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Recursive online IV method for identification of continuous-time slowly time-varying models in closed loop

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Abstract:
Model estimation of industrial processes is often done in closed loop due, for instance, to production constraints or safety reasons. On the other hand, many processes are time-varying because of aging effects or changes in the environmental conditions. In this study, a recursive estimation algorithm for linear, continuous-time, slowly time-varying systems operating in closed loop, is developed. The proposed method consists in coupling linear filter approaches to handle the time-derivative, with closed-loop instrumental variable (IV) techniques to deal with measurement noise. Simulations show the advantages of using this IV-based method.

Keywords: recursive estimation, instrumental variable method, continuous-time models, closed loop identification

1. INTRODUCTION

Recursive on-line identification, also known as real-time identification, is a well studied subject. Standard literature about the topic are (Ljung and Söderström, 1983; Goodwin and Sin, 1984; Ljung and Gunnarsson, 1990; Ljung, 1999). The estimation is usually performed considering discrete-time (DT) models instead of continuous-time (CT) models, because the latter case is more involved due to the need of computing the time-derivatives. For linear time-invariant (LTI) models, linear filter methods can be used. Such an approach consists in applying a low-pass filter to the model and obtain prefiltered time-derivatives. Nevertheless, in the linear time-varying (LTV) case, multiplication with the differentiation operator is not commutative with the time-dependent variables (see *e.g.* (Laurain et al., 2011)). Thus, it would appear at first that we are no longer able to use linear filter methods as in the LTI case. One solution would be to model the time-varying parameters in a deterministic way such that the new parameters are constant or nearly constant, as in (Jiang and Schaufelberger, 1991). For on-line estimation, the difficulty of such an approach lies in finding suitable deterministic models. Moreover, extra parameters are introduced and, as a consequence, larger variance of the estimates are obtained. In this study, assuming that the parameters are slowly varying, we are able to apply a linear filter method to circumvent the time-derivative issue.

A standard way to deal with the on-line estimation of slowly time-varying parameters is to use the forgetting factor approach, which consists in discarding old data by weighting the prediction error accordingly (Ljung and Söderström, 1983). However, if the parameters vary at different rates, other approaches are more suitable. One of them consists in modeling the parameter variations by a stochastic model and the simplest one is the generalized random walk class of models (see chapters 4 and 5 in Young, 2011) of which the simplest is the random walk itself. Due to its flexibility, in this study, a simple random walk-based method is proposed.

In real-life, experiments for model identification are often done in closed loop because there are economic or safety constraints, or the system is unstable. The issue with closed-loop identification is that the feedback mechanism introduces correlation between the disturbances and the input signal, yielding biased estimates if no special strategy is used. For LTI systems, instrumental variable (IV) techniques have been proposed to solve this problem both for DT models (Gilson and Van den Hof, 2005; Gilson et al., 2011) and CT models (Gilson et al., 2008); (see also, Chapter 9 in Young, 2011). These ideas have been also extended to the identification of DT linear parameter-varying models in (Toth et al., 2012). On the other hand, recursive identification of nonlinear CT systems in closed loop has been considered in (Landau et al., 2001). In (Padilla et al., 2016), CT LTV systems operating in open loop are identified in a recursive fashion using an IV-based method. That study is extended here for the recursive identification of linear CT slowly time-varying systems operating in closed loop.

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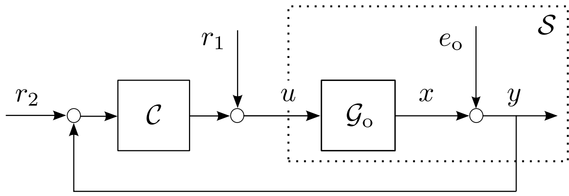

Fig. 1. Closed-loop system.

The paper is organized as follows: In Section 2, the identification problem is formulated. Then, in Section 3, the closed-loop IV identification method for LTI models is reviewed, recalling first the optimal (unbiased and minimum variance) result for off-line estimation and then considering the recursive algorithm. The proposed closed-loop IV-based identification method for linear CT slowly time-varying systems is given in Section 4 and afterwards a numerical example is presented in Section 5. Finally, the conclusions are drawn in Section 6.

2. PROBLEM FORMULATION

Let us consider a CT LTV system $\mathcal{S}$ with plant $\mathcal{G}_o$, input u and output y

$$\mathcal{S}\begin{cases} A_o(p,t)x(t) = B_o(p,t)u(t) \\ y(t_k) = x(t_k) + e_o(t_k) \end{cases} \tag{1}$$

where p is the differentiation operator; $x(t)$ is the noise-free output; and the additive term $e_o(t_k)$ is a zero-mean DT white noise sequence. The argument t_k in the second equation denotes the sampled value of the associated variable at the kth sampling instant. In the closed-loop configuration in Figure 1, the CT controller $\mathcal{C}$ can be any nonlinear and/or time-varying controller. Knowing $\mathcal{C}$, we can compute $u(t)$ as follows

$$u(t_k) = r_1(t_k) + \mathcal{C}(r_2(t_k) - y(t_k)) \tag{2}$$

where $\mathcal{C}$ is the operator form of the controller and $r_1(t_k), r_2(t_k)$ are external signals.

Assuming that the system (1) belongs to the model set $\mathcal{M}$, *i.e.* $\mathcal{S} \in \mathcal{M}$, we have

$$\mathcal{M}\begin{cases} A(p,t,\theta)x(t) = B(p,t,\theta)u(t) \\ y(t_k) = x(t_k) + e(t_k) \end{cases} \tag{3}$$

$A(p,t,\theta)$ and $B(p,t,\theta)$ are the following polynomials with time-varying parameters:

$$B(p,t,\theta) = b_0(t)p^{n_b} + b_1(t)p^{n_b-1} + \ldots + b_{n_b}(t) \tag{4}$$

$$A(p,t,\theta) = p^{n_a} + a_1(t)p^{n_a-1} + \ldots + a_{n_a}(t) \tag{5}$$

where $n_a \geq n_b$ and $e(t_k)$ is a zero-mean DT white noise. The time-varying parameters can be gathered in the parameter vector $\theta(t)$, *i.e.*

$$\theta(t) = [a_1(t) \ldots a_{n_a}(t)\, b_0(t) \ldots b_{n_b}(t)]^T \tag{6}$$

which contains $d = n_a + n_b + 1$ elements.

Suppose additionally that the following assumptions are satisfied:

A1. The polynomial degrees n_a and n_b are *a priori* known.
A2. The true time-varying parameter vector is bounded. Moreover it is slowly varying, *i.e.* $\left\|\dot{\theta}_o(t)\right\| \leq \epsilon_\theta$, where ϵ_θ is a small number.
A3. The controller $\mathcal{C}$ is known.
A4. The controller $\mathcal{C}$ ensures BIBO stability of the closed-loop system defined by (1) and (2).
A5. The reference signal $r(t_k) = r_1(t_k) + \mathcal{C}r_2(t_k)$ is persistently exciting.

Then, the identification problem is to recursively estimate the time-varying parameters that characterize the model structure given by (3), based on the sequences $\{r(t_k), u(t_k), y(t_k)\}_{k=1}^{N'}$, where N' is the number of samples which increases by one with every recursion.

3. IV ESTIMATION OF LTI SYSTEMS IN CLOSED LOOP

In this section we assume that both the plant and the controller are LTI systems, *i.e.*

$$\mathcal{G}_o : G_o(p) = \frac{B_o(p)}{A_o(p)} \tag{7}$$

$$\mathcal{C} : C_c(p) = \frac{Q_c(p)}{P_c(p)} \tag{8}$$

with the pairs (A_o, B_o) and (P_c, Q_c) assumed to be coprime. First we address the off-line estimation problem and afterwards its recursive counterpart.

3.1 Optimal off-line estimation

Informally, when the parameters in (3) are constant, a *snapshot* of the model at the kth sampling instant can be written as

$$y(t_k) = G(p,\theta)u(t_k) + e(t_k) \tag{9}$$

This can be written also as the following linear regression

$$y^{(n_a)}(t_k) = \varphi^T(t_k)\theta + v(t_k) \tag{10}$$

where

$$\varphi^T(t_k) = \left[-y^{(n_a-1)}(t_k) \cdots - y(t_k)\, u^{(n_b)}(t_k) \cdots u(t_k)\right] \tag{11}$$

and

$$v(t_k) = A(p)e(t_k) \tag{12}$$

If the time-derivatives were available, the parameters θ could be estimated by minimizing the sum of squares of

$$\epsilon(t_k) = y^{(n_a)}(t_k) - \varphi^T(t_k)\theta \tag{13}$$

i.e.,

$$\hat{\theta} = \arg\min_{\theta\in\mathbb{R}^d} \frac{1}{N'}\sum_{k=1}^{N'} \|\epsilon(t_k)\|_2^2 \tag{14}$$

However, due to the correlation between $\varphi(t_k)$ and $v(t_k)$ introduced by the feedback mechanism, the parameter estimates will be biased. A solution is to use the closed-loop IV method given by (see (Gilson et al., 2008))

$$\hat{\theta} = \arg\min_{\theta\in\mathbb{R}^d} \frac{1}{N'}\left\|\left[\sum_{k=1}^{N'} F(p)\zeta(t_k)F(p)\varphi^T(t_k)\right]\theta - \left[\sum_{k=1}^{N'} F(p)\zeta(t_k)F(p)y^{(n_a)}(t_k)\right]\right\|_W^2 \tag{15}$$

where $F(p)$ is a stable prefilter and $\|x\|_W^2 = x^T W x$, with W a positive definite weighting matrix. $\zeta(t_k)$ is the instrument vector that can be computed in different ways.

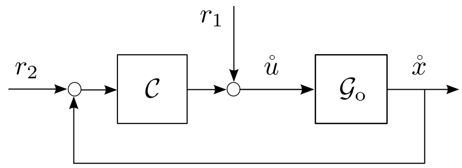

Fig. 2. Auxiliary model.

If $\mathcal{S} \in \mathcal{M}$, the estimates (15) are consistent under the following conditions [1] :

C1. $\bar{\mathbb{E}}\{F(p)\zeta(t_k)F(p)\zeta^T(t_k)\}$ is full column rank.
C2. $\bar{\mathbb{E}}\{F(p)\zeta(t_k)F(p)v_{\mathrm{o}}(t_k)\} = 0$.

Optimal estimates, *i.e.* unbiased and minimum variance estimates, can be obtained if the following additional conditions are satisfied (Gilson et al., 2008)

C3. $W = I$
C4. $F(p) = \frac{1}{A_{\mathrm{o}}(p)}$
C5. The instrument vector is computed using the auxiliary model from Figure 2 as follows

$$\zeta(t_k) = \big[-\mathring{x}^{(n_a-1)}(t_k) \; \dots \; -\mathring{x}(t_k) \quad \mathring{u}^{(n_b)}(t_k) \; \dots \; \mathring{u}(t_k) \big]^T \tag{16}$$

where

$$A_{\mathrm{o}}(p)\mathring{x}(t_k) = B_{\mathrm{o}}(p)\mathring{u}(t_k) \tag{17a}$$

$$\mathring{u}(t_k) = r_1(t_k) + \mathcal{C}(r_2(t_k) - \mathring{x}(t_k)) \tag{17b}$$

3.2 CLSRIVC algorithm

Conditions C1-C5 have to be fulfilled to obtain the optimal IV solution. The first three conditions can be readily satisfied (Söderström and Stoica, 1983). However C4-C5 require knowledge of the true unknown system, which is the usual dilemma with optimal estimation. To circumvent this problem, different estimation algorithms have been proposed, which vary depending on the choice of the instruments $\zeta(t_k)$, the filter $F(p)$ and the model structure of the true system (Gilson et al., 2008). In the following we recall the closed loop simplified refined instrumental variable method for continuous-time model estimation (CLSRIVC). This is an optimal method for COE models in closed loop and it was proposed in (Gilson et al., 2008).

CLSRIVC is an iterative method, which is based on a linear filter approach and the IV technique. The former allows us to compute prefiltered time-derivatives using a filter, and the latter considers instruments that are used to reduce bias. Both the filter and the instruments are updated in each iteration based on the parameter estimates obtained at the previous iteration. The filter is defined as follows

$$F(p, \hat{\theta}^{i-1}) = \frac{1}{\hat{A}(p, \hat{\theta}^{i-1})} \tag{18}$$

Then, the signals $y_{\mathrm{f}}^{(i)}(t_k)$ and $u_{\mathrm{f}}^{(i)}(t_k)$ correspond to the prefiltered time-derivatives of the output and input in the bandwidth of interest, *i.e.*

$$y_{\mathrm{f}}^{(i)}(t_k) = p^i F(p, \hat{\theta}^{i-1})y(t_k) \qquad i = 0, \dots, n_a \tag{19a}$$

$$u_{\mathrm{f}}^{(i)}(t_k) = p^i F(p, \hat{\theta}^{i-1})u(t_k) \qquad i = 0, \dots, n_b \tag{19b}$$

[1] The notation $\bar{\mathbb{E}}[.] = \lim_{N' \to \infty} \frac{1}{N'} \sum_{k=1}^{N'} \mathbb{E}[.]$ is adopted from the prediction error framework of (Ljung, 1999).

Applying the filter (18) to (9) yields (except for transients)

$$y_{\mathrm{f}}^{(n_a)}(t_k) + a_1 y_{\mathrm{f}}^{(n_a-1)}(t_k) + \ldots + a_{n_a} y_{\mathrm{f}}(t_k) = b_0 u_{\mathrm{f}}^{(n_b)}(t_k) + \ldots + b_{n_b} u_{\mathrm{f}}(t_k) + v_{\mathrm{f}}(t_k) \tag{20}$$

with

$$v_{\mathrm{f}}(t_k) = F(p)v(t_k) = F(p)A(p)e(t_k) \tag{21}$$

Equation (20) can be rewritten as a linear regression,

$$y_{\mathrm{f}}^{(n_a)}(t_k) = \varphi_{\mathrm{f}}^T(t_k)\theta + v_{\mathrm{f}}(t_k) \tag{22}$$

where

$$y_{\mathrm{f}}^{(n_a)}(t_k) = p^{n_a} F(p, \hat{\theta}^{i-1})y(t_k) \tag{23a}$$

$$\varphi_{\mathrm{f}}^T(t_k) = F(p, \hat{\theta}^{i-1})\varphi^T(t_k) \tag{23b}$$

$\varphi(t_k)$ is defined in (11) and its filtered version is given by

$$\varphi_{\mathrm{f}}^T(t_k) = \left[-y_{\mathrm{f}}^{(n_a-1)}(t_k) \cdots - y_{\mathrm{f}}(t_k) \; u_{\mathrm{f}}^{(n_b)}(t_k) \cdots u_{\mathrm{f}}(t_k) \right] \tag{24}$$

The filtered instrument is computed as follows

$$\begin{aligned} \zeta_{\mathrm{f}}(t_k, \hat{\theta}^{i-1}) &= F(p, \hat{\theta}^{i-1})\zeta(t_k, \hat{\theta}^{i-1}) \\ &= \Big[-\hat{\mathring{x}}_{\mathrm{f}}^{(n_a-1)}(t_k) \; \dots \; -\hat{\mathring{x}}_{\mathrm{f}}(t_k) \\ &\qquad \hat{\mathring{u}}_{\mathrm{f}}^{(n_b)}(t_k) \; \dots \; \hat{\mathring{u}}_{\mathrm{f}}(t_k) \Big]^T \end{aligned} \tag{25}$$

with

$$\hat{A}(p, \hat{\theta}^{i-1})\hat{\mathring{x}}(t_k) = \hat{B}(p, \hat{\theta}^{i-1})\hat{\mathring{u}}(t_k) \tag{26a}$$

$$\hat{\mathring{u}}(t_k) = r_1(t_k) + \mathcal{C}(r_2(t_k) - \hat{\mathring{x}}(t_k)) \tag{26b}$$

Then, the CLSRIVC estimates are

$$\hat{\theta}^i = \left[\sum_{k=1}^{N'} \zeta_{\mathrm{f}}(t_k, \hat{\theta}^{i-1})\varphi_{\mathrm{f}}^T(t_k, \hat{\theta}^{i-1}) \right]^{-1} \cdot \left[\sum_{k=1}^{N'} \zeta_{\mathrm{f}}(t_k, \hat{\theta}^{i-1}) y_{\mathrm{f}}^{(n_a)}(t_k, \hat{\theta}^{i-1}) \right]$$

3.3 Recursive estimation algorithm

The algorithm presented previously can be adapted for recursive estimation. The recursive version of CLSRIVC is given by the following algorithm, which is based on the methodology used in the DT case (see *e.g.* (Young, 2011)):

$$\hat{\theta}(t_k) = \hat{\theta}(t_{k-1}) + L(t_k)\varepsilon(t_k) \tag{27a}$$

$$\varepsilon(t_k) = y_{\mathrm{f}}^{(n_a)}(t_k) - \varphi_{\mathrm{f}}^T(t_k)\hat{\theta}(t_{k-1}) \tag{27b}$$

$$L(t_k) = \frac{P(t_{k-1})\zeta_{\mathrm{f}}(t_k)}{1 + \varphi_{\mathrm{f}}^T(t_k)P(t_{k-1})\zeta_{\mathrm{f}}(t_k)} \tag{27c}$$

$$P(t_k) = P(t_{k-1}) - L(t_k)\varphi_{\mathrm{f}}^T(t_k)P(t_{k-1}) \tag{27d}$$

with

$$y_{\mathrm{f}}^{(n_a)}(t_k) = p^{n_a} F(p, \hat{\theta}(t_{k-1}))y(t_k) \tag{28a}$$

$$\varphi_{\mathrm{f}}(t_k) = F(p, \hat{\theta}(t_{k-1}))\varphi(t_k) \tag{28b}$$

$\varphi(t_k)$ is defined in (11), and the adaptive prefilter is

$$F(p, \hat{\theta}(t_{k-1})) = \frac{1}{\hat{A}(p, \hat{\theta}(t_{k-1}))} \tag{29}$$

The instrument is given by

$$\begin{aligned} \zeta_{\mathrm{f}}(t_k) &= F(p, \hat{\theta}(t_{k-1}))\zeta(t_k, \hat{\theta}(t_{k-1})) \\ &= \Big[-\hat{\mathring{x}}_{\mathrm{f}}^{(n_a-1)}(t_k) \; \cdots \; -\hat{\mathring{x}}_{\mathrm{f}}(t_k) \\ &\qquad \hat{\mathring{u}}_{\mathrm{f}}^{(n_b)}(t_k) \; \cdots \; \hat{\mathring{u}}_{\mathrm{f}}(t_k) \Big]^T \end{aligned} \tag{30}$$

This algorithm will be called the CLRSRIVC method, where the additional 'R' stands for recursive. Note that unlike CLSRIVC for the off-line estimation case where the filter and instrument is defined using estimates from a previous iteration, here it is computed from the estimates obtained at the previous recursion time t_{k-1}.

4. IV ESTIMATION OF LTV SYSTEMS IN CLOSED LOOP

We assume now that the CT parameters of the plant are slowly time-varying.

4.1 Handling of the time-derivative issue

In theoretical terms, multiplication with the differentiation operator p is not commutative with the time-dependent variables (see *e.g.* (Laurain et al., 2011)). Thus, it would appear at first that we are no longer able to filter (3) and obtain a linear regression, as in the LTI case (see (22)).

A solution is to assume that the parameters are slowly varying (see Assumption (A2) in Section 2) in the sense specified by Lemma 1 in (Dimogianopoulos and Lozano, 2001), which states that for a certain time interval, under slow parameter variations, an LTV model can be locally approximated by an LTI model. In this study, under Assumption (A2) in the sense of Lemma 1 in (Dimogianopoulos and Lozano, 2001), we neglect the non-commutativity aspect and apply a filter to (3), which yields

$$y_{\mathrm{f}}^{(n_a)}(t_k) = \varphi_{\mathrm{f}}^T(t_k)\theta(t_k) + v_{\mathrm{f}}(t_k) \tag{31}$$

where $y_{\mathrm{f}}^{(n_a)}(t_k)$ and $\varphi_{\mathrm{f}}^T(t_k)$ are defined in (28).

4.2 Random walk-based CLRSRIVC algorithm

A solution to the LTV estimation problem when the parameters vary at different rates is to represent them through a simple random walk model, so the full model becomes:

$$\mathcal{M}\begin{cases}\theta(t_k) = \theta(t_{k-1}) + w(t_k)\\ A(p,t,\theta)x(t) = B(p,t,\theta)u(t)\\ y(t_k) = x(t_k) + e(t_k)\end{cases} \tag{32}$$

Here, $w(t_k)$ and $e(t_k)$ are independent zero-mean DT Gaussian noise processes with covariance matrix Q_w and variance σ_e^2, respectively.

Although the comments about bias in Section 3 apply to LTI systems, we should expect them to be valid when an LTV system approaches an LTI one, *i.e.* when the parameters vary slowly. A suitable solution is to consider the CLRSRIVC algorithm that can be adapted for LTV systems. In this time-varying case, the estimates are obtained by means of the following algorithm

Prediction step:

$$\hat{\theta}(t_k|t_{k-1}) = \hat{\theta}(t_{k-1}) \tag{33a}$$

$$P(t_k|t_{k-1}) = P(t_{k-1}) + Q_n \tag{33b}$$

Correction step:

$$\hat{\theta}(t_k) = \hat{\theta}(t_k|t_{k-1}) + L(t_k)\varepsilon(t_k) \tag{33c}$$

$$\varepsilon(t_k) = y_{\mathrm{f}}^{(n_a)}(t_k) - \varphi_{\mathrm{f}}^T(t_k)\hat{\theta}(t_k|t_{k-1}) \tag{33d}$$

$$L(t_k) = \frac{P(t_k|t_{k-1})\zeta_{\mathrm{f}}(t_k)}{1 + \varphi_{\mathrm{f}}^T(t_k)P(t_k|t_{k-1})\zeta_{\mathrm{f}}(t_k)} \tag{33e}$$

$$P(t_k) = P(t_k|t_{k-1}) - L(t_k)\varphi_{\mathrm{f}}^T(t_k)P(t_k|t_{k-1}) \tag{33f}$$

where $y_{\mathrm{f}}^{(n_a)}(t_k)$ and $\varphi_{\mathrm{f}}(t_k)$ are defined in (28). The normalized covariance matrix Q_n, also called *noise-variance ratio* matrix, is defined by

$$Q_n = Q_w/\sigma_e^2 \tag{34}$$

The instrument is computed as follows

$$\begin{aligned}\zeta_{\mathrm{f}}(t_k) &= F(p, \hat{\theta}(t_{k-1}))\zeta(t_k, \hat{\theta}(t_{k-1}))\\ &= \Big[-\hat{\mathring{x}}_{\mathrm{f}}^{(n_a-1)}(t_k) \ \dots \ -\hat{\mathring{x}}_{\mathrm{f}}(t_k)\\ &\qquad \hat{\mathring{u}}_{\mathrm{f}}^{(n_b)}(t_k) \ \dots \ \hat{\mathring{u}}_{\mathrm{f}}(t_k) \Big]^T\end{aligned} \tag{35}$$

with $\hat{\mathring{x}}(t_k)$ and $\hat{\mathring{u}}(t_k)$ generated from

$$\hat{A}(p, \hat{\theta}(t_{k-1}), t)\hat{\mathring{x}}(t_k) = \hat{B}(p, \hat{\theta}(t_{k-1}), t)\hat{\mathring{u}}(t_k) \tag{36a}$$

$$\hat{\mathring{u}}(t_k) = r_1(t_k) + \mathcal{C}(r_2(t_k) - \hat{\mathring{x}}(t_k)) \tag{36b}$$

where $\mathcal{C}$ is the controller.

4.3 Implementation issues

To apply the proposed estimation algorithm in practice, the following implementation issues have to be considered:

- *Initialization of the IV-based methods.* The initialization is done using the recursive least squares state-variable filter (RLSSVF) approach (Padilla et al., 2016), which is the algorithm (33) but with $\zeta_{\mathrm{f}}(t_k) = \varphi_{\mathrm{f}}(t_k)$. The estimation with RLSSVF is performed using only the measurements of the input and output $\{u(t_k), y(t_k)\}_{k=1}^{N'}$ as for open-loop operation. This corresponds to a direct estimation approach (Ljung, 1999).
- *Choice of λ_{svf} in SVF.* RLSSVF requires the use of a state-variable filter (SVF) of the form

$$F(p) = \frac{1}{(p + \lambda_{\mathrm{svf}})^{n_a}} \tag{37}$$

  with λ_{svf} the cut-off frequency of the SVF. λ_{svf} is a user parameter that should be chosen somewhat larger than the system bandwidth. In the LTV case and especially for systems with large variations in the bandwidth, the specification of λ_{svf} can be critical since the system bandwidth is time-varying.
- *Specification of the normalized covariance matrix Q_n.* The random walk model requires the specification of Q_n. The performance of the algorithm in terms of tracking ability and noise sensitivity depends on Q_n, which can be estimated, for instance, by the maximum likelihood method (Young, 2011). So far this method has not been extended to be used with the approaches proposed in this study.

5. NUMERICAL EXAMPLE

Two recursive algorithms are evaluated: CLRSRIVC and RLSSVF. The example presented in (Padilla et al., 2016) is adapted here to the closed-loop configuration shown in Figure 1, where $\mathcal{S}$ is the following second order system

$$\mathcal{S}\begin{cases}\big(p^2 + a_1^{\circ}(t)p + a_2^{\circ}(t)\big)x(t) = b_0^{\circ}(t)u(t)\\ y(t_k) = x(t_k) + e(t_k)\end{cases} \tag{38}$$

The parameters are slowly varying as follows: $a_1^{\circ}(t)$ varies between 5 and 45 in a linear fashion, $b_0^{\circ}(t)$ remains constant at 200 and $a_2^{\circ}(t)$ is given by

$$a_2^{\circ}(t) = 160 - 90\cos(2\pi t/1000)$$

The sampling time is set to 0.01 s and the total simulation time is 1000 s. Regarding the external signals (see Figure 1), $r_1(t)$ is a PRBS and $r_2(t) = 0$. $e(t_k)$ is a zero-mean DT Gaussian noise with constant variance 0.1. In this example, as a consequence of the time-varying parameters, the DC gain is decreasing towards half of the simulation time; and since the noise variance is kept constant, the signal-to-noise ratio (SNR) is decreasing around half of the simulation time. Notice that the ratio between the maximum and minimum bandwidths is nearly 10, *i.e.* the bandwidth variation is relatively large over the total simulation time.

CT filtering operations for the computation of prefiltered time-derivatives and instruments are implemented from their discretized counterparts. The discretized version of the PID controller is given by

$$\mathcal{C} : C_c(q^{-1}) = k_p + \frac{k_i}{1-q^{-1}} + k_d(1-q^{-1}) \tag{39}$$

with q^{-1} the backward shift operator and $k_p = 1.79$, $k_i = 13.8$, $k_d = 5.83\cdot 10^{-2}$. Considering (38) and (39), and assuming that the given parameters are slowly varying, the closed-loop system is exponentially stable according to the frozen-time approach (Ilchmann et al., 1987). Moreover, since the parameters are bounded, the closed-loop system is BIBO stable (Rugh, 1996).

For all the simulations, we use the same value for the hyperparameters Q_n and λ_{svf}. The former is a diagonal matrix and its diagonal elements are determined by means of a numerical search that yields 10^{-5}, 10^{-4} and 10^{-10} for a_1°, a_2° and b_0°, respectively. The value corresponding to b_0° has been chosen assuming that it is known that this parameter is constant. For the SVF, $\lambda_{\text{svf}} = 16$ rad/s is chosen, *i.e.* a value larger than the maximum bandwidth.

We present next some results for a single experiment run out of the 100 that are considered later in a Monte Carlo simulation analysis.

5.1 Single experiment analysis

In the LTI open-loop case, it is known that the RLSSVF estimates are always biased due to the noise. Even if the bias cannot be removed, it can be reduced by a proper choice of the cut off frequency λ_{svf}. In the open-loop LTV case, this is more difficult since the system bandwidth is varying, while the SVF bandwidth is constant (Padilla et al., 2016). When the system operates in closed loop there is an additional issue, namely the correlation between the input $u(t_k)$ and the noise $e_{\text{o}}(t_k)$ due to the feedback mechanism. That might result in a larger bias on the estimates. In order to illustrate the impact of the noise on RLSSVF in the closed-loop LTV situation, we compare the deterministic case with the noisy case. From Figure 3,

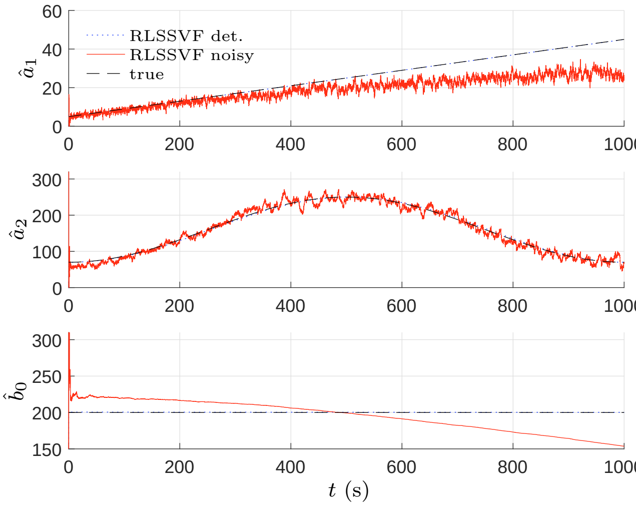

Fig. 3. True parameters and RLSSVF estimates for the deterministic (det.) case and noisy case. (The estimates for the deterministic case are matching the true values).

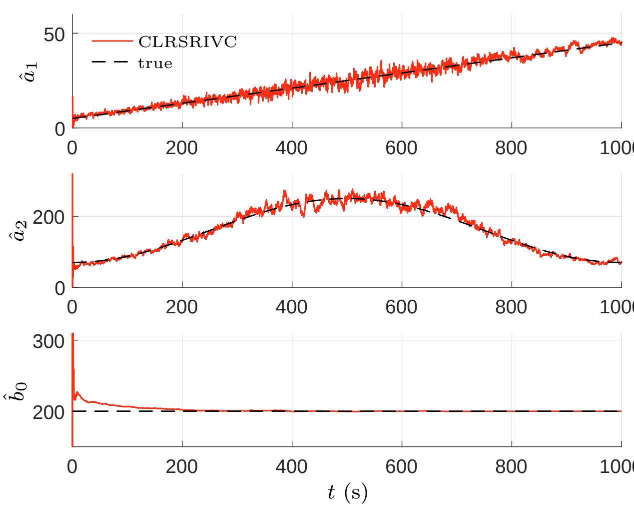

Fig. 4. True parameters and CLRSRIVC estimates.

we can see that RLSSVF is able to track the parameters only in the noise-free situation. It is important to notice that the value used for λ_{svf} is a very good choice since it is slightly higher than the maximum system bandwidth.

To circumvent this problem, the closed-loop IV approach introduced in Section 4 can be used. From Figure 4 we can see that CLRSRIVC is able to track the parameters in this situation where the parameters vary at different rates.

5.2 Monte-Carlo simulation analysis

To complete the analysis, a Monte-Carlo simulation with 100 experiments is run. The parameter relative error given by

$$\tilde{\theta}_i(t_k) = \frac{|\theta_i(t_k) - \hat{\theta}_i(t_k)|}{|\theta_i(t_k)|} \times 100 \tag{40}$$

with

$$\begin{bmatrix}\hat{\theta}_1(t_k)\\ \hat{\theta}_2(t_k)\\ \hat{\theta}_3(t_k)\end{bmatrix} = \begin{bmatrix}\hat{a}_1(t_k)\\ \hat{a}_2(t_k)\\ \hat{b}_0(t_k)\end{bmatrix}$$

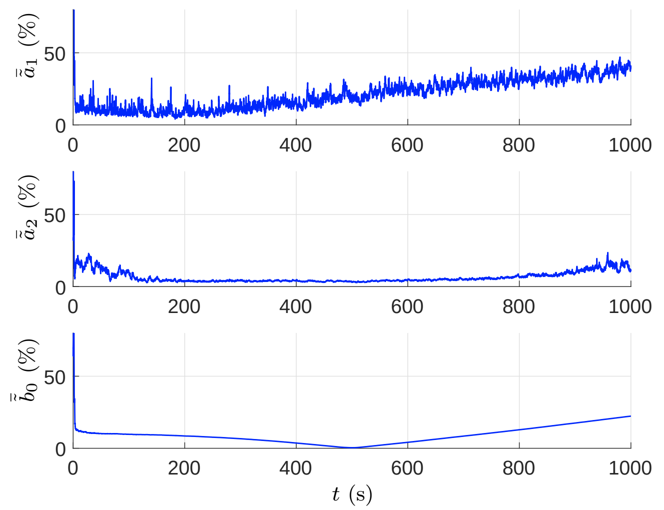

Fig. 5. Average of the relative errors for RLSSVF for a Monte-Carlo simulation with 100 runs.

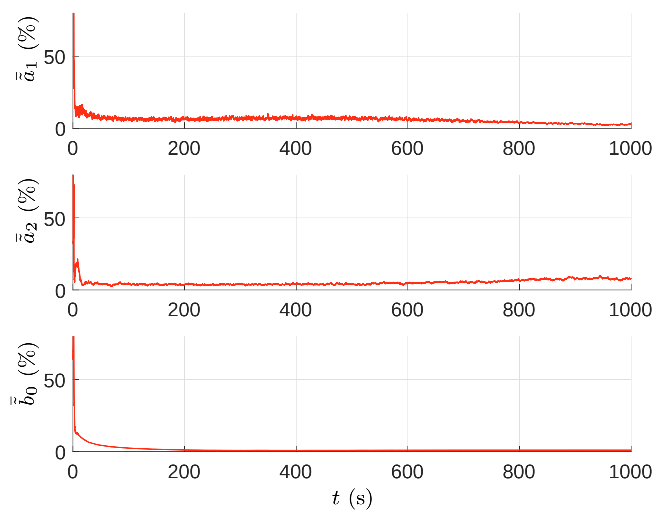

Fig. 6. Average of the relative errors for CLRSRIVC for a Monte-Carlo simulation with 100 runs.

is used for comparison purposes. Averaging the relative errors of the 100 experiments, we obtain the results presented in Figure 5 for RLSSVF and in Figure 6 for CLRSRIVC. It can be seen that the results for a single experiment hold for the Monte Carlo simulations.

6. CONCLUSIONS

In this study, the CLRSRIVC algorithm has been developed for the recursive on-line identification of linear, continuous-time, slowly time-varying systems operating in closed loop. With this approach we are able to track the time-varying parameters that vary at different rates by considering a stochastic model. For the estimation of CT models, prefiltered time-derivatives of the input and output are computed using an adaptive low-pass prefilter. The advantage of using CLRSRIVC over a direct approach like RLSSVF is illustrated by means of a Monte-Carlo simulation analysis considering a system whose bandwidth variation is large.

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