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Nuevas transiciones entre diagramas de bifurcación bajo el lente de teoría de catástrofes y singularidades

Tesis presentada por:

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RESUMEN

En esta tesis investigamos la aparición y transición entre dos estructuras de bifurcación complejas, las bifurcaciones tipo *Mushroom* e *Isola*, en el contexto de sistemas dinámicos unidimensionales, discretos y suaves.

Utilizando herramientas de la Teoría de Bifurcaciones, la Teoría de Singularidades y los Sistemas Dinámicos, construimos y analizamos una familia polinomial de grado mínimo de la forma

$$x \mapsto x + x^3 - x^2 + \lambda^2 + \alpha,$$

la cual actúa como un despliegue universal capaz de exhibir tanto bifurcaciones tipo *Mushroom* como tipo *Isola*. Demostramos que, bajo perturbaciones suaves, cualquier sistema dinámico contacto equivalente a esta familia reproduce el mismo comportamiento cualitativo cerca de las regiones críticas del espacio de parámetros. Además, extendemos este análisis mediante la introducción de un despliegue de tres parámetros que organiza la transición entre tipos de bifurcaciones y proporciona una clasificación completa de la dinámica local.

ABSTRACT

In this thesis, we investigate the emergence and transition between two complex bifurcation structures, *Mushroom* and *Isola* bifurcations, in the context of smooth, one-dimensional discrete-time dynamical systems.

Using tools from Bifurcation Theory, Singularity Theory, and Dynamical Systems, we construct and analyze a minimal-degree polynomial family of the form

$$x \mapsto x + x^3 - x^2 + \lambda^2 + \alpha,$$

which serves as a universal unfolding capable of exhibiting both Mushroom and Isola bifurcations. We prove that under smooth perturbations, any dynamical system contact-equivalent to this family will reproduce the same qualitative behavior near critical parameter regions. Furthermore, we extend this analysis by introducing a three-parameter unfolding that organizes the transition between bifurcation types and provides a complete classification of the local dynamics.

NOTATION

In this page we present some important notation used in this thesis.

1. \mathbb{R}^n : This refers to the n th-dimensional Euclidean space, meaning that $x \in \mathbb{R}^n$ is a vector with n real components.
2. $M^{n \times n}(\mathbb{R})$: The space of $n \times n$ real matrices.
3. *Germ equivalence*: Let $G : V \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$, $H : U \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$, where V, U are vicinities of the origin. We say that G is germ equivalent to H if there exists $W \subset U \cap V$ such that $H|_W \equiv G|_W$.
4. $G : (\mathbb{R}^n; 0) \rightarrow \mathbb{R}$: Equivalence class of G under germ equivalence.
5. $C_{x,\lambda}^\infty$: Set of equivalence classes of C^∞ functions $G : V \subset \mathbb{R}^{1+1} \rightarrow \mathbb{R}$ under germ equivalence. This space can be seen as either a local ring, a module over $C_{x,\lambda}^\infty$, or an \mathbb{R} -vector space.
6. C_λ^∞ : Set of equivalence classes of C^∞ functions $G : U \subset \mathbb{R} \rightarrow \mathbb{R}$ under germ equivalence. This space naturally embeds itself into $C_{x,\lambda}$.

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Chapter 1

Introduction

In the context of biomathematical models, a concept known as *multistability* may arise. This refers to the existence of multiple asymptotically stable states for the same configuration of parameters. Multistability is often associated with decision-making events, such as cell differentiation [Giri and Kar \(2021\)](#), or with oscillatory behavior [Gray and Scott \(1985\)](#). This phenomenon is typically linked to bifurcations, which refer to sudden qualitative changes in the structure of a dynamical system.

Among these complex bifurcation structures are the so-called *Mushroom* and *Isola* bifurcations. These occur in systems where multiple fold bifurcations interact in specific ways, giving rise to regions of bistability enclosed by loops or islands in the bifurcation diagram. Such structures have been observed in diverse contexts including biological regulation, memory formation, and gene expression dynamics. More recently, they have also appeared in the study of planar population models [Xu et al. \(2023\)](#), where the Poincaré return map exhibits a transition from a Mushroom to an Isola bifurcation under certain conditions.

The aim of this thesis is to investigate this latter case: the emergence and transition between Mushroom and Isola bifurcations in smooth, one-dimensional discrete-time dynamical systems, interpreted as Poincaré return maps. By combining tools from Bifurcation Theory, Dynamical Systems [Kuznetsov \(1998\)](#), and Singularity Theory [Golubitsky and Schaeffer \(1979\)](#), we propose a universal family of mappings whose bifurcation diagrams capture these phenomena. Our goal is to provide a rigorous and general mathematical framework that explains how such bifurcation structures arise and how they transition into one another. This includes characterizing the pa-

parameter regions where each bifurcation type occurs, analyzing their stability properties, and studying the dynamics near critical bifurcation points.

This thesis is organized as follows: in Chapter 2, we introduce key results from bifurcation theory and real analysis. In Chapter 3, we propose a model for a discrete dynamical system whose bifurcation diagram exhibits the desired transition between a *Mushroom* and an *Isola* bifurcation. Chapter 4 introduces tools from singularity theory that are necessary to study our transition model. In Chapter 5, we classify all possible behaviors that a smooth perturbation of our model can exhibit. Finally, Chapter 6 generalizes the results obtained in Chapter 5 to a broader family of maps.

Chapter 2

Preliminaries from analysis and dynamical systems theory

In this chapter, we introduce the fundamental concepts from bifurcation theory and real analysis that will be used throughout this work. In particular, we present the notions of *normal form* and *bifurcation conditions*, which allow us to conclude the local topological equivalence between two systems by verifying a finite set of algebraic conditions.

2.1 Elements of dynamical systems and bifurcation theory

In the context of this work, we consider mappings $x \mapsto f(x, \mu)$, where $f \in C^\infty(\mathbb{R}^{1+1}, \mathbb{R})$. Here, $x \in \mathbb{R}$ is referred to as the *spatial variable*, and $\mu \in \mathbb{R}$ as the *parameter*. When denoting Euclidean spaces of the form \mathbb{R}^n , we adopt the convention \mathbb{R}^{i+j} , where i corresponds to the number of spatial variables and j to the number of parametric variables. If it becomes necessary to restrict the domain of f , this will be explicitly indicated.

Definition 2.1.1 (Fixed Point and Bifurcation Curve). *Let $f \in C^\infty(\mathbb{R}^{1+1}, \mathbb{R})$. A point $x^* \in \mathbb{R}$ is called a fixed point of the mapping $x \mapsto f(x, \lambda)$ at $\lambda = \lambda^*$ if*

$$f(x^*, \lambda^*) = x^*.$$

Moreover, the set

$$\text{Fix}(f) = \{(x, \lambda) \in \mathbb{R}^{1+1} \mid f(x, \lambda) = x\}$$

is called the bifurcation curve of f .

Definition 2.1.2 (Orbit). Let $f \in C^\infty(\mathbb{R}^{1+1}, \mathbb{R})$ and fix $\lambda \in \mathbb{R}$. Given $x \in \mathbb{R}$, the orbit of x under the mapping $f(\cdot, \lambda)$ is the sequence

$$\{f^{(n)}(x, \lambda)\}_{n \in \mathbb{N}},$$

where $f^{(n)}(\cdot, \lambda)$ denotes the n th composition of $f(\cdot, \lambda)$ with itself.

When studying a perturbation of a mapping $x \mapsto f(x, \lambda)$, for example, a mapping of the form $x \mapsto f(x, \lambda) + \epsilon\varphi(x, \lambda)$ for sufficiently small ϵ , it is natural to ask whether the orbits of both systems are organized in a topologically similar manner. In other words, we are interested in whether there exists a one-to-one, bicontinuous correspondence between the orbits of the two mappings. This notion is formalized through the concept of *topological equivalence*.

Definition 2.1.3 (Topological Equivalence). Let $f, g \in C^\infty(\mathbb{R}^{1+1}, \mathbb{R})$. We say that f and g are topologically equivalent if there exist mappings $h : \mathbb{R}^{1+1} \rightarrow \mathbb{R}$ and $\psi : \mathbb{R} \rightarrow \mathbb{R}$ such that:

- ψ is a homeomorphism,
- for each $\lambda \in \mathbb{R}$, the map $x \mapsto h(x, \lambda)$ is a homeomorphism, and
- the following identity holds for all $(x, \lambda) \in \mathbb{R}^{1+1}$:

$$h(f(x, \lambda), \lambda) = g(h(x, \lambda), \psi(\lambda)). \quad (2.1)$$

The map h is called a topological equivalence. If equation (2.1) only holds in neighborhoods $V_1 \times P_1 \ni (x, \lambda)$ and $V_2 \times P_2 \ni (y, \mu)$, with $h : V_1 \times P_1 \rightarrow V_2$ and $\psi : P_1 \rightarrow P_2$, then we say that f and g are locally topologically equivalent near (x, λ) and (y, μ) .

If two functions are topologically equivalent, then the orbits induced by both functions are essentially the same from a qualitative point of view. The notion of *topological equivalence* generalizes the concept of *topological conjugacy* to the case where the mappings depend on a set of parameters. While topological conjugacy allows us to

classify individual dynamical systems into topological equivalence classes, topological equivalence extends this idea to families of dynamical systems. This broader framework enables the study of structural properties of such families, including structural stability and the bifurcations that arise as parameters vary.

Given a map $f : \mathbb{R}^{1+1} \rightarrow \mathbb{R}$, one can ask how does the system behaves near a fixed point. To answer this type of questions usually one invokes a simpler function \tilde{f} defined in a vicinity of the fixed point such that f is locally topologically equivalent to \tilde{f} near this neighborhood. In this section we will discuss briefly some results of this kind.

Definition 2.1.4 (Hyperbolic Fixed Point). *Let λ be arbitrary but fixed. A fixed point $x^* \in \mathbb{R}$ of the mapping $x \mapsto f(x, \lambda)$ is called hyperbolic if*

$$\left| \frac{\partial f}{\partial x}(x^*, \lambda) \right| \neq 1.$$

Furthermore:

- If $\left| \frac{\partial f}{\partial x}(x^*, \lambda) \right| < 1$, we say that x^* is a stable hyperbolic fixed point.
- If $\left| \frac{\partial f}{\partial x}(x^*, \lambda) \right| > 1$, we say that x^* is an unstable hyperbolic fixed point.

The hyperbolicity of a fixed point allows one to understand how the system behaves near said fixed point through the Hartman-Grobman Theorem.

Theorem 2.1.1 (Hartman-Grobman). (*Perko (2013)*) *Let $f \in C^1(\mathbb{R}^{1+1}, \mathbb{R})$ and let $(x_0, \mu_0) \in \mathbb{R}^{1+1}$ be a hyperbolic fixed point of f , then f is locally topologically conjugate near (x_0, μ_0) to $\frac{\partial f}{\partial x}(x_0, \mu_0)$ near the origin.*

A consequence of theorem 2.1.1 is that the behaviour of the mapping $x \mapsto f(x, \mu)$ near its equilibrium (x_0, μ_0) does not change under small changes in the parameter μ . If the equilibrium (x_0, μ_0) is not hyperbolic then the system's behaviour can change under small perturbations of the parameter μ . To study these types of equilibrium we have to use tools from Bifurcation Theory.

Theorem 2.1.2 (Fold Bifurcation). (*Kuznetsov (1998)*) *Suppose that a one-dimensional system*

$$x \mapsto f(x, \mu), \quad x \in \mathbb{R}, \mu \in \mathbb{R}, \tag{2.2}$$

with $f \in C^2(\mathbb{R}^{1+1}, \mathbb{R})$, has a fixed point (x_0, μ_0) such that

$$B1) \frac{\partial f}{\partial x}(x_0, \mu_0) = 1.$$

$$G1) \frac{\partial^2 f}{\partial x^2}(x_0, \mu_0) \neq 0.$$

$$G2) \frac{\partial f}{\partial \mu}(x_0, \mu_0) \neq 0.$$

Then f is locally topologically equivalent near (x_0, μ_0) to one of the following mappings

$$\eta \mapsto \beta + \eta \pm \eta^2. \quad (2.3)$$

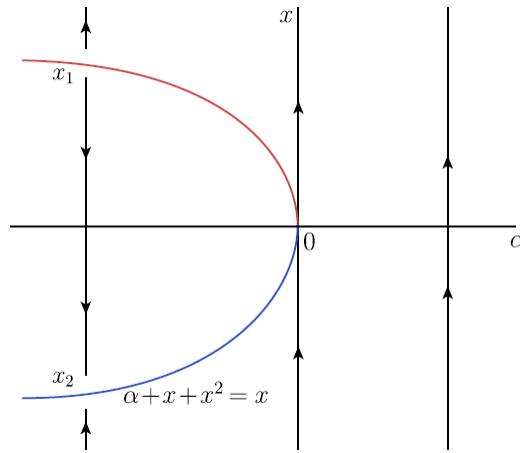


Figure 2.1. Bifurcation diagram of the mapping $x \mapsto \alpha + x + x^2$. (Kuznetsov (1998))

Observation 2.1.1. The mapping (2.3) will be called the normal form of the Fold bifurcation. In Kuznetsov (1998) and Guckenheimer and Holmes (2013) it is explained that a normal form of a bifurcation corresponds to a standard simplified expression that describes the local behaviour of a system near its bifurcation point, eliminating non essential terms to facilitate its analysis.

Theorem 2.1.3 (Transcritical Bifurcation). Suppose that a one-dimensional system

$$x \mapsto f(x, \mu), \quad x \in \mathbb{R}, \mu \in \mathbb{R}, \quad (2.4)$$

with $f \in C^2(\mathbb{R}^{1+1}, \mathbb{R})$, has a fixed point (x_0, μ_0) such that

$$B1) \frac{\partial f}{\partial x}(x_0, \mu_0) = 1.$$

$$B2) \frac{\partial f}{\partial \mu}(x_0, \mu_0) = 0.$$

$$G1) \frac{\partial^2 f}{\partial x^2}(x_0, \mu_0) \neq 0.$$

$$G2) \frac{\partial^2 f}{\partial \mu^2}(x_0, \mu_0) \frac{\partial^2 f}{\partial x^2}(x_0, \mu_0) < \frac{\partial^2 f}{\partial x \partial \mu}(x_0, \mu_0)^2.$$

Then (2.4) locally topologically equivalent near (x_0, μ_0) to one of the following normal forms

$$\eta \mapsto \eta \pm \eta^2 \mp \beta^2, \quad (2.5)$$

near the origin.

In figure 2.2 we can see the bifurcation diagram of the mapping (2.5). Although we were unable to find a complete proof of this result in the literature, it is mentioned in [Newhouse et al. \(1983\)](#) that this result is a corollary of a theorem stating that the action of a monotone mapping is equivalent to the action of the flow of a continuous dynamical system. Based on this idea, we constructed a proof of theorem 2.1.3 by leveraging the local monotonicity of the mapping.

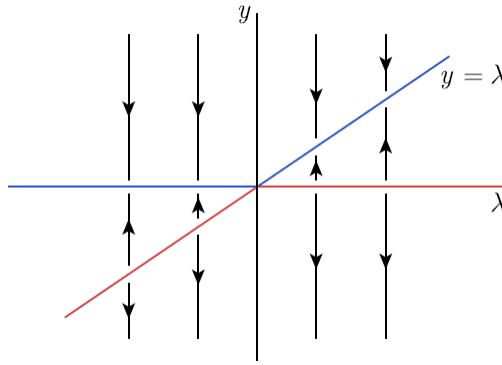


Figure 2.2. Bifurcation diagram of the mapping $x \mapsto x + \lambda x - x^2$, which is topologically equivalent to (2.5).

Proof. Let $f \in C^2(\mathbb{R}^{1+1}, \mathbb{R})$ satisfy the hypotheses of the theorem. We can always translate the fixed point (x_0, μ_0) to the origin, thus without loss of generality we may assume that $(x_0, \mu_0) = (0, 0)$. If we express f as its Taylor expansion with respect to x at $x = 0$ then we have

$$f(x, \mu) = f_0(\mu) + f_1(\mu)x + f_2(\mu)x^2 + \mathcal{O}(|x|^3). \quad (2.6)$$

The hypotheses over f translate into the next set of hypotheses over f_0 , f_1 and f_2 :

1. $f_0(0) = 0$.
2. $f_1(0) = 1$.

3. $f'_0(0) = 0$.
4. $f_2(0) \neq 0$.
5. $f''_0(0)f_2(0) < \frac{(f'_1(0))^2}{2}$.

We can define $g(\mu) = f_1(\mu) - 1$, where trivially $g(0) = 0$ and $g'(0) = f'_1(0)$. Using this we can express (2.6) as

$$f(x, \mu) = f_0(\mu) + x + g(\mu)x + f_2(\mu)x^2 + \mathcal{O}(|x|^3). \quad (2.7)$$

Let $h_1(x, \mu) = x + \delta(\mu)$, $\bar{x} := f(x, \mu)$, $\xi = x + \delta(\mu)$ and $\bar{\xi} = f(\xi - \delta(\mu), \mu) + \delta(\mu)$. Then, by expanding this last expression with respect to ξ , and expanding each coefficient with respect to $\delta(\mu)$ we obtain

$$\begin{aligned} \bar{\xi} &= (f_0(\mu) - g(\mu)\delta(\mu) + f_2(\mu)(\delta(\mu))^2 + \mathcal{O}(|\delta(\mu)|^3)) \\ &\quad + \xi + (g(\mu) - 2f_2(\mu)\delta(\mu) + \mathcal{O}(|\delta(\mu)|^2))\xi \\ &\quad + (f_2(\mu) + \mathcal{O}(\delta(\mu)))\xi^2 + \mathcal{O}(|\xi|^3). \end{aligned} \quad (2.8)$$

We can define the auxiliary function $\Phi(\mu, \delta) = g(\mu) - 2f_2(\mu)\delta + \phi(\mu, \delta)\delta^2$ where ϕ is smooth. Note that:

1. $\Phi(0, 0) = 0$.
2. $\frac{\partial \Phi}{\partial \delta}(0, 0) = -2f_2(0) \neq 0$.
3. $\frac{\partial \Phi}{\partial \mu}(0, 0) = g'(0)$.

Then by the implicit function theorem there exists a function $\delta(\mu)$ defined near the origin such that $\Phi(\mu, \delta(\mu)) = 0$ and

$$\delta(\mu) = \frac{g'(0)}{2f_2(0)}\mu + \mathcal{O}(|\mu|^2).$$

Let $\delta(\mu)$ be such function. Note that this implies that $h_1(x, \mu) = x + \delta(\mu)$ will be a bijection on x and thus the functions $f(x, \mu)$ and $f(\xi - \delta(\mu)) + \delta(\mu)$ are locally topologically equivalent near the origin.

Using these expressions on (2.8) we get that

$$\bar{\xi} = (f_0(\mu) - g(\mu)\delta(\mu) + f_2(\mu)(\delta(\mu))^2 + \mathcal{O}(|\mu|^3)) + \xi + (f_2(\mu) + \mathcal{O}(|\mu|))\xi^2 + \mathcal{O}(|\xi|^3). \quad (2.9)$$

If we define $\Gamma(\mu) = f_0(\mu) - g(\mu)\delta(\mu) + f_2(\mu)(\delta(\mu))^2 + \mathcal{O}(|\mu|^3)$, then it is easy to note that

1. $\Gamma(0) = 0$.
2. $\Gamma'(0) = 0$.
3. $\Gamma''(0) = f_0''(0) - \frac{(f_1'(0))^2}{2f_2(0)}$.

Then we are able to write (2.9) as

$$\bar{\xi} = (K + \mathcal{O}(|\mu|))\mu^2 + \xi + (f_2(\mu) + \mathcal{O}(|\mu|))\xi^2 + \mathcal{O}(|\xi|^3), \quad (2.10)$$

where

$$K = \frac{1}{2} \left(f_0''(0) - \frac{(f_1'(0))^2}{2f_2(0)} \right). \quad (2.11)$$

From our previous hypotheses we know that

$$f_0''(0)f_2(0) < \frac{(f_1'(0))^2}{2},$$

thus by multiplying (2.11) by $f_2(0)$, we either get:

- $f_0''(0) - \frac{(f_1'(0))^2}{2f_2(0)} < 0$ if $f_2(0) > 0$, or
- $f_0''(0) - \frac{(f_1'(0))^2}{2f_2(0)} > 0$ if $f_2(0) < 0$,

which implies that $\text{sgn}(K) = -\text{sgn}(f_2(0))$, thus the sign of the coefficients of ξ^2 and μ^2 in (2.10) have opposite signs and they are non zero. Let $\mu = \psi_1(\alpha) = \alpha + \gamma(\alpha)$, where

$$\gamma(\alpha) = -\alpha + \text{sgn}(\alpha) \sqrt{\alpha^2 - \frac{\alpha^2 \mathcal{O}(|\alpha|)}{K + \mathcal{O}(|\alpha|)}}.$$

Then ψ_1 is a bicontinuous bijection near the origin, then the map induced by the system (2.10) is locally topologically equivalent near the origin to

$$\bar{\xi} = K\alpha^2 + \xi + (f_2(0) + \mathcal{O}(|\alpha|))\xi^2 + \mathcal{O}(|\xi|^3). \quad (2.12)$$

Let $\eta = h_2(\xi, \alpha) = |f_2(0) + \mathcal{O}(|\alpha|)|\xi$, $\beta = \psi_2(\alpha) = \sqrt{|(f_2(0) + \mathcal{O}(|\alpha|))K|}\alpha$, then the map induced by system (2.12) will be locally topologically equivalent near the origin to

$$\bar{\eta} = -\text{sgn}(f_2(0))\beta^2 + \eta + \text{sgn}(f_2(0))\eta^2 + \mathcal{O}(|\eta|^3). \quad (2.13)$$

Depending on the sign of $f_2(0)$, the mapping (2.13) must be of the following form

$$\bar{\eta} = \mp\beta^2 + \eta \pm \eta^2 + \mathcal{O}(|\eta|^3).$$

where the sign of the term x^2 is the same as $f_2(0)$. The following lemma 2.1.1 we conclude the proof. \square

Lemma 2.1.1. *The system*

$$x \mapsto x \pm x^2 \mp \alpha^2 + O(|x|^3),$$

near the origin, is locally topologically equivalent to

$$x \mapsto x \pm x^2 \mp \alpha^2,$$

near the origin.

Proof. Without loss of generality, we can assume that $f(x, \alpha) = x - x^2 + \alpha^2$ and $\bar{f}(x, \alpha) = x - x^2 + \alpha^2 + \varphi(x)$, where φ is a C^2 function such that $\varphi(0) = \varphi'(0) = \varphi''(0) = 0$. The case with $x + x^2 - \alpha^2$ in the statement of the lemma is analogous.

Let $\epsilon > 0$ be such that

$$\bar{f}(x, \alpha) > 0 \quad \text{and} \quad \frac{\partial \bar{f}}{\partial x}(x, \alpha) > 0, \quad \forall x \in [-\epsilon, \epsilon] =: \bar{U}.$$

Note that $\nabla f(0, 0) = 0$ and the Hessian matrix of f is non-singular. Hence, by Morse's Lemma 2.2.1, there exists a diffeomorphism $j : A \rightarrow B$ with $A, B \subset \mathbb{R}^2$ neighborhoods of the origin such that

$$f \circ j(y_1, y_2) = y_1^2 - y_2^2, \quad (y_1, y_2) \in A. \quad (2.14)$$

Equation (2.14) yields two families of roots of $f \circ j$:

$$y_1^+(y_2) = y_2, \quad y_1^-(y_2) = -y_2,$$

which intersect transversally at the origin. These curves naturally define the curves

$$(\bar{x}_1(t), \alpha_1(t)) = j(y_1^+(t), t), \quad (\bar{x}_2(t), \alpha_2(t)) = j(y_1^-(t), t), \quad (2.15)$$

which also intersect transversally at $t = 0$, since j is a diffeomorphism, and satisfy that

$$f(\bar{x}_i(t), \alpha_i(t)) = 0. \quad (2.16)$$

Moreover, since $\frac{\partial f}{\partial x}(x, 0) = -2x + O(|x|^2)$ is locally linear, we know that

$$\frac{\partial(\bar{x}_i(t), \alpha_i(t))}{\partial t} \notin \langle (1, 0) \rangle,$$

which allows us to conclude that the curves defined in (2.11) intersect the axis $\{\alpha = 0\}$ transversally. Thus, without loss of generality, we obtain

$$\alpha_1(t), \alpha_2(t) < 0, \quad \text{if } t < 0,$$

$$\alpha_1(t), \alpha_2(t) > 0, \quad \text{if } t > 0,$$

and

$$\bar{x}_1(t) < x_2(t), \quad \text{if } t < 0,$$

$$\bar{x}_2(t) < x_1(t), \quad \text{if } t > 0.$$

$$\bar{x}_1(0) = \bar{x}_2(0) = 0.$$

Finally, since $\frac{\partial f}{\partial x}(x, \alpha) \neq 0$ for all $x \in \bar{U} - \{0\}$, the implicit function theorem 2.2.2 applied to equation (2.16) provides a parametrization of the curves in (2.15) in terms of α , meaning that we may assume that $\alpha_i(t) = t$. Therefore, we conclude that in the interval $\bar{U} := [-\epsilon, \epsilon]$, the mapping $x \mapsto \bar{f}(x, \alpha)$ has two fixed points that merge at $\alpha = 0$, which we can be parametrized as $\bar{x}_1 : \bar{V} \rightarrow \bar{U}$ and $\bar{x}_2 : \bar{V} \rightarrow \bar{U}$, where \bar{V} is a suitable neighborhood of the origin in the α -space such that $\bar{x}_i(\bar{V}) \subset \bar{U}$.

We can define analogous intervals U and V explicitly for the mapping $x \mapsto f(x, \alpha)$, choosing

$$U = [-1/4, 1/4] \quad \text{and} \quad V = [-1/4, 1/4] \cap \bar{V},$$

where, in this case, the fixed points of the mapping are given by $x_1 : V \rightarrow U$ and $x_2 : V \rightarrow U$, with

$$x_1(\alpha) = \alpha \quad \text{and} \quad x_2(\alpha) = -\alpha.$$

From these definitions, we will construct a continuous 1-parameter family of homeomorphism $h_\alpha : U \rightarrow \bar{U}$ such that

$$h_\alpha(f(x, \alpha)) = \bar{f}(h_\alpha(x), \alpha), \quad \forall (x, \alpha) \in U \times V.$$

Let $\alpha > 0$. Notice that both $x_1(\alpha)$ and $\bar{x}_1(\alpha)$ are unstable hyperbolic fixed points, while $x_2(\alpha)$ and $\bar{x}_2(\alpha)$ are stable hyperbolic fixed points. Using the fact that f and \bar{f} are invertible in U and \bar{U} respectively, as they are monotonous, it is then easy to verify the following:

- If $x \in U$ and $x < x_2(\alpha)$, then $f^{(n)}(x) \rightarrow x_2(\alpha)$ as $n \rightarrow \infty$.
- If $x \in U$ and $x_2(\alpha) < x < x_1(\alpha)$, then $f^{(n)}(x) \rightarrow x_2(\alpha)$ and $f^{(-n)}(x) \rightarrow x_1(\alpha)$ as $n \rightarrow \infty$.
- If $x \in U$ and $x > x_1(\alpha)$, then $f^{(-n)}(x) \rightarrow x_1(\alpha)$ as $n \rightarrow \infty$.

Analogously:

- If $x \in \bar{U}$ and $x < \bar{x}_2(\alpha)$, then $\bar{f}^{(n)}(x) \rightarrow \bar{x}_2(\alpha)$ as $n \rightarrow \infty$.
- If $x \in \bar{U}$ and $\bar{x}_2(\alpha) < x < \bar{x}_1(\alpha)$, then $\bar{f}^{(n)}(x) \rightarrow \bar{x}_2(\alpha)$ and $\bar{f}^{(-n)}(x) \rightarrow \bar{x}_1(\alpha)$ as $n \rightarrow \infty$.
- If $x \in \bar{U}$ and $x > \bar{x}_1(\alpha)$, then $\bar{f}^{(-n)}(x) \rightarrow \bar{x}_1(\alpha)$ as $n \rightarrow \infty$.

These observations, together with the invertibility of f and \bar{f} in U and \bar{U} respectively, yield the following results:

- If $x \in U$ and $x < x_2(\alpha)$, then the family of sets $\{[f^{(n)}(x), f^{(n+1)}(x)]\}_{n \in \mathbb{N}}$ forms a partition of the interval $(x, x_2(\alpha))$.
- If $x \in U$ and $x_2(\alpha) < x < x_1(\alpha)$, then the family of sets $\{[f^{(n+1)}(x), f^{(n)}(x)]\}_{n \in \mathbb{Z}}$ forms a partition of the interval $(x_2(\alpha), x_1(\alpha))$.
- If $x \in U$ and $x > x_1(\alpha)$, then the family of sets $\{[f^{(-n+1)}(x), f^{(-n)}(x)]\}_{n \in \mathbb{N}}$ forms a partition of the interval $(x_1(\alpha), x)$.

Analogously:

- If $x \in \bar{U}$ and $x < \bar{x}_2(\alpha)$, then the family of sets $\{[\bar{f}^{(n)}(x), \bar{f}^{(n+1)}(x)]\}_{n \in \mathbb{N}}$ forms a partition of the interval $(x, \bar{x}_2(\alpha))$.
- If $x \in \bar{U}$ and $\bar{x}_2(\alpha) < x < \bar{x}_1(\alpha)$, then the family of sets $\{[\bar{f}^{(n+1)}(x), \bar{f}^{(n)}(x)]\}_{n \in \mathbb{Z}}$ forms a partition of the interval $(\bar{x}_2(\alpha), \bar{x}_1(\alpha))$.
- If $x \in \bar{U}$ and $x > \bar{x}_1(\alpha)$, then the family of sets $\{[\bar{f}^{(-n+1)}(x), \bar{f}^{(-n)}(x)]\}_{n \in \mathbb{N}}$ forms a partition of the interval $(\bar{x}_1(\alpha), x)$.

We will now use these sets, together with the definition of topological equivalence, to define a homeomorphism $h : U \rightarrow \bar{U}$. Let $\xi_1(\alpha), \xi_2(\alpha), \xi_3(\alpha) \in U$ and $\bar{\xi}_1(\alpha), \bar{\xi}_2(\alpha), \bar{\xi}_3(\alpha) \in \bar{U}$ be such that:

- $\xi_1(\alpha) < x_1(\alpha)$, $x_2(\alpha) < \xi_3(\alpha)$ for all α ,
- $x_1(\alpha) < \xi_2(\alpha) < x_2(\alpha)$ if $\alpha < 0$,
- $x_2(\alpha) < \xi_2(\alpha) < \xi_1(\alpha)$ if $\alpha > 0$.

and

- $\bar{\xi}_1(\alpha) < \bar{x}_1(\alpha)$, $\bar{x}_2(\alpha) < \bar{\xi}_3(\alpha)$ for all α ,
- $\bar{x}_1(\alpha) < \bar{\xi}_2(\alpha) < \bar{x}_2(\alpha)$ if $\alpha < 0$,
- $\bar{x}_2(\alpha) < \bar{\xi}_2(\alpha) < \bar{\xi}_1(\alpha)$ if $\alpha > 0$.

This construction is illustrated in Figure 2.3.

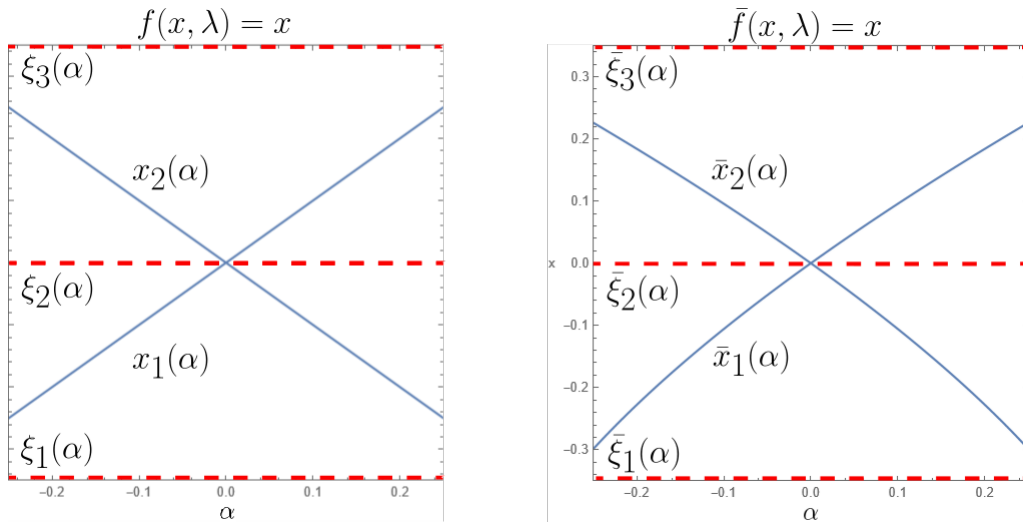


Figure 2.3. Configuration of the points ξ_i and $\bar{\xi}_i$ for $i = 1, 2, 3$ in relation with the fixed points x_i and \bar{x}_i for $i = 1, 2$, with respect to α .

Let $x \in (\xi_1, f(\xi_1))$, then there exists $t \in (0, 1)$ such that $x = t\xi_1 + (1 - t)f(\xi_1)$. Using this, we define:

$$\tilde{h}_\alpha(t\xi_1 + (1 - t)f(\xi_1)) = t\bar{\xi}_1 + (1 - t)\bar{f}(\bar{\xi}_1).$$

Analogously, we define:

$$\tilde{h}_\alpha(tf(\xi_2) + (1 - t)\xi_2) = t\bar{f}(\bar{\xi}_2) + (1 - t)\bar{\xi}_2,$$

$$\tilde{h}_\alpha(tf^{-1}(\xi_3) + (1 - t)\xi_3) = t\bar{f}^{-1}(\bar{\xi}_3) + (1 - t)\bar{\xi}_3.$$

And finally we define

$$\tilde{h}_\alpha(x_i) = \bar{x}_i, \quad i = 1, 2.$$

These intervals, along with the fixed points of our systems, will serve as anchor points for constructing a local topological equivalence. This is illustrated in figure 2.4

Let $x \in U \setminus \{x_1(\alpha), x_2(\alpha)\}$. Then, there are three possible cases:

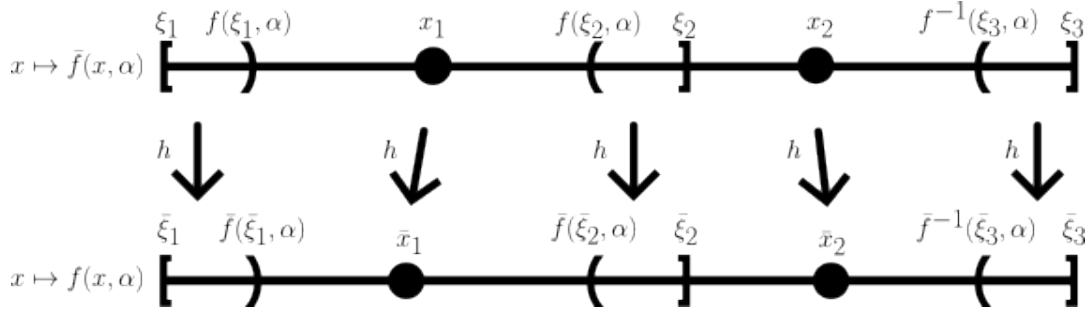


Figure 2.4. Mapping $h: U \rightarrow \bar{U}$.

- $x \in [f^{(n)}(\xi_1), f^{(n+1)}(\xi_1)]$ for some $n \in \mathbb{N}$.
- $x \in [f^{(n+1)}(\xi_2), f^{(n)}(\xi_2)]$ for some $n \in \mathbb{Z}$.
- $x \in (f^{(-n)}(\xi_3), f^{(-n+1)}(\xi_3)]$ for some $n \in \mathbb{N}$.

If $x \in [f^{(n)}(\xi_1), f^{(n+1)}(\xi_1)]$ for some $n \in \mathbb{N}$, then we define:

$$\bar{h}_\alpha(x) = \bar{f}^{(n)} \circ \bar{h}_\alpha \circ f^{(-n)}(x).$$

We can extend this definition to

$$h_\alpha(x) = \begin{cases} \bar{f}^{(n)} \circ \tilde{h} \circ f^{(-n)}, & [f^{(n)}(\xi_1), f^{(n+1)}(\xi_1)] \text{ for some } n \in \mathbb{N}, \\ \bar{f}^{(n)} \circ \tilde{h} \circ f^{(-n)}, & (f^{(n+1)}(\xi_2), f^{(n)}(\xi_2)] \text{ for some } n \in \mathbb{Z}, \\ \bar{f}^{(-n)} \circ \tilde{h} \circ f^{(n)}, & (f^{-(n+1)}(\xi_3), f^{(-n)}(\xi_3)] \text{ for some } n \in \mathbb{N}, \\ \bar{x}_1(\alpha), & x = x_1(\alpha), \\ \bar{x}_2(\alpha), & x = x_2(\alpha). \end{cases} \quad (2.17)$$

Notice that h_α is continuous because f, \bar{f} , and the auxiliary mapping \tilde{h} are continuous, and h_α trivially satisfies the topological equivalence condition.

Continuity with respect to α is ensured if the mappings $\alpha \mapsto \xi_i(\alpha)$ and $\alpha \mapsto \bar{\xi}_i(\alpha)$ are continuous for $i = 1, 2, 3$. This concludes the proof. \square

These results allow us to reduce the difficult problem of finding local topological equivalences to the verification of a finite set of algebraic conditions, whenever our map is undergoing a local bifurcation. In the following chapter, we will show that satisfying these conditions is invariant under *contact equivalence*, which will enable us to conclude the local topological equivalence of a family of mappings.

2.2 Results from Real Analysis

To prove certain results of this work, we will need to invoke certain theorems from real analysis.

Theorem 2.2.1 (Mean Value Theorem). (*Rudin (1976)*) Let $f : [a, b] \rightarrow \mathbb{R}$ be a function that is continuous on $[a, b]$ and differentiable on (a, b) . Then, there exists at least one point $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Theorem 2.2.2 (Implicit Function Theorem). (*Rudin (1976)*) Let $F : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$ be a continuously differentiable function, and let $(x_0, y_0) \in \mathbb{R}^n \times \mathbb{R}$ satisfy

$$F(x_0, y_0) = 0, \quad \text{and} \quad \frac{\partial F}{\partial y}(x_0, y_0) \neq 0.$$

Then, there exists an open neighborhood $U \subset \mathbb{R}^n$ of x_0 and a unique continuously differentiable function $g : U \rightarrow \mathbb{R}$ such that

$$F(x, g(x)) = 0, \quad \text{for all } x \in U, \quad \text{and} \quad g(x_0) = y_0.$$

Theorem 2.2.3 (Banach Fixed Point Theorem). (*Kreyszig (1978)*) Let (X, d) be a non-empty complete metric space, and let $T : X \rightarrow X$ be a contraction mapping; that is, there exists a constant $0 < \alpha < 1$ such that

$$d(T(x), T(y)) \leq \alpha d(x, y) \quad \text{for all } x, y \in X.$$

Then:

1. There exists a unique fixed point $x^* \in X$ such that $T(x^*) = x^*$.
2. For any $x_0 \in X$, the sequence defined by $x_{n+1} = T(x_n)$ converges to x^* as $n \rightarrow \infty$.
3. Moreover, the convergence is at least linear; more precisely,

$$d(x_n, x^*) \leq \frac{\alpha^n}{1 - \alpha} d(x_1, x_0).$$

Corollary 2.2.1. (*Kreyszig (1978)*) If $F : [a, b] \rightarrow [a, b]$ is a C^1 function such that

$$\left| \frac{\partial F}{\partial x}(c) \right| < \frac{1}{2}, \quad \forall c \in [a, b],$$

then there exists a unique fixed point $x^* \in [a, b]$ for the mapping F , and

$$F^{(n)}(y) \rightarrow x^*, \quad \text{as } n \rightarrow \infty, \quad \forall y \in [a, b].$$

Lemma 2.2.1 (Morse's lemma). ([Audin et al. \(2013\)](#)) Let $f : V \rightarrow \mathbb{R}$ where $V \subset \mathbb{R}^n$ is an open set. If $c \in V$ is a nondegenerate critical point of f , meaning that $(df)_0 = 0$, then there exists neighborhood U of c and a diffeomorphism $\varphi : U \rightarrow \mathbb{R}^n$ such that

$$f \circ \varphi^{-1}(x_1, x_2, \dots, x_n) = f(c) - \sum_{j=1}^i x_j^2 + \sum_{j=i+1}^n x_j^2,$$

where i is the amount of eigenvalues with negative real part, counting multiplicity, of the Hessian matrix of f at c .

Chapter 3

Singular case (Organization center)

In this chapter, we propose a 1-parameter family of mappings whose bifurcation curve acts as an organizing center for the transition between a Mushroom and an Isola bifurcation, in the sense that, under arbitrarily small perturbations, the system exhibits either of these behaviors.

We say that this family is a *normal form*, as it is the simplest polynomial mapping capable of exhibiting such a transition.

After introducing the family, we analyze its behavior using tools from bifurcation theory.

3.1 Family of normal form mappings for the singular case

To study the transition from a mushroom bifurcation to an isola bifurcation, we will first focus on the geometry of an intermediate case. For this we propose the 1-parameter mapping $x \mapsto P(x, \lambda)$ where

$$P(x, \lambda) = x + x^3 - x^2 + \lambda^2, \quad \forall (x, \lambda) \in \mathbb{R}^{1+1}. \quad (3.1)$$

The bifurcation diagram of P is shown in figure 3.1(b). This graphic exhibits a singularity at $(x, \lambda) = (0, 0)$ from where a loop is formed for $x > 0$, and two folds at $(x, \lambda) = \left(\frac{2}{3}, \pm \frac{2}{3\sqrt{3}}\right)$. From now on we will refer to this locus simply as a *loop*, as long as there is no confusion. When the mapping $x \mapsto P(x, \lambda)$ is perturbed by adding a

sufficiently small constant, the bifurcation diagram transitions into either a mushroom or an isola, depending on the sign of the perturbation; see panels (a) and (c) in Fig 3.1, respectively. This illustrates that the mapping (3.1) is an organization center for the transition between a mushroom and an isola. In what follows, we are going to focus our analysis to the case where $(x, \lambda) \in [-1.2, 1.2]^2$, as this range of values is sufficient to illustrate all the different behaviours that our system can exhibit, and most importantly it contains the connected locus that characterizes this bifurcation diagram.

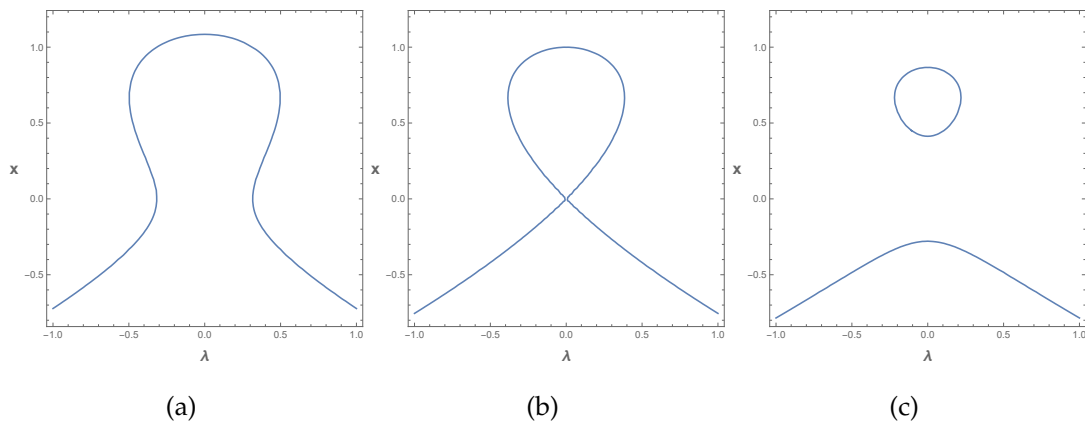


Figure 3.1. Bifurcation diagram of $x \mapsto P(x, \lambda) + \alpha$ for: $\alpha = -0.1$ in (a), $\alpha = 0$ in (b), and $\alpha = 0.1$ in (c).

Notably, the selection of an organization center is not unique. We selected the family (3.1) not only because it is a polynomial mapping but also because

$$\frac{\partial P}{\partial x}(x, \lambda) = 1 + 3x^2 - 2x \geq \frac{2}{3} > 0, \quad \forall x \in \mathbb{R}, \quad (3.2)$$

and thus the function $P(\cdot, \lambda)$ is invertible for every $\lambda \in \mathbb{R}$, allowing us to interpret it as a hypothetical Poincaré return map of a two-dimensional continuous dynamical system. Additionally, this polynomial family of mappings is minimal in degree, since the emergence of both Mushroom and Isola bifurcations requires the coexistence of at least three fixed points for some values of the parameter λ , which means that the equation

$$P(x, \lambda) = x,$$

must allow three different solutions for some values of λ . If we ignore the mapping $P(x, \lambda) = 0$, we conclude that the minimal degree that this polynomial map may

have is 3. Hence, (3.1) can be thought of as a normal form for the singular case of the transition between mushrooms and isolas.

Further along in this thesis, we will define the notion of contact equivalence between two bifurcation diagrams, which enables us to distinguish bifurcation diagrams up to changes of coordinates. This, in turn, will allow us to extend the main results obtained for the family of mappings defined in (3.1) to all mappings whose bifurcation diagrams are contact equivalent to those of $x \mapsto P(x, \lambda)$. Consequently, choosing a family of mappings that is algebraically simple will prove crucial for facilitating the derivation of forthcoming results.

3.2 Study of the family of normal forms

We aim to understand the dynamical system induced by the family of mappings P , thus we are first going to study the stability of the fixed points of P . Figure 3.2 shows the phase portraits of P for each $\lambda \in [-1.2, 1.2]$. As seen from this image, we will now formalize the stability of each fixed point and the nature of the bifurcations one encounters along this locus.

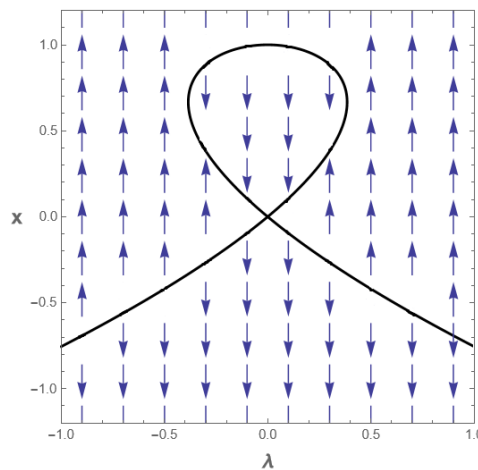


Figure 3.2. Phase portrait of the mapping $x \mapsto P(x, \lambda)$ for $\lambda \in [-1, 1]$.

Theorem 3.2.1. *If $x^* \in \mathbb{R}$ is a fixed point of $x \mapsto P(x, \lambda^*) = x + x^3 - x^2 + (\lambda^*)^2$ for some $\lambda^* \in \mathbb{R}$, then*

1. *it is hyperbolic and stable if $0 < x^* < \frac{2}{3}$.*

2. it is hyperbolic and unstable if $x^* < 0$ or $x^* > \frac{2}{3}$.

3. it is non hyperbolic and semistable if $x^* = 0$ or $x^* = \frac{2}{3}$.

Proof. Let x^* be a fixed point of $x \mapsto P(x, \lambda^*)$ for some $\lambda^* \in \mathbb{R}$, i.e.

$$P(x^*, \lambda^*) = x^* + (x^*)^3 - (x^*)^2 + (\lambda^*)^2 = x^*. \quad (3.3)$$

Let us study the differential of P ,

$$\frac{\partial P}{\partial x}(x, \lambda) = 1 + 3x^2 - 2x, \quad (3.4)$$

to determine the stability of the fixed points. First note that the value of (3.4) does not depends on λ .

$x \in$	$(-\infty, 0)$	$\{0\}$	$(0, \frac{2}{3})$	$\{\frac{2}{3}\}$	$(\frac{2}{3}, +\infty)$
$\frac{\partial P}{\partial x}(x, \lambda) \in$	$(1, +\infty)$	$\{1\}$	$(\frac{2}{3}, 1)$	$\{1\}$	$(1, +\infty)$

Table 3.1. Values of $\frac{\partial P}{\partial x}(x, \lambda)$ for different ranges of values of x .

Using theorem 2.1.1, it follows from table 3.1 that a fixed point x^* of P will be hyperbolic and stable if $x^* \in (0, \frac{2}{3})$, and will be hyperbolic and unstable if $x^* \in (-\infty, 0) \cup (\frac{2}{3}, +\infty)$.

If $x^* = 0$, then from equation (3.3) we have $\lambda^* = 0$. Notice that for $x < 0$

$$P(x, 0) = x + x^3 - x^2 = x(x^2 - x) < x.$$

Thus, for $x < 0$, the succession $\{f^{(n)}(x)\}_{n \in \mathbb{N}}$ is monotonically decreasing. Notice that said succession can only converge to a fixed point that is strictly less than 0, thus we conclude that $f^{(n)}(x) \rightarrow -\infty$ as $n \rightarrow \infty$.

On the other hand, if $0 < x < 1$ then

$$P(x, 0) = x + x^3 - x^2 < x,$$

and

$$P(x, 0) = x + x^3 - x^2 > 0,$$

which means that for $0 < x < 1$ the succession $\{f^{(n)}(x)\}_{n \in \mathbb{N}}$ is monotonically decreasing and bounded from below, hence it is convergent. Furthermore, it must converge

to a fixed point of f , which allows us to conclude that it must converge to 0. With this we conclude that for $\lambda = 0$, $x^* = 0$ is a semistable fixed point, as it is stable from above and unstable from below.

The case when $x^* = \frac{2}{3}$ is analogous and leads to the conclusion that if $x^* = \frac{2}{3}$ is a fixed point of $x \mapsto P(x, \lambda^*)$, then it is stable from below and unstable from above. \square

Theorem 3.2.2. *The 1-parameter mapping $x \mapsto P(x, \lambda) = x + x^3 - x^2 + \lambda^2$ exhibits the following bifurcations:*

1. A fold bifurcation on both $(x, \lambda) = \left(\frac{2}{3}, \pm \frac{2}{3\sqrt{3}}\right)$.
2. A transcritical bifurcation at $(x, \lambda) = (0, 0)$.

Proof. If $x^* = \frac{2}{3}$ is a fixed point of $x \mapsto P(x, \lambda^*)$, then from equation (3.3), we find that $\lambda^* = \pm \frac{2}{3\sqrt{3}}$. We will now prove that if $x^* = \frac{2}{3}$ and $\lambda^* = \pm \frac{2}{3\sqrt{3}}$, then the bifurcation conditions stated in Theorem 2.1.2 for a generic fold bifurcation are satisfied.

$$\text{B1) } \frac{\partial P}{\partial x} \left(\frac{2}{3}, \pm \frac{2}{3\sqrt{3}} \right) = (1 + 3x^2 - 2x) \Big|_{(x,\lambda)=\left(\frac{2}{3}, \pm \frac{2}{3\sqrt{3}}\right)} = 1.$$

$$\text{G1) } \frac{\partial^2 P}{\partial x^2} \left(\frac{2}{3}, \pm \frac{2}{3\sqrt{3}} \right) = (6x - 2) \Big|_{(x,\lambda)=\left(\frac{2}{3}, \pm \frac{2}{3\sqrt{3}}\right)} = 2 \neq 0.$$

$$\text{G2) } \frac{\partial P}{\partial \lambda} \left(\frac{2}{3}, \pm \frac{2}{3\sqrt{3}} \right) = (2\lambda) \Big|_{(x,\lambda)=\left(\frac{2}{3}, \pm \frac{2}{3\sqrt{3}}\right)} = \pm \frac{4}{3\sqrt{3}} \neq 0.$$

By Theorem 2.1.2, the mapping $x \mapsto P(x, \lambda)$ exhibits folds bifurcations at points $(x, \lambda) = \left(\frac{2}{3}, \pm \frac{2}{3\sqrt{3}}\right)$.

We will now prove that if $x^* = 0$ and $\lambda^* = 0$, then the genericity conditions for a transcritical bifurcation stated in Theorem 2.1.3 are satisfied.

$$\text{B1) } \frac{\partial P}{\partial x}(0, 0) = (1 + 3x^2 - 2x) \Big|_{(x,\lambda)=(0,0)} = 1.$$

$$\text{B2) } \frac{\partial P}{\partial \lambda}(0, 0) = (2\lambda) \Big|_{(x,\lambda)=(0,0)} = 0.$$

$$\text{G1) } \frac{\partial^2 P}{\partial x^2}(0, 0) = (6x - 2) \Big|_{(x,\lambda)=(0,0)} = -2 \neq 0.$$

$$\text{G2) } \frac{\partial^2 P}{\partial \lambda^2}(0, 0) \frac{\partial^2 P}{\partial x^2}(0, 0) - \left(\frac{\partial^2 P}{\partial x \partial \lambda}(0, 0) \right)^2 = ((2)(6x - 2) - (0)) \Big|_{(x,\lambda)=(0,0)} = -4.$$

By Theorem 2.1.3, the mapping $x \mapsto P(x, \lambda)$ undergoes a transcritical bifurcation at $(x^*, \lambda^*) = (0, 0)$. \square

As mentioned before, the selection of the mapping $x \mapsto P(x, \lambda)$ is not unique. For example, the mapping

$$x \mapsto Q(x, \lambda) = x + 8x^3 + (12\lambda - 4)x^2 + (6\lambda^2 - 4\lambda)x + \lambda^3,$$

exhibits the same bifurcations and has a similar bifurcation curve as that of the mapping $x \mapsto P(x, \lambda)$, as shown in Figure 3.3

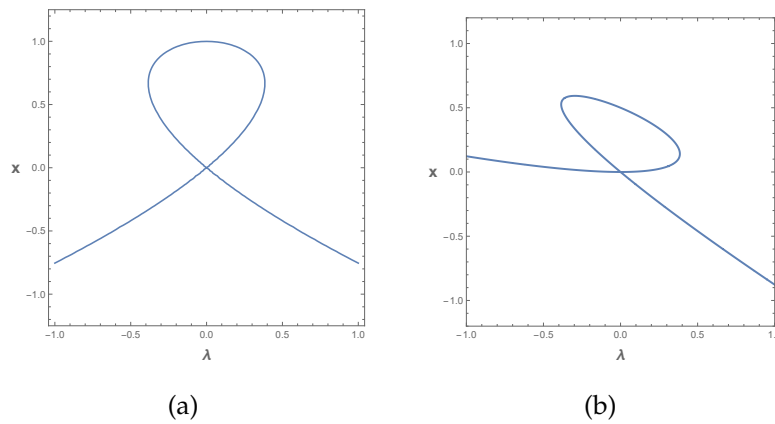


Figure 3.3. (a) Bifurcation curve of the mapping $x \mapsto P(x, \lambda)$. (b) Bifurcation curve of the mapping $x \mapsto Q(x, \lambda)$.

If we define the functions $G(x, \lambda) := P(x, \lambda) - x$ and $H(x, \lambda) = Q(x, \lambda) - x$, then the bifurcation curves of their respective mappings will be the zero-level sets of said functions. In this particular scenario, we have that if we define $\rho(x, \lambda) = 2x + \lambda$, then

$$H(x, \lambda) = G(\rho(x, \lambda), \lambda).$$

Thus the function H is equivalent to the function G under a change of coordinates. In chapter 4 this idea will be formalized with the definition of *contact equivalence* between functions.

Chapter 4

Elements from singularity theory

4.1 Preliminaries

To study the organization center associated with the transition between a Mushroom and an Isola we propose to use tools from Singularity Theory, which allows us to study in which ways the set of roots of a function change locally.

Definition 4.1.1 (Golubitsky and Schaeffer (1979)). *Given two functions $H, G : \mathbb{R}^{1+1} \rightarrow \mathbb{R}$, we say that they are equivalent at a germ level if*

$$G(x, \lambda) = H(x, \lambda), \quad \forall (x, \lambda) \in V,$$

where V is some open set containing the origin.

It is easy to note that if G and H are equivalent at a germ level, then their derivatives will also be equivalent at a germ level. We will write $G : (\mathbb{R}^{1+1}; 0) \rightarrow (\mathbb{R}; 0)$ to denote the equivalence class of G under germ equivalence, and we will denote the set of said equivalence classes as $C_{x,\lambda}^\infty$. We define the space $C_\lambda^\infty \subset C_{x,\lambda}^\infty$ as the subset of germs that only depend on the variable λ . Notably, if we endow $C_{x,\lambda}^\infty$ with the usual sum and multiplication, it can be seen as a **local ring**¹, a module over either $C_{x,\lambda}^\infty$ or C_λ^∞ , or a real vector space.

¹A ring $(A, +, \cdot)$ is said to be a local ring if it only has one maximal ideal.

4.2 Contact equivalence

From this point forward, we will assume that $G \in C_{x,\lambda}^\infty$ is such that $G(0, 0) = 0$. Note that this is unambiguous, as the origin is contained in all of its vicinities, thus the statement $G(0, 0) = 0$ is independent of the representative chosen. We can interpret the set of $(x, \lambda) \in (\mathbb{R}^{1+1}; 0)$ such that $G(x, \lambda) = 0$ as the bifurcation curve of the map $x \mapsto x + G(x, \lambda)$. Note that by using this, we lose the ability to understand G globally, but we are still able to compute the derivatives of G at the origin, and thus can understand the dynamics of $x \mapsto x + G(x, \lambda)$ through the bifurcation conditions mentioned in chapter 2.

Definition 4.2.1 (Contact equivalence). (*Golubitsky and Schaeffer (1979)*) Let $G, H \in C_{x,\lambda}^\infty$ such that $G(0, 0) = H(0, 0) = 0$. We say that G and H are contact equivalent if

$$H(x, \lambda) = \tau_{x,\lambda} \cdot G(\rho(x, \lambda), \Lambda(\lambda))$$

where

1. $\tau_{\cdot, \cdot} : (\mathbb{R}^{1+1}; 0) \rightarrow M^{n \times n}(\mathbb{R})$ is C^∞ in (x, λ) , and $\tau_{x,\lambda}$ is invertible.
2. $\rho : (\mathbb{R}^{1+1}; 0) \rightarrow (\mathbb{R}; 0)$ is a C^∞ diffeomorphism such that $\rho(0, 0) = 0$ and $\rho(\cdot, \lambda)$ preserves orientation.
3. $\Lambda : (\mathbb{R}; 0) \rightarrow (\mathbb{R}; 0)$ is a C^∞ diffeomorphism such that $\Lambda(0) = 0$ and preserves orientation.

This definition allows us to understand when two bifurcation curves are, in essence, the same curve. The notion of contact equivalence was proposed by [Golubitsky and Schaeffer \(1979\)](#) and it was chosen because it accommodates the possibility of imperfect bifurcations.

Given that this equivalence notion allows us to compute the derivatives of the functions associated with bifurcation curves, one can expect these bifurcation conditions to be invariant under contact equivalence, giving us a way to relate mappings by relating their bifurcation curves through contact equivalence. Now we prove the following original result.

Theorem 4.2.1. Let $G, H : (\mathbb{R}^{1+1}; 0) \rightarrow (\mathbb{R}; 0)$ be contact-equivalent functions such that $G(0, 0) = H(0, 0) = 0$. Then:

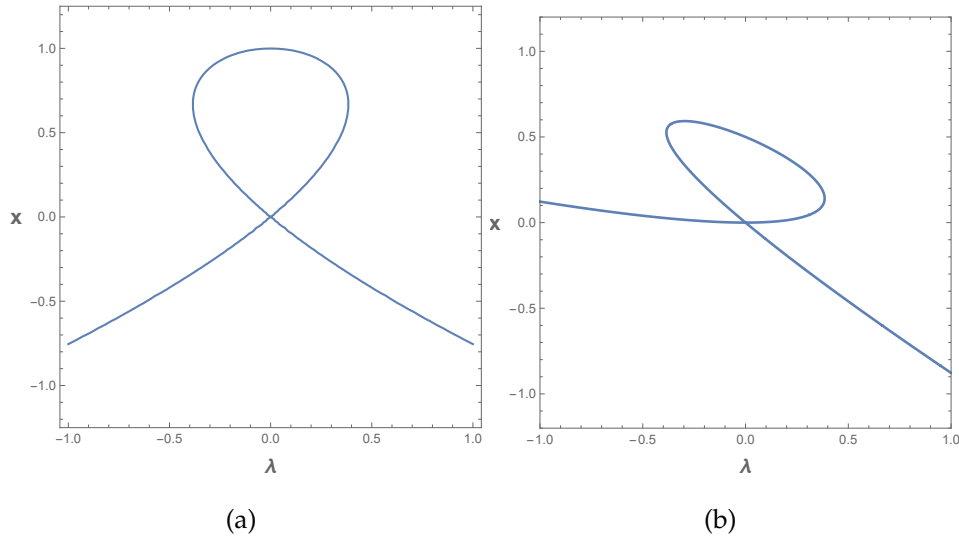


Figure 4.1. (a) Bifurcation curve $G(x, \lambda) = x^3 - x^2 + \lambda^2 = 0$. (b) Bifurcation curve $G(\rho(x, \lambda), \lambda) = 0$ with $\rho(x, \lambda) = 2x + \lambda$.

- a) If $x \mapsto x + G(x, \lambda)$ exhibits a generic fold bifurcation at the origin, then $x \mapsto x + H(x, \lambda)$ also does.
- b) If $x \mapsto x + G(x, \lambda)$ exhibits a generic transcritical bifurcation at the origin, then $x \mapsto x + H(x, \lambda)$ also does.
- c) If the origin is a stable (resp. unstable) hyperbolic fixed point for the system $x \mapsto x + G(x, \lambda)$ such that $\frac{\partial G}{\partial x}(0, 0), \frac{\partial H}{\partial x}(0, 0) > -1$, then the origin is a stable (resp. unstable) hyperbolic fixed point for the system $x \mapsto x + H(x, \lambda)$.

Moreover, in any of these cases, the systems $x \mapsto x + G(x, \lambda)$ and $x \mapsto x + H(x, \lambda)$ are locally topologically equivalent at the origin.

Proof. Theorem 2.1.2 states that a mapping $\hat{G}(x, \lambda) = x + G(x, \lambda)$ undergoes a non-degenerate saddle-node bifurcation if:

B1) $G(0, 0) = 0$.

B2) $\frac{\partial G}{\partial x}(0, 0) = 0$.

G1) $\frac{\partial^2 G}{\partial x^2}(0, 0) \neq 0$.

G2) $\frac{\partial G}{\partial \lambda}(0, 0) \neq 0$.

We will verify that the mapping $\hat{H}(x, \lambda) = x + G(x, \lambda)$ satisfies these conditions. Since G and H are contact equivalent, we have:

$$H(x, \lambda) = \tau_{x,\lambda} G(\rho(x, \lambda), \Lambda(\lambda)). \quad (4.1)$$

Differentiating with respect to x we obtain

$$\frac{\partial H}{\partial x}(x, \lambda) = \frac{\partial \tau_{x,\lambda}}{\partial x} G(\rho(x, \lambda), \Lambda(\lambda)) + \tau_{x,\lambda} \frac{\partial G}{\partial x}(\rho(x, \lambda), \Lambda(\lambda)) \cdot \frac{\partial \rho}{\partial x}(x, \lambda). \quad (4.2)$$

Evaluating at the origin results in

$$\frac{\partial H}{\partial x}(0, 0) = \left. \frac{\partial \tau_{x,\lambda}}{\partial x} \right|_{(x,\lambda)=(0,0)} G(0, 0) + \tau_{0,0} \frac{\partial G}{\partial x}(0, 0) \cdot \frac{\partial \rho}{\partial x}(0, 0).$$

Since $G(0, 0) = 0$ and $\frac{\partial G}{\partial x}(0, 0) = 0$, we get

$$\frac{\partial H}{\partial x}(0, 0) = 0.$$

If we differentiate equation (4.2) with respect to x again, we obtain

$$\begin{aligned} \frac{\partial^2 H}{\partial x^2}(x, \lambda) &= \frac{\partial^2 \tau_{x,\lambda}}{\partial x^2} G(\rho(x, \lambda), \Lambda(\lambda)) + 2 \frac{\partial \tau_{x,\lambda}}{\partial x} \frac{\partial G}{\partial x}(x, \lambda) \frac{\partial \rho}{\partial x}(x, \lambda) \\ &\quad + \tau_{x,\lambda} \frac{\partial^2 G}{\partial x^2}(x, \lambda) \left(\frac{\partial \rho}{\partial x}(x, \lambda) \right)^2 + \tau_{x,\lambda} \frac{\partial G}{\partial x}(x, \lambda) \frac{\partial^2 \rho}{\partial x^2}(x, \lambda). \end{aligned}$$

Evaluating at the origin results in

$$\begin{aligned} \frac{\partial^2 H}{\partial x^2}(0, 0) &= \left. \frac{\partial^2 \tau_{x,\lambda}}{\partial x^2} \right|_{(x,\lambda)=(0,0)} G(\rho(0, 0), \Lambda(0)) + 2 \left. \frac{\partial \tau_{x,\lambda}}{\partial x} \right|_{(x,\lambda)=(0,0)} \frac{\partial G}{\partial x}(0, 0) \frac{\partial \rho}{\partial x}(0, 0) \\ &\quad + \tau_{x,\lambda} \left. \frac{\partial^2 G}{\partial x^2} \right|_{(x,\lambda)=(0,0)}(0, 0) \left(\frac{\partial \rho}{\partial x}(0, 0) \right)^2 + \tau_{0,0} \frac{\partial G}{\partial x}(0, 0) \frac{\partial^2 \rho}{\partial x^2}(0, 0). \end{aligned}$$

Since $G(0, 0) = \frac{\partial G}{\partial x}(0, 0) = 0$, we get

$$\frac{\partial^2 H}{\partial x^2}(0, 0) = \tau_{0,0} \frac{\partial^2 G}{\partial x^2}(0, 0) \left(\frac{\partial \rho}{\partial x}(0, 0) \right)^2.$$

Because $\tau_{x,\lambda} > 0$ for all x, λ , $\frac{\partial \rho}{\partial x}(x, \lambda) > 0$ as $\rho(\cdot, \lambda)$ is a diffeomorphism that preserves orientation for all λ , and $\frac{\partial^2 G}{\partial x^2}(0, 0) \neq 0$, we conclude that $\frac{\partial^2 H}{\partial x^2}(0, 0)$ is non zero and has the same sign as $\frac{\partial^2 G}{\partial x^2}(0, 0)$. Finally, if we differentiate equation (4.1) with respect to λ , we obtain

$$\begin{aligned} \frac{\partial H}{\partial \lambda}(x, \lambda) &= \frac{\partial \tau_{x,\lambda}}{\partial \lambda} G(\rho(x, \lambda), \Lambda(\lambda)) \\ &\quad + \tau_{x,\lambda} \left(\frac{\partial G}{\partial x}(\rho(x, \lambda), \Lambda(\lambda)) \frac{\partial \rho}{\partial \lambda} + \frac{\partial G}{\partial \lambda}(\rho(x, \lambda), \Lambda(\lambda)) \frac{\partial \Lambda}{\partial \lambda}(\lambda) \right). \end{aligned}$$

If we evaluate at the origin we obtain

$$\begin{aligned} \frac{\partial H}{\partial \lambda}(0, 0) &= \left. \frac{\partial \tau_{x,\lambda}}{\partial \lambda} \right|_{(x,\lambda)=(0,0)} G(0, 0) \\ &\quad + \tau_{0,0} \left(\frac{\partial G}{\partial x}(0, 0) \frac{\partial \rho}{\partial \lambda}(0, 0) + \frac{\partial G}{\partial \lambda}(0, 0) \frac{\partial \Lambda}{\partial \lambda}(0) \right). \end{aligned}$$

Since $G(0, 0) = \frac{\partial G}{\partial x}(0, 0) = 0$, we get

$$\frac{\partial H}{\partial \lambda}(0, 0) = \tau_{0,0} \frac{\partial G}{\partial \lambda}(0, 0) \frac{\partial \Lambda}{\partial \lambda}(0).$$

Because $\tau_{x,\lambda} > 0$ for all x, λ , $\frac{\partial \Lambda}{\partial \lambda}(\lambda) > 0$ as Λ is a diffeomorphism that preserves orientations, and $\frac{\partial G}{\partial \lambda}(0, 0) \neq 0$, we conclude that $\frac{\partial H}{\partial \lambda}(0, 0) \neq 0$ and it has the same sign as $\frac{\partial G}{\partial \lambda}(0, 0)$. Theorem 2.1.2 allows us to conclude that H also undergoes a non-degenerate saddle-node bifurcation at the origin. The other bifurcations conditions and hyperbolicity conditions follow similarly. \square

4.3 Unfoldings

To model the transition between two different bifurcation curves, we use the notion of unfolding, which allows us to add an extra set of parameters with the intent of changing the topological type of the bifurcation curve.

Definition 4.3.1 (Unfolding). (*Golubitsky and Schaeffer (1979)*) Let $G \in C_{x,\lambda}^\infty$ such that $G(0, 0) = 0$. We say that $F : (\mathbb{R}^{1+1+l}; 0) \rightarrow (\mathbb{R}; 0)$ is an l -parameter unfolding of G if F is C^∞ in (x, λ, α) and $F(x, \lambda, 0) = G(x, \lambda)$. We may denote $F_\alpha(x, \lambda) := F(x, \lambda, \alpha)$.

An unfolding of a given bifurcation problem is a generalization of the original problem. For example, the function

$$F_\alpha(x, \lambda) = x^3 - x^2 + \lambda^2 + \alpha,$$

is an unfolding of

$$G(x, \lambda) = x^3 - x^2 + \lambda^2.$$

On figure 4.2 we see how the diagram of F_α changes as α increases.

It is easy to note that every bifurcation problem admits an unfolding with as many unfolding parameters as desired. However, adding more parameters does not necessarily generate new problems, which motivates the definition of unfolding factorizations and universal unfoldings.

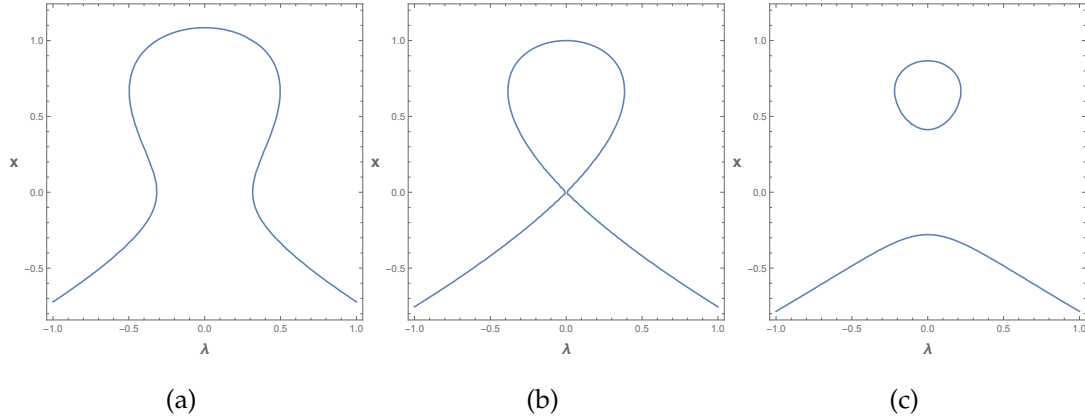


Figure 4.2. Bifurcation diagram of $x \mapsto P(x, \lambda) + \alpha$ for: $\alpha = -0.1$ in (a), $\alpha = 0$ in (b), and $\alpha = 0.1$ in (c).

Definition 4.3.2 (Unfolding factorization). (*Golubitsky and Schaeffer (1979)*) Let G be a bifurcation problem, and let F, H be an l -parameter and k -parameter unfolding of G respectively. We say that H factors through F if there exists $\psi : (\mathbb{R}^k : 0) \rightarrow (\mathbb{R}^l : 0)$ such that $\psi(0) = 0$ and H_β is contact equivalent (4.2.1) to $F_{\psi(\beta)}$, such that the mappings ρ, Λ and τ are smooth with respect to β . This means there exist $\tau_{x,\lambda,\beta}, \rho_\beta(x, \lambda), \Lambda_\beta(\lambda)$ satisfying

$$H_\beta(x, \lambda) = \tau_{x,\lambda,\beta} \cdot F_{\psi(\beta)}(\rho_\beta(x, \lambda), \Lambda_\beta(\lambda)).$$

Note that necessarily $\tau_{\cdot,\cdot,0}, \rho_0, \Lambda_0$ are their respective identities. If every unfolding of G factors through F , we say that F is a universal unfolding.

Having a universal unfolding of a problem allows us to understand how a curve can bifurcate and, consequently, what local bifurcations the associated discrete dynamical system undergoes. Therefore, it is essential to determine if a problem has a universal unfolding and, if so, how to obtain one with the minimum number of additional unfolding parameters such that it fully describes all the possible perturbations of G .

Definition 4.3.3 (Tangent space of a diagram). (*Golubitsky and Schaeffer (1979)*) Let $G \in C_{x,\lambda}^\infty$. We can define the following spaces

$$\widehat{TG} = \left\langle G, \frac{dG}{dx} \right\rangle_{C_{x,\lambda}^\infty} \subset C_{x,\lambda}^\infty,$$

and

$$TG = \widehat{TG} + \left\langle \frac{\partial G}{\partial \lambda} \right\rangle_{C_\lambda^\infty},$$

where the brackets $\langle v_1, v_2, \dots, v_n \rangle_{\mathbb{K}}$ denote the span of the elements v_1, v_2, \dots, v_n under multiplication by the ring \mathbb{K} .

In [Golubitsky and Schaeffer \(1979\)](#) is shown how these spaces can be understood as a tangent space of a diagram G by taking 1-parameter unfoldings of G .

Definition 4.3.4 (Codimension of a diagram). ([Golubitsky and Schaeffer \(1979\)](#)) Let $G \in C_{x,\lambda}^{\infty}$. We define the codimension of the diagram G as

$$\text{codim}(G) = \dim_{\mathbb{R}} C_{x,\lambda}^{\infty}/TG,$$

where $\dim_{\mathbb{R}}$ is the dimension of the quotient as an \mathbb{R} -vector space.

The following theorem allows us to relate the tangent spaces stated in definition 4.3.3 of a diagram G to a universal unfolding of it.

Theorem 4.3.1. ([Golubitsky and Schaeffer \(1979\)](#)) Let $G \in C_{x,\lambda}^{\infty}$ such that $G(0,0) = 0$ and $\text{codim}(G) < \infty$. Let F_{α} be an l -unfolding of G . Then F_{α} is a universal unfolding of G if and only if

$$TG + \left\langle \frac{\partial F}{\partial \alpha_1} + \dots + \frac{\partial F}{\partial \alpha_l} \right\rangle_{\mathbb{R}} = C_{x,\lambda}^{\infty}.$$

Moreover, the minimum number of unfolding parameters required for F_{α} to be a universal unfolding of G is $\text{codim}(G)$.

As a consequence, if $p_1, p_2, \dots, p_l \in C_{x,\lambda}^{\infty}$ project onto a basis of $C_{x,\lambda}^{\infty}/TG$, then

$$F_{\alpha}(x, \lambda) = G(x, \lambda) + \sum_{i=1}^l \alpha_i p_i(x, \lambda), \quad \alpha \in (\mathbb{R}^l; 0) \quad (4.3)$$

will be a universal unfolding of G . From all these, we can derive the following original result that will be useful later.

Theorem 4.3.2. Let $G \in C_{x,\lambda}^{\infty}$ such that $G(0,0) = 0$ and $\text{codim}(G) < \infty$. Let F be an l -parameter unfolding of G , and φ_{β} be a k -parameter unfolding of the zero function, that is to say that φ_{β} is a C^{∞} function in $2 + k$ variables, such that

$$\varphi_0(x, \lambda) = 0, \quad \forall x, \lambda \in (\mathbb{R}^2; 0),$$

then there are vicinities of their respective origins $V_x, V_{\lambda} \subset \mathbb{R}$ and $V_{\beta} \in \mathbb{R}^k$ such that

$$G(x, \lambda) + \varphi_{\beta}(x, \lambda) = \tau_{x,\lambda,\beta} F_{\psi(\beta)}(\rho_{\beta}(x, \lambda), \Lambda_{\beta}(\lambda)), \quad (x, \lambda, \beta) \in V_x \times V_{\lambda} \times V_{\beta}$$

for τ, ρ, Λ and ψ as stated in theorem 4.3.2.

Proof. Notice that the function

$$H_\beta(x, \lambda) = G(x, \lambda) + \varphi_\beta(x, \lambda)$$

is a k -parameter unfolding of G . Since F_α is a universal unfolding of G , there exist functions ρ, Λ, τ and ψ defined in a vicinity of the origin $V \subset \mathbb{R}^{1+1+k}$, which without generality we can assume it to be of the form $V = V_x \times V_\lambda \times V_\beta$ such that

$$G(x, \lambda) + \varphi_\beta(x, \lambda) = \tau_{x,\lambda,\beta} F(\rho_\beta(x, \lambda), \Lambda(\lambda)), \quad (x, \lambda, \beta) \in V_x \times V_\lambda \times V_\beta.$$

□

We can understand the function φ_β as an arbitrarily small perturbation at the origin, since thanks to the continuity of φ_β with respect to β , for any $\epsilon > 0$, we can choose a vicinity of the origin $V_\beta \subset \mathbb{R}^k$ small enough such that

$$|\varphi_\beta(x, \lambda)| < \epsilon, \quad (x, \lambda, \beta) \in V_x \times V_\lambda \times V_\beta.$$

4.4 Unstable diagrams

Let $G \in C_{x,\lambda}^\infty$ and let φ_t be a 1-parameter unfolding of the zero function such that for some arbitrarily small $t \neq 0$,

$$G(x, \lambda) + \varphi_t(x, \lambda)$$

is not contact equivalent to $G(x, \lambda)$. In other words, φ_t induces a genuinely new bifurcation diagram for arbitrarily small t . In this case, we say that G is an *unstable diagram*, since its qualitative structure can be changed by arbitrarily small perturbations. First we will formalize what we will understand as an unstable diagram.

Definition 4.4.1 (Unstable diagram). *Let $G \in C_{x,\lambda}^\infty$ and let H_α be an l -parameter unfolding of G . We will say that α^* induces an unstable diagram if there exists φ_t , a 1-parameter unfolding of the zero function, such that, for arbitrarily small, non zero t , we have*

$$H_{\alpha^*}(x, \lambda) + \varphi_t(x, \lambda)$$

is not contact equivalent to $H_{\alpha^}(x, \lambda)$.*

By Theorem 4.3.2, if F_α is an l -parameter universal unfolding of G , then there exist smooth functions $\rho_\beta, \Lambda_\beta, \tau_{x,\lambda,\beta}$, and ψ , such that

$$G(x, \lambda) + \varphi_\beta(x, \lambda) = \tau_{x,\lambda,\beta} F_{\psi(\beta)}(\rho_\beta(x, \lambda), \Lambda_\beta(\lambda)).$$

Hence, to understand all possible qualitative changes of the diagram of G under small perturbations, it suffices to analyze its universal unfolding F_α and classify which parameter values produce unstable diagrams. Thus we want to classify the l -parameter space of F_α into sets of parameters that induce unstable diagrams, and those that do not. The following theorem makes said partition of the l -parameter space explicit.

Theorem 4.4.1. (*Golubitsky and Schaeffer (1979)*) *Let $G \in C_{x,\lambda}^\infty$ and let F_α be an l -parameter unfolding of G . Then, if $\alpha^* \in \mathbb{R}^l$ is such that F_{α^*} is an unstable diagram then $\alpha^* \in \mathcal{C} = \mathcal{B} \cup \mathcal{H} \cup DL \subset \mathbb{R}^l$, where*

$$\mathcal{B} = \{\alpha \in \mathbb{R}^l : \exists(x, \lambda) : F_\alpha(x, \lambda) = \frac{\partial F_\alpha}{\partial x}(x, \lambda) = \frac{\partial F_\alpha}{\partial \lambda}(x, \lambda) = 0\},$$

$$\mathcal{H} = \{\alpha \in \mathbb{R}^l : \exists(x, \lambda) : F_\alpha(x, \lambda) = \frac{\partial F_\alpha}{\partial x}(x, \lambda) = \frac{\partial^2 F_\alpha}{\partial x^2}(x, \lambda) = 0\},$$

$$DL = \{\alpha \in \mathbb{R}^l : \exists(x_1, x_2, \lambda) : x_1 \neq x_2, F_\alpha(x_i, \lambda) = \frac{\partial F}{\partial x}(x_i, \lambda) = 0, \text{ for } i=1,2\}.$$

Additionally, if $\alpha \in \mathcal{B}$ we say that α is a bifurcation parameter, if $\alpha \in \mathcal{H}$ we say that α is a hysteresis parameter, and if $\alpha \in DL$ then we say that α induces a double limit point.

The set \mathcal{C} induces a partition of l -parameter space. It seems reasonable that if $\alpha^* \notin \mathcal{C}$, then F_{α^*} would be contact equivalent to $F_{\alpha^* + \epsilon\beta}$ for any $\beta \in \mathbb{R}^l$, provided that ϵ is sufficiently small. The following definition and theorem state this result.

Definition 4.4.2. (*Golubitsky and Schaeffer (1979)*) *Let $U \subset \mathbb{R}^{1+1}$ and $V \subset \mathbb{R}^l$ be vicinities of their respective origins, and $G, H : U \rightarrow \mathbb{R}$ be C^∞ . (a) Given a neighborhood of the origin $U' \subset U$ we shall say that G is equivalent to H on U' if G is contact equivalent to H on U' .*

(b) If $\alpha \in V$, we say that $F_\alpha : U \times V \rightarrow \mathbb{R}$ is (F, U') -stable if F_α is equivalent to F_β on U' for all β in a sufficiently small neighborhood of α .

Theorem 4.4.2. (*Golubitsky and Schaeffer (1979)*) *In terms of the notation above, there exist open neighborhoods of the origin $U' \subset \mathbb{R}^{1+1}$ and $V' \subset \mathbb{R}^l$, with $U' \subset U$ and $V' \subset V$, such that $F(\cdot, \cdot, \alpha)$ is (F, U') -stable for all $\alpha \in V' \setminus \mathcal{C}$.*

This in particular implies the following consequence

Corollary 4.4.1. (*Golubitsky and Schaeffer (1979)*) *Let $G \in C_{x,\lambda}^\infty$ and F_α be an l -parameter unfolding of G . If $\alpha \notin \mathcal{C}$, then F_α is equivalent to F_β if α and β lie in the same connected component of the partition of \mathbb{R}^l induced by the set \mathcal{C} .*

With these results, we are able to classify the space of parameters into finitely many connected components such that on each connected component, all the diagrams are contact equivalent.

Chapter 5

A universal unfolding

In this section we will prove that $F(x, \lambda, \alpha) = x^3 - x^2 + \lambda^2 + \alpha$ is a universal unfolding of $G(x, \lambda) = x^3 - x^2 + \lambda^2$, which means that any sufficiently small perturbation of G is contact equivalent to F near the origin.

We will also extend this property of F to a fixed compact vicinity of the origin that contains the Mushroom and Isola parts of the previously presented diagrams. After this, we will study the dynamical properties of the mapping $x \mapsto x + F(x, \lambda, \alpha)$ for different values of (λ, α) and finalize by proving that this mapping is an unfolding of the mapping $x \mapsto x + G(x, \lambda)$ in the classical dynamical sense.

5.1 Universal unfolding of G

In this section we will use the definitions of contact equivalence, unfolding and universal unfolding presented in Chapter 5.

Using theorem 4.3.1, we can prove the following result.

Theorem 5.1.1. *The function*

$$F : \mathbb{R}^{1+1+1} \rightarrow \mathbb{R}, \quad F(x, \lambda; \alpha) = x^3 - x^2 + \lambda^2 + \alpha,$$

is a universal unfolding of $G(x, \lambda) = x^3 - x^2 + \lambda^2$, in a vicinity of the origin.

Proof. Note that

$$G(x, \lambda) = x^3 - x^2 + \lambda^2, \quad \frac{\partial G}{\partial x}(x, \lambda) = 3x^2 - 2x.$$

Hence, using the notation of Theorem 4.3.1, we have

$$\widehat{TG} = \langle x^3 - x^2 + \lambda^2, 3x^2 - 2x \rangle_{C_{x,\lambda}^\infty}.$$

Given that we are studying these functions in an arbitrarily small vicinity of the origin, the factor $(3x - 2)$ is invertible, and thus

$$\begin{aligned} \widehat{TG} &= \langle x^3 - x^2 + \lambda^2, 3x^2 - 2x \rangle_{C_{x,\lambda}^\infty} \\ &= \langle x^3 - x^2 + \lambda^2, x \rangle_{C_{x,\lambda}^\infty} \\ &= \langle \lambda^2, x \rangle_{C_{x,\lambda}^\infty}. \end{aligned}$$

Furthermore,

$$TG = \langle \lambda^2, x \rangle_{C_{x,\lambda}^\infty} + \langle \lambda \rangle_{C_\lambda^\infty},$$

which only leaves the constant functions out of the space TG . Finally

$$TG + \left\langle \frac{\partial F}{\partial \alpha}(x, \lambda; \alpha) \right\rangle_{\mathbb{R}} = TG + \langle 1 \rangle_{\mathbb{R}} = C_{x,\lambda}^\infty,$$

which concludes the proof by Theorem 4.3.1. □

5.1.1 Towards a global universal unfolding

Theorem 5.1.1 along with theorem 4.4.1 would allow us to know all the different behaviors that a perturbation of G can exhibit near the origin, understanding a perturbation as an unfolding of G . However, this result is only local and does not provide information on the global consequences of small perturbations along the entire bifurcation curve of G away from the origin.

Example 5.1.1. *Using*

$$\tau_{x,\lambda} = (1 - x)^{-1}, \rho(x, \lambda) = x \text{ and } \Lambda(\lambda) = \lambda\sqrt{1 - x},$$

which satisfy the hypothesis of definition 4.2.1, we obtain that G is contact equivalent to

$$-x^2 + \lambda^2 = \tau_{x,\lambda}G(\rho(x, \lambda), \Lambda(\lambda)). \tag{5.1}$$

Figure 5.1 shows that zero set of equation (5.1) does not exhibit the desired loop, thus we need to find an additional tool capable of capturing non-local phenomena, such as our loop.

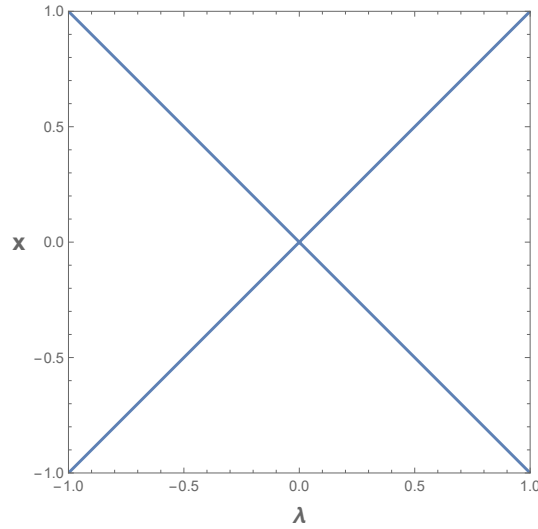


Figure 5.1. Diagram of the curve $-x^2 + \lambda^2 = 0$.

In the next section we are going to conclude that F is a universal unfolding of G in a global sense. Our strategy will be to shrink the loop associated with G into a point, resulting in a cusp, which would exhibit the desired loop under arbitrarily small perturbations; this allows us to understand G and F as a 1-parameter unfolding and 2-parameter unfolding of the cusp respectively, which will in turn allow us to conclude a sort of universal unfolding property of F through the local universal unfolding property of the cusp, thus allowing us to bypass the locality problem shown in example 5.1.1.

5.2 Study of the cusp

Consider the family of functions

$$U_\epsilon(x, \lambda) = x^3 - \epsilon x^2 + \lambda^2,$$

where $(x, \lambda) \in \mathbb{R}^2$ and $\epsilon \in \mathbb{R}$. This family is well-defined for $\epsilon \in \mathbb{R}$, and is such that $U_1(x, \lambda) = G(x, \lambda)$. In Figure 5.2, we show the zero set of $U_\epsilon(x, \lambda)$ for several values of ϵ . As the parameter ϵ decreases, the loop shrinks toward the origin. In the singular limit $\epsilon = 0$, the loop collapses into a cusp.

Now, define the functions

$$\rho_\epsilon^1(x, \lambda) = \epsilon x, \quad \Lambda_\epsilon^1(\lambda) = \epsilon^{3/2} \lambda, \quad \tau_{x, \lambda, \epsilon}^1 = \frac{1}{\epsilon^3}, \quad (5.2)$$

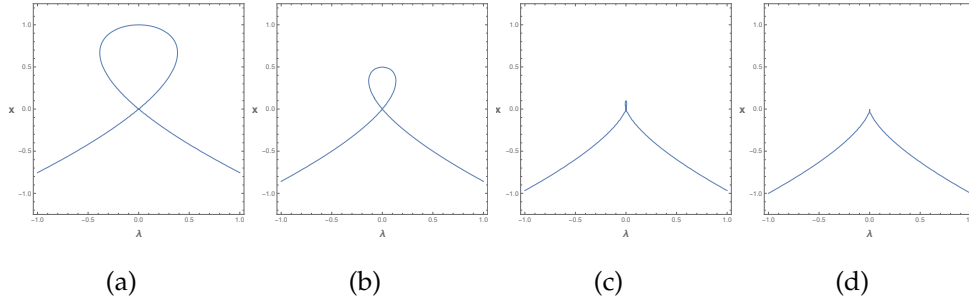


Figure 5.2. Plot of the set $U_\epsilon(x, \lambda) = 0$ for (a) $\epsilon = 1$, (b) $\epsilon = 0.5$, (c) $\epsilon = 0.1$, and (d) $\epsilon = 0$.

for every $\epsilon > 0$. It is easy to verify that

$$G(x, \lambda) = \tau_{x, \lambda, \epsilon}^1 U_\epsilon(\rho_\epsilon^1(x, \lambda), \Lambda_\epsilon^1(\lambda)), \quad (x, \lambda) \in \mathbb{R}^2, \quad \epsilon > 0. \quad (5.3)$$

This means that U_ϵ is globally contact-equivalent to G for every positive value of ϵ . Although the change of coordinates by (5.2)-(5.3) are not defined when $\epsilon = 0$, it remains valid for any $\epsilon > 0$ arbitrarily close to zero. Therefore, we may interpret G as an unfolding of the cusp $C(x, \lambda) = x^3 + \lambda^2$. This insight suggests a strategy for constructing a universal unfolding S for C , which we can then use to derive a universal unfolding of G by lifting along the contact equivalence (5.3).

Observation 5.2.1. Notice that, if $(x, \lambda) \in [-1.2, 1.2]^2$ then $(\rho_\epsilon(x, \lambda), \Lambda_\epsilon(\lambda)) \in [-\epsilon, \epsilon] \times [-\epsilon^{3/2}, \epsilon^{3/2}]$, therefore the loops that we see in figure 5.2 are contained in $[-\epsilon, \epsilon] \times [-\epsilon^{3/2}, \epsilon^{3/2}]$.

Theorem 5.2.1. The function

$$S(x, \lambda, \alpha, \beta, \gamma) = x^3 + \lambda^2 + \alpha + \beta x + \gamma x \lambda$$

is a universal unfolding of the function $C(x, \lambda) = x^3 + \lambda^2$ near the origin.

Proof. Using the notation presented in theorem 4.3.1, we have that

$$\begin{aligned} \hat{TC} &= \left\langle C, \frac{\partial C}{\partial x} \right\rangle_{C_{x, \lambda}^\infty} \\ &= \langle x^3 - \lambda^2, 3x^2 \rangle_{C_{x, \lambda}^\infty} \\ &= \langle \lambda^2, x^2 \rangle_{C_{x, \lambda}^\infty}. \end{aligned}$$

Then we have that

$$\begin{aligned} TC &= \langle \lambda^2, x^2 \rangle_{C_{x, \lambda}^\infty} + \left\langle \frac{\partial C}{\partial \lambda} \right\rangle_{C_\lambda^\infty} \\ &= \langle \lambda^2, x^2 \rangle_{C_{x, \lambda}^\infty} + \langle 2\lambda \rangle_{C_\lambda^\infty}. \end{aligned}$$

Finally we have that

$$C_{x,\lambda}^\infty/TC = \langle 1, x, x\lambda \rangle_{\mathbb{R}}.$$

And thus the function

$$S(x, \lambda, \alpha, \beta, \gamma) = x^3 + \lambda^2 + \alpha + \beta x + \gamma x \lambda$$

is a universal unfolding of C near the origin. \square

Let us consider an unfolding of the cusp $Q_\epsilon : \mathbb{R}^{1+1+l} \rightarrow \mathbb{R}$ of the form

$$Q_\epsilon(x, \lambda, \delta) = U_\epsilon(x, \lambda) + \Omega(x, \lambda, \delta), \quad (x, \lambda) \in [-1.2, 1.2], \quad \epsilon \in \mathbb{R}, \quad \delta \in \mathbb{R}^l, \quad (5.4)$$

such that $\Omega(x, \lambda, 0) = 0$. In particular, if we impose that $\epsilon > 0$, then we can apply identity (5.3) to obtain the following equality:

$$G(x, \lambda) + \tau_{x,\lambda,\epsilon}^1 \Omega(\rho_\epsilon^1(x, \lambda), \Lambda_\epsilon^1(\lambda), \delta) = \tau_{x,\lambda,\epsilon}^1 Q_\epsilon(\rho_\epsilon^1(x, \lambda), \Lambda_\epsilon^1(\lambda), \delta), \quad (5.5)$$

where

$$\tilde{\Omega}(x, \lambda, \delta) = \tau_{x,\lambda,\epsilon}^1 \Omega(\rho_\epsilon^1(x, \lambda), \Lambda_\epsilon^1(\lambda), \delta).$$

Because of the fact that the changes of coordinates defined in (5.2) are invertible, we can assure that $\tilde{\Omega}$ can be any C^∞ function.

Notice that from identity (5.4) that Q_ϵ is an $(l + 1)$ -parameter unfolding of C , then from theorem 5.2.1 it follows that there exist functions $\rho_{\epsilon,\delta}^2$, $\Lambda_{\epsilon,\delta}^2$, $\tau_{x,\lambda,\epsilon,\delta}^2$ and $\psi : \mathbb{R}^{(l+1)} \rightarrow \mathbb{R}^3$ such that

$$Q_\epsilon(x, \lambda, \delta) = \tau_{x,\lambda,\epsilon}^2 S(\rho_\epsilon^2(x, \lambda), \Lambda_\epsilon^2(\lambda), \psi(\epsilon, \delta)), \quad (5.6)$$

where this equality holds for $(x, \lambda, \epsilon, \delta) \in \mathbb{R}^{1+1+(l+1)}$ in a vicinity of the origin (without loss of generality we can assume it to be of the form $V \times V_\epsilon \times V_\delta$, where $(x, \lambda) \in V$, $\epsilon \in V_\epsilon$ and $\delta \in V_\delta$). If we also impose that $\epsilon > 0$, we can apply identity (5.5) and obtain the following equality

$$G(x, \lambda) + \tilde{\Omega}(x, \lambda, \delta) = \tau_{x,\lambda,\epsilon}^3 S(\rho_\epsilon^3(x, \lambda), \Lambda_\epsilon^3(\lambda), \psi(\epsilon)), \quad \forall (x, \lambda) \in V, \quad \epsilon \in V_\epsilon \cap \mathbb{R}^+, \quad \delta \in \mathbb{R}^l, \quad (5.7)$$

where ρ_ϵ^3 , Λ_ϵ^3 and $\tau_{x,\lambda,\epsilon}^3$ correspond to the appropriate compositions of the functions ρ_ϵ^i , Λ_ϵ^i and $\tau_{x,\lambda,\epsilon}^i$ for $i = 1, 2$:

$$\rho_\epsilon^3(x, \lambda) = \rho_\epsilon^2(\rho_\epsilon^1(x), \Lambda_\epsilon^1(\lambda)),$$

$$\Lambda_\epsilon^3(\lambda) = \Lambda_\epsilon^2(\Lambda_\epsilon^1(\lambda)),$$

$$\tau_{x,\lambda,\epsilon}^3 = \tau_{\rho_\epsilon(x,\Lambda_\epsilon^1(\lambda)),\Lambda_\epsilon^1(\lambda),\epsilon}^2 \cdot \tau_{x,\lambda,\epsilon}^1.$$

As shown in observation 5.2.1, for any neighborhood V of $(x, \lambda) = (0, 0)$ we can choose $\epsilon^* > 0$ sufficiently small such that $[-\epsilon^*, \epsilon^*] \times [-(\epsilon^*)^{3/2}, (\epsilon^*)^{3/2}] \subset V$ and $\epsilon^* \in V_\epsilon$, which allow us to conclude that for $\epsilon < \epsilon^*$ the loop is entirely contained in V . We will prove that for $\epsilon = \epsilon^*$ and $\delta \in \mathbb{R}^l$ sufficiently close to the origin, the diagram of $S_{\epsilon^*, \delta}$ can only be either a Mushroom, an Isola, or the Loop, which then equation (5.7) allow us to lift this property to G for $(x, \lambda) \in [-1.2, 1.2]^2$. This in turn implies that any universal unfolding of G , in particular F , is able to capture how small perturbations affect G for $(x, \lambda) \in [-1.2, 1.2]^2$.

By fixing the domain $(x, \lambda, \epsilon, \delta) \in V \times V_\epsilon \times V_\delta \subset \mathbb{R}^{1+1+1+l}$, we can now study the set of unfolding parameters $(\alpha, \beta, \gamma) \in \mathbb{R}^3$ of the cusp map that yield unstable diagrams $S(\cdot, \cdot, \alpha, \beta, \gamma)$. As posed in theorem 4.4.1, the set of unfolding parameters that induce unstable diagrams are

$$\mathcal{B} = \{\alpha \in \mathbb{R}^l : \exists(x, \lambda) : F_\alpha(x, \lambda) = \frac{\partial F_\alpha}{\partial x}(x, \lambda) = \frac{\partial F_\alpha}{\partial \lambda}(x, \lambda) = 0\}, \quad (5.8)$$

$$\mathcal{H} = \{\alpha \in \mathbb{R}^l : \exists(x, \lambda) : F_\alpha(x, \lambda) = \frac{\partial F_\alpha}{\partial x}(x, \lambda) = \frac{\partial^2 F_\alpha}{\partial x^2}(x, \lambda) = 0\}, \quad (5.9)$$

$$DL = \{\alpha \in \mathbb{R}^l : \exists(x_1, x_2, \lambda) : x_1 \neq x_2, F_\alpha(x_i, \lambda) = \frac{\partial F}{\partial x}(x_i, \lambda) = 0, \text{ for } i=1,2\}. \quad (5.10)$$

This in turn implies that if $(\alpha^*, \beta^*, \gamma^*) \in \mathbb{R}^3$ are such that $S(x, \lambda, \alpha^*, \beta^*, \gamma^*)$ induce an unstable diagram, then $(\alpha^*, \beta^*, \gamma^*) \in \mathcal{C}$, where

$$\mathcal{C} = \mathcal{B} \cup \mathcal{H} \cup DL. \quad (5.11)$$

Because identity (5.7) relates a perturbation of G and S , and because G is in itself an unstable diagram, we know that for every $\epsilon > 0$, $\psi(\epsilon, 0) \in \mathcal{C}$. We will now prove that for any $\epsilon > 0$, $\psi(\epsilon, 0) \in \mathcal{B}$.

Lemma 5.2.1. *The function ψ from identity (5.7) satisfies that,*

$$\forall \epsilon > 0, \psi(\epsilon, 0) \in \mathcal{B} \cap \mathcal{H}^c \cap DL^c.$$

Proof. We need to prove that for $\epsilon > 0$, the function $S(x, \lambda, \psi(\epsilon, 0))$ satisfies the conditions defining \mathcal{B} while failing to satisfy the conditions defining \mathcal{H} and DL . Notice that the conditions defining \mathcal{B} , \mathcal{H} , and DL are invariant under contact equivalence;

hence, instead of studying the conditions for $S(x, \lambda, \psi(\epsilon, 0))$, we can study them for G .

Thus, to prove that $\psi(\epsilon, 0) \in \mathcal{B}$, $\forall \epsilon > 0$, we have to show that

$$\forall \epsilon > 0, \exists (x, \lambda) \in \mathbb{R}^2, G(x, \lambda) = 0 = \frac{\partial G}{\partial x}(x, \lambda) = \frac{\partial G}{\partial \lambda}(x, \lambda); \quad (5.12)$$

to prove that $\psi(\epsilon, \delta) \notin \mathcal{H}$, we have to show that

$$\forall \epsilon > 0, \nexists (x, \lambda) \in \mathbb{R}^2, G(x, \lambda) = \frac{\partial G}{\partial x}(x, \lambda) = \frac{\partial^2 G}{\partial x^2}(x, \lambda); \quad (5.13)$$

and to prove that $\psi(\epsilon, \delta) \notin DL$, we have to show that

$$\forall \epsilon > 0, \nexists (x_1, x_2, \lambda) \in \mathbb{R}^3, G(x_i, \lambda) = 0 = \frac{\partial G}{\partial x}(x_i, \lambda), \text{ for } i = 1, 2, \text{ and } x_1 \neq x_2. \quad (5.14)$$

We begin by proving statement (5.12). Notice that:

$$\frac{\partial G}{\partial x}(x, \lambda) = 3x^2 - 2\epsilon x = x(3x - 2\epsilon), \quad \frac{\partial G}{\partial \lambda}(x, \lambda) = 2\lambda.$$

Thus, $\lambda = 0$, and either $x = 0$ or $x = \frac{2}{3}\epsilon$.

To determine whether these points lie on the zero level set of G , we evaluate $G(x, \lambda)$ at each candidate:

- At $(x, \lambda) = (0, 0)$, we find $G(0, 0) = 0$, so this point lies on the set $\{G = 0\}$.
- At $(x, \lambda) = (\frac{2}{3}\epsilon, 0)$, we compute

$$G\left(\frac{2}{3}\epsilon, 0\right) = \left(\frac{2}{3}\epsilon\right)^3 - \epsilon\left(\frac{2}{3}\epsilon\right)^2 = \frac{8}{27}\epsilon^3 - \frac{4}{9}\epsilon^3 = -\frac{4}{27}\epsilon^3,$$

which is negative for $\epsilon > 0$. Hence, this point does not belong to the zero level set.

Therefore, $(x, \lambda) = (0, 0)$ satisfies the equation posed in statement (5.12) for all $\epsilon > 0$, hence we conclude that $\psi(\epsilon) \in \mathcal{B}$, $\forall \epsilon > 0$. This result is also true if we change the domain of (x, λ) to an arbitrary vicinity of the origin.

Now, we check that the conditions stated in (5.13) are not met for any $\epsilon > 0$:

$$\frac{\partial G}{\partial x}(x, \lambda) = 0 \Leftrightarrow 3x^2 - 2x\epsilon = 0 \Leftrightarrow x = 0 \text{ or } x = \frac{2}{3}\epsilon,$$

$$\frac{\partial^2 G}{\partial x^2}(x, \lambda) = 0 \Leftrightarrow 6x - 2\epsilon = 0 \Leftrightarrow x = \frac{1}{3}\epsilon.$$

These conditions cannot simultaneously hold for any $\epsilon > 0$; therefore, $\psi(\epsilon, 0)$ does not belong to \mathcal{H} for any $\epsilon > 0$.

Similarly, we conclude that $\psi(\epsilon, 0)$ does not belong to DL because the conditions defining DL are not satisfied; specifically, G does not exhibit two distinct folds for the same value of λ . This concludes the proof of the lemma. \square

This lemma allows us to focus our study of the universal unfolding S to values of $(\alpha, \beta, \gamma) \in \mathcal{B}$. The set of conditions that determine \mathcal{B} will form a manifold of codimension 1 in the (α, β, γ) -space. We will now find the algebraic expressions that $(\alpha, \beta, \gamma) \in \mathcal{B}$ must satisfy.

Lemma 5.2.2. *Let $(\alpha, \beta, \gamma) \in \mathbb{R}^3$, then $(\alpha, \beta, \gamma) \in \mathcal{B}$ for the universal unfolding S if and only if*

$$2(864\alpha + 72\beta\gamma^2 - \gamma^6) + 2(48\beta - \gamma^4)\sqrt{\gamma^4 - 48\beta} = 0, \quad (5.15)$$

or

$$2(864\alpha + 72\beta\gamma^2 - \gamma^6) - 2(48\beta - \gamma^4)\sqrt{\gamma^4 - 48\beta} = 0. \quad (5.16)$$

Proof. Let $(\alpha, \beta, \gamma) \in \mathcal{B}$, this implies that there exists (x^*, λ^*) such that

$$S(x^*, \lambda^*, \alpha, \beta, \gamma) = 0 = \frac{\partial S}{\partial x}(x^*, \lambda^*, \alpha, \beta, \gamma) = \frac{\partial S}{\partial \lambda}(x^*, \lambda^*, \alpha, \beta, \gamma).$$

First notice that

$$\begin{aligned} \frac{\partial S}{\partial \lambda}(x^*, \lambda^*, \alpha, \beta, \gamma) &= 0 \\ \Leftrightarrow 2\lambda^* + \gamma x^* &= 0 \\ \Leftrightarrow \lambda^* &= -\frac{\gamma x^*}{2}, \end{aligned}$$

then

$$\begin{aligned} \frac{\partial S}{\partial x}(x^*, \lambda^*, \alpha, \beta, \gamma) &= 0 \\ \Leftrightarrow 3(x^*)^2 + \beta + \gamma\lambda^* &= 0 \\ \Leftrightarrow 3(x^*)^2 + \beta - \frac{\gamma^2}{2}x^* &= 0 \\ \Leftrightarrow x^* &= \frac{\gamma^2 \pm \sqrt{\gamma^4 - 48\beta}}{12}. \end{aligned}$$

If we substitute the expressions for x^* and λ^* in the equation $S(x^*, \lambda^*, \alpha, \beta, \gamma) = 0$, we obtain the following

$$2(864\alpha + 72\beta\gamma^2 - \gamma^6) \pm 3(48\beta - \gamma^4)\sqrt{\gamma^4 - 48\beta} \pm \left(\sqrt{\gamma^4 - 48\beta}\right)^3 = 0.$$

which by factoring the square root we obtain that (α, β, γ) satisfies one of the two equations posed in the lemma. \square

Both expressions (5.15) and (5.16) represent manifolds in the (α, β, γ) -space. Notably, if $\gamma^4 - 48\beta \neq 0$, then the left hand side of both equations is differentiable, as both square roots are being evaluated in the interior of their domain. If we suppose that $(\alpha^*, \beta^*, \gamma^*) \in \mathcal{B}$ satisfies either identity (5.15) or (5.16), but not both at the same time, then using the implicit function theorem we can assure that locally the manifold \mathcal{B} divides the space in two connected components. We can interpret this as there being an unstable diagram, that serves as an organization center, that bifurcates into one of two distinct behaviours.

We will prove that for $\epsilon > 0$, $\psi(\epsilon, 0)$ satisfies only one of the two equations posed in lemma 5.2.2, this will allow us to conclude that sufficiently small perturbations of $S(x, \lambda, \psi(\epsilon))$ can only exhibit one of two stable diagrams, which we will then prove are contact equivalent to a Mushroom or to an Isola.

Lemma 5.2.3. *The function ψ from identity (5.7) is such that, for all $\epsilon > 0$, $\psi(\epsilon, 0)$ satisfies only one of the two equations presented in lemma 5.2.2. Furthermore the set \mathcal{B} divides the (α, β, γ) -space in two different connected components in a vicinity of $\psi(\epsilon, 0)$.*

Proof. We know from lemma 5.2.1 that for $\epsilon > 0$, $\psi(\epsilon, 0) \in \mathcal{B}$, which then lemma 5.2.2 tells us that $\psi(\epsilon, 0)$ satisfies (5.15) or (5.16).

Suppose that $\psi(\epsilon, 0) = (\alpha^*, \beta^*, \gamma^*)$ satisfies both equations (5.15) and (5.16), then if we subtract equation (5.15) from (5.16) we obtain that

$$(\gamma^*)^4 - 48\beta^* = 0. \tag{5.17}$$

Given that $S(x, \lambda, \psi(\epsilon, 0))$ is contact equivalent to $G(x, \lambda)$ for $\epsilon > 0$, then the singularity of G at the origin must be mapped to a singularity (x^*, λ^*) of S . We know from the proof of lemma 5.2.2 that said singularity (x^*, λ^*) must satisfy the equations

$$\lambda^* = -\frac{\gamma x}{2}, \quad x^* = \frac{\gamma^2 \pm \sqrt{\gamma^4 - 48\beta}}{12}.$$

If we apply the identity (5.17) to this equation, we obtain that

$$x^* = \frac{\gamma^2}{12} > 0.$$

Notice that

$$\frac{\partial^2 S}{\partial x^2}(x^*, \lambda^*, \alpha^*, \beta^*, \gamma^*) = 6x^* > 0,$$

which contradicts the fact that the sign of the second derivative of a singularity is preserved under contact equivalence, as $\frac{\partial^2 G}{\partial x^2}(0, 0) < 0$. Hence, we can conclude that $\psi(\epsilon, 0)$ can only satisfy one of the two equations of lemma 5.2.2.

Without loss of generality assume that $\psi(\epsilon, 0) = (\alpha^*, \beta^*, \gamma^*)$ satisfies equation (5.15) for some $\epsilon > 0$. If we define the function

$$L(\alpha, \beta, \gamma) = 2(864\alpha + 72\beta\gamma^2 - \gamma^6) + 2(48\beta - \gamma^4)\sqrt{\gamma^4 - 48\beta},$$

because $(\gamma^*)^4 - 48\beta^* > 0$, then L will be differentiable at $(\alpha^*, \beta^*, \gamma^*)$, furthermore

$$\frac{\partial L}{\partial \alpha}(\alpha, \beta, \gamma) = 1728 \neq 0, \quad \forall (\alpha, \beta, \gamma) \in \mathbb{R}^3,$$

which by the implicit function theorem we know that locally the graph of the function L divides the space in two connected components near $(\alpha^*, \beta^*, \gamma^*)$. \square

This lemma allows us to conclude the main result of this section, which states that any sufficiently small perturbation of G is contact equivalent to either, a Mushroom, an Isola, or our Organization Center. Before proving this result we need one last technical lemma that allows us to effectively choose any small enough perturbation of G .

Lemma 5.2.4. *Let $\Omega(x, \lambda, \delta) \in C^\infty(\mathbb{R}^{1+l}, \mathbb{R})$ have a compact support and is such that*

$$\Omega(x, \lambda, 0) = 0,$$

then the function $\tilde{\Omega}(x, \lambda, \delta, \xi, \kappa)$ defined by

$$\tilde{\Omega}(x, \lambda, \delta, \xi, \kappa) = e^{-\frac{1}{\xi^2}} e^{-\frac{1}{\kappa^2}} \Omega\left(\frac{1}{\xi}x, \frac{1}{\kappa}\lambda, \delta\right), \quad (5.18)$$

where $\epsilon^{-\frac{1}{x^2}}|_{x=0} = 0$, is in $C^\infty(\mathbb{R}^{1+l+2})$ and satisfies that

$$\tilde{\Omega}(x, \lambda, \delta, 0, 0) = 0.$$

Proof. Let $\Omega(x, \lambda, \delta) \in C^\infty(\mathbb{R}^{1+1+l}, \mathbb{R})$ and have a compact support. Then we have that

$$\lim_{\xi \rightarrow 0} \tilde{\Omega}(x, \lambda, \delta, \xi, \kappa) = \lim_{\xi \rightarrow 0} e^{-\frac{1}{\xi^2}} e^{-\frac{1}{\kappa^2}} \Omega\left(\frac{x}{\xi}, \frac{\lambda}{\kappa}, \delta\right) = 0,$$

as $\Omega\left(\frac{x}{\xi}, \frac{\lambda}{\kappa}, \delta\right)$ is bounded (because it has a compact support) and $e^{-\frac{1}{\xi^2}} \rightarrow 0$ as $\xi \rightarrow 0$. Therefore, $\tilde{\Omega}$ is at least continuous.

To prove that this function is infinitely differentiable we use a similar argument, together with the fact that

$$\lim_{\xi \rightarrow 0} \frac{e^{-\frac{1}{\xi^2}}}{\xi^n} = 0, \quad \forall n \in \mathbb{N}.$$

□

Theorem 5.2.2. *Let $\Omega \in C^\infty(\mathbb{R}^{1+1+l}, \mathbb{R})$ have compact support, then there exists a vicinity of the origin $V_\delta \subset \mathbb{R}^l$ such that for $\delta \in V_\delta$, there exists a constant $\mu > 0$, and changes of coordinates $\tau_{x,\lambda}$, ρ and Λ such that*

$$G(x, \lambda) + \mu\Omega(x, \lambda, \delta) = \tau_{x,\lambda}F(\rho(x, \lambda), \Lambda(\lambda), \alpha^*), \quad \forall (x, \lambda) \in [-1.2, 1.2]^2$$

for $|\alpha^*|$ small enough.

Proof. Let $\Omega \in C^\infty(\mathbb{R}^{1+1+l}, \mathbb{R})$ such that it has compact support and $\Omega(x, \lambda, 0) = 0$. Using lemma 5.2.4 we know that the function $\tilde{\Omega}(x, \lambda, \delta, \xi, \kappa)$ defined by

$$\tilde{\Omega}(x, \lambda, \delta, \xi, \kappa) = e^{-\frac{1}{\xi^2}} e^{-\frac{1}{\kappa^2}} \Omega\left(\frac{x}{\xi}, \frac{\lambda}{\kappa}, \delta\right),$$

is in $C^\infty(\mathbb{R}^{1+1+l+2}, \mathbb{R})$ and is such that $\tilde{\Omega}(x, \lambda, 0, 0) = 0$. Using theorem 5.2.1 we know that there exist vicinities of the origin $V \subset \mathbb{R}^2$, $V_\epsilon \subset \mathbb{R}$, $V_\delta \subset \mathbb{R}^{l+1}$ and $U \subset \mathbb{R}^2$ such that the following equality holds

$$C(x, \lambda) - \epsilon x^2 + \delta_0 + \tilde{\Omega}(x, \lambda, \delta, \xi, \kappa) = \tau_{x,\lambda,\Theta}S(\rho_\Theta(x, \lambda), \Lambda_\Theta(\lambda), \psi(\Theta)), \quad (5.19)$$

where $(x, \lambda) \in V$, $\epsilon \in V_\epsilon$, $(\delta_0, \delta) \in V_\delta$ and $(\xi, \kappa) \in U$, and $\Theta = (\epsilon, \delta_0, \delta, \xi, \kappa)$.

Let $\epsilon \in V_\epsilon \cap \mathbb{R}^+$, using lemma 5.2.3, we know that for $(\delta_0, \delta, \xi, \kappa) \in V_\delta \times U$ close enough to the origin, the function $S(x, \lambda, \psi(\epsilon, \delta_0, \delta, \xi, \kappa))$ can exhibit one of three different non contact equivalent diagrams. We know that those diagrams are either a Mushroom, an Isola or the organization center, because if in expression (5.19) we set $\delta = 0$ we get

$$C(x, \lambda) - \epsilon x^2 + \delta_0,$$

which is contact equivalent to a Mushroom if $\delta_0 < 0$, an Isola if $\delta_0 > 0$ or the organization center when $\delta_0 = 0$. Then lemma 5.2.3 allows us to conclude that if $(\delta_0, \delta, \xi, \kappa) \in V_\delta \times U$ are close enough to the origin, then

$$C(x, \lambda) - \epsilon x^2 + \tilde{\Omega}(x, \lambda, \delta) \text{ is contact equivalent to } C(x, \lambda) - \epsilon x^2 + \delta_0, \quad (5.20)$$

where δ_0 must be either positive, negative or zero.

Let us fix $\delta_0 = 0$, $\epsilon \in V_\epsilon \cap \mathbb{R}^+$ and $\delta \in V_\delta$. If we apply the change of coordinates (5.3) to equation (5.20) we obtain

$$G(x, \lambda) + \frac{1}{\epsilon^3} \tilde{\Omega}(\epsilon x, \epsilon^{3/2} \lambda, \delta, \xi, \kappa) = F\left(x, \lambda, \frac{\delta_0}{\epsilon^3}\right), \quad (5.21)$$

where now $(x, \lambda) \in [-1.2, 1.2]^2$. We can choose $(\xi, \kappa) \in U$ and $\epsilon \in V_\epsilon \cap \mathbb{R}^+$ small enough such that

$$\epsilon = \xi, \text{ and } \epsilon^{3/2} = \kappa, \quad (5.22)$$

which finally implies that

$$\frac{1}{\epsilon^3} \tilde{\Omega}(\epsilon x, \epsilon^{3/2} \lambda, \delta, \xi, \kappa) = \frac{e^{-\left(\frac{1}{\xi^2} + \frac{1}{\kappa^2}\right)}}{\epsilon^3} \Omega(x, \lambda, \delta) =: \mu \Omega(x, \lambda, \delta). \quad (5.23)$$

With this we conclude that for $\delta \in V_\delta$ sufficiently small, there exists a constant μ , and there are C^∞ functions $\tau_{x,\lambda}$, ρ and Λ , such that

$$G(x, \lambda) + \mu \Omega(x, \lambda, \delta) = \tau_{x,\lambda} F(\rho(x, \lambda), \Lambda(\lambda)), \quad (x, \lambda) \in [-1.2, 1.2]^2.$$

□

Theorem 5.2.2 allows us to finally conclude that the unfolding $F(x, \lambda, \alpha) = x^3 - x^2 + \lambda^2$ is a universal unfolding of $G(x, \lambda) = x^3 - x^2 + \lambda^2$ globally. With this we conclude that the only two stable behaviours that a perturbation of G can exhibit are either a Mushroom, and Isola, or the organization center.

Observation 5.2.2. *The imposition that the perturbation $\Omega \in C^\infty(\mathbb{R}^{1+1+l}, \mathbb{R})$ does not really restricts the types of perturbations we can choose, as any bounded C^∞ perturbation defined in a closed set can be extended to the whole space by a function with compact support.*

5.3 Structural stability of the mapping $x \mapsto x + F(x, \lambda, \alpha)$

In the previous section we proved that the unfolding F is able to capture any perturbation of G in a non-local setting. In this section we will show that F is also a universal unfolding of G dynamically, which will be understood as there being a local topological equivalence between the fixed points of C^∞ perturbations of the mapping $x \mapsto x + G(x, \lambda)$ and $x \mapsto x + F(x, \lambda, \alpha)$, which are induced by the contact equivalence shown in theorem 5.2.2.

Theorem 5.3.1. *Let $G(x, \lambda) = x^3 - x^2 + \lambda^2$, $F(x, \lambda, \alpha) = G(x, \lambda) + \alpha$, and $\Omega \in C^\infty(\mathbb{R}^{1+1+l}, \mathbb{R})$ such that*

$$\Omega(x, \lambda, 0) = 0.$$

Then, there exists a positive constant $\mu > 0$, such that for $\delta \in \mathbb{R}^l$ sufficiently close to the origin, there exist α^ near zero such that the mapping $x \mapsto x + G(x, \lambda) + \mu\Omega(x, \lambda, \delta)$ for $(x, \lambda) \in [-1.2, 1.2]^2$ is topologically equivalent to*

$$x \mapsto x + F(x, \lambda, \alpha^*), \text{ for } (x, \lambda) \in [-1.2, 1.2]^2.$$

Moreover, the topological equivalence $h : [-1.2, 1.2]^2 \rightarrow [-1.2, 1.2]^2$ is such that if x^ is a fixed point of $x \mapsto x + G(x, \lambda^*) + \mu\Omega(x, \lambda^*, \delta)$, then $h(x^*, \lambda^*) = (\rho(x^*, \lambda^*), \Lambda(\lambda^*))$, where the functions ρ and Λ come from the contact equivalence*

$$G(x, \lambda) + \mu\Omega(x, \lambda, \delta) = \tau_{x,\lambda}F(\rho(x, \lambda), \Lambda(\lambda)),$$

given by theorem 5.2.2.

Proof. As stated in theorem 5.2.2, there exists a positive constant $\mu > 0$, such that for $\delta \in \mathbb{R}^l$ sufficiently close to the origin, there are functions $\tau_{x,\lambda}$, Λ , and ρ such that

$$G(x, \lambda) + \mu\Omega(x, \lambda, \delta) = \tau_{x,\lambda}F(\rho(x, \lambda), \Lambda(\lambda), \alpha^*), \quad (x, \lambda) \in [-1.2, 1.2]^2, \quad (5.24)$$

for some α^* near zero. We can further restrict $\delta \in \mathbb{R}^l$ such that the mapping $x \mapsto x + G(x, \lambda) + \mu\Omega(x, \lambda, \delta)$ is monotonous for all λ . This last condition can be written as

$$\delta \in \mathbb{R}^l : \max_{(x,\lambda) \in \mathbb{R}^2} \left\{ 1 + \frac{\partial G}{\partial x}(x, \lambda) + \mu\Omega(x, \lambda, \delta) \right\} > 0,$$

Notice that this last condition is open, as the argument of the maximum function is uniformly continuous.

Let $(x^*, \lambda^*) \in [-1.2, 1.2]^2$ be such that x^* is a fixed point for the mapping $x \mapsto x + G(x, \lambda^*) + \Omega(x, \lambda^*, \delta)$; this implies that

$$G(x^*, \lambda^*) + \mu\Omega(x^*, \lambda^*, \delta) = 0,$$

which then follows from equation (5.24) that

$$\tau_{x^*, \lambda^*} F(\rho(x^*, \lambda^*), \Lambda(\lambda), \alpha^*) = 0,$$

and thus the pair $(\rho(x^*, \lambda^*), \Lambda(\lambda^*))$ are such that $\rho(x^*, \lambda^*)$ is a fixed point of the mapping $x \mapsto x + \tau_{x, \lambda^*} F(x, \Lambda(\lambda^*), \alpha^*)$. Notice that trivially the function $\tau_{x, \lambda} F(x, \Lambda(\lambda), \alpha^*)$ is contact equivalent to $F(x, \lambda, \alpha^*)$ and thus exhibits the same type of fixed points, which by theorems 3.2.1 and 3.2.2 we can conclude that are either hyperbolic, exhibit a generic transcritical bifurcation, or exhibit a generic fold bifurcation, which theorem 4.2.1 tells us that these types of fixed points are preserved under contact equivalence.

This implies that the mappings $x \mapsto x + G(x, \lambda^*) + \mu\Omega(x, \lambda^*, \delta)$ near x^* and $x \mapsto x + F(x, \Lambda(\lambda^*))$ near $\rho(x^*, \lambda^*)$ are both locally topologically equivalent to the same corresponding normal form, and thus are both locally topologically equivalent. We can extend this local topological equivalence to a topological equivalence by using the fact that both mappings are monotonous, and arguments similar to those in the proof of lemma 2.1.1. \square

From this theorem and thanks to the fact that the mapping Λ preserves orientation, we can assure that for ϵ sufficiently small, the mapping $x \mapsto x + G(x, \lambda) + \Omega(x, \lambda, \delta)$ either exhibits:

- Four folds at (x_1, λ_1) , (x_2, λ_2) , (x_3, λ_3) and (x_4, λ_4) such that $\lambda_1 < \lambda_2 < \lambda_3 < \lambda_4$. This corresponds to a mushroom.
- Two folds at (x_1, λ_1) and (x_3, λ_3) , and a transcritical bifurcation at (x_2, λ_2) , such that $\lambda_1 < \lambda_2 < \lambda_3$. This corresponds to the diagram presented as the organization center.
- Two folds at (x_1, λ_1) and (x_2, λ_2) where $\lambda_1 < \lambda_2$. This corresponds to an Isola.

We cannot guaranty that there will be any type of order on the x_i values. For instance, in figure 5.3 there are two examples of an organization center type of diagram. On

panel (a) we have $x_1 = x_3 > x_2$, while panel (b) shows a case where $x_1 = x_3 < x_2$. (Notice that, in panel (b), x_1 and x_3 are very close to the local minima of the function.)

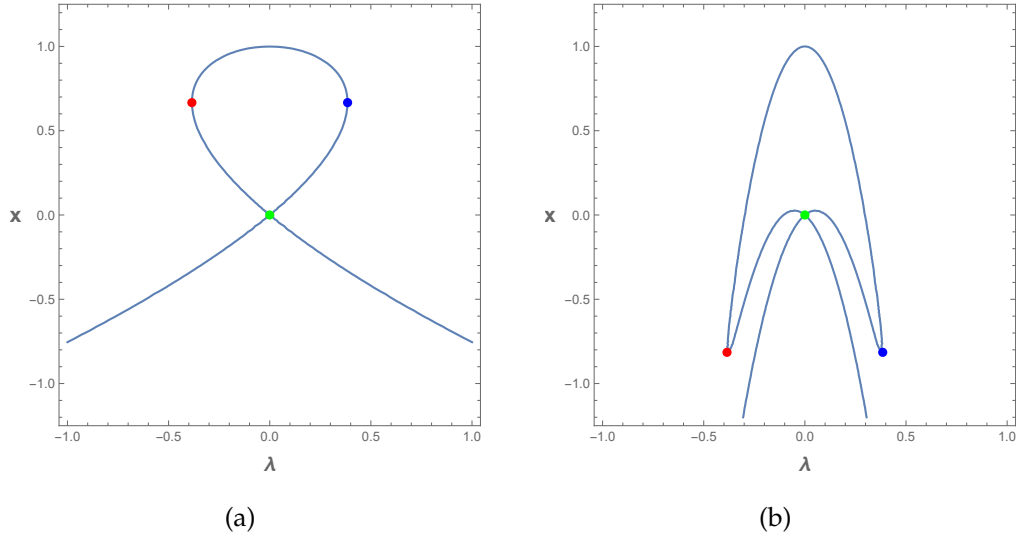


Figure 5.3. (a) Bifurcation curve $G(x, \lambda) = x^3 - x^2 + \lambda^2 = 0$. (b) Bifurcation curve $G(\rho(x, \lambda), \lambda) = 0$ with $\rho(x, \lambda) = x - 10\lambda^2$. The red and blue dots represent the folds of the diagram, while the green dot represents the transcritical bifurcation.

5.4 Bifurcation analysis of F

Theorem 5.3.1 allows us to focus our study on the discrete dynamics induced by the function $F(x, \lambda, \alpha^*)$ for different values of α .

Given $\alpha \in \mathbb{R}$, the non-hyperbolic fixed points of this mapping satisfy the following system of equations

$$\begin{cases} F(x, \lambda; \alpha) = 0 \\ \frac{\partial F}{\partial x}(x, \lambda; \alpha) = 0. \end{cases}$$

The solutions of this system are

$$(x, \lambda) = (0, \sqrt{-\alpha}), \quad (x, \lambda) = \left(\frac{2}{3}, \sqrt{\frac{4}{27} - \alpha} \right), \quad (5.25)$$

from where we can conclude that $\alpha = 0$ and $\alpha = \frac{4}{27}$ are critical values in our analysis.

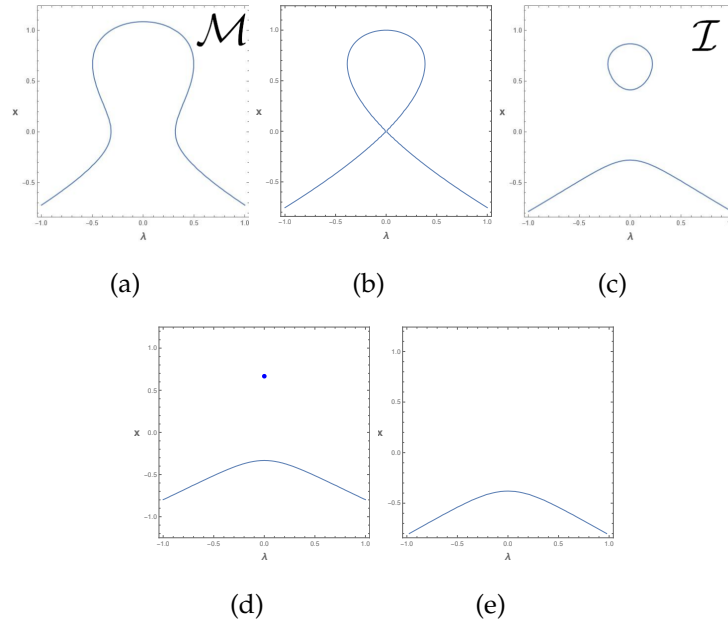


Figure 5.4. Solutions of $F(x, \lambda, \alpha) = 0$, for (a) $\alpha = -0.1 < 0$, (b) $\alpha = 0$, (c) $0 < \alpha = 0.1 < 4/27$, (d) $\alpha = 4/27$ and (e) $\alpha = 0.2 > 4/27$.

Panels (a), (b) and (c) of Figure 5.4 represent the transition from a Mushroom to an Isola. For increasing $0 < \alpha < 4/27$, the closed component of the Isola shrinks to a point at $\alpha = 4/27$, shown in the figure (d), and then it disappears completely for $\alpha > 4/27$, as illustrated in panel (e).

Interpreting α as a second parameter of the mapping F allows us to visualize the two-parameter curves for the fold bifurcations, which, from (5.25), are given by $\alpha = -\lambda^2$ and $\alpha = \frac{4}{27} - \lambda^2$ respectively.

For each fixed value of α , one can determine the nature of the bifurcation curve simply by counting the number of folds in its locus; see figure 5.4 for comparison.

It is important to note that in figure 5.5, the two parabolas do not intersect each other, but they do meet at two points at infinity in the (λ, α) -space. Let us consider a compactification of the (λ, α) -space, if we homogenize the polynomials $\alpha + \lambda^2$ and $\alpha + \lambda^2 - \frac{4}{27}$ by replacing $\alpha = \frac{X}{Z}$, $\lambda = \frac{Y}{Z}$, we get the following expressions

$$\begin{cases} XZ + Y^2 = 0, \\ XZ + Y^2 - \frac{4}{27}Z^2 = 0, \end{cases} \quad (5.26)$$

where $Z \rightarrow 0$ as $\alpha, \lambda \rightarrow \infty$. If $Z \neq 0$, then equations (5.26) can never hold simultane-

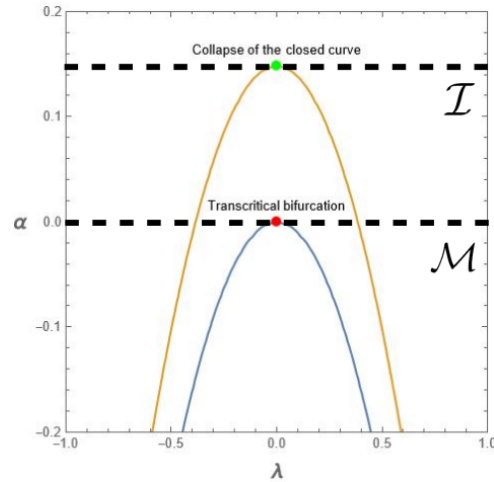


Figure 5.5. Two-parameter bifurcation diagram of $F(x, \lambda, \alpha)$.

ously, but if $Z = 0$, then equations (5.26) reduce to

$$Y^2 = 0,$$

which we can interpret as there being an intersection of both quadratics at infinity in the (λ, α) -plane, and that said intersection has algebraic multiplicity two. Thus we can interpret this as there being an intersection of both families of folds at two points, and thus there being two cusp bifurcations of $F(x, \lambda, \alpha)$ at infinity in the parameter plane.

Chapter 6

Study of the 3-parameter unfolding

In chapter 5, we constructed a three-parameter universal unfolding S of the cusp C , which, through Theorem 5.2.2, allowed us to lift the local universal unfolding property of S into a global¹ universal unfolding F of G . In this section, we study the unfolding S further, with the aim of identifying other curves from which one can similarly construct a universal unfolding using S .

6.1 Study of the unstable diagrams of S

In theorem 4.4.1 we presented the sets of parameters that induce an unfolding to exhibit an unstable diagram. Let us consider the universal unfolding of the cusp

$$S(x, \lambda, \alpha, \beta, \gamma) = x^3 + \lambda^2 + \alpha + \beta x + \gamma x \lambda.$$

We will now find the sets B , H and DL for the unfolding $S_{\alpha, \beta, \gamma}$.

Theorem 6.1.1. *Consider the unfolding $S_{\alpha, \beta, \gamma}$ of the cusp. Then, the sets described in Theorem 4.4.1 corresponding to the unfolding $S_{\alpha, \beta, \gamma}$ are given by:*

$$DL = \emptyset;$$

$$\mathcal{B} = \left\{ (\alpha, \beta, \gamma) \in \mathbb{R}^3 : \alpha = -\frac{1}{12}\beta(\gamma^2 \pm \sqrt{-48\beta + \gamma^4}) + \frac{1}{576}\gamma^2(\gamma^2 \pm \sqrt{-48\beta + \gamma^4})^2 - \frac{1}{1728}(\gamma^2 \pm \sqrt{-48\beta + \gamma^4})^3 \right\};$$

¹In the sense that it was in a fixed domain.

$$\mathcal{H} = \left\{ (\alpha, \beta, \gamma) \in \mathbb{R}^3 : \begin{array}{ll} \alpha\gamma^2 = -\beta^2, & \gamma \neq 0, \\ \beta = 0, \alpha \in \mathbb{R}_- \cup \{0\}, & \gamma = 0 \end{array} \right\}.$$

Proof. The expression for set \mathcal{B} was stated and proved in lemma 5.2.2. We will now prove that the unfolding $S_{\alpha,\beta,\gamma}$ cannot exhibit double limit points for any (α, β, γ) . Let $(x_1, x_2, \lambda) \in \mathbb{R}^3$ be such that $x_1 \neq x_2$ and

$$S_{\alpha,\beta,\gamma}(x_i, \lambda) = 0, \quad \frac{\partial S_{\alpha,\beta,\gamma}}{\partial x}(x_i, \gamma) = 0. \quad (6.1)$$

Let us first assume that $\gamma \neq 0$, then the second equation of (6.1) implies that

$$\lambda = \frac{-\beta - 3x_i^2}{\gamma}, \quad i = 1, 2. \quad (6.2)$$

Equation (6.2) implies that $x_1^2 = x_2^2$, and because $x_1 \neq x_2$, we conclude that $x_1 = -x_2$. If we plug the expression for λ from equation (6.2) into the first equation of (6.1) we obtain that

$$\alpha = \frac{-\beta^2 - 6\beta x_i^2 + 2\gamma^2 x_i^3 - 9x_i^4}{\gamma^2}, \quad i = 1, 2. \quad (6.3)$$

If we equal the expressions for α in terms of x_1 and x_2 from equation (6.3), we obtain

$$\frac{9(-x_1^4 + x_2^4) - 6\beta(x_1^2 - x_2^2) + 2\gamma^2(x_1^3 - x_2^3)}{\gamma} = 0. \quad (6.4)$$

If we use the fact that $x_1 = -x_2$ in equation (6.4) we are finally able to obtain that

$$x_1^3 = 0,$$

which contradicts our supposition that $x_1 \neq x_2$. The case where $\gamma = 0$ is analogous. Thus we conclude that for the unfolding $S_{\alpha,\beta,\gamma}$, the set $DL = \emptyset$.

Finally, to characterize the set \mathcal{H} , let $(x, \lambda) \in \mathbb{R}^2$ be such that

$$S_{\alpha,\beta,\gamma}(x, \lambda) = 0, \quad \frac{\partial S_{\alpha,\beta,\gamma}}{\partial x}(x, \lambda) = 0, \quad = \frac{\partial^2 S_{\alpha,\beta,\gamma}}{\partial x^2}(x, \lambda) = 0. \quad (6.5)$$

From the third equation of (6.5) we get that

$$x = 0, \quad (6.6)$$

Let us assume that $\gamma \neq 0$. If we apply (6.5) to the second equation of (6.5) we get that

$$\lambda = -\frac{\beta}{\gamma}. \quad (6.7)$$

By replacing the expressions for x and λ from equations (6.6) and (6.7) into the first equation of (6.5) we finally obtain that

$$\alpha = -\frac{\beta^2}{\gamma^2}. \quad (6.8)$$

Thus, if $\gamma \neq 0$, the set of parameters $(\alpha, \beta, \gamma) = \left(-\frac{\beta^2}{\gamma^2}, \beta, \gamma\right) \in \mathcal{H}$. If $\gamma = 0$, then following a similar process we can obtain that $\beta = 0$ and $\alpha \in \mathbb{R}_- \cup \{0\}$, which concludes the proof. \square

Let us define the following functions

$$\begin{aligned} B_-(\beta, \gamma) &= -\frac{1}{12}\beta(\gamma^2 - \sqrt{-48\beta + \gamma^4}) + \frac{1}{576}\gamma^2(\gamma^2 - \sqrt{-48\beta + \gamma^4})^2 \\ &\quad - \frac{1}{1728}(\gamma^2 - \sqrt{-48\beta + \gamma^4})^3, \\ B_+(\beta, \gamma) &= -\frac{1}{12}\beta(\gamma^2 + \sqrt{-48\beta + \gamma^4}) + \frac{1}{576}\gamma^2(\gamma^2 + \sqrt{-48\beta + \gamma^4})^2 \\ &\quad - \frac{1}{1728}(\gamma^2 + \sqrt{-48\beta + \gamma^4})^3, \\ H(\beta, \gamma) &= \frac{-\beta^2}{\gamma^2}, \end{aligned}$$

which have the following properties

$$\mathcal{B} = \{(B_-(\beta, \gamma), \beta, \gamma) \in \mathbb{R}^3 : \gamma^4 \geq 48\beta\} \cup \{(B_+(\beta, \gamma), \beta, \gamma) \in \mathbb{R}^3 : \gamma^4 \geq 48\beta\},$$

$$\mathcal{H} = \{(H(\beta, \gamma), \beta, \gamma) : \gamma \neq 0\} \cup \{(\alpha, \beta, \gamma) \in \mathbb{R}^3 : \alpha \leq 0, \gamma = \beta = 0\}.$$

Figure 6.1 shows the graphs of B_- , B_+ , and the set \mathcal{H} .

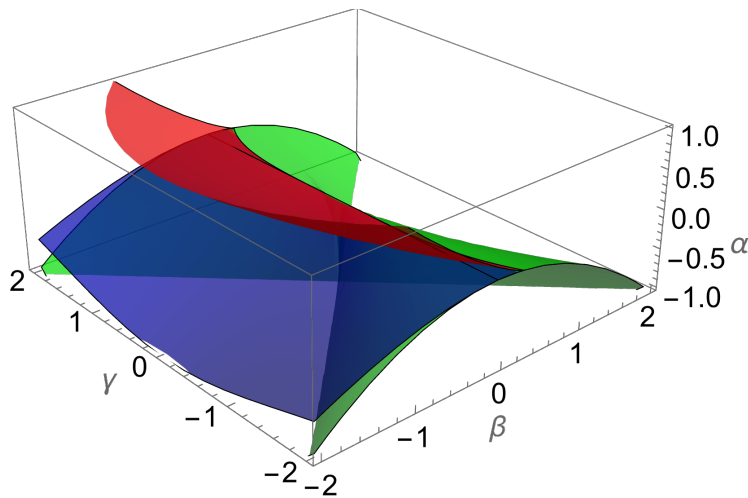


Figure 6.1. Graphs of the functions B_- in blue, B_+ in red, and the set \mathcal{H} in green.

Theorem 4.4.2 tells us that we must study the different connected components of the partition that these sets induce.

Theorem 6.1.2. *The sets B and H induce a partition of \mathbb{R}^3 into 7 connected open sets, and $B \cup H$. Said seven connected open sets are of the form*

$$\begin{aligned} \mathcal{R}_1 &= \left\{ (\alpha, \beta, \gamma) \in \mathbb{R}^3 : 0 < \gamma, \left\{ \begin{array}{l} \beta < \frac{\gamma^4}{48}, \alpha < \min(B_-(\beta, \gamma), H(\beta, \gamma)) \\ \frac{\gamma^4}{54} < \beta < \frac{\gamma^4}{48}, B_+(\beta, \gamma) < \alpha < H(\beta, \gamma) \\ \frac{\gamma^4}{48} < \beta, \alpha < H(\beta, \gamma) \end{array} \right. \right. , or \left. \right\}, \\ \mathcal{R}_2 &= \left\{ (\alpha, \beta, \gamma) \in \mathbb{R}^3 : \gamma < 0, \left\{ \begin{array}{l} \beta < \frac{\gamma^4}{48}, \alpha < \min(B_-(\beta, \gamma), H(\beta, \gamma)) \\ \frac{\gamma^4}{54} < \beta < \frac{\gamma^4}{48}, \max\{B_-(\beta, \gamma), B_+(\beta, \gamma)\} < \alpha < H(\beta, \gamma) \\ \frac{\gamma^4}{48} < \beta, \alpha < H(\beta, \gamma) \end{array} \right. \right. , or \left. \right\}, \\ \mathcal{R}_3 &= \left\{ (\alpha, \beta, \gamma) \in \mathbb{R}^3 : 0 < \gamma, 0 < \beta < \frac{\gamma^4}{48}, B_-(\beta, \gamma) < \alpha < \min\{B_+(\beta, \gamma), H(\beta, \gamma)\} \right\}, \\ \mathcal{R}_4 &= \left\{ (\alpha, \beta, \gamma) \in \mathbb{R}^3 : \gamma < 0, 0 < \beta < \frac{\gamma^4}{48}, B_-(\beta, \gamma) < \alpha < \min\{B_+(\beta, \gamma), H(\beta, \gamma)\} \right\}, \\ \mathcal{R}_5 &= \left\{ (\alpha, \beta, \gamma) \in \mathbb{R}^3 : \gamma \in \mathbb{R}, \beta < \frac{\gamma^4}{54}, \max\{B_-(\beta, \gamma), H(\beta, \gamma)\} < \alpha < B_+(\beta, \gamma) \right\}, \\ \mathcal{R}_6 &= \left\{ (\alpha, \beta, \gamma) \in \mathbb{R}^3 : \gamma \in \mathbb{R}, \beta < 0, H(\beta, \gamma) < \alpha < B_-(\beta, \gamma) \right\}, \\ \mathcal{R}_7 &= \left\{ (\alpha, \beta, \gamma) \in \mathbb{R}^3 : \gamma \in \mathbb{R}, \beta \in \mathbb{R}, \max\{H(\beta, \gamma), B_+(\beta, \gamma)\} < \alpha \right\}. \end{aligned}$$

Proof. We are going to characterize the seven open sets by studying the graphs of the functions B_- , B_+ and H . First we will identify the set of points (β, γ) where the graphs of B_- , B_+ and H intersect. Notice that

- $B_-(\beta, \gamma) = B_+(\beta, \gamma)$ if $\beta = \frac{\gamma^4}{48}$.
- $B_-(\beta, \gamma) = H(\beta, \gamma)$ if $\beta = 0$.
- $B_+(\beta, \gamma) = H(\beta, \gamma)$ if $\beta = \frac{\gamma^4}{54}$.

These curves induce a partition in the (β, γ) -space that have the following properties:

- a) If $\gamma > 0$ and $0 < \beta < \frac{\gamma^4}{54}$, then $B_-(\beta, \gamma) < H(\beta, \gamma) < B_+(\beta, \gamma)$, which divides the space into four connected components.

- b) If $\gamma > 0$ and $\frac{\gamma^4}{54} < \beta < \frac{\gamma^4}{48}$, then $B_-(\beta, \gamma) < B_+(\beta, \gamma) < H(\beta, \gamma)$, which divides the space into four connected components.
- c) If $\gamma \in \mathbb{R}$ and $\frac{\gamma^4}{48} < \beta$, then the functions B_- and B_+ are not defined. In this case the graph of H divides the space into two connected components.
- d) If $\gamma < 0$ and $0 < \beta < \frac{\gamma^4}{54}$, then $B_-(\beta, \gamma) < H(\beta, \gamma) < B_+(\beta, \gamma)$, which divides the space into four connected components.
- e) If $\gamma < 0$ and $\frac{\gamma^4}{54} < \beta < \frac{\gamma^4}{48}$, then $B_-(\beta, \gamma) < B_+(\beta, \gamma) < H(\beta, \gamma)$, which divides the space into four connected components.
- f) If $\gamma \in \mathbb{R}$ and $\beta < 0$, then $H(\beta, \gamma) < B_-(\beta, \gamma) < B_+(\beta, \gamma)$, which divides the space into four connected components.

The four connected components of case a) are of the form

$$U_1 = \left\{ (\alpha, \beta, \gamma) \in \mathbb{R}^3 : \gamma > 0, 0 < \beta < \frac{\gamma^4}{54}, \alpha < B_-(\beta, \gamma) \right\},$$

$$U_2 = \left\{ (\alpha, \beta, \gamma) \in \mathbb{R}^3 : \gamma > 0, 0 < \beta < \frac{\gamma^4}{54}, B_-(\beta, \gamma) < \alpha < H(\beta, \gamma) \right\},$$

$$U_3 = \left\{ (\alpha, \beta, \gamma) \in \mathbb{R}^3 : \gamma > 0, 0 < \beta < \frac{\gamma^4}{54}, H(\beta, \gamma) < \alpha < B_+(\beta, \gamma) \right\},$$

$$U_4 = \left\{ (\alpha, \beta, \gamma) \in \mathbb{R}^3 : \gamma > 0, 0 < \beta < \frac{\gamma^4}{54}, H(\beta, \gamma) < \alpha \right\},$$

and the four connected components of case b) are of the form

$$V_1 = \left\{ (\alpha, \beta, \gamma) \in \mathbb{R}^3 : \gamma > 0, \frac{\gamma^4}{54} < \beta < \frac{\gamma^4}{48}, \alpha < H(\beta, \gamma) \right\},$$

$$V_2 = \left\{ (\alpha, \beta, \gamma) \in \mathbb{R}^3 : \gamma > 0, \frac{\gamma^4}{54} < \beta < \frac{\gamma^4}{48}, H(\beta, \gamma) < \alpha < B_-(\beta, \gamma) \right\},$$

$$V_3 = \left\{ (\alpha, \beta, \gamma) \in \mathbb{R}^3 : \gamma > 0, \frac{\gamma^4}{54} < \beta < \frac{\gamma^4}{48}, B_-(\beta, \gamma) < \alpha < B_+(\beta, \gamma) \right\},$$

$$V_4 = \left\{ (\alpha, \beta, \gamma) \in \mathbb{R}^3 : \gamma > 0, \frac{\gamma^4}{54} < \beta < \frac{\gamma^4}{48}, B_+(\beta, \gamma) < \alpha \right\}.$$

Notice that the sets U_1 and V_1 are actually connected because we can write its union as

$$U_1 \cup V_1 = \left\{ (\alpha, \beta, \gamma) \in \mathbb{R}^3 : \gamma > 0, 0 < \beta < \frac{\gamma^4}{48}, \alpha < \min\{B_-(\beta, \gamma), H(\beta, \gamma)\} \right\}.$$

Similarly, we can conclude that the sets U_3 and V_3 are connected, and the sets U_4 and V_4 are connected. If we continue this process, we can conclude that there are 7 connected open sets, which take the forms presented in theorem 6.1.2. \square

Theorem 4.4.2 tells us that if two set of parameters in the (α, β, γ) -space belong to the same \mathcal{R}_i for some $i \in \{1, \dots, 7\}$, then the resulting diagrams are contact equivalent. In figures 6.2, 6.3, 6.4, 6.5, 6.6, 6.7, 6.8 we show the sets \mathcal{R}_i for $i = 1, \dots, 7$ and a representative diagram for each region.

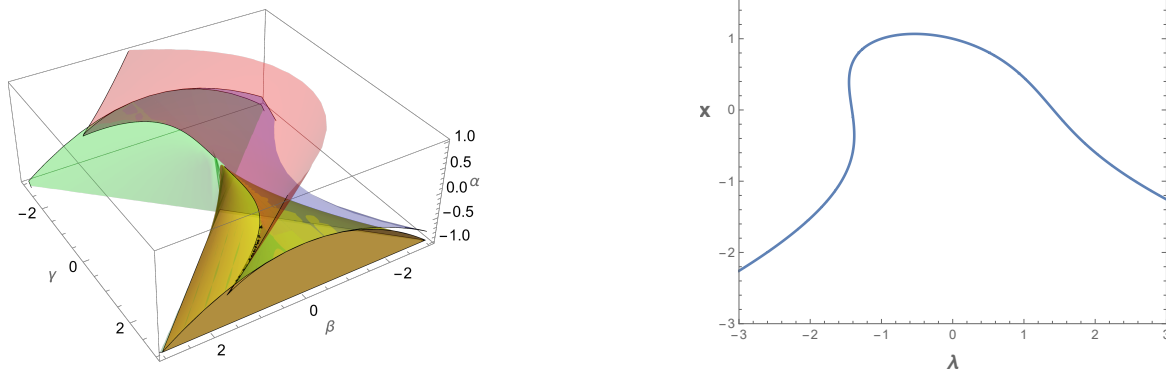


Figure 6.2. On the left, the solid region \mathcal{R}_1 (yellow) bounded by the graphs of B_- (blue), B_+ (red) and H (green), and on the right, a plot of the solutions of $S_{\alpha,\beta,\gamma}(x, \lambda) = 0$ for $\alpha = -2$, $\beta = 1$ and $\gamma = 1$.

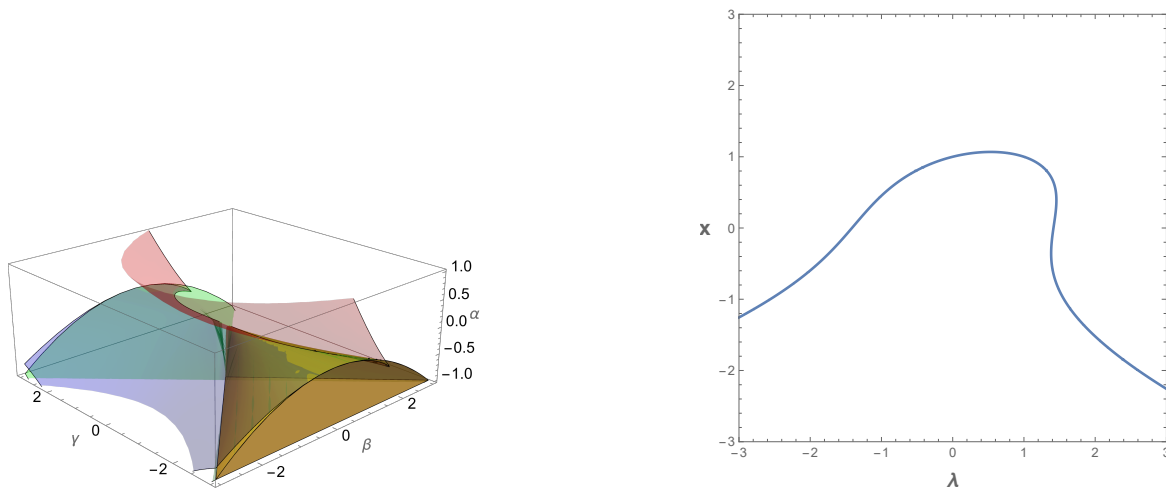


Figure 6.3. On the left, the solid region \mathcal{R}_2 (yellow) bounded by the graphs of B_- (blue), B_+ (red) and H (green), and on the right, a plot of the solutions of $S_{\alpha,\beta,\gamma}(x, \lambda) = 0$ for $\alpha = -2$, $\beta = 1$ and $\gamma = -1$.

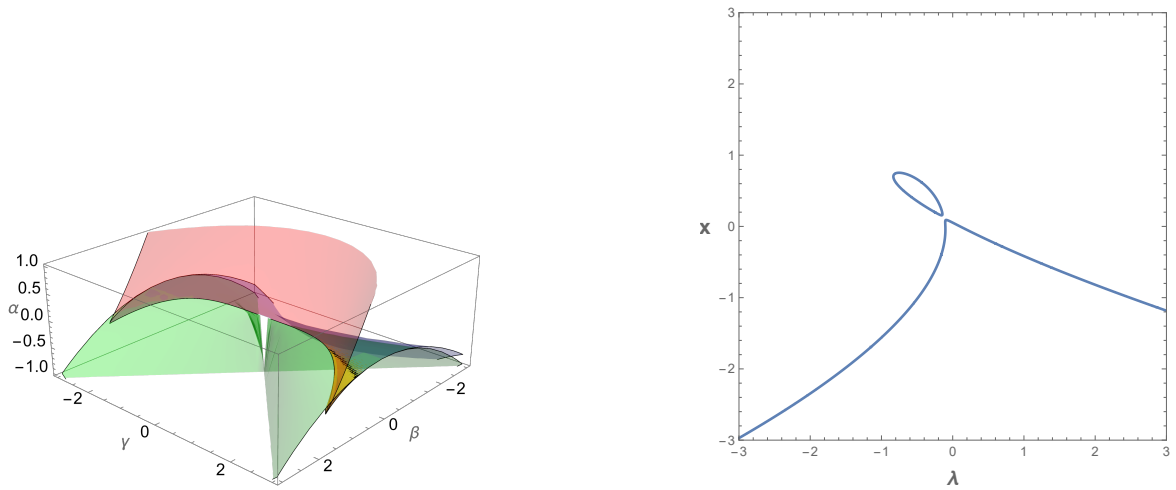


Figure 6.4. On the left, the solid region \mathcal{R}_3 (yellow) bounded by the graphs of B_- (blue), B_+ (red) and H (green), and on the right, a plot of the solutions of $S_{\alpha, \beta, \gamma}(x, \lambda) = 0$ for $\alpha = -0.01$, $\beta = 1/5$ and $\gamma = 2$.

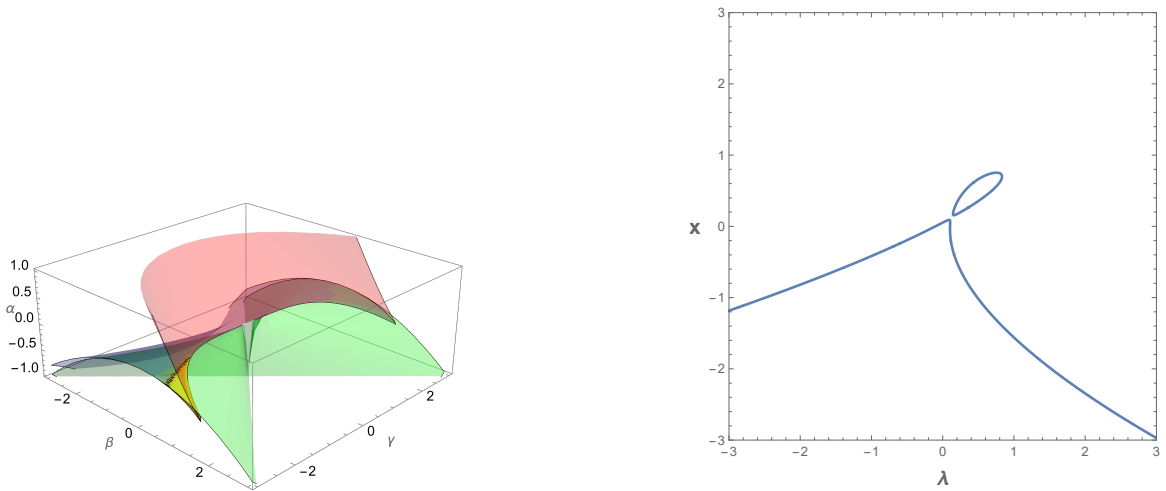


Figure 6.5. On the left, the solid region \mathcal{R}_4 (yellow) bounded by the graphs of B_- (blue), B_+ (red) and H (green), and on the right, a plot of the solutions of $S_{\alpha, \beta, \gamma}(x, \lambda) = 0$ for $\alpha = -0.01$, $\beta = 1/5$ and $\gamma = -2$.

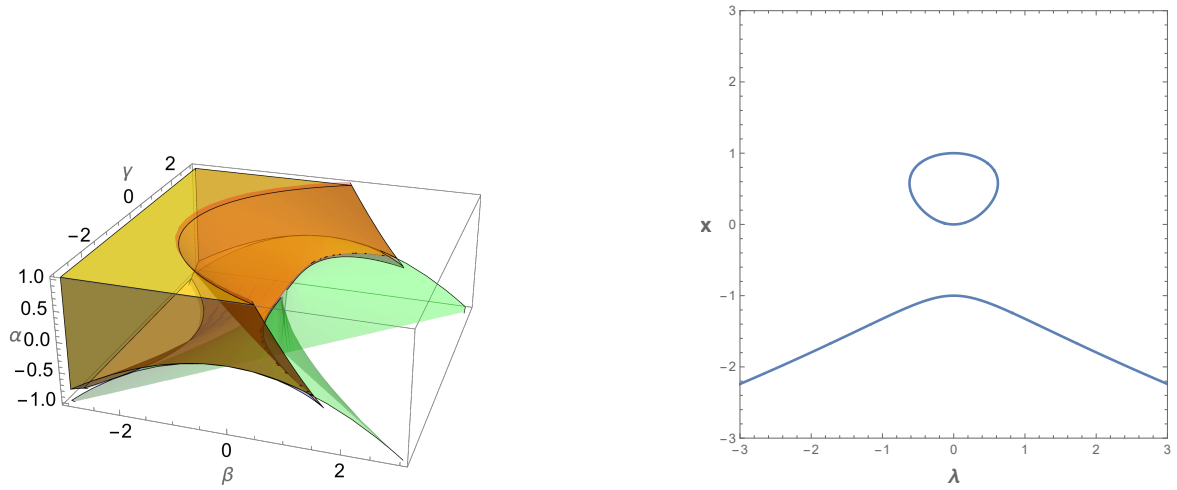


Figure 6.6. On the left, the solid region \mathcal{R}_5 (yellow) bounded by the graphs of B_- (blue), B_+ (red) and H (green), and on the right, a plot of the solutions of $S_{\alpha,\beta,\gamma}(x, \lambda) = 0$ for $\alpha = 0$, $\beta = -1$ and $\gamma = 0$.

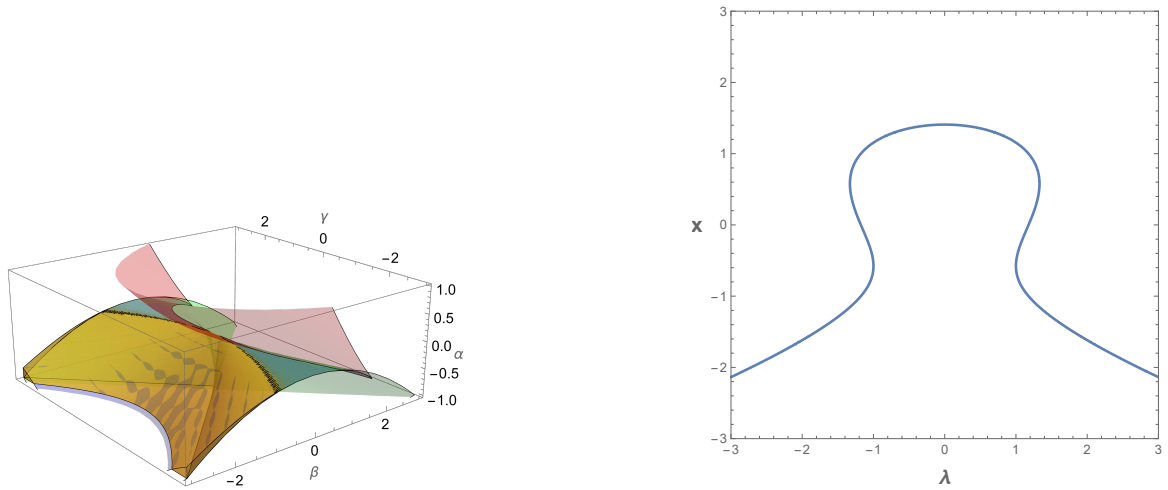


Figure 6.7. On the left, the solid region \mathcal{R}_6 (yellow) bounded by the graphs of B_- (blue), B_+ (red) and H (green), and on the right, a plot of the solutions of $S_{\alpha,\beta,\gamma}(x, \lambda) = 0$ for $\alpha = -1.38$, $\beta = -1$ and $\gamma = 0$.

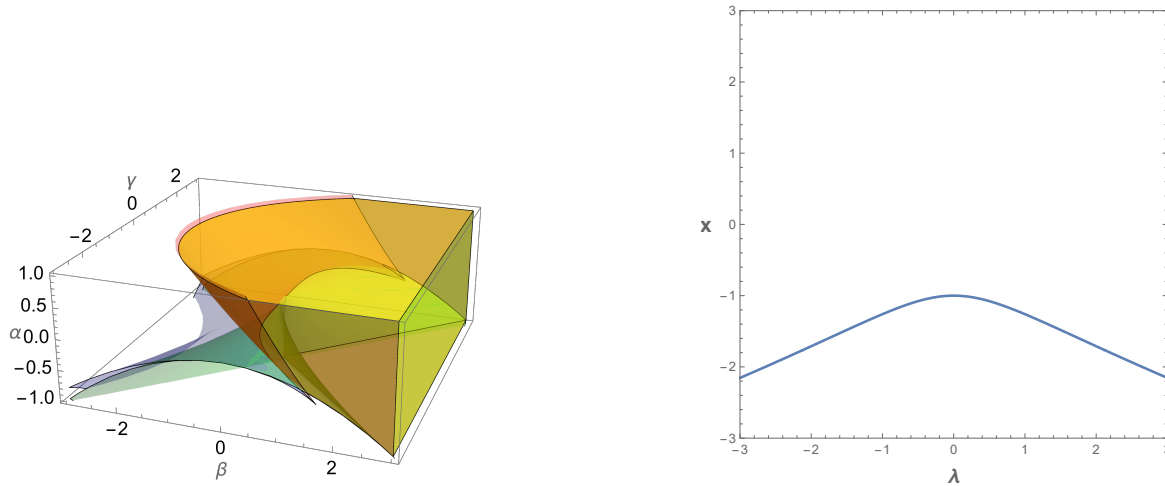


Figure 6.8. On the left, the solid region \mathcal{R}_7 (yellow) bounded by the graphs of B_- (blue), B_+ (red) and H (green), and on the right, a plot of the solutions of $S_{\alpha, \beta, \gamma}(x, \lambda) = 0$ for $\alpha = 1$, $\beta = 0$ and $\gamma = 0$.

By parametrizing curves in the (α, β, γ) -space that cross from one region \mathcal{R}_i to another \mathcal{R}_j with $i \neq j$, their associated diagrams must pass through an unstable diagram whose parameters lie in the set \mathcal{C} . For example, the transition from region \mathcal{R}_6 to region \mathcal{R}_5 corresponds to the transition from a mushroom to an isola. Notice that the transition from region \mathcal{R}_6 to region \mathcal{R}_7 corresponds to the scenario described in Section 5.4, where the closed loop in the bifurcation diagram shrinks to a point and subsequently disappears. In Section 5.4, we needed to compactify the parameter space in order to force F_α to exhibit this type of bifurcation. On the other hand, the unfolding $S_{\alpha, \beta, \gamma}$ is capable of exhibiting it within arbitrarily small neighborhoods in parameter space, without any need to resort to compactification.

6.2 Study of the family of mappings

$$x \mapsto x + S_{\alpha, \beta, \gamma}(x, \lambda)$$

Using techniques analogous to those presented in Theorem 5.3.1, we conclude the following:

Theorem 6.2.1. *Let f be a smooth, one-dimensional, discrete, and monotonous dynamical system whose bifurcation diagram is contact-equivalent to $S_{\alpha^*, \beta^*, \gamma^*}$ for some $(\alpha^*, \beta^*, \gamma^*) \in \mathbb{R}^3$. Then, the behavior of smooth perturbations of f can be studied by analyzing the family*

of dynamical systems defined by

$$x \mapsto x + S_{\alpha, \beta, \gamma}(x, \lambda), \quad (6.9)$$

where the parameters (α, β, γ) are taken sufficiently close to $(\alpha^*, \beta^*, \gamma^*)$.

This result allows us to reduce the local analysis of the perturbed system to the study of a universal unfolding, providing a unified framework to understand nearby dynamical behaviors. We now present the study of the dynamics of the mapping $x \mapsto x + S_{\alpha, \beta, \gamma}(x, \lambda)$ for $(\alpha, \beta, \gamma) \in \mathbb{R}_i$ for $i = 1, \dots, 7$.

6.2.1 Case 1: $(\alpha, \beta, \gamma) \in \mathcal{R}_1$.

If $(\alpha, \beta, \gamma) \in \mathcal{R}_1$, then system (6.9) exhibits two fold points at λ_1 and λ_2 with $\lambda_1 < \lambda_2$.

The system's dynamics as λ increases are as follows:

- For $\lambda < \lambda_1$, the system has one unstable fixed point $x_1(\lambda)$.
- When λ crosses λ_1 , a fold bifurcation occurs and two more fixed points $x_2(\lambda)$ and $x_3(\lambda)$ appear, where

$$x_1 < x_2 < x_3,$$

with:

- x_1 : unstable fixed point,
- x_2 : stable fixed point,
- x_3 : unstable fixed point.
- Finally, when λ crosses λ_2 , another fold occurs, and the fixed points $x_1(\lambda)$ and $x_2(\lambda)$ disappear. Only the unstable fixed point $x_3(\lambda)$ remains.

6.2.2 Case 2: $(\alpha, \beta, \gamma) \in \mathcal{R}_2$.

If $(\alpha, \beta, \gamma) \in \mathcal{R}_2$, then system (6.9) exhibits two fold points at λ_1 and λ_2 with $\lambda_1 < \lambda_2$.

The system's dynamics as λ increases are as follows:

- For $\lambda < \lambda_1$, the system has one unstable fixed point $x_3(\lambda)$.

- When λ crosses λ_1 , a fold bifurcation occurs and two more fixed points $x_1(\lambda)$ and $x_2(\lambda)$ appear, where

$$x_1 < x_2 < x_3,$$

with:

- x_1 : unstable fixed point,
 - x_2 : stable fixed point,
 - x_3 : unstable fixed point.
- Finally, when λ crosses λ_2 , another fold occurs, and the fixed points $x_3(\lambda)$ and $x_2(\lambda)$ disappear. Only the unstable fixed point $x_1(\lambda)$ remains.

6.2.3 Case 3: $(\alpha, \beta, \gamma) \in \mathcal{R}_3$.

If $(\alpha, \beta, \gamma) \in \mathcal{R}_3$, then system (6.9) exhibits four fold points at λ_i for $i = 1, 2, 3, 4$, with $\lambda_i < \lambda_{i+1}$.

The system's dynamics as λ increases are as follows:

- For $\lambda < \lambda_1$, the system has one unstable fixed point $x_1(\lambda)$.
- When λ crosses λ_1 , a fold bifurcation occurs and two more fixed points $x_4(\lambda)$, $x_5(\lambda)$ appear, where

$$x_1 < x_4 < x_5,$$

with:

- x_1 : unstable fixed point,
 - x_4 : stable fixed point,
 - x_5 : unstable fixed point.
- When λ crosses λ_2 , a fold bifurcation occurs and the fixed points $x_4(\lambda)$ and $x_5(\lambda)$ disappear, thus the graphs of $x_4(\lambda)$ and $x_5(\lambda)$ make a close loop.
 - When λ crosses λ_3 , a fold bifurcation occurs and the fixed points $x_2(\lambda)$ and $x_3(\lambda)$ appear, where

$$x_1 < x_2 < x_3,$$

with:

- x_1 : unstable fixed point,
 - x_2 : stable fixed point,
 - x_3 : unstable fixed point.
- Finally, when λ crosses λ_4 a fold bifurcation occurs and the fixed points $x_1(\lambda)$ and $x_2(\lambda)$ disappear. Only the unstable fixed point $x_3(\lambda)$ remains.

6.2.4 Case 4: $(\alpha, \beta, \gamma) \in \mathcal{R}_4$.

If $(\alpha, \beta, \gamma) \in \mathcal{R}_4$, then system (6.9) exhibits four fold points at λ_i for $i = 1, 2, 3, 4$, with $\lambda_i < \lambda_{i+1}$.

The system's dynamics as λ increases are as follows:

- For $\lambda < \lambda_1$, the system has one unstable fixed point $x_3(\lambda)$.
- When λ crosses λ_1 , a fold bifurcation occurs and the fixed points $x_2(\lambda)$ and $x_1(\lambda)$ appear, where

$$x_1 < x_2 < x_3,$$

with:

- x_1 : unstable fixed point,
 - x_2 : stable fixed point,
 - x_3 : unstable fixed point.
- When λ crosses λ_2 a fold bifurcation occurs and the fixed points $x_3(\lambda)$ and $x_2(\lambda)$ disappear. Only the unstable fixed point $x_1(\lambda)$ remains.
 - When λ crosses λ_3 , a fold bifurcation occurs and two more fixed points $x_4(\lambda)$, $x_5(\lambda)$ appear, where

$$x_1 < x_4 < x_5,$$

with:

- x_1 : unstable fixed point,
- x_4 : stable fixed point,
- x_5 : unstable fixed point.

- Finally, when λ crosses λ_2 , a fold bifurcation occurs and the fixed points $x_4(\lambda)$ and $x_5(\lambda)$ disappear, thus the graphs of $x_4(\lambda)$ and $x_5(\lambda)$ make a close loop.

6.2.5 Case 5: $(\alpha, \beta, \gamma) \in \mathcal{R}_5$.

If $(\alpha, \beta, \gamma) \in \mathcal{R}_4$, then system (6.9) exhibits two fold points at λ_i for $i = 1, 2$, with $\lambda_i < \lambda_{i+1}$.

The system's dynamics as λ increases are as follows:

- For $\lambda < \lambda_1$, the system has one unstable fixed point $x_1(\lambda)$.
- When λ crosses λ_1 , a fold bifurcation occurs and the fixed points $x_2(\lambda)$ and $x_3(\lambda)$ appear, where

$$x_1 < x_2 < x_3,$$

with:

- x_1 : unstable fixed point,
- x_2 : stable fixed point,
- x_3 : unstable fixed point.
- Finally, When λ crosses λ_2 a fold bifurcation occurs and the fixed points $x_2(\lambda)$ and $x_3(\lambda)$ disappear, thus the graphs of $x_2(\lambda)$ and $x_3(\lambda)$ make a close loop.

6.2.6 Case 6: $(\alpha, \beta, \gamma) \in \mathcal{R}_6$.

If $(\alpha, \beta, \gamma) \in \mathcal{R}_6$, then system (6.9) exhibits four fold points at λ_i for $i = 1, 2, 3, 4$, with $\lambda_i < \lambda_{i+1}$.

The system's dynamics as λ increases are as follows:

- For $\lambda < \lambda_1$, the system has one unstable fixed point $x_1(\lambda)$.
- When λ crosses λ_1 , a fold bifurcation occurs and the fixed points $x_3(\lambda)$ and $x_4(\lambda)$ appear, where

$$x_1 < x_3 < x_4,$$

with:

- x_1 : unstable fixed point,
 - x_3 : stable fixed point,
 - x_4 : unstable fixed point.
- When λ crosses λ_2 a fold bifurcation occurs and the fixed points $x_1(\lambda)$ and $x_3(\lambda)$ disappear. Only the unstable fixed point $x_4(\lambda)$ remains.
 - When λ crosses λ_3 , a fold bifurcation occurs and two more fixed points $x_2(\lambda)$, $x_3(\lambda)$ appear, where

$$x_1 < x_2 < x_4,$$

with:

- x_1 : unstable fixed point,
 - x_2 : stable fixed point,
 - x_4 : unstable fixed point.
- Finally, when λ crosses λ_4 , a fold bifurcation occurs and the fixed points $x_4(\lambda)$ and $x_2(\lambda)$ disappear.

6.2.7 Case 7: $(\alpha, \beta, \gamma) \in \mathcal{R}_7$.

If $(\alpha, \beta, \gamma) \in \mathcal{R}_6$, then system (6.9) exhibits doesn't exhibit any local bifurcation, and has one unstable fixed point x_1 for all λ .

Chapter 7

Conclusions

In this study, we sought to deepen our understanding of the transition between *Mushroom* and *Isola* bifurcation structures in one-dimensional discrete dynamical systems. Our main objectives were:

- To present a normal form for this transition.
- To classify all dynamical behaviors arising from small perturbations of that normal form.
- To extend these results to a broader family of related mappings.

The use of tools from *Singularity Theory* and *Bifurcation Theory*, allowed us to establish Theorem 5.3.1, which states that any smooth perturbation of the mapping

$$x \mapsto x + x^3 - x^2 + \lambda^2 = x + G(x, \lambda), \quad (x, \lambda) \in [-1.2, 1.2]^2, \quad (7.1)$$

is topologically equivalent to

$$x \mapsto x + x^3 - x^2 + \lambda^2 + \alpha = x + F_\alpha(x, \lambda), \quad (x, \lambda) \in [-1.2, 1.2]^2, \quad (7.2)$$

for either $\alpha < 0$ or $0 < \alpha < 4/27$. Thus (7.2) serves as a normal form for the transition between a Mushroom bifurcation and an Isola bifurcation, and also a dynamical classification of small perturbations of the mapping (7.1). The proof of theorem 5.3.1 required relating $G(x, \lambda)$ to the more singular cusp

$$C(x, \lambda) = x^3 + \lambda^2, \quad (7.3)$$

and then studying its three-parameter universal unfolding

$$S_{\alpha,\beta,\gamma}(x, \lambda) = x^3 + \lambda^2 + \alpha + \beta x + \gamma x\lambda. \quad (7.4)$$

By factoring small perturbations of G through $S_{\alpha,\beta,\gamma}$, we showed that any map

$$y \mapsto f(y, \mu), \quad (y, \mu) \in [-1.2, 1.2]^2, \quad (7.5)$$

such that $f(y, \mu) - y$ is contact equivalent to $S_{\alpha^*,\beta^*,\gamma^*}$, is itself topologically equivalent to

$$x \mapsto x + S_{\alpha^*,\beta^*,\gamma^*}(x, \lambda).$$

Moreover, since G can be understood as an unfolding of the cusp C , global perturbations of G can be analyzed by performing germ-level analysis of $S_{\alpha,\beta,\gamma}$. We also showed that the parameter space decomposes into seven connected open regions, with two of them corresponding to Mushroom and Isola diagrams. Also, the boundaries of these connected components are codimension 1 submanifolds in the unfolding parameter space and correspond to the set of parameters that induce unstable diagrams of $S_{\alpha,\beta,\gamma}$.

Although our normal form F_α in (7.2) does not itself exhibit the bistability found in the literature (e.g. [Giri and Kar \(2021\)](#), [Otero \(2023\)](#), [Xu et al. \(2023\)](#)), its invertibility allows us to consider its inverse map which shares the same fixed points but reverses their stability. By applying the inverse function rule of differentiation and the Hartman-Grobman theorem 2.1.1, we recover the desired bistability alongside the Mushroom Isola transition. This allows us to conclude that despite our mapping not exhibiting the desired bistability, it is able to show all the possible geometric transitions a Mushroom or Isola bifurcation may exhibit.

For further work, one promising avenue is to derive explicit algebraic conditions guaranteeing that a given $P(x, \lambda) - x$ is contact equivalent to $G(x, \lambda)$ or even to the cusp $C(x, \lambda)$. This would align with the constructive approach of Theorem 4.4.2. Ultimately, implementing these conditions in numerical software could enable automatic detection and continuation of unstable diagrams, bringing us closer to explicit Poincaré return maps for planar limit cycle models exhibiting Mushroom and Isola phenomena.

Chapter 8

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