-\ell-h} \frac{\beta h^{1+s}}{|\eta|^{1+2s}} d\eta \leq \int_{-\ell-h}^{-\ell} \frac{\left[\varepsilon(\ell+h+\eta)^s - \beta h^{1+s}\right]}{|\eta|^{1+2s}} d\eta,$$

For this, we estimate

$$\begin{aligned}\int_{-\ell-h}^{-\ell} \frac{\left[\varepsilon(\ell+h+\eta)^s - \beta h^{1+s}\right]}{|\eta|^{1+2s}} d\eta &\geq \frac{1}{(\ell+h)^{1+2s}} \int_{-\ell-h}^{-\ell} \left[\varepsilon(\ell+h+\eta)^s - \beta h^{1+s}\right] d\eta\\ &= \frac{1}{(\ell+h)^{1+2s}} \left(\frac{\varepsilon}{1+s} h^{s+1} - \beta h^{2+s}\right).\end{aligned}$$

This holds analogously for ℓ'. Hence,

$$\begin{aligned}\Delta_\infty^s \tilde{u}_h^\delta\left(x,t\right) \geq& \frac{h^{s+1}}{\left(\ell'+h\right)^{2s}} \left(\frac{\varepsilon}{1+s} \frac{1}{\ell'+h} - \frac{\beta}{2s} - \frac{\beta h}{\ell'+h}\right)\\ &+ \frac{h^{1+s}}{(\ell+h)^{2s}} \left(\frac{\varepsilon}{1+s} \frac{1}{\ell+h} - \frac{\beta}{2s} - \frac{\beta h}{\ell+h}\right).\end{aligned}$$

By (4.21) we have,

$$\Delta_\infty^s \tilde{u}_h^\delta(x,t) \geq h^{1+s} \left(\frac{C_\theta}{M}\right)^{2s} \left(\frac{2\varepsilon}{1+s} \frac{C_\theta}{M} - \frac{\beta}{s} - \frac{\beta h}{\ell+h} - \frac{\beta h}{\ell'+h}\right)$$

and using that $\frac{h}{\ell+h}, \frac{h}{\ell'+h} < 1$ we get,

$$\Delta_\infty^s \tilde{u}_h^\delta(x,t) \geq h^{1+s} \left(\frac{C_\theta}{M}\right)^{2s} \left(\frac{2\varepsilon}{1+s} \frac{C_\theta}{M} - \frac{\beta(2s+1)}{s}\right)$$

Then, choosing $\beta > 0$ such that,

$$\beta \leq \left(\frac{2\varepsilon}{1+s} \frac{C_\theta}{M}\right) \left(\frac{s}{2s+1}\right)$$

we get

$$\Delta_\infty^s \tilde{u}_h^\delta(x,t) \geq 0$$

Then, by (4.20), this implies

$$\partial_t \tilde{u}_h^\delta(x,t) - \Delta_\infty^s \tilde{u}_h^\delta(x,t) \leq 0.$$

It follows that, choosing β_0 as in (ii), we can take $\beta < \min\left\{\beta_0, \left(\frac{2\varepsilon}{1+s} \frac{C_\theta}{M} \frac{s}{2s+1}\right)\right\}$ to ensure $u_h^\delta(x,0) \leq u_0(x)$ on $\mathbb{R}^N$, and conclude that u_h^δ is a subsolution. This concludes the proof when $\nabla\phi\left(x,t\right) \neq 0$.

A analogous argument holds when $\nabla\phi\left(x,t\right) = 0$. Indeed, if the supremum in the definition of $\Delta_\infty^s \tilde{u}_h^\delta$ is reached at $y \in S^{N-1}$, then $C_\theta \leq y \cdot e_1$ as a consequence of the monotony of u^δ in C^+. Similarly, if the infimum in the definition is reached at $z \in S^{N-1}$, then $z \in C^-$. Let ℓ' be the distance between x and $\partial\Omega^+$ along the line with direction y, and let ℓ be the distance between x and $\{z : d\left(z, \partial\Omega^-\right) \leq h\}$ along the line with direction z. Then (4.21) still holds and we can apply the same argument used previously to establish $\Delta_\infty^s \tilde{u}_h^\delta(x,t) \geq 0$ and conclude in the same way that we did previously. □

This monotonicity result implies that if a test function touches U from above or below, it will have non zero gradient.

Lemma 4.11. *Let u be a uniformly monotone function for some $\alpha < 2$. If φ is a test function that touches u from above or below at $x_0 \in \Omega$, then $\nabla\varphi(x_0) \neq 0$.*

Proof. Suppose that φ touches u from above at x_0 with $\nabla\varphi(x_0) = 0$. Then, by the monotonicity and the $C^{1,1}$ regularity of φ we have,

$$\beta h^\alpha \leq u(x_0 + hy) - u(x_0) \leq \varphi(x_0 + hy) - \varphi(x_0) \leq M h^2$$

For some small constants $M > 0$ and $h > 0$. Since $\alpha < 2$ this is a contradiction for small h. If φ touches u from below the argument is analogous. □

As a direct consequence of this Lemma, we have the following Corollary,

Corollary 4.12. *If ϕ is a test function that touches U from above or below at some point $(x,t) \in \Omega \times (0,T)$. Then $\nabla\phi(x,t) \neq 0$.*

Our next goal is to prove that U is a subsolution. To achieve this, we will establish a new regularity result which, combined with Lemma 3.14, will enable us to construct an equicontinuous sequence of subsolutions. This, in turn, will allow us to apply the stability Theorem 3.15.

Lemma 4.13. *Let $x_0 \in \partial\Omega^+$ be arbitrary but fixed, and consider*

$$w(x) := 1 - m_0^{-\alpha_0}|x - x_0|^{\alpha_0} \qquad \forall x \in \mathbb{R}^N$$

where α_0 is as in assumption (iv). Then $w \in \mathcal{F}$.

Proof. Clearly $w \leq 1$ on $\mathbb{R}^N$, then, by (i) and (iv) one has $w \leq u_0$ on $\mathbb{R}^N$. If $x \in \Omega^{c,-}$ one has $|x - x_0| \geq m_0$ which implies $w(x) < 0$ on $\Omega^{c,-}$, therefore $w \leq g$ on Ω^c.

Let $x \in \Omega$, by the regularity of w in this case, we can evaluate $\Delta^s_\infty w(x)$ directly. Since $\alpha_0 \in (0, 2s-1)$, by Lemma 3.10,

$$\Delta^s_\infty w(x) = -m_0^{-\alpha_0}|x - x_0|^{\alpha_0 - 2s} F(\alpha_0)$$

where $F(\alpha_0) < 0$, which implies,

$$\Delta^s_\infty w(x) > 0 \qquad \text{on } \Omega.$$

Thus,

$$\partial_t w(x) - \Delta^s_\infty w(x) < 0 \qquad \text{on } \Omega \times (0,T).$$

Concluding $w \in \mathcal{F}$. □

Remark 4.14. As a consequence of the previous result and Lemma 3.14, the function

$$\mathcal{C}(x) = \sup_{x_0 \in \partial\Omega^+} \left\{1 - m_0^{-\alpha_0}|x - x_0|^{\alpha_0}\right\}$$

is in $\mathcal{F}$.

Theorem 4.15. *Let $u \in \mathcal{F}$ such that*

$$\sup_{z \in \partial\Omega^+} \left\{1 - m_0^{-\alpha_0}|x - z|^{\alpha_0}\right\} \leq u(x,t) \tag{4.22}$$

for all $(x,t) \in \Omega \times (0,T)$, where α_0 is as in (iv). Then, for any $\mathcal{L} > ||\Delta^s_\infty u_0||_\infty$, it holds that

$$|u(x,t) - u(x_0,t_0)| \leq \mathcal{L}|t - t_0| + m_0^{-\alpha_0}|x - x_0|^{\alpha_0}$$

for all $(x,t), (x_0,t_0) \in \Omega \times (0,T)$.

Proof. Denote $\lambda := ||\Delta^s_\infty u_0||_\infty$. Fix $(x_0,t_0) \in \Omega \times (0,T)$, and define

$$C(x,t) = B + \mathcal{L}|t - t_0| + m_0^{-\alpha_0}|x - x_0|^{\alpha_0}$$

for $(x,t) \in \mathbb{R}^N \times (0,T)$, where $B, \mathcal{L} > 0$ will be chosen conveniently. Recall that $w^-(x,t) = u_0(x) - \lambda t$ is in $\mathcal{F}$, therefore, by replacing u by $\max\{u, w\}$ we have $u(x,t) \geq u_0(x) - \lambda t$, and we can assume $u(x,0) = u_0(x)$ on $\mathbb{R}^N$.

By (4.22), taking $B = u(x_0,t_0)$ yields $C > g$ on $\Omega^c \times (0,T)$, and

$$\begin{aligned} C(x,0) &= u(x_0,t_0) + \mathcal{L}t_0 + m_0^{-\alpha_0}|x-x_0|^{\alpha_0} \\ &\geq u_0(x_0) - \lambda t_0 + \mathcal{L}t_0 + m_0^{-\alpha_0}|x-x_0|^{\alpha_0} \\ &= u_0(x_0) + m_0^{-\alpha_0}|x-x_0|^{\alpha_0} + (\mathcal{L}-\lambda)t_0 \\ &\geq u_0(x) + (\mathcal{L}-\lambda)t_0 \end{aligned}$$

Hence, taking $\mathcal{L} > \lambda$ gives $C(x,0) > u_0(x)$ on $\mathbb{R}^N$. This implies that, if we take $B = 1$ first, so $C \geq u$ on $\mathbb{R}^N \times [0,T)$, and then lower B until C touches u at a point $(\bar{x},\bar{t})$, then $(\bar{x},\bar{t}) \in \Omega \times (0,T)$. We claim $\bar{x} = x_0$ and $\bar{t} = t_0$. Let's suppose by contradiction that the claim is not true. In this case we can use C as a test function on $(\bar{x},\bar{t})$. Then, recalling that $F(\alpha_0) < 0$ we have,

$$\begin{aligned} 0 &\geq \partial_t \tilde{u}(\bar{x},\bar{t}) - \Delta_\infty^s \tilde{u}(\bar{x},\bar{t}) \\ &\geq \partial_t C(\bar{x},\bar{t}) - \Delta_\infty^s C(\bar{x},\bar{t}) \\ &= \mathcal{L}\frac{(\bar{t}-t_0)}{|\bar{t}-t_0|} - F(\alpha_0) m_0^{-\alpha_0}|\bar{x}-x_0|^{\alpha_0-2s} \\ &\geq -\mathcal{L} - F(\alpha_0) m_0^{-\alpha_0}|\bar{x}-x_0|^{\alpha_0-2s} \end{aligned} \tag{4.23}$$

Observe that, since $u \in \mathcal{F}$ and $u \leq 1$, we have $C(\bar{x},\bar{t}) = u(\bar{x},\bar{t}) \leq 1$, from this we get $(m_0^{-\alpha_0})|\bar{x} - x_0|^{\alpha_0} \leq 1$, which implies $|\bar{x} - x_0| \leq (m_0^{-\alpha_0})^{-1/\alpha_0}$. Moreover, note that $\alpha_0 - 2s < 0$. Therefore, going back to (4.23), we have

$$\begin{aligned} 0 &\geq -\mathcal{L} - F(\alpha_0) m_0^{-\alpha_0}|\bar{x}-x_0|^{\alpha_0-2s} \\ &\geq -\mathcal{L} - F(\alpha_0) m_0^{-\alpha_0}(m_0^{-\alpha_0})^{\frac{2s-\alpha_0}{\alpha_0}} \\ &= -\mathcal{L} - F(\alpha_0) m_0^{-2s} \end{aligned}$$

Recall that $\lambda < -F(\alpha_0)m_0^{-2s}$, therefore choosing $\mathcal{L}$ such that

$$\lambda < \mathcal{L} < -F(\alpha_0)m_0^{-2s}$$

we get

$$0 \geq \partial_t \tilde{u}(\bar{x},\bar{t}) - \Delta_\infty^s \tilde{u}(\bar{x},\bar{t}) \geq -\mathcal{L} - F(\alpha_0)m_0^{-2s} > 0$$

which contradicts the fact that $u \in \mathcal{F}$. Hence, $(\bar{x},\bar{t}) = (x_0,t_0)$, which implies,

$$u(x,t) \leq C(x,t) = u(x_0,t_0) + \mathcal{L}|t-t_0| + m_0^{-\alpha_0}|x-x_0|^{\alpha_0}$$

for $\mathcal{L} > \lambda$. Finally, by repeating the argument changing the roles of x and x_0, we conclude the result. □

First, we prove that U is a subsolution, we do this is by constructing sequences that converge to U, and using Theorem 4.15 and Lemma 3.14, we get equicontinuous sequences so that we can envoque the Arzelá-Ascoli Theorem and finally take advantage of the stability Theorem 3.15 to conclude that U is a subsolution.

Proposition 4.16. *U is a subsolution of $\mathcal{P}_\mathcal{S}$. Moreover, $U(\cdot,t) \in C^{0,2s-1}(\mathbb{R}^N)$ for all $t \in (0,T)$, and for any $\mathcal{L} > ||\Delta_\infty^s u_0||_\infty$,*

$$|U(x,t) - U(x_0,t_0)| \leq \mathcal{L}|t-t_0| + m_0^{-\alpha_0}|x-x_0|^{\alpha_0}. \tag{4.24}$$

for all $(x,t), (x_0,t_0) \in \Omega \times (0,T)$.

Proof. Note that by Lemma 3.10 and Remark 4.14, if U is a subsolution, it would satisfy the conditions necessary to apply Theorems 3.11 and 4.15 directly, so the regularity results follow immediately once we prove that U is a subsolution. Furthermore, by Remark 4.3, the initial and exterior conditions of $\mathcal{P}_{\mathcal{S}}$ are met by U.

To simplify the notation, we denote $\mathcal{Q} = \Omega \times (0, T)$ and $C(x) = \sup_{x_0 \in \partial\Omega^+}\{1 - m_0^{-\alpha_0}|x - x_0|^{\alpha_0}\}$.

Let $D = \{\xi_1, \xi_2, ...\} \subset \mathcal{Q}$ be a dense and countable set. By the definition of U, for each $\xi_i \in D$ there exists a sequence $\{u_k^{\xi_i}\}_{k\in\mathbb{N}} \subset \mathcal{F}$ such that

$$u_k^{\xi_i}(\xi_i) \to U(\xi_i), \qquad \text{as} \qquad k \to \infty.$$

For each $\xi_i \in D$, define the sequence

$$w_k^{\xi_i}(z) = \max\{0, u_k^{\xi_i}(z), C(x)\}, \qquad x \in \mathbb{R}^N, z \in \mathcal{Q}.$$

Recall that $C(x) \in \mathcal{F}$, therefore by Lemma 3.14, $w_k^{\xi_i} \in \mathcal{F}$ for each ξ_i, it follows that $u_k^{\xi_i}(z) \leq w_k^{\xi_i}(z) \leq U(z)$ for $z \in \mathcal{Q}$. This yields

$$w_k^{\xi_i}(\xi_i) \to U(\xi_i) \tag{4.25}$$

By replacing $w_k^{\xi_i}(x)$ for $\max\{w_1^{\xi_i}(x), w_2^{\xi_i}(x), ..., w_k^{\xi_i}(x)\}$, we can assume without loss of generality that the sequences $\{w_k^{\xi_i}\}_{k\in\mathbb{N}}$ are non decreasing, that is, $w_k^{\xi_i} \leq w_{k+1}^{\xi_i}$ on $\mathcal{Q}$, for each ξ_i. Moreover, by construction and Theorem 4.15, the sequences are equicontinuous. Then, since $0 \leq w_k^{\xi_i} \leq 1$, for each $k \in \mathbb{N}$ and $\xi \in D$, by the Arzelá-Ascoli Theorem there exists a subsequence $\{w_{k_j}^{\xi_i}\}_{j\in\mathbb{N}}$ that converges pointwise to a continuous function W^{ξ_i}, and the convergence will be uniform on compact subsets of $\mathcal{Q}$. Therefore, by Theorem 3.15, for each $\xi_i \in D$, W^{ξ_i} is a subsolution on non zero gradient points. Moreover, from (4.25), it follows that

$$W^{\xi_i}(\xi_i) = U(\xi_i).$$

Denote $W_k = W^{\xi_k}$, and define the sequence $\{U_k\}_{k\in\mathbb{N}}$ as follows,

$$\begin{aligned} U_1(z) &= \max\{C(z), W_1(z)\}, \\ U_2(z) &= \max\{C(z), W_1(z), W_2(z)\}, \\ &\vdots \\ U_k(z) &= \max\{C(z), W_1(z), \ldots, W_k(z)\}, \end{aligned}$$

It follows that, by Theorem 4.15 and Lemma 3.14, the sequence $\{U_k\}_{k\in\mathbb{N}}$ is equicontinuous, and U_k is a subsolution at non zero gradient points for each $k \in \mathbb{N}$. Note that for each fixed $\bar{k} \in \mathbb{N}$,

$$U_k(\xi_{\bar{k}}) = U(\xi_{\bar{k}}) \qquad \forall k \geq \bar{k}.$$

Therefore, for each $\xi \in D$, the sequence $\{U_k(\xi)\}_{k\in\mathbb{N}}$ converges to $U(\xi)$, since $U_k \leq U_{k+1} \leq \ldots \leq U$ for each $k \in \mathbb{N}$.

Once again, by Arzelá-Ascoli, there exists a subsequence $\{U_{k_j}\}_{j\in\mathbb{N}}$ that converges pointwise to a continuous function $\overline{U}$. Note that, since $U_k(\xi) \to U(\xi)$, we have

$$\overline{U}(\xi) = U(\xi) \qquad \forall \xi \in D,$$

and, by Theorem 3.15, $\overline{U}$ is a subsolution on non zero gradient points.

To conclude the proof it only remains to check that $U = \overline{U}$ on $\mathcal{Q}$.

Let $z_0 \in \mathcal{Q}$ such that $z_0 \notin D$. Then, there exists a subsequence $\{v_k\}_{k\in\mathbb{N}} \subset \mathcal{F}$ such that

$$v_k(z_0) \to U(z_0).$$

Again, arguing in the same way that we did with $\{w_k^{\xi_i}\}_{k\in\mathbb{N}}$, we can assume that $\{v_k\}_{k\in\mathbb{N}}$ is equicontinuous and non decreasing.

Replacing $v_k(z)$ for

$$\max\{C(z), v_k(z), \overline{U}(z)\},$$

we get $v_k(\xi) = \overline{U}(\xi) = U(\xi)$ for each $\xi \in D$. Again, by Arzelá-Ascoli, v_k converges to a continuous function v up to a subsequence. This yields,

$$v(\xi) = \overline{U}(\xi) = U(\xi) \text{ and } v(z_0) = U(z_0).$$

Since D is dense in $\mathcal{Q}$ and $z_0 \notin D$, there exists a sequence $\{\xi_k\}_{k\in\mathbb{N}} \subset D$ such that

$$\xi_k \to z_0.$$

Then, by the continuity of v and $\overline{U}$, and the uniqueness of the limit,

$$\overline{U}(z_0) = \lim_{k\to\infty} \overline{U}(\xi_k) = \lim_{k\to\infty} v(\xi_k) = v(z_0) = U(z_0).$$

Thus, by the arbitrariness of x_0, we conclude $\overline{U} = U$ on $\mathcal{Q}$. Finally, U is a subsolution of $\mathcal{P}_{\mathcal{S}}$ on non zero gradient points. Note that, Corollary 4.12 implies that U is a subsolution on non zero gradient points on every point of $\mathcal{Q}$, subsequently, U is a subsolution on $\mathcal{Q}$. ☐

Theorem 4.17. *U is a solution of $\mathcal{P}_{\mathcal{S}}$.*

Proof. Assume by contradiction that U is not a supersolution. Since $U(x,t) = g(x)$ on $\Omega^c \times (0,T)$ and $U(x,0) = u_0(x)$ on $\mathbb{R}^N$, the contradiction has to come from the equation on $\Omega \times (0,T)$ in the viscosity sense. Then, there exists a point $(x_0,t_0) \in \Omega \times (0,T)$, a test function ψ that touches U from below at (x_0,t_0), and constants $r > 0$, $\rho < 0$ such that

$$\partial_t \hat{U}(x_0,t_0) - \Delta^s_\infty \hat{U}(x_0,t_0) \le 2\rho < 0$$

where

$$\hat{U}(x,t) = \begin{cases} \psi(x,t) & \text{if } (x,t) \in B_r(x_0,t_0), \\ U(x,t) & \text{if } (x,t) \notin B_r(x_0,t_0). \end{cases} \tag{4.26}$$

We claim that there exists $\delta_0, r_0 > 0$ such that, for all $\delta \in (0,\delta_0)$,

$$w_\delta(x,t) = U(x,t) \vee [(\psi(x,t) + \delta)\mathbb{1}_{B_{r_0}(x_0,t_0)}] \tag{4.27}$$

is a subsolution.

Clearly $w_\delta(x_0,t_0) > U(x_0,t_0)$, which would contradict the definition of U, and show that U is a supersolution if w_δ was a subsolution.

Note that, Corollary 4.12 gives $\nabla\psi(x_0,t_0) \neq 0$, and as a consequence of the regularity of $\psi, \nabla\psi$ and U, we have $\partial_t \hat{U}(x,t) - \Delta^s_\infty \hat{U}(x,t)$ is continuous near (x_0,t_0), therefore there exists $r_0 < \frac{r}{2}$ such that

$$\partial_t \hat{U}(x,t) - \Delta^s_\infty \hat{U}(x,t) < \rho \quad \text{and} \quad \nabla\psi(x,t) \neq 0 \quad \forall (x,t) \in B_{r_0}(x_0,t_0) \tag{4.28}$$

From the assumption $\psi < U$ on $B_r(x_0,t_0) \setminus \{(x_0,t_0)\}$, it follows that there exists $\delta_1 > 0$ small enough such that $\psi(x,t) + \delta_1 < U(x,t)$ for all $(x,t) \in B_r(x_0,t_0) \setminus B_{r_0}(x_0,t_0)$.

Now we consider w_δ with $\delta < \delta_1$. If ϕ is a test function that touches w_δ from above at $(\hat{x},\hat{t}) \in \Omega \times (0,T)$, then it touches either U or $\psi + \delta$. In the first case, using that U is a subsolution and compute,

$$0 \ge \partial_t \tilde{U}(\hat{x},\hat{t}) - \Delta^s_\infty \tilde{U}(\hat{x},\hat{t}) \ge \partial_t \tilde{w}(\hat{x},\hat{t}) - \Delta^s_\infty \tilde{w}(\hat{x},\hat{t})$$

since $\partial_t \tilde{w}_\delta = \partial_t \tilde{U}$ and $w_\delta \geq U$, where $\tilde{w}_\delta$ and $\tilde{U}$ are defined as in (3.3). If ϕ touches $\psi + \delta$, then $(\hat{x}, \hat{t}) \in B_{r_0}(x_0, t_0)$, and $\partial_t \phi(\hat{x}, \hat{t}) = \partial_t \psi(\hat{x}, \hat{t})$. By (4.28), and possibly choosing smaller r_0, we compute,

$$\begin{aligned}
\partial_t \tilde{w}_\delta - \Delta^s_\infty \tilde{w}_\delta(\hat{x}, \hat{t}) \leq \partial\psi(\hat{x}, \hat{t}) &- \int_0^{r_0} \frac{\phi(\hat{x} + \eta v, \hat{t}) + \phi(\hat{x} - \eta v, \hat{t}) - 2\phi(\hat{x}, \hat{t})}{\eta^{1+2s}} d\eta \\
&- \int_{r_0}^{\infty} \frac{U(\hat{x} + \eta v, \hat{t}) + U(\hat{x} - \eta v, \hat{t}) - 2\phi(\hat{x}, \hat{t})}{\eta^{1+2s}} d\eta \\
&= \partial\psi(\hat{x}, \hat{t}) - \int_0^{r_0} \frac{\psi(\hat{x} + \eta v, \hat{t}) + \delta + \psi(\hat{x} - \eta v, \hat{t}) + \delta - 2\psi(\hat{x}, \hat{t}) - 2\delta}{\eta^{1+2s}} d\eta \\
&- \int_{r_0}^{\infty} \frac{U(\hat{x} + \eta v, \hat{t}) + U(\hat{x} - \eta v, \hat{t}) - 2\psi(\hat{x}, \hat{t}) - 2\delta}{\eta^{1+2s}} d\eta \\
&= \partial\hat{U}(\hat{x}, \hat{t}) - \Delta^s_\infty \hat{U}(\hat{x}, \hat{t}) + \int_{r_0}^{\infty} \frac{2\delta}{\eta^{1+2s}} d\eta \\
&< \rho + \int_{r_0}^{\infty} \frac{2\delta}{\eta^{1+2s}} d\eta \\
&= \rho + \frac{r_0^{2s}\delta}{s},
\end{aligned}$$

where $v := \nabla\phi(\hat{x}, \hat{t}) = \nabla\psi(\hat{x}, \hat{t}) \neq 0$. Finally, it is enough to choose $\delta_0 = -s\rho r_0^{-2s}$ to deduce $\partial_r \tilde{w}_\delta - \Delta^s_\infty \tilde{w}_\delta \leq 0$. If necessary, one can choose a smaller r_0, such that $B_{r_0}(x_0, t_0) \subset \Omega \times (0, T)$. This way one gets $w_\delta(x, 0) = u_0(x)$ on $\mathbb{R}^N$ and $w_\delta(x, t) = g(x)$ on $\Omega^c \times (0, T)$. Hence w_δ is a subsolution, and we obtained the desired contradiction. □

To conclude this chapter, we prove a comparison principle that yields the uniqueness of a solution to $\mathcal{P}_\mathcal{S}$. To prove this result, instead of relying on the compactness of Ω like Theorem 3.8, we ask for at least one of the functions to be uniformly monotone so we can take advantage of the stability of Δ^s_∞ when the limit function cannot be touched by a test function with zero derivative.

Theorem 4.18. *Let Ω be bounded in the e_1 direction (i.e., $\Omega \subset \{-M \leq x_1 \leq M\}$ for some $M > 0$). Consider two functions $u, w : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ such that*

- *u (resp. w) is a subsolution (resp. supersolution) at non-zero gradient points of $\mathcal{P}_\mathcal{S}$,*
- *$u, w \in C\left((0, T); C^{0,2s-1}(\mathbb{R}^N)\right)$,*
- *u or w is uniformly monotone along e_1 away from $\partial\Omega$ for some $\alpha < 2$.*

Then $u \leq w$ in Ω.

Proof. By way of contradiction, we assume there is a point $(\bar{x}, \bar{t}) \in \Omega \times (0, T)$ such that $u(\bar{x}, \bar{t}) > w(\bar{x}, \bar{t})$. Recalling the proof of Theorem 3.8, by replacing u and w by $u^{\varepsilon,\kappa}$ and $w_{\varepsilon,\kappa}$, we have $u^{\varepsilon,\kappa}(\bar{x}, \bar{t}) - w_{\varepsilon,\kappa}(\bar{x}, \bar{t}) \geq c > 0$ for ε, κ sufficiently small. Moreover, since $u, w \in C\left((0, T); C^{0,2s-1}(\mathbb{R}^N)\right)$, the uniform continuity of u and w, together with the assumptions $u \leq w$ on $\Omega^c \times (0, T)$ and $u \leq w$ on $\mathbb{R}^N \times \{0\}$, imply that $(u^{\varepsilon,\kappa} - w_{\varepsilon,\kappa}) \vee 0 \to 0$ as $\varepsilon \to 0$ uniformly on $\Omega^c \times (0, T) \cup \mathbb{R}^N \times \{0\}$. So, assume that ε, κ are small enough that $u^{\varepsilon,\kappa} - w_{\varepsilon,\kappa} \leq c/2$ on $\Omega^c \times (0, T) \cup \mathbb{R}^N \times \{0\}$, and define

$$\begin{aligned}
\delta_0 &= \inf\left\{\delta : \delta \geq u^{\varepsilon,\kappa}(x, t) - w_{\varepsilon,\kappa}(x, t) \quad \forall (x, t) \in \mathbb{R}^N \times [0, T)\right\} \\
&= \inf\left\{\delta : \delta \geq u^{\varepsilon,\kappa}(x, t) - w_{\varepsilon,\kappa}(x, t) \quad \forall (x, t) \in \Omega \times (0, T)\right\}.
\end{aligned}$$

The last inequality follows from $u^{\varepsilon,\kappa} - w_{\varepsilon,\kappa} \leq c/2$ on $\Omega^c \times (0, T) \cup \mathbb{R}^N \times \{0\}$ and $u^{\varepsilon,\kappa}(\bar{x}, \bar{t}) - w_{\varepsilon,\kappa}(\bar{x}, \bar{t}) \geq c > 0$, where $(\bar{x}, \bar{t}) \in \Omega \times (0, T)$. Moreover, observe that $\delta_0 \geq c > 0$.

If there exists a point $(x_0, t_0) \in \Omega \times (0, T)$ such that $\delta_0 = u^{\varepsilon,\kappa}(x_0, t_0) - w_{\varepsilon,\kappa}(x_0, t_0)$, an argument analogous to that in the proof of Theorem 3.8 yields a contradiction. If no such (x_0, t_0) exists, let $(x_n, t_n) \in \Omega \times (0, T)$ be a sequence such that $\delta_0 \leq u^{\varepsilon,\kappa}(x_n, t_n) - w_{\varepsilon,\kappa}(x_n, t_n) + \frac{1}{n}$. Let $\tau_n \in \{0\} \times \mathbb{R}^{N-1}$ be such that $\tilde{x}_n := x_n - \tau_n = (e_1 \cdot x_n) e_1$, that is τ_n is the translation for which $\tilde{x}_n \in \mathbb{R} \times \{(0, \ldots, 0)\}$. The sequence $\tilde{x}_n$ lies in a bounded set of $\mathbb{R}^N$ since Ω is bounded in the e_1 direction, so we may extract a subsequence with a limit $\tilde{x}_0$. On the other hand, since $u \leq w$ on $\mathbb{R}^N \times \{t = 0\}$ it follows that there exists $\tilde{t}_0 \in (0, T)$ such that $t_n \to \tilde{t}_0$. Given that subsolutions and supersolutions are invariant under translations, $u_n(x,t)=u^{\varepsilon,\kappa}(x+\tau_n,t)$ and $w_n(x,t)=w_{\varepsilon,\kappa}(x+\tau_n,t)$ form families of sub and supersolutions at non-zero gradient points which are uniformly equicontinuous and bounded. Therefore, by the Arzelà-Ascoli Theorem, there exist two functions $\tilde{u}$ and $\tilde{w}$ such that, up to subsequences, $u_n(x,t)\to\tilde{u}(x,t)$ and $w_n(x,t)\to\tilde{w}(x,t)$ uniformly on compact sets. As a consequence of Theorem 3.15, $\tilde{u}$ (resp. $\tilde{w}$) is a subsolution (resp. supersolution) at non-zero gradient points. Moreover $\tilde{u}(x,t)\le\tilde{w}(x,t)+\delta_0$ for all $(x,t)\in\mathbb{R}^N\times(0,T)$ and $\tilde{u}(x_0,t_0)=\tilde{w}(x_0,t_0)+\delta_0$.

Finally, we can conclude the proof with a similar argument to that of the proof of Theorem 3.8; that is, thanks to the $C^{1,1}$ bounds from below and above of $u^{\varepsilon,\kappa}$ and $w_{\varepsilon,\kappa}$ respectively, we have that $\tilde{u}$ and $\tilde{w}$ also posses these bounds. Additionally, of we assume for instance that u is uniformly monotone in the e_1 direction with constant α, then also $\tilde{u}$ is uniformly monotone along e_1 for the same vale of α. Then, both functions are $C^{1,1}$ at (x_0,t_0), so Lemma 4.11 applied with $u=\tilde{u}$ and $\phi=\tilde{w}$ implies $\nabla\tilde{w}(x_0,t_0)\neq 0$, and the subsolution and supersolution conditions at (x_0,t_0) give a contradiction, which concludes the proof. □

Chapter 5

The problem in an annular domain

In this chapter we study the well posedness of $\mathcal{P}_{\mathcal{S}}$ considering $\Omega = B_R \setminus \overline{B_r}$, with $0 < r < R$. The choice of Ω as an annular domain is motivated by its similarity to the strip domain: since we have two disjoint connected components of Ω^c, we can replicate the growth and decay estimates for U from the first setting, by following the exact same approach as in the previous chapter.

To maintain consistent notation with chapter 4, we denote:

- $\Omega^{c,+} := B_R^c$ and $\Omega^{c,-} := \overline{B_r}$,
- $\partial\Omega^- = \{x \in \mathbb{R}^N : |x| = r\}$, and $\partial\Omega^+ = \{x \in \mathbb{R}^N : |x| = R\}$

Consider the problem,

$$\left\{ \begin{array}{rcll} \partial_t u(x,t) - \Delta_\infty^s u(x,t) & = & 0 & \text{if } (x,t) \in \Omega \in (0,T) \\ u(x,t) & = & g(x) & \text{if } (x,t) \in \Omega^c \times (0,T), \\ u(x,0) & = & u_0(x) & \text{on } \mathbb{R}^N. \end{array} \right. \tag{$\mathcal{P}_{\mathcal{A}}$}$$

where,

$$g(x,t) = \left\{ \begin{array}{ll} 0 & \text{if } x \in \Omega^{c,-}, \\ 1 & \text{if } x \in \Omega^{c,+}. \end{array} \right.$$

In this case, as we will see below, the direction of the gradients $\nabla d(x, \partial\Omega^-)$ and $\nabla d(x, \partial\Omega^+)$ are always orthogonal to $\partial\Omega$, so there is no need to consider the neighborhoods $\mathcal{N}$ and $\mathcal{M}$ from Remark 4.1. Because of this, the assumptions on u_0 are slightly different from the ones in the previous chapter. In this chapter we assume:

($\mathrm{i}_{\mathcal{A}}$) $u_0 \in C^{1,1}(\Omega) \cap BC(\mathbb{R}^N)$, $u_0 \geq 0$ and $u_0(x) = g(x)$ on Ω^c.

($\mathrm{ii}_{\mathcal{A}}$) u_0 is uniformly monotone in the radial direction. Precisely, there exists $\beta_0 > 0$ such that for all $x \in \mathbb{R}^N$ there exists $h_0 > 0$ such that

$$u_0(x) + \beta_0 h^{1+s} \leq u_0\left(x + h\frac{x}{|x|}\right) \qquad \forall h < h_0.$$

($\mathrm{iii}_{\mathcal{A}}$) There exist $\varepsilon_0 > 0$ and $\kappa_0 > 0$ such that:

- $\varepsilon_0 d^s(x, \partial\Omega^-) \leq u_0(x)$ for all $x \in \{x \in \Omega : r < |x| < r + \kappa_0\}$.
- $u_0(x) \leq 1 - \varepsilon_0 d^s(x, \partial\Omega^+)$ for all $x \in \{x \in \Omega : R - \kappa_0 < |x| < R\}$.

($\mathrm{iv}_{\mathcal{A}}$) There exists $\alpha_0 \in (0, 2s-1)$ such that $u_0 \in C^{0,\alpha_0}(\mathbb{R}^N)$ with constant $m_0^{-\alpha_0}$, that is

$$|u_0(x) - u_0(y)| \leq m_0^{-\alpha_0}|x-y|^{\alpha_0} \qquad \forall x, y \in \mathbb{R}^N.$$

In addition, we assume that the radii R and r are such that,

$$-F(\alpha_0)(R-r)^{-2s} > ||\Delta_\infty^s u_0||_\infty. \tag{5.1}$$

This last assumption is analogous to (4.5).

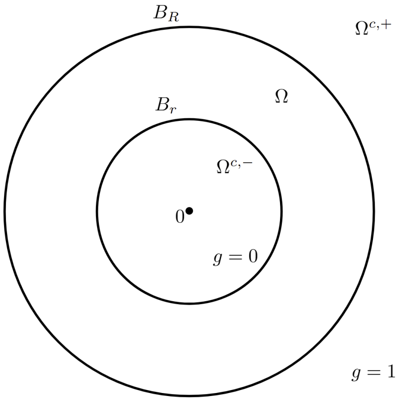

FIGURE 5.1: Parabolic problem in an annular domain.

These assumptions serve the same purpose as the ones in the previous chapter. Note that in assumption ($\text{iii}_{\mathcal{A}}$), the sets $\{x \in \Omega : r < |x| < r + \kappa_0\}$ and $\{x \in \Omega : R - \kappa_0 < |x| < R\}$ are analogous to $\Omega^-_{\kappa_0}$ and $\Omega^+_{\kappa_0}$ respectively.

Once again we follow Perron's Method. In this chapter we consider the family of subsolution defined as,

$$\mathcal{F}_{\mathcal{A}} := \{u : u \text{ is a subsolution of } \mathcal{P}_{\mathcal{A}}, \text{ it is a radial function, and } u \leq 1\}.$$

And analogously we define,

$$U(x,t) = \sup\{u(x,t) : u \in \mathcal{F}_{\mathcal{A}}\}. \tag{5.2}$$

Note that $0 \in \mathcal{F}_{\mathcal{A}}$ and $\mathbb{1}_{\Omega^{c,+}} \in \mathcal{F}_{\mathcal{A}}$, then, arguing analogously as in the beginning of chapter 4, we have that U is well defined, $U = g$ on $\Omega^c \times (0,T)$, $0 \leq U \leq 1$ and $U(x,0) = u_0(x)$ on $\mathbb{R}^N$. Moreover, since the functions on $\mathcal{F}_{\mathcal{A}}$ are radial, U is radial.

To prove existence we will follow the same approach that we did in the previous chapter, that is, we want to prove the uniform monotonicity of U on Ω. The uniform monotonicity that we seek to prove in this case is radial, precisely, we want to prove that U satisfies the following:

Definition 5.1. We say that a function $u : \Omega \to \mathbb{R}$ is uniformly monotone in the radial direction with exponent $\alpha > 0$ and constant $\beta > 0$ if the following holds: For all $x \in \Omega$ there exists $h_0 > 0$ such that

$$u(x) + \beta h^\alpha \leq u\left(x + h\frac{x}{|x|}\right) \qquad \forall 0 \leq h \leq h_0.$$

Once we prove the uniform monotonicity of U, the main result follows from results that were proved at the end of the previous chapter, that are independent of the domain Ω. Moreover, since Ω is a bounded set in this setting, we can use the Comparison Principle, Theorem 3.8, to obtain uniqueness.

5.1 Radial Monotonicity

First, we prove radial monotonicity of U. This result is analogous to Proposition 4.5, and it has similar implications on the values of the gradients of test functions that touch U at a point.

Proposition 5.2. *U is non-decreasing in the radial direction, that is, for all $\lambda > 0$,*

$$U(x,t) \leq U\left(x + \lambda\frac{x}{|x|}, t\right)$$

for all $(x,t) \in \mathbb{R}^N \times (0,T)$.

Proof. This proof follows a similar idea that of the proof of Proposition 4.5.

Fix $0 < \lambda < r$ and $(x_0, t_0) \in \Omega \times (0, T)$. Let $u^\delta \in \mathcal{F}_\mathcal{A}$ be such that $U(x_0, t_0) \leq u^\delta(x_0, t_0) + \delta$ as $\delta \to 0$. As in Proposition 4.5, we can assume that $u^\delta \geq 0$, and

$$u^\delta(x,t) = \begin{cases} 1, & \text{if } x \in \Omega^{c,+} \times (0,T), \\ 0, & \text{if } x \in \Omega^{c,-} \times (0,T). \end{cases} \tag{5.3}$$

Arguing analogously, we can ensure that the function

$$u_\lambda^\delta(x,t) = \begin{cases} u^\delta\left(x - \lambda\frac{x}{|x|}, t\right), & \text{if } (x,t) \in \overline{\Omega} \times [0,T), \\ 1, & \text{if } (x,t) \in \Omega^{c,+} \times [0,T), \\ 0, & \text{if } (x,t) \in \Omega^{c,-} \times [0,T) \end{cases} \tag{5.4}$$

is a subsolution of $\mathcal{P}_\mathcal{A}$. Note that, by assumption (ii$_\mathcal{A}$), $u_\lambda^\delta(x, 0) \leq u_0(x)$ on $\mathbb{R}^N$. This yields,

$$U(x_0,t_0) \leq u^\delta(x_0,t_0) + \delta \leq u_\lambda^\delta(x_0 + \lambda\frac{x_0}{|x_0|}, t_0) + \delta \leq U(x_0 + \lambda\frac{x_0}{|x_0|}, t_0) + \delta.$$

By taking $\delta \to 0$ we conclude the proof for $0 < \lambda < r$. The result for all $\lambda > 0$ follow from transitivity. ☐

In this setting, the consequence of this monotonicity is: If ϕ touches U from above at a point $(x_0, t_0) \in \Omega \times (0, T)$, then $\nabla\phi(x_0, t_0) = 0$ or $0 < v \cdot \frac{x_0}{|x_0|} \leq 1$, where $v \in S^{N-1}$ is the direction of $\nabla\phi(x_0, t_0)$. Indeed,

$$\begin{aligned} \nabla\phi(x_0,t_0) \cdot \frac{x_0}{|x_0|} &= \lim_{h\to 0} \frac{\phi(x_0 + h\frac{x_0}{|x_0|}, t_0) - \phi(x_0,t_0)}{h} \\ &\geq \limsup_{h\to 0} \frac{U\left(x_0 + h\frac{x_0}{|x_0|}, t_0\right) - U(x_0,t_0)}{h} \\ &\geq 0. \end{aligned}$$

A similar argument holds for test functions that touch U from below.

5.2 Barriers

Following our framework, we want to construct appropriate barriers to show that there exists a constant $\varepsilon > 0$ such that,

$$\varepsilon d^s(x, \partial\Omega^-) \leq U \leq 1 - \varepsilon d^s(x, \partial\Omega^+). \tag{5.5}$$

Once again, this estimate for U, stems from the fact that $\mathbb{R}^+ \ni \eta \mapsto (\eta^+)^s$ satisfies,

$$\Delta^s(\eta^+)^s = 0 \qquad \text{on } (0,\infty). \tag{5.6}$$

In this setting we can prove that the functions,

$$f_\varepsilon(x) = \begin{cases} \varepsilon d^s(x,\partial\Omega^-) & \text{if } x \in \overline{\Omega}, \\ 1 & \text{if } x \in \Omega^{c,+}, \\ 0 & \text{if } x \in \Omega^{c,-}. \end{cases} \quad \text{and,} \quad f^\varepsilon(x) = \begin{cases} 1 - \varepsilon d^s(x,\partial\Omega^+) & \text{if } x \in \overline{\Omega}, \\ 1 & \text{if } x \in \Omega^{c,+}, \\ 0 & \text{if } x \in \Omega^{c,-}. \end{cases} \tag{5.7}$$

are sub and super solutions, respectively, in the whole Ω. This implies that we can establish estimate (5.5) on the whole Ω without the need to prove a result analogous to Lemma 4.8. This is because, for $x \in \Omega$ we can write,

$$d(x,\partial\Omega^-) = (|x| - r)^+ \qquad \text{and} \qquad d(x,\partial\Omega^+) = (R - |x|)^+. \tag{5.8}$$

Recall that $d(x, \partial\Omega^-)$ and $d(x, \partial\Omega^+)$ are equal to zero for $x \notin \overline{\Omega}$. The proof follows a similar idea that of Lemma 4.9: For $x \in \Omega$, we can take use (5.6) to obtain a lower bound on $\Delta_\infty^s d^s(x, \partial\Omega^-)$ so we can estimate the difference between $\Delta_\infty^s f_\varepsilon(x)$ and $\Delta_\infty^s d^s(x, \partial\Omega^-)$. Since f_ε and $\varepsilon d^s(x, \partial\Omega^-)$ coincide inside Ω, the difference is represented by a gain and a loss that comes from changing the value of f_ε

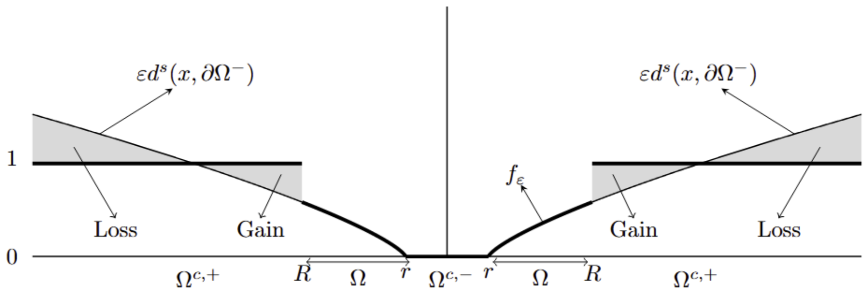

Figure 5.2: Gain and Loss from $\Delta^s_\infty f_\varepsilon - \Delta^s_\infty \varepsilon d^s$

outside Ω. By proving that the gain is greater than the loss, we obtain that f_ε satisfies $\Delta^s_\infty f_\varepsilon \geq 0$ on Ω. An analogous idea holds when proving that f^ε is a supersolution.

Lemma 5.3. *There exists a constant $\varepsilon > 0$ such that $\varepsilon d^s(x,\partial\Omega^-) \leq U(x,t) \leq 1 - \varepsilon d^s(x,\partial\Omega^+)$ on $\Omega \times (0,T)$.*

Proof. Consider f_ε and f^ε as defined in (5.7). To prove the result, it suffices to prove that f_ε and f^ε are sub and super solution, and use the Comparison Principle to conclude. We assume $\varepsilon^{-1/s} > (R-r)$ so $f_\varepsilon < 1$ and $f^\varepsilon > 0$ on Ω.

We start by proving that f_ε is a subsolution. Fix $x \in \Omega$. By the regularity of f_ε on Ω, we can evaluate $\Delta^s_\infty f_\varepsilon$ directly without appealing to test functions. We denote by v the direction of ∇f_ε, we have

$$v := \frac{\nabla f_\varepsilon(x)}{|\nabla f_\varepsilon(x)|} = \frac{\nabla d^s(x,\partial\Omega^-)}{|\nabla d^s(x,\partial\Omega^-)|} = \frac{x}{|x|}.$$

This implies that the line $\{x+\eta v\}_{\eta\in\mathbb{R}}$ intersects $\partial\Omega^-$ orthogonally, and for $\eta < 0$ it goes through the origin and through Ω again. Let τ (resp. τ') be the distance between x and $\partial\Omega^-$ (resp. $\partial\Omega^+$) along this line. Note that $(|x|-r) > 0$, so owing to (5.6) we have,

$$0 = \int_0^\infty \frac{\left((|x|-r+\eta)^+\right)^s + \left((|x|-r)^+ - \eta\right)^s - 2\left((|x|-r)^+\right)^s}{|\eta|^{1+2s}}\, d\eta.$$

On the other hand, by (5.8),

$$\Delta^s_\infty d^s(x,\partial\Omega^-) = \int_0^\infty \frac{\left((||x|+\eta|-r)^+\right)^s + \left((||x|-\eta|-r)^+ - \eta\right)^s - 2\left((|x|-r)^+\right)^s}{|\eta|^{1+2s}}\, d\eta.$$

Since for all $\eta > 0$ we have $|x|-\eta \leq ||x|-\eta|$, this yields,

$$0 \leq \varepsilon\Delta^s_\infty d^s(x,\partial\Omega^-) = \Delta^s_\infty \varepsilon d^s(x,\partial\Omega^-)$$

which implies,

$$\Delta^s_\infty f_\varepsilon(x) - \Delta^s_\infty \varepsilon d^s(x,\partial\Omega^-) \leq \Delta^s_\infty f_\varepsilon(x).$$

Expanding the difference on the left hand side yields,

$$\begin{aligned}\Delta^s_\infty f_\varepsilon(x) \geq \int_{\tau'}^{\varepsilon^{-\frac{1}{s}}-\tau} \frac{1-\varepsilon(\tau+\eta)^s}{\eta^{1+2s}} d\eta + \int_{R-r+\tau}^{\varepsilon^{-1/s}+2r+\tau} \frac{1-\varepsilon\left(\eta-(2r+\tau)\right)^s}{\eta^{1+2s}} d\eta \\ - \int_{\varepsilon^{-\frac{1}{s}}-\tau}^{\infty} \frac{\varepsilon(\tau+\eta)^s-1}{\eta^{1+2s}} d\eta - \int_{\varepsilon^{-1/s}+2r+\tau}^{\infty} \frac{\varepsilon\left(\eta-(2r+\tau)\right)^s-1}{\eta^{1+2s}} d\eta\end{aligned}$$

This is a similar bound to the one obtained in the proof of Lemma 4.9. However, the difference is that here we have two integrals that represent the gain and two that represent the loss. This is due to the fact that, along the line $\{x+\eta v\}_{\eta\in\mathbb{R}}$, the function f_ε changes its value twice when it goes in $\Omega^{c,+}$. In this case, the first and second integrals represent the gain, where $\varepsilon(\tau+\eta)^s \leq 1$, when the value of f_ε changes to 1 and $\varepsilon d^s < f_\varepsilon$. While the third and fourth integrals represent the loss, where

$\varepsilon(\tau'+\eta)^s \geq 1$, when εd^s becomes greater than 1, therefore $\varepsilon d^s > f_\varepsilon$ (See Figure 5.2). Now we prove that the righ hand side is positive for ε small enough. Since for $\eta > \tau'$ it holds $\frac{\eta}{\tau'} > 1$, we have

$$\eta + \tau < \left(\frac{\tau+\tau'}{\tau'}\right)\eta. \tag{5.9}$$

Note that an analogous bound to (5.9) is also true for $\eta > \tau$ with the roles of τ and τ swapped. By (5.9),

$$\begin{aligned}
\int_{\tau'}^{\varepsilon^{-\frac{1}{s}}-\tau} \frac{1-\varepsilon(\tau+\eta)^s}{\eta^{1+2s}}d\eta &\geq \int_{\tau'}^{\varepsilon^{-\frac{1}{s}}-\tau} \frac{1}{\eta^{1+2s}} - \varepsilon\left(\frac{\tau+\tau'}{\tau'}\right)^s \frac{\eta^s}{\eta^{1+2s}}d\eta \\
&= \int_{\tau'}^{\varepsilon^{-\frac{1}{s}}-\tau} \frac{1}{\eta^{1+2s}} - \varepsilon\left(\frac{\tau+\tau'}{\tau'}\right)^s \frac{1}{\eta^{1+s}}d\eta \\
&= -\frac{1}{2s}\left((\varepsilon^{-1/s}-\tau)^{-2s} - (\tau')^{-2s}\right) \\
&\quad + \frac{1}{s}\left(\varepsilon\left(\frac{\tau+\tau'}{\tau'}\right)^s\right)\left((\varepsilon^{-1/s}-\tau)^{-s} - (\tau')^{-s}\right) \\
&= -\frac{1}{2s}\frac{1}{(\varepsilon^{-1/s}-\tau)^{2s}} + \frac{1}{2s}\frac{1}{(\tau')^{2s}} + \frac{\varepsilon}{s}\left(\frac{\tau+\tau'}{\tau'}\right)^s \frac{1}{(\varepsilon^{-1/s}-\tau)^s} \\
&\quad - \frac{\varepsilon}{s}\left(\frac{\tau+\tau'}{\tau'}\right)^s \frac{1}{(\tau')^s}.
\end{aligned}$$

Also,

$$\begin{aligned}
\int_{\varepsilon^{-\frac{1}{s}}-\tau}^{\infty} \frac{\varepsilon(\tau+\eta)^s - 1}{\eta^{1+2s}}d\eta &\leq \int_{\varepsilon^{-\frac{1}{s}}-\tau}^{\infty} \varepsilon\left(\frac{\tau+\tau'}{\tau'}\right)^s \frac{\eta^s}{\eta^{1+2s}} - \frac{1}{\eta^{1+2s}}\, d\eta \\
&= \frac{\varepsilon}{s}\left(\frac{\tau+\tau'}{\tau'}\right)^s \frac{1}{(\varepsilon^{-1/s}-\tau)^s} - \frac{1}{2s}\frac{1}{(\varepsilon^{-1/s}-\tau)^{2s}}.
\end{aligned}$$

Putting these last two bounds together we obtain,

$$\int_{\tau'}^{\varepsilon^{-\frac{1}{s}}-\tau} \frac{1-\varepsilon(\tau+\eta)^s}{\eta^{1+2s}}d\eta - \int_{\varepsilon^{-\frac{1}{s}}-\tau}^{\infty} \frac{\varepsilon(\tau+\eta)^s - 1}{\eta^{1+2s}}d\eta > \frac{1}{(\tau')^{2s}}\left(\frac{1}{2s} - \frac{\varepsilon}{s}(\tau+\tau')^s\right)$$

Note that $\tau + \tau' = R - r$. Then, taking $\varepsilon(R-r)^s < \frac{1}{2}$, we get

$$\int_{\tau'}^{\varepsilon^{-\frac{1}{s}}-\tau} \frac{1-\varepsilon(\tau+\eta)^s}{\eta^{1+2s}}d\eta - \int_{\varepsilon^{-\frac{1}{s}}-\tau}^{\infty} \frac{\varepsilon(\tau+\eta)^s - 1}{\eta^{1+2s}}d\eta > \frac{1}{s(R-r)^{2s}}\left(\frac{1}{2} - \varepsilon(R-r)^s\right) > 0 \tag{5.10}$$

On the other hand,

$$\begin{aligned}
\int_{R-r+\tau}^{\varepsilon^{-1/s}+2r+\tau} \frac{1-\varepsilon\left(\eta-(2r+\tau)\right)^s}{\eta^{1+2s}}d\eta &> \int_{R-r+\tau}^{\varepsilon^{-1/s}+2r+\tau} \frac{1-\varepsilon\eta^s}{\eta^{1+2s}}d\eta \\
&= \int_{R-r+\tau}^{\varepsilon^{-1/s}+2r+\tau} \frac{1}{\eta^{1+2s}} - \frac{\varepsilon}{\eta^{1+s}}d\eta \\
&= -\frac{1}{2s}\frac{1}{(\varepsilon^{-1/s}+2r+\tau)^{2s}} + \frac{1}{2s}\frac{1}{(R+r+\tau)^{2s}} + \\
&\quad + \frac{\varepsilon}{s}\frac{1}{(\varepsilon^{-1/s}+2r+\tau)^s} - \frac{\varepsilon}{s}\frac{1}{(R+r+\tau)^s}.
\end{aligned}$$

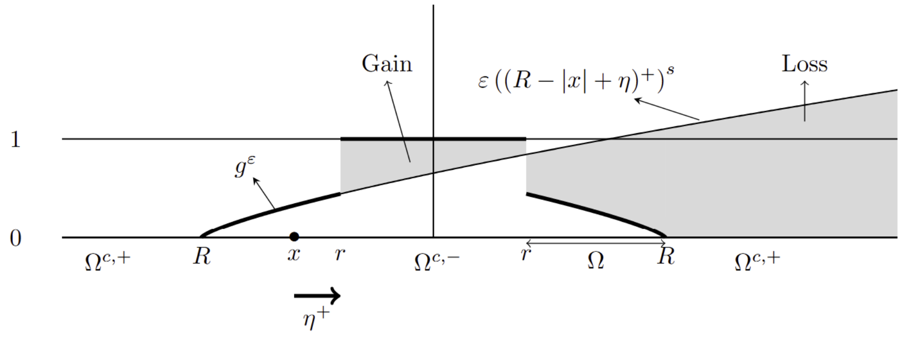

FIGURE 5.3: Gain and loss for $\Delta^s_\infty g^\varepsilon - K$

Also,

$$\begin{aligned}\int_{\varepsilon^{-1/s}+2r+\tau}^{\infty} \frac{\varepsilon\left(\eta-(2r+\tau)\right)^s-1}{\eta^{1+2s}}d\eta &< \int_{\varepsilon^{-1/s}+2r+\tau}^{\infty} \frac{\varepsilon\eta^s-1}{\eta^{1+2s}}d\eta\\ &= \int_{\varepsilon^{-1/s}+2r+\tau}^{\infty} \frac{\varepsilon}{\eta^{1+s}}-\frac{1}{\eta^{1+2s}}d\eta\\ &= \frac{\varepsilon}{s}\frac{1}{(\varepsilon^{-1/s}+2r+\tau)^s}-\frac{1}{2s}\frac{1}{(\varepsilon^{-1/s}+2r+\tau)^{2s}}\end{aligned}$$

Putting both together,

$$\begin{aligned}\int_{R-r+\tau}^{\varepsilon^{-1/s}+2r+\tau} \frac{1-\varepsilon\left(\eta-(2r+\tau)\right)^s}{\eta^{1+2s}}d\eta - \int_{\varepsilon^{-1/s}+2r+\tau}^{\infty} \frac{\varepsilon\left(\eta-(2r+\tau)\right)^s-1}{\eta^{1+2s}}d\eta&\\ > \frac{1}{2s}\frac{1}{(R+r+\tau)^{2s}}-\frac{\varepsilon}{s}\frac{1}{(R+r+\tau)^s}&\\ > \frac{1}{2s}\frac{1}{(2R)^{2s}}-\frac{\varepsilon}{s}\frac{1}{(R+r)^s}&.\end{aligned}$$

Taking $\varepsilon>0$ such that,

$$\frac{1}{2s}\frac{1}{(2R)^{2s}} > \frac{\varepsilon}{s}\frac{1}{(R+r)^s} \tag{5.11}$$

we get

$$\int_{R-r+\tau}^{\varepsilon^{-1/s}+2r+\tau} \frac{1-\varepsilon\left(\eta-(2r+\tau)\right)^s}{\eta^{1+2s}}d\eta - \int_{\varepsilon^{-1/s}+2r+\tau}^{\infty} \frac{\varepsilon\left(\eta-(2r+\tau)\right)^s-1}{\eta^{1+2s}}d\eta > 0 \tag{5.12}$$

Finally, taking $\varepsilon>0$ satisfying (5.10) and (5.11), and with $\varepsilon<\varepsilon_0$, where ε_0 is as in assumption (iii$_{\mathcal{A}}$) so that $f_\varepsilon<u_0$, we conclude that $f_\varepsilon\in\mathcal{F}$. Therefore,

$$\varepsilon d^s(x,\partial\Omega^-)\le U(x,t) \quad \forall (x,t)\in\Omega\times(0,T).$$

Now we move on to proving that f^ε is a supersolution. Keeping $x\in\Omega$ fixed, we maintain the notation of τ and τ', and denote

$$v := \frac{\nabla f^\varepsilon(x)}{|\nabla f^\varepsilon(x)|} = \frac{-\nabla d^s(x,\partial\Omega^+)}{|\nabla d^s(x,\partial\Omega^+)|} = -\frac{x}{|x|}.$$

Once again, we use $\Delta^s(\eta^+)^s=0$ for $\eta>0$, it gives,

$$0=\int_0^\infty \frac{((R-|x|+\eta)^+)^s+((R-|x|-\eta)^+)^s-2((R-|x|)^+)^s}{|\eta|^{1+2s}}\,d\eta.$$

We denote,

$$K:=\int_0^\infty \frac{((R-|x|+\eta)^+)^s+((R-|x|-\eta)^+)^s-2((R-|x|)^+)^s}{|\eta|^{1+2s}}\,d\eta.$$

Define the function,

$$g^{\varepsilon}(x) = \begin{cases} \varepsilon((R-|x|)^{+})^{s} & \text{if } x \in \overline{\Omega}, \\ 1 & \text{if } x \in \Omega^{c,-}, \\ 0 & \text{if } x \in \Omega^{c,+}. \end{cases}$$

Owing to (5.8), $g^{\varepsilon} = 1 - f^{\varepsilon}$. We prove that g_{ε} is a subsolution, which will imply that f^{ε} is a supersolution. Here we will estimate the difference between $\Delta_{\infty}^{s} g^{\varepsilon}$ and K. Taking ε such that $\varepsilon(R+2r)^{s} < 1$, we have,

$$\begin{aligned} \Delta_{\infty}^{s} g^{\varepsilon}(x) = & \int_{0}^{\tau} \frac{\varepsilon(\tau'+\eta)^{s} - \varepsilon\tau'^{s}}{|\eta|^{1+2s}}\, d\eta + \int_{\tau}^{2r+\tau} \frac{1-\varepsilon\tau'^{s}}{|\eta|^{1+2s}}\, d\eta + \int_{2r+\tau}^{R+r+\tau} \frac{\varepsilon(R+r+\tau-\eta)^{s} - \varepsilon\tau'^{s}}{|\eta|^{1+2s}}\, d\eta \\ & + \int_{R+r+\tau}^{\infty} \frac{-\varepsilon\tau'^{s}}{|\eta|^{1+2s}}\, d\eta + \int_{0}^{\tau'} \frac{\varepsilon(\tau'-\eta)^{s} - \varepsilon\tau'^{s}}{|\eta|^{1+2s}}\, d\eta + \int_{\tau'}^{\infty} \frac{-\varepsilon\tau'^{s}}{|\eta|^{1+2s}}\, d\eta \end{aligned}$$

and,

$$\begin{aligned} 0 = K = & \int_{0}^{\tau} \frac{\varepsilon(\tau'+\eta)^{s} - \varepsilon\tau'^{s}}{|\eta|^{1+2s}}\, d\eta + \int_{\tau}^{2r+\tau} \frac{\varepsilon(\tau'+\eta)^{s} - \varepsilon\tau'^{s}}{|\eta|^{1+2s}}\, d\eta + \int_{2r+\tau}^{R+r+\tau} \frac{\varepsilon(\tau'+\eta)^{s} - \varepsilon\tau'^{s}}{|\eta|^{1+2s}}\, d\eta \\ & + \int_{R+r+\tau}^{\infty} \frac{\varepsilon(\tau'+\eta)^{s} - \varepsilon\tau'^{s}}{|\eta|^{1+2s}}\, d\eta + \int_{0}^{\tau'} \frac{\varepsilon(\tau'-\eta)^{s} - \varepsilon\tau'^{s}}{|\eta|^{1+2s}}\, d\eta + \int_{\tau'}^{\infty} \frac{-\varepsilon\tau'^{s}}{|\eta|^{1+2s}}\, d\eta \end{aligned}$$

This yields,

$$\begin{aligned} \Delta_{\infty}^{s} g^{\varepsilon}(x) - K = & \int_{\tau}^{2r+\tau} \frac{1-\varepsilon(\tau'+\eta)^{s}}{|\eta|^{1+2s}}\, d\eta - \int_{2r+\tau}^{R+r+\tau} \frac{\varepsilon(\tau'+\eta)^{s} - \varepsilon(R+r+\tau-\eta)^{s}}{|\eta|^{1+2s}}\, d\eta \\ & - \int_{R+r+\tau}^{\infty} \frac{\varepsilon(\tau'+\eta)^{s}}{|\eta|^{1+2s}}\, d\eta \\ \geq & \int_{\tau}^{2r+\tau} \frac{1-\varepsilon(\tau'+\eta)^{s}}{|\eta|^{1+2s}}\, d\eta - \int_{2r+\tau}^{\infty} \frac{\varepsilon(\tau'+\eta)^{s}}{|\eta|^{1+2s}}\, d\eta. \end{aligned}$$

The first integral in the left hand side represents the gain when $1 = g^{\varepsilon} > ((R-|x|)^{+})^{s}$, and the second integral represents the loss when $g^{\varepsilon} < ((R-|x|)^{+})^{s}$. By (5.9),

$$\int_{\tau}^{2r+\tau} \frac{1-\varepsilon(\tau'+\eta)^{s}}{|\eta|^{1+2s}}\, d\eta - \int_{2r+\tau}^{\infty} \frac{\varepsilon(\tau'+\eta)^{s}}{|\eta|^{1+2s}}\, d\eta \geq -\frac{1}{2s}\left(\frac{1}{(2r+\tau)^{2s}} - \frac{1}{\tau^{2s}}\right) - \frac{\varepsilon}{s}\left(\frac{\tau+\tau'}{\tau}\right)^{s} \frac{1}{\tau^{s}}$$

Recalling that $(\tau+\tau') = (R-r)$. By taking ε such that $\varepsilon(R-r)^{s} < \frac{1}{2}\left[1-\left(\frac{R}{R+2r}\right)^{2s}\right]$ we get

$$\begin{aligned} -\frac{1}{2s}\left(\frac{1}{(2r+\tau)^{2s}} - \frac{1}{\tau^{2s}}\right) - \frac{\varepsilon}{s}\left(\frac{\tau+\tau'}{\tau}\right)^{s} \frac{1}{\tau^{s}} & = \frac{1}{\tau^{2s}}\left(\frac{1}{2s} - \frac{\varepsilon}{s}(R-r)^{s}\right) - \frac{1}{2s}\frac{1}{(2r+\tau)^{2s}} \\ & \geq \frac{1}{\tau^{2s}}\left(\frac{1}{2s} - \frac{1}{2s}\left[1-\left(\frac{R}{R+2r}\right)^{2s}\right]\right) - \frac{1}{2s}\frac{1}{(2r+\tau)^{2s}} \\ & = \frac{1}{\tau^{2s}}\left(\frac{1}{2s} + \left(\frac{R}{R+2r}\right)^{2s}\right) - \frac{1}{2s}\frac{1}{(2r+\tau)^{2s}} \\ & = \frac{1}{2s}\underbrace{\left(\frac{1}{\tau^{2s}} - \frac{1}{(2r+\tau)^{2s}}\right)}_{>0} + \frac{1}{\tau^{2s}}\left(\frac{R}{R+2r}\right)^{2s} \\ & > \frac{1}{R^{2s}}\left(\frac{R}{R+2r}\right)^{2s} \\ & = \frac{1}{(R+2r)^{2s}} > 0 \end{aligned}$$

Therefore,

$$\Delta_\infty^s g^\varepsilon(x) - K > 0$$

Since $K = 0$, we get

$$\Delta_\infty^s g^\varepsilon(x) > 0.$$

Hence $\Delta_\infty^s f^\varepsilon(x) < 0$. Finally, taking $\varepsilon < \varepsilon_0$ so $f^\varepsilon \geq u_0$ on Ω, we conclude that f^ε is a supersolution.

By the Comparison principle on bounded sets we get,

$$U(x,t) \leq 1 - \varepsilon d^s(x, \partial\Omega^+) \quad \forall (x,t) \in \Omega \times (0,T),$$

and taking ε sufficiently small, we get the desired bound,

$$\varepsilon d^s(x, \partial\Omega^-) \leq U(x,t) \leq 1 - \varepsilon d^s(x, \partial\Omega^+) \quad \forall (x,t) \in \Omega \times (0,T).$$

□

5.3 Radial Uniform Monotonicity

Lemma 5.4. *There exists $\beta > 0$ such that $U(\cdot, t)$ is uniformly monotone in the radial direction with $\alpha = s+1$. Precisely, there exists $h_0 > 0$ such that, for all $x \in \Omega$,*

$$U(x,t) + \beta h^{1+s} \leq U\left(x + h\frac{x}{|x|}, t\right) \qquad h < \min\{d(x.\partial\Omega), h_0\}$$

for all $t \in (0,T)$.

Proof. The idea of this proof is analogous to that of Lemma 4.10, as we will see below, the major difference comes when estimating the gain and loss.

Fix $(x_0, t_0) \in \Omega \times (0,T)$, and consider a sequence of subsolutions $u^\delta \in \mathcal{F}$ such that $U(x_0,t_0) \leq u^\delta(x_0,t_0) + \delta$. Arguing as in 4.10, we can assume that u^δ is monotone in the radial direction and $u^\delta = 1$ on $\Omega^{c,+}$ and $u^\delta = 0$ on $\Omega^{c,-}$. Moreover, we assume that u^δ satisfies

$$\varepsilon d^s(x, \partial\Omega^-) \leq u^\delta(x,t) \leq 1 - \varepsilon d^s(x, \partial\Omega^+). \tag{5.13}$$

For all $\beta > 0$ and $h < d(x_0, \partial\Omega)$ define the function,

$$u_h^\delta(x,t) = \begin{cases} u^\delta(x,t), & \text{if } (x,t) \in \Omega \times (0,T) \text{ and } d(x, \partial\Omega^-) \leq h \\ u^\delta(x,t) \vee (u^\delta(x - h\frac{x}{|x|}, t) + \beta h^{1+s}), & \text{if } (x,t) \in \Omega \times (0,T) \text{ and } d(x, \partial\Omega^-) > h \\ 1, & \text{if } (x,t) \in \Omega^{c,+} \times (0,T), \\ 0, & \text{if } (x,t) \in \Omega^{c,-} \times (0,T). \end{cases}$$

We prove that there exists $\beta > 0$ such that, for small h, u_h^δ is a subsolution of $\mathcal{P}$, which would imply the result.

It suffices to prove $\partial_t \tilde{u}_h^\delta(x,t) - \Delta_\infty^s \tilde{u}_h^\delta(x,t) \leq 0$ in the case $u^\delta(x - he_1, t) + \beta h^{1+s} > u^\delta(x,t)$. For this we use

$$\partial_t\left(\tilde{u}^\delta\left(x - h\frac{x}{|x|}, t\right) + \beta h^{1+s}\right) - \Delta_\infty^s\left(\tilde{u}^\delta\left(x - h\frac{x}{|x|}, t\right) + \beta h^{1+s}\right) \leq 0, \tag{5.14}$$

and

$$\partial_t\left(\tilde{u}^\delta\left(x - h\frac{x}{|x|}, t\right) + \beta h^{1+s}\right) = \partial_t \tilde{u}_h^\delta(x,t) \tag{5.15}$$

to prove $\Delta_\infty^s \tilde{u}_h^\delta(x,t) - \Delta_\infty^s \tilde{u}^\delta(x - he_1, t) + \beta h^{1+s} \geq 0$.

In this domain, the gain and loss appear because of the change in value of u_h^δ in Ω^c, and the growth and decay estimate (5.13). But, we have more terms that will represent them, this is due to the fact that that when we evaluate the functions in the operator, we will consider a line that will give us the "worst" case, this line goes through the origin and passes through Ω again. This would be the worst case because we would have the most amount of loss possible that comes from the change of value in $\Omega^{c,-}$.

Let ϕ be a test function that touches $u^\delta(\cdot - \frac{x}{|x|}h, \cdot) + \beta h^{1+s}$ from above at (x,t). Note that $\phi(\cdot,\cdot) - \beta h^{1+s}$ touches u^δ from above at $(x - \frac{x}{|x|}h, t)$. Then, by the monotonicity of u^δ in the radial direction we have $0 \leq v \cdot \frac{x}{|x|}$, where v is the direction of $\nabla\phi(x,t)$.

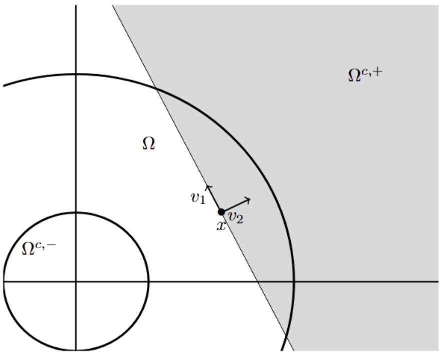

FIGURE 5.4: Possible directions for $\nabla\phi \neq 0$

Consider first the case $\nabla\phi(x) \neq 0$. Here, we divide the estimation of the gain and loss in two cases (Figure 5.4). The first case is when the line with direction v does not go through $\Omega^{c,-}$, here we have an analogous case to that of the strip, since there are two terms that represent the gain and two that represent the loss, with the difference being that both gain and loss come from the change of value in $\Omega^{c,+}$ and the behavior of u_h^δ near $\partial\Omega^+$. What happens in $\Omega^{c,-}$ and near $\partial\Omega^-$ is not considered in this case since the line in which we are integrating does not go through $\Omega^{c,-}$. Because of this similarity, the expressions that represent the gain and loss will be the same as in the proof of Lemma 4.10, so we can replicate that bound and conclude the result.

The second case is when the line goes through $\Omega^{c,-}$. Here we consider the worst case, just like we did in Lemma 4.10. This worst case happens when the line in which we are integrating goes through the origin, since in this line we would have the most amount of loss possible from the change of u_h^δ on $\Omega^{c,-}$, while the gain from the growth on $\partial\Omega^-$ would be smaller (Figure 5.5). We estimate this quantities following the same ideas outlined in 4.3.

Let $\ell > 0$ be the distance between x and $\{z : d(z, \partial\Omega^-) \leq h\}$ along the line that goes through x and the origin, and let $\ell' > 0$ be the distance between x and $\partial\Omega^+$ along this same line. Then,

$$\ell + h, \ell' + h < R - r. \tag{5.16}$$

Using (5.13) we estimate,

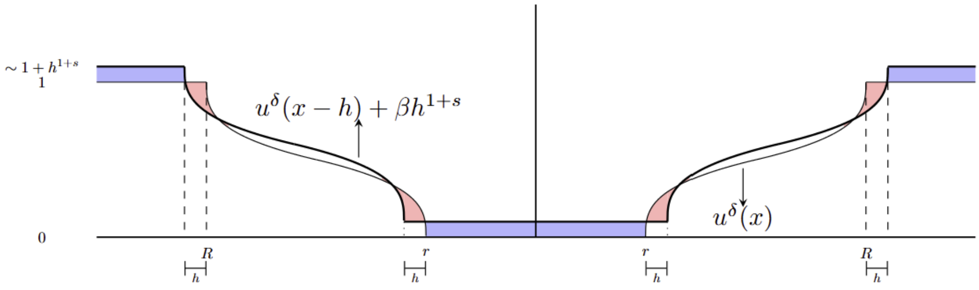

FIGURE 5.5: Representation of the gain and loss in Ω.

$$\begin{aligned}
\Delta^s_\infty \tilde{u}^\delta_h(x,t) \geq{}& \Delta^s_\infty \tilde{u}^\delta\left(x-\frac{x}{|x|}h,t\right) - \underbrace{\int_{-(\ell+h+2r)}^{-(\ell+h)} \frac{\beta h^{1+s}}{|\eta|^{1+2s}}\,d\eta}_{I} - \underbrace{\int_{\ell'+h}^{\infty} \frac{\beta h^{1+s}}{|\eta|^{1+2s}}\,d\eta}_{II} \\
& - \underbrace{\int_{-\infty}^{-(\ell+2h+r+R)} \frac{\beta h^{1+s}}{|\eta|^{1+2s}}\,d\eta}_{III} \\
& + \underbrace{\int_{-\ell-h}^{-\ell} \frac{\varepsilon(\eta+\ell+h)^s-\beta h^{1+s}}{|\eta|^{1+2s}}\,d\eta}_{IV} + \underbrace{\int_{\ell'}^{\ell'+h} \frac{\varepsilon(\ell'+h-\eta)^s-\beta h^{1+s}}{|\eta|^{1+2s}}\,d\eta}_{V} \\
& + \underbrace{\int_{2r+\ell+h}^{2r+\ell+2h} \frac{\varepsilon(\eta-(\ell+h+2r))^s-\beta h^{1+s}}{|\eta|^{1+2s}}\,d\eta}_{VI} \\
& + \underbrace{\int_{R+r+\ell+h}^{R+r+\ell+2h} \frac{\varepsilon(R+r+\ell+2h-\eta)^s-\beta h^{1+s}}{|\eta|^{1+2s}}\,d\eta}_{VII}.
\end{aligned}$$

Here, the first three integrals on the right hand side represent the loss, and the rest represent the gain. Precisely, the first is the change in $\Omega^{c,-}$, the second and third is the change $\Omega^{c,+}$, fourth and fifth come from estimate (5.13) near $\partial\Omega^-$ and the two remaining come from estimate (5.13) near $\partial\Omega^+$. We have,

$$\begin{aligned}
I &= \frac{-\beta h^{1+s}}{2s}\left(\frac{1}{(\ell+h+2r)^{2s}}-\frac{1}{(\ell+h)^{2s}}\right) \\
II &= \frac{\beta h^{1+s}}{2s}\frac{1}{(\ell'+h)^{2s}} \\
III &= \frac{\beta h^{1+s}}{2s}\frac{1}{(\ell+2h+R+r)^{2s}} \\
IV &\geq \frac{1}{(\ell+h)^{1+2s}}\int_{-\ell-h}^{-\ell} \varepsilon(\eta+\ell+h)^s-\beta h^{1+s}\,d\eta = \frac{\left(\frac{\varepsilon}{1+s}h^{s+1}-\beta h^{2+s}\right)}{(\ell+h)^{1+2s}} \\
V &\geq \frac{1}{(\ell'+h)^{1+2s}}\left(\frac{\varepsilon}{1+s}h^{s+1}-\beta h^{2+s}\right) \\
VI &\geq \frac{1}{(R+r+\ell+2h)^{1+2s}}\left(\frac{\varepsilon}{1+s}h^{s+1}-\beta h^{2+s}\right) \\
VII &\geq \frac{1}{(R+r+\ell+2h)^{1+2s}}\left(\frac{\varepsilon}{1+s}h^{s+1}-\beta h^{2+s}\right)
\end{aligned}$$

Putting everything together,

$$\begin{aligned}
\Delta^s_\infty \tilde{u}^\delta_h(x,t) \geq{}& \frac{-\beta h^{1+s}}{2s}\left(\frac{1}{(\ell+h)^{2s}}+\frac{1}{(\ell'+h)^{2s}}+\frac{1}{(\ell+2h+R+r)^{2s}}\right) \\
& + \left(\frac{\varepsilon}{1+s}h^{s+1}-\beta h^{2+s}\right)\left(\frac{1}{(\ell+h)^{1+2s}}+\frac{1}{(\ell'+h)^{1+2s}}+\frac{2}{(R+r+\ell+2h)^{1+2s}}\right) \\
\geq{}& \frac{h^{s+1}}{(\ell+h)^{2s}}\left(\frac{\varepsilon}{s+1}\frac{1}{\ell+h}-\frac{\beta h}{\ell+h}-\frac{\beta}{2s}\right) \\
& + \frac{h^{s+1}}{(\ell'+h)^{2s}}\left(\frac{\varepsilon}{s+1}\frac{1}{\ell'+h}-\frac{\beta h}{\ell'+h}-\frac{\beta}{2s}\right) \\
& + \frac{h^{s+1}}{(R+r+\ell+2h)^{2s}}\left(\frac{\varepsilon}{s+1}\frac{2}{R+r+\ell+2h}-\frac{2\beta h}{R+r+\ell+2h}-\frac{\beta}{2s}\right)
\end{aligned}$$

Note that, since $h < d(x_0, \partial\Omega) = \min\{R - |x_0|, |x_0| - r\}$, then $2h < R - r$. Moreover, $R + r + \ell + 2h \leq 3R + R - r = 4R - r$, therefore,

$$\begin{aligned}\Delta^s_\infty \tilde{u}^\delta_h(x,t) \geq{}& \frac{h^{s+1}}{(4R-r)^{2s}}\left(\frac{\varepsilon}{s+1}\frac{1}{4R-r} - \frac{\beta h}{\ell + h} - \frac{\beta}{2s}\right)\\ &+ \frac{h^{s+1}}{(4R-r)^{2s}}\left(\frac{\varepsilon}{s+1}\frac{1}{4R-r} - \frac{\beta h}{\ell' + h} - \frac{\beta}{2s}\right)\\ &+ \frac{h^{s+1}}{(4R-r)^{2s}}\left(\frac{\varepsilon}{s+1}\frac{2}{4R-r} - \frac{2\beta h}{R + r + \ell + 2h} - \frac{\beta}{2s}\right)\end{aligned}$$

Since $\frac{h}{\ell+h}, \frac{h}{\ell'+h}, \frac{h}{R+r+\ell+2h} < 1$, we get

$$\begin{aligned}\Delta^s_\infty \tilde{u}^\delta_h(x,t) &\geq 3\frac{h^{s+1}}{(4R-r)^{2s}}\left(\frac{\varepsilon}{s+1}\frac{1}{4R-r} - \beta - \frac{\beta}{2s}\right)\\ &= 3\frac{h^{s+1}}{(4R-r)^{2s}}\left(\frac{\varepsilon}{s+1}\frac{1}{4R-r} - \beta\frac{2s+1}{2s}\right)\end{aligned}$$

Hence, taking $\beta > 0$ tal such that

$$\frac{\varepsilon}{s+1}\frac{1}{4R-r}\frac{2s}{2s+1} > \beta$$

We get $\Delta^s_\infty \tilde{u}^\delta_h(x,t) > 0$. Furthermore, taking $\beta < \min\{\beta_0, \frac{\varepsilon}{s+1}\frac{1}{4R-r}\frac{2s}{2s+1}\}$ we can ensure $u^\delta_h(x,0) \leq u_0(x)$ on $\mathbb{R}^N$, and conclude that u^δ_h is a subsolution, which proves the result when $\nabla\phi(x,t) \neq 0$.

An analogous argument holds when $\nabla\phi\,(x,t) = 0$. Indeed, if the supremum in the definition of $\Delta^s_\infty \tilde{u}^\delta_h$ is reached at $y \in S^{N-1}$, then $0 \leq y \cdot \frac{x}{|x|}$, as a consequence of the monotony of u^δ in the radial direction. Analogously, if $z \in S^{N-1}$ is the direction for the infimum, then $z \cdot \frac{x}{|x|} \leq 0$. Let ℓ' be the distance between x and $\partial\Omega^+$ along the line with direction y, and let ℓ be the distance between x and $\{z : d\,(z, \partial\Omega^-) \leq h\}$ along the line with direction z. Then (5.16) still holds and we can apply the same argument as the previous case to obtain $\Delta^s_\infty \tilde{u}^\delta_h(x,t) \geq 0$, and conclude in the same way. ☐

5.4 Existence and Uniqueness

Now that we have that U is uniformly monotone on Ω, Corollary 4.12 holds in this domain. On the other hand, Theorem 4.15 is not domain dependent, so it still holds in this setting. Moreover, note that w, as defined in Lemma 4.13, is a subsolution in this setting for any choice of $x_0 \in \partial\Omega^+$. Therefore, since these were the main results used in the proof of Proposition 4.16, we get that U is a subsolution in an analogous way that we did in the first setting.

Furthermore, we know that $U = g$ on $\Omega \times (0,T)$ and arguing in an analogous way as Remark 4.3, we obtain that U satisfies the exterior and initial conditions of $\mathcal{P}_\mathcal{A}$.

Lemma 5.5. *U is a subsolution of $\mathcal{P}_\mathcal{A}$. Furthermore, $U(\cdot,t) \in C^{0,2s-1}(\mathbb{R}^N)$ for all $t \in (0,T)$, and for any $\mathcal{L} > ||\Delta^s_\infty u_0||_\infty$,*

$$|U(x,t) - U(x_0,t_0)| \leq \mathcal{L}|t - t_0| + m_0^{-\alpha_0}|x - x_0|^{\alpha_0}. \tag{5.17}$$

for all $(x,t), (x_0,t_0) \in \Omega \times (0,T)$.

Proof. The proof is analogous to that of Proposition 4.16. ☐

Finally, we can replicate exactly the proof of Theorem 4.17, to obtain that U is a solution, while the uniqueness follows immediately from the Comparison Principle, Theorem 3.8. This way, the following result is proven.

Theorem 5.6. *U is the only solution of $\mathcal{P}_\mathcal{A}$.*

Appendix A

Viscosity Solutions

Viscosity solutions were first introduced in the 1980's by Crandall and Lions [16]. The term "viscosity solutions" originates from the "vanishing viscosity method", but it is not necessarily related to this method. Viscosity solutions constitute a general theory of "weak" (i.e. non-differentiable) solutions which applies to certain fully nonlinear Partial Differential Equations (PDE) of 1st and 2nd order.

Consider the PDE

$$F\left(\cdot, u, Du, D^2u\right) = 0$$

where

$$F : \Omega \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{S}_N \to \mathbb{R}$$

and $\mathbb{S}_N$ denotes the set of symmetric $N \times N$ matrices. The idea behind Viscosity Solutions is to use the maximum principle in order to "pass derivatives to smooth test functions". This idea allows us to consider operators in non divergence form. We will assume that F is degenerate elliptic, that is, F satisfy

$$X \leq Y \text{ in } \mathbb{S}_N \Longrightarrow F(x,r,p,X) \geq F(x,r,p,Y)$$

for all $(x,r,p) \in \Omega \times \mathbb{R} \times \mathbb{R}^N$. Now, let us motivate the definition of viscosity solution. Suppose that $u \in C^2(\Omega)$ is a classical solution of the PDE

$$F\left(x, u(x), Du(x), D^2u(x)\right) = 0, \quad x \in \Omega.$$

Assume further that at some $x_0 \in \Omega, u$ can be "touched from above" by some smooth function $\psi \in C^2\left(\mathbb{R}^N\right)$ at x_0. That is

$$\psi - u \geq 0 = (\psi - u)(x_0)$$

on a ball $B_r(x_0)$. Since $\psi - u$ attains a minimum at x_0 we have

$$D(\psi - u)(x_0) = 0 \quad \text{and} \quad D^2(\psi - u)(x_0) \leq 0.$$

By using that u is a solution and the ellipticity of F, we obtain

$$0 = F\left(x_0, u(x_0), Du(x_0), D^2u(x_0)\right) \geq F\left(x_0, \psi(x_0), D\psi(x_0), D^2\psi(x_0)\right).$$

We have proved that if u is a solution to the equation and ψ "touches from above" u then

$$0 \geq F\left(x_0, \psi(x_0), D\psi(x_0), D^2\psi(x_0)\right).$$

Analogously, it can be seen that if ϕ "touches from below" u then

$$0 \leq F\left(x_0, \phi(x_0), D\phi(x_0), D^2\phi(x_0)\right).$$

Now, with this result in mind, we are ready to give the definition of viscosity solution to the equation

$$F\left(\cdot, u, \nabla u, D^2u\right) = 0. \tag{A.1}$$

Definition A.1. A lower semi-continuous function u is a viscosity supersolution of (A.1) if for every $\phi \in C^2$ such that ϕ touches u at $x \in \Omega$ strictly from below (that is, $u - \phi$ has a strict minimum at x with $u(x) = \phi(x)$), we have

$$F\left(x, \phi(x), \nabla\phi(x), D^2\phi(x)\right) \geq 0.$$

An upper semi-continuous function u is a subsolution of (A.1) if for every $\psi \in C^2$ such that ψ touches u at $x \in \Omega$ strictly from above (that is, $u - \psi$ has a strict maximum at x with $u(x) = \psi(x)$), we have

$$F\left(x, \phi(x), \nabla\phi(x), D^2\phi(x)\right) \leq 0.$$

Finally, u is a viscosity solution of (A.1) if it is both a sub- and a supersolution.

Observe that we have required $u - \phi$ to have a strict minimum. We have done this since in general this is the definition that we use along the thesis. If we only require the difference to have a minimum we obtain an equivalent definition.

In general we assume that F is continuous, that is, for sequences $x_k \to x$ in Ω, $u_k \to u$ in $\mathbb{R}, \xi_k \to \xi$ in $\mathbb{R}^N$ and $M_k \to M$ in $\mathbb{S}_N$, we have

$$F\left(x_k, r_k, p_k, X_k\right) \to F(x, r, p, X) \quad \text{as } k \to \infty.$$

Although, discontinuous operators arise along the thesis and we are interested in operators as the homogeneous p-laplacian and the ∞-laplacian that are not defined when the gradient vanishes. In order to be able to handle these cases, we need to consider the lower semicontinous, F_*, and upper semicontinous, F^*, envelopes of F. These functions are given by

$$\begin{aligned} F^*(x, r, p, X) &= \limsup_{(y,s,w,Y)\to(x,r,p,X)} F(y, s, w, Y), \\ F_*(x, r, p, X) &= \liminf_{(y,s,w,Y)\to(x,r,p,X)} F(y, s, w, Y). \end{aligned}$$

These functions coincide with F at every point of continuity of F and are lower and upper semicontinous respectively.

Definition A.2. A lower semi-continuous function u is a viscosity supersolution of (A.1) if for every $\phi \in C^2$ such that ϕ touches u at $x \in \Omega$ strictly from below (that is, $u - \phi$ has a strict minimum at x with $u(x) = \phi(x)$), we have

$$F^*\left(x, \phi(x), \nabla\phi(x), D^2\phi(x)\right) \geq 0.$$

An upper semi-continuous function u is a subsolution of (A.1) if for every $\psi \in C^2$ such that ψ touches u at $x \in \Omega$ strictly from above (that is, $u - \psi$ has a strict maximum at x with $u(x) = \psi(x)$), we have

$$F_*\left(x, \phi(x), \nabla\phi(x), D^2\phi(x)\right) \leq 0$$

Finally, u is a viscosity solution of (A.1) if it is both a sub- and supersolution.

Here we have required supersolutions to be lower semi-continuous and subsolutions to be upper semi-continuous. To extend this concept we consider the lower semicontinuous envelope, u_*, and the upper semicontinuous envelope, u^*, of u, that is,

$$u_*(x) = \sup_{r>0} \inf_{y \in B_r(x)} u(y) \quad \text{and} \quad u^*(x) = \inf_{r>0} \sup_{y \in B_r(x)} u(y).$$

As stated before for F, these functions coincide with u at every point of continuity of u and are lower and upper semicontinous respectively. Now we give the more general definition of viscosity solution involving these functions.

Definition A.3. A function u is a viscosity supersolution of (A.1) if for every $\phi \in C^2$ such that ϕ touches u_* at $x \in \Omega$ strictly from below (that is, $u_* - \phi$ has a strict minimum at x with $u_*(x) = \phi(x)$), we have

$$F^*\left(x, \phi(x), \nabla\phi(x), D^2\phi(x)\right) \geq 0.$$

A function u is a subsolution of (A.1) if for every $\psi \in C^2$ such that ψ touches u^* at $x \in \Omega$ strictly from above (that is, $u^* - \psi$ has a strict maximum at x with $u^*(x) = \psi(x)$), we have

$$F_* \left(x, \phi(x), \nabla \phi(x), D^2 \phi(x)\right) \leq 0.$$

Finally, u is a viscosity solution of (A.1) if it is both a sub- and supersolution.

The definitions given above are going to be consider depending on the context (whether we are considering a continuous F or not, if u is continuous or not know a priori, etc). Another possible way to state the definition of viscosity solution, that we do not include here, is to define the Super-Jets and Sub-Jets, that play the role of the derivatives of u, and give later the definition of viscosity solution referring to them.

Appendix B

Convolutions

Definition B.1. Let $v \in C(\overline{\Omega} \times [0,T])$ and $\varepsilon, \kappa > 0$. Define

$$v_{\varepsilon,\kappa}(x,t) = \inf_{(y,s)\in\mathbb{R}^{N+1}} \left(v(y,s) + \frac{1}{2\varepsilon}|x-y|^2 + \frac{1}{2\kappa}|t-s|^2 \right),$$
$$v^{\varepsilon}(x,t) = \sup_{y\in\mathbb{R}^N} \left(v(y,t) - \frac{1}{2\varepsilon}|x-y|^2 \right).$$

The convolutions $v^{\varepsilon,\kappa}$ and v_{ε} are defined similarly.

The following result (Proposition 4.2 in [24]) gathers a series of known properties about the inf and sup convolutions.

Proposition B.2. *Assume that $u \in C(\bar{\Omega} \times [0,T])$, and let $\varepsilon, \kappa, \delta > 0$.*

1. *Both operations preserve both pointwise upper and lower bounds, i.e.,*

$$\inf u \le u_{\epsilon,\kappa} \le \sup u,$$
$$\inf u \le u^{\epsilon} \le \sup u,$$

where inf and sup are taken over $\Omega \times (0,T)$.

2. *Let $\epsilon^* = 2\sqrt{\epsilon\|u\|_\infty}, \kappa^* = 2\sqrt{\kappa\|u\|_\infty}, \Omega^{\epsilon^*} = \{x \in \Omega \mid d(x,\partial\Omega) > \epsilon^*\}$. For all $(x,t) \in \Omega^{\epsilon^*} \times (\kappa^*, T-\kappa^*)$, there exist $(y,s) \in \Omega \times (0,T)$ such that*

$$u_{\epsilon,\kappa}(x,t) = u(y,s) + \frac{1}{2\epsilon}|x-y|^2 + \frac{1}{2\kappa}|t-s|^2.$$

In other words, the sup and inf in the definition of the convolutions are achieved, provided we are at a sufficient distance from the boundary.

3. *Both $u_{\epsilon,\kappa}$ and $u^{\epsilon,\kappa}$ are Lipschitz continuous in x with constant $\frac{K}{\sqrt{\epsilon}}$, where $K = 2\|u\|_\infty$. That is,*

$$\sup_{\substack{x,y\in\Omega \\ t\in[0,T]}} \frac{|u(x,t)-u(y,t)|}{|x-y|} \le \frac{K}{\sqrt{\epsilon}}$$

Similarly, they are Lipschitz continuous in t with constant $\frac{K}{\sqrt{\kappa}}$.

4. *$u^{\epsilon,\kappa}, u_{\epsilon,\kappa} \to u$ uniformly as $\epsilon, \kappa \to 0$, and similarly for u^{ϵ}.*

5. *$u^{\epsilon,\kappa}, u_{\epsilon,\kappa}$ are respectively semiconvex and semiconcave. In particular, they are twice differentiable a.e. That is, there are measurable functions $a\colon \Omega \times [0,T] \to \mathbb{R}, q : \Omega \times [0,T] \to \mathbb{R}^n, M : \Omega \times [0,T] \to S(n)$ such that*

$$\begin{aligned} u^{\epsilon,\kappa}(y,s) =& u^{\epsilon,\kappa}(x,t) + a(x,t)(s-t) + \langle q(x,t), y-x\rangle \\ &+ \langle M(x,t)(y-x), y-x\rangle + o\left(|y-x|^2 + |s-t|\right) \end{aligned}$$

We will denote $a = (u^{\epsilon,\kappa})_t$, $q = Du^{\epsilon,\kappa}$, $M = D^2u^{\epsilon,\kappa}$ *for simplicity. The same goes for* $u_{\epsilon,\kappa}$.

6. *With the notation above,*

$$D^2u_{\epsilon,\kappa} \leq \frac{1}{\epsilon}I \quad \text{and} \quad D^2u^{\epsilon,\kappa} \geq -\frac{1}{\epsilon}I \quad \text{a.e. in } \Omega \times [0,T]$$

7. $(u_{\epsilon,\kappa})_\delta = u_{\epsilon+\delta,\kappa}$.

8. $(u_{\epsilon+\delta,\kappa})^\delta \leq u_{\epsilon,\kappa}$.

9. *The operation* $u \mapsto (u_\delta)^\delta$ *preserves semiconcavity, i.e., if* u *is* $\frac{1}{2\epsilon}$*-semiconcave, then* $(u_\delta)^\delta$ *is* $\frac{1}{2\epsilon}$*-semiconcave.*

Lemma B.3. *If* $u : \mathbb{R}^N \times (0,T) \to \mathbb{R}$ *is such that* $\partial_t u - \Delta^s_\infty u \leq f$ *on* $\Omega \times (0,T)$ *(in the sense of Definition 3.6), then*

$$\partial_t u^{\varepsilon,\kappa} - \Delta^s_\infty u^{\varepsilon,\kappa} \leq f + d_{\varepsilon,\kappa},$$

on $\Omega \times (0,T)$ *and, if* $w : \mathbb{R}^N \times (0,T) \to \mathbb{R}$ *is such that* $\partial_t w - \Delta^s_\infty w \geq f$ *on* $\Omega \times (0,T)$*, then,*

$$\partial_t w_{\varepsilon,\kappa} - \Delta^s_\infty w_{\varepsilon,\kappa} \geq f - d_{\varepsilon,\kappa},$$

on $\Omega \times (0,T)$*, where* $d_{\varepsilon,\kappa}$ *depends o* ω*, the modulus of continuity of* f.

Proof. Assume that u, w are sub and super solutions of $\mathcal{P}$ on $(x_0,t_0) \in \Omega \times (0,T)$ and let φ be a test function that touches $u^{\varepsilon,\kappa}$ from above at (x_0,t_0) on $B_r(x_0,t_0)$. Define $\psi(y,s) := \varphi(y + x_0 - t_0, s + t_0 - s_0)$, where $(y_0,s_0) \in \mathbb{R}^N \times (0,T)$ is such that

$$u^{\varepsilon,\kappa}(x_0,t_0) = u(y_0,s_0) - \frac{|x_0-y_0|^2}{2\varepsilon} - \frac{|t_0-s_0|^2}{2\kappa}$$

Then, $\psi \in C^1((0,T); C^{1,1}(y_0) \cap C(\overline{B_r(y_0)}))$, $\nabla\psi(y_0,s_0) = \nabla\varphi(x_0,t_0)$, $\partial_t\psi(y_0,s_0) = \partial_t\varphi(x_0,t_0)$ and $u - \psi$ has a local maximum at (y_0,s_0). Moreover,

$$\psi(y_0,s_0) = v(y_0,s_0), \quad \psi(y,s) > v(y,s), \quad \forall B_r(y_0,s_0) \setminus \{(y_0,s_0)\}$$

where $v(y) := u(y + x_o - t_0, s + t_0 - s_0)$ is a subsolution of $\mathcal{P}$ at (y_0,s_0), since u is a subsolution at (x_0,t_0).

It follows that,

$$\partial_t\tilde{v}(y_0,s_0) - \Delta^s_\infty\tilde{v}(y_0,s_0) \leq f(y_0,s_0) \Longrightarrow \partial_t\tilde{u}(x_0,t_0) - \Delta^s_\infty\tilde{u}(x_0,t_0) \leq f(y_0,s_0),$$

where $\tilde{v}$ and $\tilde{u}$ are defined as in Definition 3.6. On the other hand, we have

$$|f(x_0,t_0) - f(y_0,s_0)| \leq \omega(|x_0-y_0| + |s_0-t_0|)$$

where ω is the modulus of continuity of f, this yields,

$$\partial_t\varphi(x_0,t_0) - \Delta^s_\infty\varphi(x_0,t_0) \leq f(x_0,t_0) + \omega(|x_0-y_0| + |t_0-s_0|)$$

By Proposition B.2 we have $|x_0-y_0| \leq C\varepsilon^{1/2}$ and $|t_0-s_0| \leq C\kappa^{1/2}$, and since ω is monotone increasing,

$$\partial_t\varphi(x_0,t_0) - \Delta^s_\infty\varphi(x_0,t_0) \leq f(x_0,t_0) + \omega(C\varepsilon^{1/2} + C\kappa^{1/2})$$

then,

$$\partial_t u^{\varepsilon,\kappa}(x_0,t_0) - \Delta^s_\infty u^{\varepsilon,\kappa}(x_0,t_0) \leq f(x_0,t_0) + \omega(C\varepsilon^{1/2} + C\kappa^{1/2})$$

that is, $d_{\varepsilon,\kappa} = \omega(C\varepsilon^{1/2} + C\kappa^{1/2})$. The proof for supersolutions is analogous by changing the corresponding inequalities. □

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