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EXPLORING NON-HOLOMORPHIC MODULAR FLAVOR SYMMETRIES IN NEUTRINO MASS MODELS

THESIS TO OBTAIN THE DEGREE OF
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Abstract

The Standard Model (SM) is the theoretical framework that describes the elementary particles and the fundamental interactions that govern them. Among the particles described by the model are neutrinos, whose interaction with the rest of the particles occurs exclusively through the weak interaction. In its original formulation, the SM assumes that neutrinos do not possess a right-handed chiral component, and therefore predicts them to be massless particles. However, the study of the solar neutrino problem revealed that neutrinos have the property of oscillating between different flavors as they propagate. This phenomenon is only possible if neutrinos have mass, which constitutes clear evidence of physics beyond the Standard Model. Following this discovery, several mechanisms were proposed to explain the origin of neutrino masses. One of the most well-known is the Type-I seesaw mechanism, which introduces very heavy right-handed neutrinos. In this scenario, the smallness of the light neutrino masses is explained by the presence of a very high mass scale associated with these new states. As an alternative, radiative models have been proposed, in which neutrino masses are generated through loop corrections. In these models, the mass term is forbidden at tree level and appears only at the loop level, so that the smallness of neutrino masses is naturally explained by the suppression factors associated with these radiative corrections. In this work we focus on a particular radiative model known as the Cocktail model, and we introduce modular flavor symmetries in order to explain the structure observed in the leptonic mixing parameters. In particular, we study seesaw scenarios under a non-holomorphic modular symmetry A_4, while for the Cocktail model we explore a non-holomorphic modular symmetry S_3. For the seesaw scenario, we are able to fit the leptonic observables within 3σ. For the Cocktail model, we also obtain a 3σ fit of the observables; however, we were not able to find solutions compatible with all the constraints from charged lepton flavor violation (CLFV), and therefore the search for a fully consistent solution remains an open problem.

Dedicado a Catalina

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Contents

Chapter 1

Introduction

1.1 Standard model

The Standard Model (SM) [1, 2] of particle physics describes the strong, electromagnetic, and weak interactions among the elementary particles. It is based on the local gauge symmetry group

$$SU(3)_C \times SU(2)_L \times U(1)_Y \ , \tag{1.1}$$

where the subscripts C, L, and Y denote color, left-handed chirality, and weak hypercharge, respectively. In the strong sector, there are eight massless gluons, which mediate the strong interaction. In contrast, the electroweak sector is mediated by four gauge bosons: three massive ones ($W^{\pm}$ and Z), and the massless photon (γ). In this work, we restrict our attention to the electroweak sector of the Standard Model, which is based on the gauge symmetry

$$SU(2)_L \times U(1)_Y \ , \tag{1.2}$$

and governs the interactions of neutrinos, among other particles. To analyze the electroweak sector, we construct the most general Lagrangian consistent with the gauge symmetry. This requires specifying both the structure of the gauge group and the way the fields transform under it.

The non-Abelian group $SU(2)_L$, known as weak isospin, acts non-trivially only on the left-handed chiral components of fermionic fields, while the right-handed components are invariant under this symmetry. A derivation of the chiral components can be found in Appendix A. The Lie algebra of $SU(2)_L$ is generated by three operators I_k (where $k = 1, 2, 3$), satisfying the commutation relations

$$[I_l, I_m] = i\varepsilon_{lmn} I_n \ , \tag{1.3}$$

where ε_{lmn} is the totally antisymmetric Levi-Civita tensor. On the other hand, the Abelian group $U(1)_Y$ corresponds to weak hypercharge. Its transformations are generated by the hypercharge operator Y, which is related to I_3 and the electric charge operator Q through the Gell-Mann-Nishijima relation, which is given by

$$Q = I_3 + \frac{Y}{2} \ . \tag{1.4}$$

This relation determines the action of the hypercharge operator on fermionic fields and reflects the unification of weak and electromagnetic interactions within the electroweak framework. Now that we have introduced the groups, we could proceed to discuss how the fields transform under these symmetries. However, before doing so, it is necessary to introduce a new concept of derivative.

To construct a theory invariant under local gauge transformations, the ordinary derivative must be promoted to a covariant derivative. To this end, three vector gauge fields A_k^μ ($k = 1, 2, 3$), associated with the generators of $SU(2)_L$, and one vector gauge field B^μ, associated with $U(1)_Y$, are introduced. The covariant derivative is defined as

$$D_\mu = \partial_\mu + ig\, \underline{A}_\mu \cdot \underline{I} + ig' B_\mu \frac{Y}{2} \ , \tag{1.5}$$

where

$$\underline{A}_\mu \equiv (A_1^\mu, A_2^\mu, A_3^\mu), \qquad \underline{I} \equiv (I_1, I_2, I_3),$$

and

$$\underline{A}_\mu \cdot \underline{I} \equiv \sum_{k=1}^{3} A_k^\mu I_k \ .$$

Here; g and g' denote the coupling constants associated with $SU(2)_L$ and $U(1)_Y$, respectively. Having introduced the covariant derivative, we now specify the gauge representations of the lepton fields (the quark sector follows analogously and we will not discuss it here, from this point onward we will refer exclusively to leptonic fields). The choice of representations is motivated by the V-A structure of the weak interactions and the two-component description of the neutrino, which successfully reproduce the observed phenomenology. In this framework, the left-handed chiral components of the lepton fields are grouped into weak isospin doublets,

$$L_{eL} = \begin{pmatrix} \nu_{eL} \\ e_L \end{pmatrix}, \qquad L_{\mu L} = \begin{pmatrix} \nu_{\mu L} \\ \mu_L \end{pmatrix}, \qquad L_{\tau L} = \begin{pmatrix} \nu_{\tau L} \\ \tau_L \end{pmatrix}.$$

With this choice, the generators of $SU(2)_L$ are given by $I_k = \tau_k/2$, so that

$$\underline{I} L_{\alpha L} = \frac{\underline{\tau}}{2} L_{\alpha L} \ , \quad (\alpha = e, \mu, \tau) \ , \tag{1.6}$$

where $\underline{\tau} = (\tau_1, \tau_2, \tau_3)$ are the Pauli matrices. In order to determine the hypercharge we just have to use the Gell-Mann-Nishijima relation, given in Eq. (1.4),

$$Y L_{\alpha L} = -L_{\alpha L} \ , \quad (\alpha = e, \mu, \tau) \ . \tag{1.7}$$

Note that each left-handed lepton doublet carries hypercharge $Y = -1$. An element g of the group $SU(2)_L \times U(1)_Y$ can be parametrized by four space-time dependent functions $(\underline{\alpha}(x), \beta(x))$, with $\underline{\alpha}(x) = (\alpha_1(x), \alpha_2(x), \alpha_3(x))$, that is,

$$g(\underline{\alpha}(x), \beta(x)) \in SU(2)_L \times U(1)_Y \ ,$$

with unitary representation of $g(\underline{\alpha}(x), \beta(x))$ on the space of the fields

$$U(\underline{\alpha}(x), \beta(x)) = \exp\left(i\underline{\alpha}(x) \cdot \underline{I} + i\beta(x)\frac{Y}{2}\right) = \exp\left(i\underline{\alpha}(x) \cdot \underline{I}\right) \exp\left(i\beta(x)\frac{Y}{2}\right) . \tag{1.8}$$

Accordingly, the left-handed lepton doublet transforms as follows

$$L_{\alpha L} \longrightarrow L'_{\alpha L} = \exp\left(\frac{i}{2}\underline{\alpha}(x) \cdot \underline{\tau} - \frac{i}{2}\beta(x)\right) L_{\alpha L} \ . \tag{1.9}$$

In the SM, we assume that neutrinos do not posses right-handed components [3, 4, 5], whereas charged leptons do. It is natural to assign the right-handed charged leptons to weak isospin singlets,

$$e_R \ , \qquad \mu_R \ , \qquad \tau_R \ .$$

Being singlets under $SU(2)_L$, they satisfy

$$\underline{I}\,\ell_{\alpha R} = 0 \ , \qquad (\alpha = e, \mu, \tau) \ , \tag{1.10}$$

The hypercharges of the right-handed leptonic fields are given by

$$Y\,\ell_{\alpha R} = -2\,\ell_{\alpha R} \ , \qquad (\alpha = e, \mu, \tau) \ . \tag{1.11}$$

Therefore, the transformation of the right-handed component is

$$\ell_R \longrightarrow \ell'_R = \exp\left(-i\beta(x)\right) \ell_R \ , \qquad (\alpha = e, \mu, \tau) \ . \tag{1.12}$$

All these ingredients allow us to construct the most general Lagrangian invariant under the local gauge group $SU(2)_L \times U(1)_Y$. We consider the three lepton generations, whose contributions share the same gauge structure and differ only through their flavor indices and Yukawa couplings

$$\begin{aligned}
\mathcal{L} = {} & i \sum_{\alpha=e,\mu,\tau} \overline{L_{\alpha L}} \not{D} L_{\alpha L} + i \sum_{\alpha=e,\mu,\tau} \overline{\ell_{\alpha R}} \not{D} \ell_{\alpha R} \\
& - \frac{1}{4}\underline{A}_{\mu\nu}\underline{A}^{\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} \\
& + (D_\mu\phi)^\dagger(D^\mu\phi) - \mu^2\phi^\dagger\phi - \lambda(\phi^\dagger\phi)^2 \\
& - \sum_{\alpha,\beta=e,\mu,\tau} \left(Y^\ell_{\alpha\beta}\overline{L'_{\alpha L}}\phi\ell_{\beta R} + Y^{\ell *}_{\alpha\beta}\overline{\ell_\beta R}\phi^\dagger L_{\alpha L}\right)
\end{aligned} \tag{1.13}$$

Here, $\not{D} \equiv \gamma^\mu D_\mu$, where γ^μ denote the Dirac matrices, and the adjoint femion field is defined as $\overline{f} \equiv f^\dagger \gamma^0$. These definitions are necessary in order for the Lagrangian to be Lorentz invariant. We will not explain the construction of these symmetries nor the structure of the Poincaré group; it suffices to know that Dirac fields transform under the Lorentz group as

$$\psi(x) \longrightarrow \psi'(x') = \mathcal{S}(\Lambda)\,\psi(x)\ , \tag{1.14}$$

where $\mathcal{S}(\Lambda)$ is a 4×4 matrix satisfying

$$\mathcal{S}^{-1}(\Lambda)\,\gamma^\mu \mathcal{S}(\Lambda) = \Lambda^\mu{}_\nu \gamma^\nu\ .$$

From the transformation of the Dirac field, it follows naturally that its adjoint transforms as

$$\overline{\psi}(x) \longrightarrow \overline{\psi'}(x') = \overline{\psi}(x)\mathcal{S}^{-1}(\Lambda)\ . \tag{1.15}$$

Meanwhile, couplings of the form $B_{\mu\nu}B^{\mu\nu}$ are manifestly Lorentz invariant, since all Lorentz indices are fully contracted. Returning to the gauge sector, the derivative transforms as

$$D_\mu \longrightarrow D'_\mu = U(\underline{\alpha}(x), \beta(x))\, D_\mu\, U^{-1}(\underline{\alpha}(x), \beta(x)).$$

This implies that the gauge bosons transform as

$$\underline{A}_\mu \cdot \underline{I} \longrightarrow \underline{A}'_\mu \cdot \underline{I} = U(\underline{\alpha}(x)) \left[\underline{A}_\mu \cdot \underline{I} - \frac{i}{g}\partial_\mu\right] U^{-1}(\underline{\alpha}(x)),$$

$$B_\mu \frac{Y}{2} \longrightarrow B'_\mu \frac{Y}{2} = U(\beta(x)) \left[B_\mu \frac{Y}{2} - \frac{i}{g'}\partial_\mu\right] U^{-1}(\beta(x)).$$

The second line of the Lagrangian in Eq. (1.13) contains the kinetic and self-interaction terms of the gauge bosons. The third line corresponds to the Higgs sector, responsible for triggering the spontaneous symmetry breaking. The fourth line describes the Yukawa interactions between leptons and the Higgs field, which generate the charged-lepton masses after the symmetry breaking. Since the left-handed and right-handed components of the leptonic fields transform differently under the gauge symmetry, explicit Dirac mass terms of the form

$$\overline{\ell}\ell = \overline{\ell}_L \ell_R + \overline{\ell}_R \ell_L$$

are forbidden by gauge invariance. Such terms can only arise after spontaneous symmetry breaking via the Higgs mechanism. The way we will develop this phenomenon (and the calculation) is based on [6].

1.2 Higgs mechanism

In the Standard Model, the masses of leptons (and quarks), as well as those of the gauge bosons, arise through the Higgs mechanism [7, 8, 9, 10, 11, 12]. This mechanism is implemented by introducing a Higgs doublet

$$\phi(x) = \begin{pmatrix} \phi^+(x) \\ \phi^0(x) \end{pmatrix},$$

where $\phi^+(x)$ is a charged complex scalar field and $\phi^0(x)$ is a neutral complex scalar field. This doublet transforms under unitary representations of $SU(2)_L \times U(1)_Y$, parametrized by $(\underline{\alpha}(x), \beta(x))$, in a similar way as the left-handed lepton doublets

$$\phi \longrightarrow \phi' = \exp\left(\frac{i}{2}\,\underline{\alpha}(x)\cdot\underline{\tau} + \frac{i}{2}\,\beta(x)\right)\phi, \tag{1.16}$$

which corresponds to the hypercharge assignment $Y = +1$. By imposing gauge invariance, we can construct the most general Lagrangian describing the Higgs sector

$$\mathcal{L}_{\text{Higgs}} = (D_\mu\phi)^\dagger(D^\mu\phi) - \mu^2\phi^\dagger\phi - \lambda(\phi^\dagger\phi)^2. \tag{1.17}$$

The associated scalar potential is therefore

$$\mathcal{V} = \mu^2\phi^\dagger\phi + \lambda(\phi^\dagger\phi)^2. \tag{1.18}$$

For the potential to be bounded from below, the quartic coupling must satisfy $\lambda > 0$, while the quadratic coupling must fulfill $\mu^2 < 0$. In order to minimize the potential, we impose the condition

$$\frac{\partial\mathcal{V}}{\partial\left(\phi^\dagger\phi\right)} = 0\,.$$

From this condition, it follows that the potential is minimized when

$$\phi^\dagger\phi = -\frac{\mu^2}{2\lambda} \equiv \frac{v^2}{2}\ . \tag{1.19}$$

Equivalently,

$$\left|\phi^+\right|^2 + \left|\phi^0\right|^2 = \frac{v^2}{2}\,.$$

This relation defines a circle containing all the field configurations that minimize the potential. All these configurations are connected by gauge transformations. In quantum field theory, the minimum of the potential corresponds to the vacuum state, namely the state of lowest energy of the system. The quantized excitations of the fields around this vacuum are interpreted as physical particle states. In order to preserve invariance under spatial rotations, fields with nonzero spin must have vanishing vacuum expectation values; otherwise, the vacuum would single out a preferred spatial direction, thereby breaking rotational symmetry. Furthermore, the vacuum must be electrically neutral. Consequently, any electrically charged scalar field must have a vanishing vacuum expectation value. On the other hand, neutral scalar fields may acquire a nonzero vacuum expectation value. This quantity is known as the vacuum expectation value (VEV). From the condition above, we conclude that, in order to preserve the electrical neutrality of the vacuum, the VEV of the Higgs doublet must be associated with its neutral component ϕ^0. Therefore,

$$\langle\phi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v \end{pmatrix} . \tag{1.20}$$

The symmetry $SU(2)_L \times U(1)_Y$ is spontaneously broken by the VEV

$$I_1\langle\phi\rangle = \frac{\tau_1}{2}\langle\phi\rangle = \frac{1}{2\sqrt{2}}\begin{pmatrix} v \\ 0 \end{pmatrix} \neq 0,$$

$$I_2\langle\phi\rangle = \frac{\tau_2}{2}\langle\phi\rangle = -\frac{i}{2\sqrt{2}}\begin{pmatrix} v \\ 0 \end{pmatrix} \neq 0,$$

$$I_3\langle\phi\rangle = \frac{\tau_3}{2}\langle\phi\rangle = -\frac{1}{2\sqrt{2}}\begin{pmatrix} 0 \\ v \end{pmatrix} \neq 0,$$

$$Y\langle\phi\rangle = \langle\phi\rangle \neq 0$$

Using the Gell-Mann-Nishijima relation, the electric charge operator is

$$Q = I_3 + \frac{Y}{2}.$$

Acting on the vacuum, one finds

$$Q\langle\phi\rangle = \left(I_3 + \frac{Y}{2}\right)\langle\phi\rangle = 0.$$

Therefore, the vacuum is invariant under the $U(1)_Q$ group generated by Q,

$$e^{i\alpha Q}\langle\phi\rangle = \langle\phi\rangle \ .$$

This guarantees the existence of a massless gauge boson associated with the unbroken symmetry $U(1)_Q$, identified with the photon. Consequently, the spontaneous symmetry breaking pattern is

$$SU(2)_L \times U(1)_Y \longrightarrow U(1)_Q.$$

To make the physical particle content explicit, it is convenient to parametrize the Higgs doublet as

$$\phi(x) = \frac{1}{\sqrt{2}}\exp\left(\frac{i}{2v}\underline{\xi}(x)\cdot\underline{\tau}\right)\begin{pmatrix} 0 \\ v + H(x) \end{pmatrix} ,$$

where $\underline{\xi}(x) = (\xi_1(x), \xi_2(x), \xi_3(x))$ and $H(x)$ are four real scalar fields. The field $H(x)$ corresponds to the physical Higgs boson, describing fluctuations of the neutral scalar component around the vacuum. The fields $\underline{\xi}(x)$ are the Goldstone bosons associated with the broken generators. By performing a gauge transformation with parameters

$$\underline{\alpha}(x) = -\frac{1}{v}\underline{\xi}(x) \ , \qquad \beta(x) = 0 \ ,$$

the Goldstone fields can be removed. This choice defines the unitary gauge, in which only the physical scalar degree of freedom remains explicit. In this gauge, the Higgs doublet takes the form

$$\phi(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + H(x) \end{pmatrix} \ .$$

In this way, three of the original four scalar degrees of freedom become the longitudinal components of the massive $W^{\pm}$ and Z bosons, while the remaining scalar corresponds to the Higgs boson.

The gauge boson masses can be obtained straightforwardly by inserting the Higgs vacuum expectation value into the covariant derivative of the scalar doublet in the Lagrangian and identifying the resulting vector field bilinear terms as their mass terms. However, since our primary interest at this stage concerns the lepton sector, we shall focus exclusively on the generation of charged lepton masses. As discussed previously, fermion mass terms in the Lagrangian must couple left-handed and right-handed chiral components. The Yukawa interaction responsible for charged lepton masses is given by

$$-\mathcal{L}_Y = \sum_{\alpha,\beta=e,\mu,\tau} Y'^{\ell}_{\alpha\beta} \overline{L'_{\alpha L}} \phi \, \ell'_{\beta R} + h.c. \tag{1.21}$$

The matrix formed by the coefficients $Y'^{\ell}_{\alpha\beta}$ is known as the Yukawa matrix. In general, it is a complex, non-diagonal 3×3 matrix. Working in the unitary gauge, we insert the Higgs doublet in the form previously derived and compute the product $\overline{L}'_{\alpha L}\phi$. This yields

$$-\mathcal{L}_Y = \left(\frac{v + H}{\sqrt{2}} \right) \sum_{\alpha,\beta=e,\mu,\tau} Y'^{\ell}_{\alpha\beta} \overline{\ell'_{\alpha L}} \ell'_{\beta R} + \text{h.c.} \tag{1.22}$$

Two distinct contributions appear in this expression. The term proportional to v generates mass terms for the charged fermions, while the term proportional to H describes a trilinear coupling between the Higgs boson and the lepton fields. It is then clear that the Higgs coupling to a fermion is always proportional to the fermion mass, an important test of the Higgs nature. Since the Yukawa matrix is not diagonal, the primed fields do not correspond to mass eigenstates. To determine the physical lepton masses, the Yukawa matrix must be diagonalized. For this purpose, let us express this formulation in matrix notation. The lepton fields form column vectors as:

$$\ell'_L \equiv \begin{pmatrix} e'_L \\ \mu'_L \\ \tau'_L \end{pmatrix} \ , \qquad \ell'_R \equiv \begin{pmatrix} e'_R \\ \mu'_R \\ \tau'_R \end{pmatrix} \ . \tag{1.23}$$

Using this notation, the Yukawa interaction can be written compactly as

$$-\mathcal{L}_Y = \left(\frac{v+H}{\sqrt{2}}\right) \overline{\ell'_L} Y'^{\ell} \ell'_R + \text{h.c.} \tag{1.24}$$

The Yukawa matrix can be diagonalized by a biunitary transformation,

$$V_L^{\ell\dagger} Y'^{\ell} V_R^{\ell} = Y^{\ell} \ , \tag{1.25}$$

where $Y^{\ell}_{\alpha\beta} = y^{\ell}_{\alpha}\delta_{\alpha\beta}$ for $\alpha, \beta = e, \mu, \tau$. The matrices V_L^{ℓ} and V_R^{ℓ} are unitary 3×3 matrices. This diagonalization allows us to rewrite the Lagrangian as

$$-\mathcal{L}_Y = \left(\frac{v+H}{\sqrt{2}}\right) \overline{\ell_L} Y^{\ell} \ell_R + \text{h.c.} \ , \tag{1.26}$$

where we have introduced the arrays containing the left-handed and right-handed componentes of the charged lepton fields with definite masses

$$\ell_L = V_L^{\ell\dagger} \ell'_L \equiv \begin{pmatrix} e_L \\ \mu_L \\ \tau_L \end{pmatrix} , \qquad \ell_R = V_R^{\ell\dagger} \ell'_R \equiv \begin{pmatrix} e_R \\ \mu_R \\ \tau_R \end{pmatrix} . \tag{1.27}$$

The charged lepton mass term becomes

$$-\mathcal{L} = \sum_{\alpha=e,\mu,\tau} \frac{y_\alpha v}{\sqrt{2}} \overline{\ell_\alpha} \ell_\alpha \ , \tag{1.28}$$

where

$$\ell_\alpha \equiv \ell_{\alpha L} + \ell_{\alpha R} \ , \qquad (\alpha = e, \mu, \tau) \ ,$$

are the charged lepton fields with definite masses. Therefore, the mass of the charged leptons are

$$m_\alpha = \frac{y^{\ell}_\alpha v}{\sqrt{2}} \ , \qquad (\alpha = e, \mu, \tau) \ . \tag{1.29}$$

It is important to emphasize that the coefficients y^{ℓ}_α are free parameters of the Standard Model. The theory provides no explanation for the strong hierarchy observed among the charged lepton masses. Understanding the origin of these Yukawa couplings remains one of the open questions in particle physics, called *the flavor problem.*

Now that we have introduced new arrays that couple in order to generate mass terms, it is natural to ask whether these couplings have implications beyond the mass terms themselves. To address this question, we must recall that in the weak interaction there are two currents responsible for determining the processes: the charged weak current and the neutral weak current. The interaction Lagrangian for the charged weak current is given by

$$\mathcal{L}_{\text{CC}} = -\frac{g}{2\sqrt{2}} j^{\rho}_W W_\rho + \text{h.c.} \ ,$$

where g is the weak coupling constant. Meanwhile, the interaction Lagrangian for the neutral weak current is given by

$$\mathcal{L}_{\mathrm{NC}} = -\frac{g}{2\cos\theta_W}\, j_Z^\rho Z_\rho + \text{h.c.} \;,$$

where θ_W is the Weinberg angle. For the moment, we will focus exclusively on what happens in the charged current interaction.Just as we did with the charged lepton fields, we can arrange the neutral leptons in a column vector

$$\nu'_L \equiv \begin{pmatrix} \nu'_{eL} \\ \nu'_{\mu L} \\ \nu'_{\tau L} \end{pmatrix} \;.$$

With this definition, the charged current can be written as

$$j_W^\rho = 2\overline{\nu'_L}\gamma^\rho \ell'_L = 2\overline{\nu'_L}\gamma^\rho V_L^\ell\, \ell_L \;.$$

Since we can freely transform the massless neutrino fields as

$$\nu_L = V_L^{\ell\dagger}\nu'_L \equiv \begin{pmatrix} \nu_{eL} \\ \nu_{\mu L} \\ \nu_{\tau L} \end{pmatrix} \;,$$

the leptonic charged weak current can be written as

$$j_W^\rho = 2\overline{\nu_L}\gamma^\rho \ell_L = 2\sum_{\alpha=e,\mu,\tau} \overline{\nu_{\alpha L}}\gamma^\rho \ell_{\alpha L} \;. \tag{1.30}$$

This expression is written in terms of the neutrino fields $\nu_{\alpha L}$ and the charged leptons $\ell_{\alpha L}$ in the mass basis. The fields $\nu_{\alpha L}$ are known as *flavor neutrino fields*, since they couple directly to their corresponding charged leptons. By substituting the charged current expression into the interaction Lagrangian, one obtains the trilinear couplings between leptons and the W gauge bosons, which can be represented by the corresponding Feynman diagrams in Figure 1.1.

As can be seen from these diagrams, the charged current j_W^ρ couples each lepton to the neutrino of the corresponding flavor. Consequently, the flavor lepton numbers, namely the electron, muon, and tau lepton numbers L_e, L_μ, and L_τ assigned in Table 1.1 are conserved. More generally, this implies the conservation of the total lepton number,

$$L = L_e + L_\mu + L_\tau \,. \tag{1.31}$$

The conservation of each lepton number is not accidental, but rather follows directly, through Noether's theorem, from the invariance of the Lagrangian under an accidental global $U(1)$ gauge symmetry. Under this symmetry, the leptonic fields of each family transform as

$$\nu_{\alpha L} \longrightarrow e^{i\varphi_\alpha}\, \nu_{\alpha L} \;, \qquad \ell_{\alpha L} \longrightarrow e^{i\varphi_\alpha}\, \ell_{\alpha L} \;, \qquad \ell_{\alpha R} \longrightarrow e^{i\varphi_\alpha}\, \ell_{\alpha R} \;,$$

where φ_α denotes the parameter associated with the transformation.

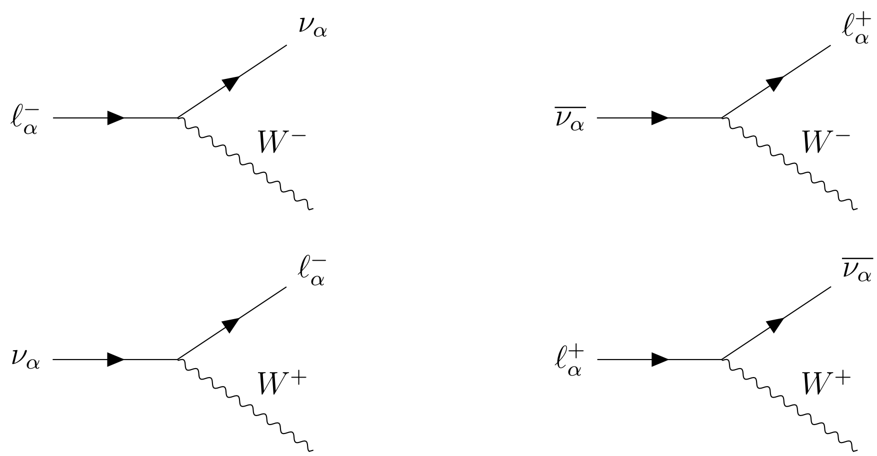


Figure 1.1: Leptonic charged current vertices mediated by the $W^\pm$ boson.

Table 1.1: Assignment of lepton numbers in the Standard Model.

Particle	L_e	L_μ	L_τ	L
e^-, ν_e	$+1$	0	0	$+1$
μ^-, ν_μ	0	$+1$	0	$+1$
τ^-, ν_τ	0	0	$+1$	$+1$
$e^+, \overline{\nu_e}$	-1	0	0	-1
$\mu^+, \overline{\nu_\mu}$	0	-1	0	-1
$\tau^+, \overline{\nu_\tau}$	0	0	-1	-1

1.3 Experimental evidence for neutrino masses

As discussed in the previous section, the generation of fermionic mass terms requires the coupling between the left-handed and right-handed chiral components of the field. However, in the SM no right-handed neutrinos are present, and therefore, it is not possible to construct a mass term for the neutrinos. Such a prediction would be consistent provided it agreed with experimental observations. However, compelling experimental evidence has established that neutrinos are massive and that lepton flavors mix. The first indication arose from the solar neutrino problem, namely the observed deficit in the flux of electron neutrinos produced in the Sun compared to theoretical predictions. This anomaly was conclusively resolved by the observation of neutrino flavor conversion, first confirmed by experiments such as Super-Kamiokande and the Sudbury Neutrino Observatory (SNO). These results demonstrated that neutrinos produced with a definite flavor can be detected later with a different flavor. The phenomenon of neutrino oscillations [13, 14] can be understood if flavor states are not identical to mass states. Denoting the flavor states by $|\nu_\alpha\rangle$, with $\alpha = e, \mu, \tau$, and the mass states by $|\nu_i\rangle$, with $i = 1, 2, 3$, the two bases are related by a unitary transformation,

$$|\nu_\alpha\rangle = \sum_i U^*_{\alpha i} |\nu_i\rangle \, , \tag{1.32}$$

where U is the leptonic mixing matrix. In the mass basis, the free Hamiltonian is diagonal, and the mass states evolve in time as plane waves,

$$|\nu_i(t)\rangle = e^{-iE_i t} |\nu_i\rangle \, . \tag{1.33}$$

As a consequence, the probability of a neutrino produced with flavor α being detected with flavor β after propagating a distance x is given by

$$P(\nu_\alpha \to \nu_\beta) = \left| \sum_i U^*_{\alpha i} U_{\beta i} e^{-iE_i x} \right|^2 . \tag{1.34}$$

For illustrative purposes, we consider the simplified two-flavor case, and the transition probability takes the well-known form

$$P(\nu_\alpha \to \nu_\beta) = \sin^2 2\theta \, \sin^2\left(\frac{\Delta m^2}{4E} x \right) , \tag{1.35}$$

where θ is the mixing angle and $\Delta m^2 = m_2^2 - m_1^2$ is the squared mass difference. The crucial feature of this expression is that flavor oscillations occur only if at least one neutrino masses are non zero. Therefore, neutrino oscillation experiments provide direct evidence that neutrinos are massive and that lepton flavors mix. This experimental fact contradicts the prediction of the Standard Model and implies the necessity of extending it. In the following sections, we explore mechanisms capable of generating neutrino masses in a theoretically consistent framework.

1.4 Massive neutrinos

In the previous section, we established that neutrinos must be massive, a feature that is not predicted by the Standard Model. This discrepancy between experimental observations and theoretical predictions motivates us to consider extensions of the Standard Model. In this section, we introduce the concepts of Dirac and Majorana neutrinos, discussing their respective mass terms and the physical implications associated with each possibility. Subsequently, we review the most well-known mechanisms proposed in the literature to explain how neutrinos acquire masses. Finally, we discuss the Flavor Puzzle, which constitutes the main motivation of this work. This will allow us to explain why it is convenient to search for an underlying symmetry that is behind the structures presented throughout this section.

1.4.1 Dirac neutrinos

As discussed above, in order to generate neutrino masses through the Higgs mechanism it is necessary to introduce right-handed neutrinos. The minimal extension of the Standard Model that allows this consists of adding three right-handed neutrinos, $\nu_{\alpha R}$ (with $\alpha = e, \mu, \tau$). This setup is commonly referred to as the minimal extension of the Standard Model. These right-handed neutrinos are singlets under $SU(2)_L$, just like the other right-handed leptons. However, in contrast to the charged right-handed leptons, they carry zero hypercharge. Consequently, they do not participate in weak interactions and are therefore often called *sterile neutrinos*. With the addition of three right-handed neutrinos, the Yukawa sector is enlarged as

$$-\mathcal{L}_Y = \sum_{\alpha,\beta=e,\mu,\tau} Y'^{\nu}_{\alpha\beta}\, \overline{L'_{\alpha L}}\, \tilde{\phi}\, \nu'_{\beta R} + \text{h.c.} \tag{1.36}$$

where $\tilde{\phi} \equiv i\tau_2\phi^*$ and $Y'^{\nu}_{\alpha\beta}$ denote the elements of the neutrino Yukawa matrix, which, as previously discussed, is complex, non-diagonal, and three dimensional in flavor space. To make the mass term explicit, we move to the unitary gauge. In this gauge, the Lagrangian takes the form

$$-\mathcal{L}_Y = \left(\frac{v+H}{\sqrt{2}}\right) \overline{\nu'_L} Y'^{\nu} \nu'_R + \text{h.c.} \tag{1.37}$$

Proceeding as in the charged lepton sector, we introduce an array for the new right-handed chiral components of the neutrino fields

$$\nu'_R = \begin{pmatrix} \nu'_{eR} \\ \nu'_{\mu R} \\ \nu'_{\tau R} \end{pmatrix}.$$

The Yukawa matrix can be diagonalized through a biunitary transformation,

$$V_L^{\nu\dagger} Y'^{\nu} V_R^{\nu} = Y^{\nu},$$

where Y^{ν} is diagonal, $Y^{\nu}_{\alpha\beta} = y^{\nu}_{\alpha}\delta_{\alpha\beta}$, and V_L^{ν}, V_R^{ν} are unitary 3×3 matrices. Now we define the array for the fields with defined masses as

$$n_L = V_L^{\nu\dagger}\nu'_L \equiv \begin{pmatrix} \nu_{1L} \\ \nu_{2L} \\ \nu_{3L} \end{pmatrix}, \qquad n_R = V_R^{\nu\dagger}\nu'_R \equiv \begin{pmatrix} \nu_{1R} \\ \nu_{2R} \\ \nu_{3R} \end{pmatrix},$$

the Yukawa Lagrangian becomes

$$\begin{aligned} -\mathcal{L}_Y &= \left(\frac{v+H}{\sqrt{2}}\right) \overline{n_L} Y^{\nu} n_R + \text{h.c.} \\ &= \left(\frac{v+H}{\sqrt{2}}\right) \sum_{k=1}^{3} y^{\nu}_k \overline{\nu_{kL}} \nu_{kR} + \text{h.c.} \end{aligned}$$

Since we have now introduced the right-handed neutrino components, we can construct the neutrino field as a standard Dirac field,

$$\nu_k = \nu_{kL} + \nu_{kR}, \qquad (k = 1, 2, 3).$$

The Dirac mass term becomes

$$-\mathcal{L}_m^D = \sum_{k=1}^{3} \frac{y_k^\nu v}{\sqrt{2}} \overline{\nu_k} \nu_k \ . \tag{1.38}$$

Therefore, the Dirac neutrino masses are

$$m_{Dk} = \frac{y_k^\nu v}{\sqrt{2}}, \qquad (k = 1, 2, 3) \ . \tag{1.39}$$

Although we have obtained an explicit expression for the neutrino masses, they are proportional to the vacuum expectation value v, just as in the charged lepton sector. We know that neutrino masses are extremely small. This implies that the Yukawa couplings y_k^ν must be very small, several orders of magnitude smaller than those of the charged leptons. Within this minimal framework there is no explanation for such tiny couplings. This is often interpreted as a naturalness problem. Even if one does not adopt naturalness as a guiding principle, the large hierarchy between charged lepton and neutrino masses remains unexplained.

In the case of Dirac neutrinos, leptonic mixing occurs in a way that is completely analogous to the mixing in the quark sector of the Standard Model. In particular, the charged weak current can be written as

$$j_W^\rho = \overline{\nu_L'} \gamma^\rho \ell_L' = \overline{n_L} V_L^{\nu\,\dagger} \gamma^\rho V_L^\ell \ell_L = \overline{n_L} V_L^{\nu\,\dagger} V_L^\ell \gamma^\rho \ell_L \, .$$

In this way, the charged current depends on the product of the rotation matrices of the left-handed leptons that diagonalize the mass matrices in the leptonic sector

$$U = V_L^{\ell\,\dagger} V_L^\nu \, , \tag{1.40}$$

where U is known as the leptonic mixing matrix. Now, it is possible to express the leptonic current in terms of the mixing matrix

$$j_W^\rho = \overline{n_L} U^\dagger \gamma^\rho \ell_L \ . \tag{1.41}$$

We then find a new definition for the charged current, which depends on the mixing matrix. It is natural to define the array of left-handed neutrino flavor fields as

$$\nu_L = U n_L = V_L^{\ell\dagger} \nu_L' \ , \qquad \text{with} \qquad \nu_L \equiv \begin{pmatrix} \nu_{eL} \\ \nu_{\mu L} \\ \nu_{\tau L} \end{pmatrix} \ . \tag{1.42}$$

This definition allows us to write the charged leptonic current in the form

$$j_W^\rho = 2\,\overline{\nu_L}\,\gamma^\rho\,\ell_L = 2\sum_{\alpha=e,\mu,\tau}\overline{\nu_{\alpha L}}\gamma^\rho\ell_{\alpha L}\ . \tag{1.43}$$

Note that (1.42) establishes an explicit relation between the so-called flavor states and the mass states. This relation is precisely what gives rise to the phenomenon of neutrino oscillations, under the assumption that flavor states are linear superpositions of mass states. It is important to note that in some references the mixing matrix is defined in the opposite order. This does not lead to any physical inconsistency, since physical observables do not depend on U alone, but rather on combinations such as $UU^\dagger$. Consequently, writing the current in terms of U or $U^\dagger$ is purely a matter of convention.

1.4.2 Majorana neutrinos

It is also possible to introduce a Majorana mass term. To construct it, we have to recall the two-component spinor formalism [3, 4, 5]. Before explicitly writing down this mass term, it is useful to analyze the physical implications of a particle being of Majorana nature. To this end, we start from the Dirac equation and consider its chiral projections using the operators P_L and P_R. By applying these projectors to the Dirac equation, one obtains two coupled equations that describe the dynamics of the left-handed and right-handed chiral components of the spinor

$$P_R\,(i\gamma^\mu\partial_\mu - m)\,\psi = 0 \longrightarrow i\gamma^\mu\partial_\mu\psi_L = m\psi_R,$$

$$P_L\,(i\gamma^\mu\partial_\mu - m)\,\psi = 0 \longrightarrow i\gamma^\mu\partial_\mu\psi_R = m\psi_L.$$

The components are coupled through the mass term. In the massless limit, the equations decouple

$$i\gamma^\mu\partial_\mu\psi_L = 0, \tag{1.44}$$

$$i\gamma^\mu\partial_\mu\psi_R = 0. \tag{1.45}$$

Therefore, a massless fermion can be described using a single chiral component (either left-handed or right-handed), which possesses only two independent components. These decoupled equations are known as the Weyl equations, and the fields ψ_L and ψ_R are referred to as Weyl spinors. Once it is established that a massless fermion can be described by a two component spinor, it is natural to ask whether a similar description is possible in the massive case, that is, whether massive fermions can also be formulated in terms of two-component spinors. The answer is affirmative, a massive fermion can indeed be described without explicitly introducing a four-component spinor. The key idea is to assume that ψ_L and ψ_R are not independent. The relations obtained in this way must be equivalent to the Weyl equations, but now including a mass term. The two resulting equations should therefore be different ways of writing the same equation for a single independent quiral field. To obtain Eq. (1.44) from Eq. (1.45), one takes the Hermitian conjugate of the latter and then multiplies it on the right by γ^0, yielding thus

$$-i\partial_\mu \overline{\psi_R}\gamma^\mu = m\overline{\psi_L} \; .$$

Taking the transpose and multiplying on the left by the charge conjugation matrix $\mathcal{C}$, and using

$$\mathcal{C}\gamma_\mu^T\mathcal{C}^{-1} = -\gamma_\mu,$$

one finds

$$i\gamma^\mu\partial_\mu \mathcal{C}\overline{\psi_R}^T = m\mathcal{C}\overline{\psi_L}^T \; .$$

This equation has the same structure as Eq. (1.44). They can be identified provided we impose

$$\psi_R = \mathcal{C}\overline{\psi_L}^T \; . \tag{1.46}$$

The previous equation corresponds to the Majorana relation between ψ_L and ψ_R. This is not unexpected, since the charge conjugation operator flips the chirality of the fields. Substituting this relation into (1.44), we obtain the Majorana equation for the chiral field ψ_L,

$$i\gamma^\mu\partial_\mu\psi_L = m\,\mathcal{C}\overline{\psi_L}^T \; .$$

The most important consequence of this result becomes apparent when we decompose the fermionic field into the sum of its chiral components,

$$\psi = \psi_L + \psi_R \; ,$$

and impose the Majorana condition

$$\psi = \psi_L + \mathcal{C}\overline{\psi_L}^T \; . \tag{1.47}$$

It can be shown that $\mathcal{C}\overline{\psi_L}^T$ is identical, up to a possible phase factor, to the charge conjugated field ψ_L^c. Since weak interactions violate charge conjugation symmetry, and neutrinos interact exclusively through the weak interaction, the phase associated with charge conjugation of neutrino fields has no physical significance and can be chosen arbitrarily. Now, we can write the relation

$$\psi = \psi_L + \psi_L^c \; ,$$

and take the charge conjugation in all terms to obtain

$$\psi^c = \psi_L^c + (\psi_L^c)^c = \psi_L^c + \psi_L \; .$$

Finally, the Majorana condition takes the form

$$\psi = \psi^c \, . \tag{1.48}$$

This condition expresses the equality between particle and antiparticle. Consequently, only electrically neutral fermions can be described by Majorana fields. Among the elementary particles of the Standard Model, the neutrino is the only neutral fermion, making it the natural candidate to be a Majorana particle. Now that we have established what is meant by a Majorana neutrino, we can proceed to construct its mass term. For simplicity, we will consider only a single neutrino species, ν. A Majorana mass term can be constructed using a single chiral field. As a first step, we construct the mass term for the left–handed chiral component. Let us first recall the Dirac mass term,

$$\mathcal{L} = -m\overline{\nu}\nu = -m\overline{\nu_R}\nu_L + \text{h.c.} \tag{1.49}$$

As discussed previously, spinors transform under the Lorentz group through a matrix $\mathcal{S}$. Consequently, their chiral components transform as

$$\nu_L \longrightarrow \nu'_L = \mathcal{S}\nu_L \; , \qquad \nu_R \longrightarrow \nu'_R = \mathcal{S}\nu_R \, , \tag{1.50}$$

while their Dirac adjoints transform according to

$$\overline{\nu_L} \longrightarrow \overline{\nu'_L} = \overline{\nu_L}\mathcal{S}^{-1} \; , \qquad \overline{\nu_R} \longrightarrow \overline{\nu'_R} = \overline{\nu_R}\mathcal{S}^{-1} \, . \tag{1.51}$$

In order to construct a Majorana mass term using only ν_L, we need a right–handed object written in terms of ν_L that transforms under Lorentz transformations in the same way as ν_L. This would allow us to replace ν_R in Eq. (1.49) and obtain a term depending exclusively on ν_L. The quantity that satisfies these requirements is the charge conjugated field,

$$\nu_L^c = \mathcal{C}\overline{\nu_L}^T \, .$$

Since ν_L^c has right–handed chirality, the bilinear $\overline{\nu_L^c}\nu_L$ does not vanish (whereas it is straightforward to verify that $\overline{\nu_L}\nu_L = 0$). Let us now analyze its behavior under Lorentz transformations. Using the explicit definition,

$$\nu_L^c = \mathcal{C}\left(\nu_L^\dagger\gamma^0\right)^T \; ,$$

and applying a Lorentz transformation, we obtain

$$\nu_L^c \longrightarrow \mathcal{C}\left(\nu_L'^\dagger\gamma^0\right)^T = \mathcal{C}\left(\mathcal{S}^{-1}\right)^T\mathcal{C}^{-1}\nu_L^c \, .$$

Using the identity

$$\mathcal{C}\left(\mathcal{S}^{-1}\right)^T\mathcal{C}^{-1} = \mathcal{S} \, ,$$

we find that

$$\nu_L^c \longrightarrow \mathcal{S}\nu_L^c\,,$$

while

$$\overline{\nu_L^c} \longrightarrow \overline{\nu_L^c}\mathcal{S}^{-1}\,.$$

We therefore observe that ν_L^c transforms in the same way as ν_L, and its adjoint transforms like $\overline{\nu_L}$. Hence, ν_L^c possesses precisely the properties required to replace ν_R in the Dirac mass term, leading to the Majorana mass term

$$-\mathcal{L}_{\nu_R}^M = \frac{1}{2}m\,\overline{\nu_L^c}\nu_L + \text{h.c.} \tag{1.52}$$

The factor of 1/2 is introduced to avoid double counting, since ν_L^c and $\overline{\nu_L}$ are not independent fields ($\nu_L^c = \mathcal{C}\overline{\nu_L}^T$). At first sight, this result appears satisfactory: no additional right–handed components are required, and one could in principle generate a mass term using only the Standard Model field content. However, this conclusion is premature. Recall that the left–handed chiral components of the leptons are arranged into $SU(2)_L$ doublets. Consequently, it is not possible to construct a Majorana mass term for ν_L that is invariant under the Standard Model gauge symmetry. Throughout the previous construction we have imposed only Lorentz invariance, while gauge invariance has been ignored. If we wish to preserve gauge symmetry, this term must therefore be forbidden. The situation changes if we consider the right–handed chiral components. These are singlets under $SU(2)_L$ and carry zero hypercharge. Therefore, it is possible to construct a Majorana mass term for them that is gauge invariant. This term takes the form

$$-\mathcal{L}_m^M = \frac{1}{2}M_R\,\overline{\nu_R^c}\nu_R + \text{h.c.} \tag{1.53}$$

1.4.3 Type-I seesaw mechanism

A simple and natural mechanism to generate neutrino masses arises from considering both Dirac and Majorana mass terms, the latter involving only the right-handed neutrino fields. In this way, we perform a minimal extension of the Standard Model by introducing right-handed neutrinos, while preserving gauge invariance. The most general gauge-invariant mass Lagrangian for the three generations of neutrinos is given by

$$-\mathcal{L}_m^{D+M} = \sum_{\alpha,\beta=e,\mu,\tau} Y_{\alpha\beta}^{\prime\nu}\,\overline{L_{\alpha L}'}\tilde{\phi}\nu_{\beta R}' + \frac{1}{2}\sum_{\alpha,\beta=e,\mu,\tau}(M_R)_{\alpha\beta}\,\overline{\nu_{\alpha R}^{\prime c}}\nu_{\beta R}' + \text{h.c.} \tag{1.54}$$

After electroweak symmetry breaking, the Higgs doublet acquires a vacuum expectation value,

$$\phi \longrightarrow \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v+H \end{pmatrix}\,, \quad \tilde{\phi} \longrightarrow \frac{1}{\sqrt{2}}\begin{pmatrix} v+H \\ 0 \end{pmatrix}\,.$$

Substituting this expression into (1.54), the Dirac Yukawa term generates a mass term for the neutrinos. The mass Lagrangian then takes the form

$$-\mathcal{L}_m^{D+M} = \overline{\nu_L'}\, m_D\, \nu_R' + \frac{1}{2}\, \overline{\nu_R'^c}\, M_R\, \nu_R' + \text{h.c.}\ , \tag{1.55}$$

where the Dirac mass matrix is defined as

$$m_D = \frac{v}{\sqrt{2}}\, Y'^{\nu}. \tag{1.56}$$

We can write a mass matrix that contains all the contributions present in the Lagrangian. In the basis (ν_L, ν_R^c), it takes the form

$$-\mathcal{L}_m = \frac{1}{2} \begin{pmatrix} \overline{\nu_L'} & \overline{\nu_R'^c} \end{pmatrix} \begin{pmatrix} 0 & m_D \\ m_D^T & M_R \end{pmatrix} \begin{pmatrix} \nu_L'^c \\ \nu_R' \end{pmatrix} + \text{h.c.} \tag{1.57}$$

where 0, m_D, and M_R are 3×3 blocks, since we have introduced three right-handed neutrinos. It is worth mentioning that some models consider a different number of right-handed neutrinos; in such cases, the dimension of the blocks within the mass matrix changes accordingly. The next question is how to diagonalize this mass matrix. This is not entirely straightforward, since we are dealing with a 6×6 matrix in flavor space. At this point, the key assumption of the model enters: one assumes that the block M_R corresponds to a mass scale much larger than that associated with m_D, that is, the eigenvalues of M_R are much larger than those of m_D. Under this assumption, the mass matrix can be approximately diagonalized [15, 16], leading to

$$m_\nu \simeq -m_D M_R^{-1} m_D^T\ ,$$

where m_ν has been defined as the effective mass matrix for the light neutrinos. The most relevant feature of the expression above is that the neutrino masses are *suppressed* by the heavy scale M_R. Thus, the smallness of neutrino masses is no longer imposed by hand, but instead arises naturally from the presence of a very large mass scale. This inverse relationship is precisely the reason for the name *seesaw*, since the smallness of one mass is associated with the largeness of another.

1.4.4 More models

According to the discussion presented above, the seesaw mechanism generates small neutrino masses by introducing a new extremely high mass scale. In this framework, the smallness of neutrino masses is explained as the result of the suppression induced by this heavy scale. However, the seesaw mechanism is not the only way to generate neutrino masses. There also exist the radiative models, in which neutrino masses are generated at the loop level. In these models, the mass term is forbidden at tree level and appears only through radiative corrections. As a consequence, the smallness of neutrino masses is explained by the suppression factors associated with loop contributions, rather than by the presence of an extremely high mass scale. Some of the most famous examples of radiative neutrino mass models are the Zee model [17, 18] and the Zee-Babu model [19, 20]. An attractive feature of these mechanisms is that the new fields introduced in the theory can often have masses close to energy scales that are potentially testable in experiments. The accessible energy scale is not the only advantage of these models. Another appealing feature is that the new particles running inside the loops may provide viable dark matter candidates. This is particularly relevant since dark matter plays a crucial role in our understanding of the structure and evolution of the universe. In this sense, radiative models offer an attractive framework in which two fundamental problems of modern physics can be addressed simultaneously: the origin of neutrino masses and the nature of dark matter. In this work we will study a particular radiative model known as the Cocktail model [21].

1.5 Flavor Puzzle

At the beginning of this section, we emphasized that the Yukawa couplings are free parameters of the Standard Model, and that there is no explanation for why, despite being arbitrary, they lead to such a pronounced hierarchy in the charged-lepton masses. This suggests that there may be an underlying structure that constrains the values of these masses. It is therefore natural to think that an underlying symmetry governs the flavor sector, and that the observed mass spectrum is the consequence of the spontaneous breaking of such a symmetry.

However, the strong mass hierarchies are not the only reason to suspect the existence of a hidden symmetry. Consider, for instance, the Type-I seesaw mechanism. In many treatments, one derives the neutrino mass matrix and stops there. Yet this is not the end of the story. In order to obtain the physical neutrino masses, one must diagonalize the mass matrix. The rotation matrix that diagonalizes the neutrino mass matrix plays a crucial role, since together with the rotation matrix of the charged leptons it gives rise to the leptonic mixing matrix, whose elements are experimentally measurable. Remarkably, the entries of the leptonic mixing matrix are not arbitrary. Their magnitudes appear to follow a specific pattern, commonly referred to as a mixing pattern. Just as in the case of mass hierarchies, there is no fundamental explanation within the Standard Model for the emergence of such a pattern.

A possible interpretation is that a symmetry governs the structure of the Yukawa couplings. Since these couplings determine the entries of the mass matrices, which in turn fix the rotation matrices and ultimately the mixing matrix, all these quantities are deeply interconnected. We therefore seek a symmetry capable of constraining the Yukawa sector in such a way that the observed mass hierarchies and mixing pattern arise naturally. This constitutes the central motivation of the present work and justifies the introduction of the theoretical framework developed thus far. We now proceed to present the mathematical structure of modular symmetry and explore how it can be embedded within the Standard Model and its extensions.

Chapter 2

Modular symmetry

2.1 Modular group

The full modular group is the invariance group of a lattice in the complex plane

$$\Lambda = \{m_1\omega_1 + m_2\omega_2 \mid m_{1,2} \in \mathbb{Z}\},$$

where ω_1 and ω_2 are linearly independent over $\mathbb{C}$. We define the modulus as $\tau \equiv \omega_1/\omega_2$, and, without loss of generality, we assume that $\mathrm{Im}(\tau) > 0$, since one may always interchange ω_1 and ω_2 if it is necessary. Consider now two lattices

$$\Lambda = \{m_1\omega_1 + m_2\omega_2 \mid m_{1,2} \in \mathbb{Z}\} \ , \qquad \Lambda' = \{m_1'\omega_1' + m_2'\omega_2' \mid m_{1,2}' \in \mathbb{Z}\} \ .$$

They are identical if and only if they generate the same set of points in the complex plane. In particular, the bases $\{\omega_1, \omega_2\}$ and $\{\omega_1', \omega_2'\}$ must be related by a linear transformation

$$\begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = A \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}, \qquad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{2.1}$$

Since ω_1' and ω_2' belong to Λ, they must be integer linear combinations of ω_1 and ω_2. In Figure (2.1), two different bases are shown, ω_i (red) and ω_i' (purple), both describing the same set of points. Therefore, $a, b, c, d \in \mathbb{Z}$, so that $A \in M_{2\times 2}(\mathbb{Z})$. Moreover, because both bases generate the same lattice, the transformation must be invertible over $\mathbb{Z}$. Hence there exists a matrix $B \in M_{2\times 2}(\mathbb{Z})$ such that

$$\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = B \begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix}.$$

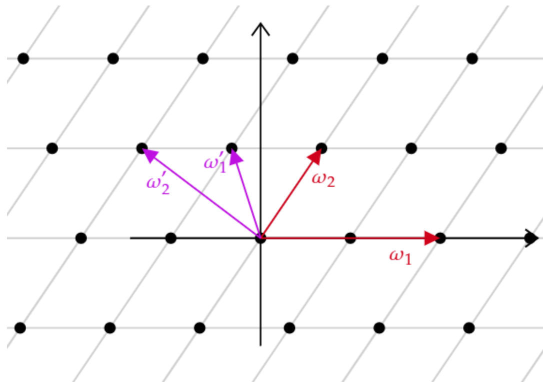


Figure 2.1: Two equivalent lattices generated by the base vectors $\{\omega_1, \omega_2\}$ and $\{\omega_1', \omega_2'\}$

Substituting into Eq. (2.1) gives

$$\begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = AB \begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix},$$

which implies

$$AB = \mathbf{1}_2.$$

Taking determinants,

$$\det(A)\det(B) = 1.$$

Since $\det(A)$ and $\det(B)$ are integers, the only possibility is

$$\det(A) = \det(B) = \pm 1. \tag{2.2}$$

Thus, $A \in GL(2, \mathbb{Z})$. Let us now determine how the modulus transforms. From Eq. (2.1),

$$\omega_1' = a\omega_1 + b\omega_2 \ , \qquad \omega_2' = c\omega_1 + d\omega_2 \ .$$

Therefore,

$$\tau' = \frac{\omega_1'}{\omega_2'} = \frac{a\omega_1 + b\omega_2}{c\omega_1 + d\omega_2}.$$

Dividing numerator and denominator by ω_2, we obtain

$$\tau' = \frac{a\tau + b}{c\tau + d} \ . \tag{2.3}$$

It can be shown with a little algebra that

$$\mathrm{Im}\left(\frac{a\tau + b}{c\tau + d}\right) = \frac{\mathrm{Im}(\tau)}{|c\tau + d|^2}\det(A) \ .$$

Since $|c\tau + d|^2 > 0$ and we have assumed $\mathrm{Im}(\tau) > 0$, the condition $\mathrm{Im}(\tau') > 0$ requires

$$\det(A) > 0.$$

Combining this with Eq. (2.2), we conclude that

$$\det(A) = 1. \tag{2.4}$$

We have therefore shown that the matrices relating equivalent lattice bases are precisely the integer matrices with unit determinant, namely

$$A \in SL(2, \mathbb{Z}).$$

Having determined that the matrices relating equivalent lattices are precisely the elements of $SL(2, \mathbb{Z})$, we can now summarize how the modulus τ transforms under such changes of basis. As shown above, if

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}),$$

then the modulus transforms as

$$\tau \longrightarrow \tau' \equiv \gamma\tau = \frac{a\tau + b}{c\tau + d}, \qquad \gamma \in SL(2, \mathbb{Z}), \qquad \mathrm{Im}(\tau) > 0. \tag{2.5}$$

This transformation is a linear fractional (or Möbius) transformation. Therefore, the action of $SL(2, \mathbb{Z})$ on the upper half–plane $\mathbb{H} = \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}$ is realized through Möbius transformations that preserve the positivity of the imaginary part. It is important to notice that the matrices γ and $-\gamma$ induce exactly the same transformation on τ, since

$$\frac{(-a)\tau + (-b)}{(-c)\tau + (-d)} = \frac{a\tau + b}{c\tau + d}.$$

Thus, the action is not faithful if we work directly with $SL(2, \mathbb{Z})$. In order to obtain a faithful group action, one must identify matrices that differ by an overall sign. This leads naturally to the projective special linear group

$$\bar{\Gamma} \equiv PSL(2, \mathbb{Z}) \cong SL(2, \mathbb{Z})/\{\mathbf{1}_2, -\mathbf{1}_2\}, \tag{2.6}$$

where $\mathbf{1}_2$ denotes the 2×2 identity matrix. The group $\bar{\Gamma}$ is what is commonly referred to as the modular group. Both $\Gamma \cong SL(2, \mathbb{Z})$ and $\bar{\Gamma} \cong PSL(2, \mathbb{Z})$ are generated by only two elements,

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{2.7}$$

These matrices generate $SL(2, \mathbb{Z})$ under matrix multiplication. A direct computation shows that

$$S^2 = -\mathbf{1}_2, \qquad (ST)^3 = -\mathbf{1}_2,$$

so that in $SL(2,\mathbb{Z})$ one has the relations

$$S^4 = \mathbf{1}_2, \qquad (ST)^3 = -\mathbf{1}_2. \tag{2.8}$$

Passing to the projective group $PSL(2,\mathbb{Z})$, where matrices differing by an overall sign are identified, the element $-\mathbf{1}_2$ becomes the identity. Consequently, the defining relations reduce to

$$S^2 = \mathbf{1}_2, \qquad (ST)^3 = \mathbf{1}_2. \tag{2.9}$$

Now that we have established the structure of the modular groups, we turn to the study of their subgroups. In particular, we introduce the principal congruence subgroups of level N. For the full modular group $\Gamma \cong SL(2,\mathbb{Z})$, the principal congruence subgroup of level N is defined as

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \;\middle|\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod N \right\}. \tag{2.10}$$

This subgroup consists of all elements of $SL(2,\mathbb{Z})$ that are congruent to the identity matrix modulo N. It is straightforward to verify that $\Gamma(N)$ is a normal subgroup of Γ. We now define the corresponding principal congruence subgroup for the projective modular group $\overline{\Gamma} \cong PSL(2,\mathbb{Z})$. The crucial observation is that the element $-\mathbf{1}_2$ belongs to $\Gamma(N)$ only for $N = 1, 2$, while for $N > 2$ it is no longer contained in $\Gamma(N)$. As a consequence, the principal congruence subgroup of $\overline{\Gamma}$ is given by

$$\overline{\Gamma}(N) = \begin{cases} \Gamma(N)/\{\mathbf{1}_2, -\mathbf{1}_2\}\ , & N = 1, 2 \\ \Gamma(N)\ , & N > 2\,. \end{cases} \tag{2.11}$$

The importance of these subgroups lies in the fact that they allow the construction of finite modular groups through quotient operations. Since $\Gamma(N)$ and $\overline{\Gamma}(N)$ are normal subgroups, the corresponding quotients are actually groups. In particular, we define the finite inhomogeneous modular group of level N as

$$\Gamma_N \equiv \overline{\Gamma}/\overline{\Gamma}(N)\,, \tag{2.12}$$

This group can be generated by two elements S and T satisfying

$$S^2 = (ST)^3 = T^N = \mathbf{1}_2\ .$$

Analogously, we define the finite homogeneous modular group as

$$\Gamma'_N \equiv \Gamma/\Gamma(N)\,. \tag{2.13}$$

This group can be obtained from Γ_N by including another generator R which is related to $-\mathbf{1}_2 \in SL(2,\mathbb{Z})$ and commutes with all elements of the $SL(2,\mathbb{Z})$. The generators S, T and R of Γ'_N satisfy the following relations

$$S^2 = R\ , \quad (ST)^3 = T^N = R^2 = 1\ , \quad RT = TR\ .$$

Having introduced the principal congruence subgroups and the corresponding quotient groups, we now proceed to make their structure more explicit. In particular, we would like to determine whether the finite modular groups defined above are isomorphic to any well-known finite groups. To illustrate the procedure, let us consider the case $N = 2$. Let's define the following homomorphism

$$f : SL(2,\mathbb{Z}) \longrightarrow SL(2,\mathbb{F}_2)\,,$$

where $\mathbb{F}_2 = \{0,1\}$ denotes the finite field with two elements. By construction, the kernel of this map consists precisely of those matrices in $SL(2,\mathbb{Z})$ that are congruent to the identity modulo 2, namely

$$\ker f = \Gamma(2).$$

The First Isomorphism Theorem then immediately implies

$$SL(2,\mathbb{Z})/\Gamma(2) \cong SL(2,\mathbb{F}_2). \tag{2.14}$$

An analogous construction can be performed for the projective modular group. Consider the homomorphism

$$g : PSL(2,\mathbb{Z}) \longrightarrow SL(2,\mathbb{F}_2),$$

In this case, the kernel is

$$\ker g = \Gamma(2)/\{\mathbf{1}_2, -\mathbf{1}_2\} = \overline{\Gamma}(2),$$

and therefore

$$PSL(2,\mathbb{Z})/\overline{\Gamma}(2) \cong PSL(2,\mathbb{F}_2). \tag{2.15}$$

The elements of $SL(2,\mathbb{F}_2)$ are explicitly given by

$$\left\{\begin{pmatrix}1&0\\0&1\end{pmatrix},\begin{pmatrix}0&1\\1&0\end{pmatrix},\begin{pmatrix}1&1\\0&1\end{pmatrix},\begin{pmatrix}1&0\\1&1\end{pmatrix},\begin{pmatrix}1&1\\1&0\end{pmatrix},\begin{pmatrix}0&1\\1&1\end{pmatrix}\right\}.$$

We first observe that the group $SL(2,\mathbb{F}_2)$ contains six elements. Indeed, since $\mathbb{F}_2 = \{0,1\}$, one may verify explicitly that there are exactly six 2×2 matrices with determinant equal to 1 over $\mathbb{F}_2$, and therefore $|SL(2,\mathbb{F}_2)| = 6$. Moreover, the group multiplication is not commutative, so $SL(2,\mathbb{F}_2)$ is non Abelian. Up to isomorphism, the only non-abelian group of order 6 is the symmetric group S_3. It follows that

$$SL(2,\mathbb{F}_2) \cong S_3.$$

Using the isomorphisms derived in Eq. (2.14) and Eq. (2.15), we conclude

$$SL(2,\mathbb{Z})/\Gamma(2) \cong S_3, \qquad PSL(2,\mathbb{Z})/\overline{\Gamma}(2) \cong S_3.$$

In other words, the finite modular group of level $N = 2$ is isomorphic to S_3. In this particular case one finds

$$\Gamma_2 \cong \Gamma'_2,$$

since the element $-\mathbf{1}_2$ becomes trivial after reduction modulo 2. This same procedure can be generalized to higher levels. For $N > 2$, the central element $-\mathbf{1}_2$ is no longer contained in $\Gamma(N)$, and one finds in general that the finite inhomogeneous modular group is obtained from the homogeneous one by quotienting out its center,

$$\Gamma_N \cong \Gamma'_N/\{\pm\mathbf{I}_2\}.$$

Consequently, their orders are related by

$$|\Gamma_N| = \frac{1}{2}\,|\Gamma'_N|.$$

For small values of N, the finite modular groups are isomorphic to familiar permutation groups. In particular [22],

$$\Gamma_2 \cong S_3, \qquad \Gamma_3 \cong A_4, \qquad \Gamma_4 \cong S_4, \qquad \Gamma_5 \cong A_5.$$

These isomorphisms can be established by studying the corresponding reduction homomorphisms modulo N and determining the structure and order of the resulting finite groups.

2.1.1 Holomorphic modular forms

Now that we understand the structure of modular groups, congruence subgroups, and finite modular groups, we can proceed to define the objects that transform under these groups. In this context, we define the modular forms $f(\tau)$ of weight k and level N as holomorphic functions of the complex variable τ, which transform under $\Gamma(N)$ according to

$$f(\gamma\tau) = (c\tau + d)^k f(\tau), \qquad \forall\, \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(N), \tag{2.16}$$

where k is a positive integer. This transformation property is known as the modularity condition. From the previous section, we recall that for levels $N = 1, 2$, the generator S satisfies $S^2 = -\mathbf{1}_2 \in \Gamma(N)$. If we consider $\gamma = S^2$, the modularity condition implies

$$f(\tau) = (-1)^k f(\tau).$$

It follows that if k is odd, then

$$f(\tau) = -f(\tau),$$

which implies that non-vanishing modular forms of odd weight do not exist for $N = 1$ and $N = 2$. However, for $\Gamma(N)$ with $N > 2$, non-zero modular forms of odd weight may exist. The modular forms of weight k and level N form a vector space denoted by $\mathcal{M}_k(\Gamma(N))$. Its dimension is given by [23]

$$\dim \mathcal{M}_{2k}(\Gamma(2)) = k + 1, \qquad N = 2, \quad k \geq 1,$$

and

$$\dim \mathcal{M}_k(\Gamma(N)) = \frac{N(k-1)+6}{24} N^2 \prod_{p|N} \left(1 - \frac{1}{p^2}\right), \qquad N \geq 2, \quad k \geq 2.$$

Since modular forms generate a vector space, one can choose a set of linearly independent basis elements, which we denote by $f_i(\tau)$, with $i = 1, 2, \ldots, \dim \mathcal{M}_k(\Gamma(N))$. Furthermore, for any element $\gamma \in \Gamma$, we define

$$F_{i\gamma}(\tau) \equiv J^{-k}(\gamma, \tau)\, f_i(\gamma\tau),$$

where the automorphy factor is defined by

$$J(\gamma, \tau) = c\tau + d.$$

Let us derive some useful identities. Consider two elements of the modular group,

$$\gamma_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \qquad \gamma_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}, \qquad \gamma_1,\, \gamma_2 \in \Gamma.$$

We now compute the automorphy factor associated with the product $\gamma_1\gamma_2$

$$\begin{aligned} J(\gamma_1\gamma_2, \tau) &= (c_1 a_2 + d_1 c_2)\tau + (c_1 b_2 + d_1 d_2) \\ &= c_1(a_2\tau + b_2) + d_1(c_2\tau + d_2). \end{aligned}$$

Factoring the expression, we obtain

$$J(\gamma_1\gamma_2, \tau) = \left(c_1 \left(\frac{a_2\tau + b_2}{c_2\tau + d_2} \right) + d_1 \right) (c_2\tau + d_2).$$

This can be written in the compact form

$$J(\gamma_1\gamma_2, \tau) = J(\gamma_1, \gamma_2\tau)\, J(\gamma_2, \tau), \qquad \gamma_1,\, \gamma_2 \in \Gamma, \tag{2.17}$$

which constitutes the first important property of the automorphy factor. To derive a second property, consider the inverse element

$$\gamma^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \qquad \gamma^{-1} \in \Gamma.$$

We compute the corresponding automorphy factor,

$$J(\gamma^{-1}, \gamma\tau) = \frac{-c}{ad - bc} \left(\frac{a\tau + b}{c\tau + d} \right) + \frac{a}{ad - bc}.$$

A direct calculation shows that

$$J(\gamma^{-1}, \gamma\tau) = \frac{1}{c\tau + d} = (c\tau + d)^{-1} = J^{-1}(\gamma, \tau), \tag{2.18}$$

which provides the second key identity. Having established these properties, let us consider a generic modular transformation $h \in \Gamma(N)$ acting on the function $F_{i\gamma}(\tau)$ defined previously. We first obtain

$$F_{i\gamma}(h\tau) = J^{-k}(\gamma, h\tau)\, f_i(\gamma h\tau) = J^{-k}(\gamma, h\tau)\, f_i(\gamma h\gamma^{-1}\gamma\tau).$$

Using the modularity condition,

$$f_i(\gamma h\gamma^{-1}\gamma\tau) = J^k(\gamma h\gamma^{-1}, \gamma\tau) f_i(\gamma\tau),$$

we obtain

$$F_{i\gamma}(h\tau) = J^{-k}(\gamma, h\tau) J^k(\gamma h\gamma^{-1}, \gamma\tau) f_i(\gamma\tau).$$

Applying property (2.17),

$$F_{i\gamma}(h\tau) = J^{-k}(\gamma, h\tau) J^k(\gamma, h\tau) J^k(h\gamma^{-1}, \gamma\tau) f_i(\gamma\tau).$$

Using property (2.18),

$$F_{i\gamma}(h\tau) = J^k(h, \tau) J^{-k}(\gamma, \tau) f_i(\gamma\tau).$$

Finally, replacing the definition of $F_{i\gamma}(\tau)$, we arrive at

$$F_{i\gamma}(h\tau) = J^k(h, \tau) F_{i\gamma}(\tau). \tag{2.19}$$

This implies that the general function $F_{i\gamma}(\tau)$ defined previously satisfies the modularity condition. Therefore, it is a modular form of weight k and level N. Moreover, since $F_{i\gamma}(\tau)$ fulfills this condition, it can be expressed as a linear combination of a basis of modular forms $f_i(\tau)$, namely

$$F_{i\gamma}(\tau) = \rho_{ij}(\gamma)\, f_j(\tau)\,.$$

This immediately leads to the following transformation

$$f_i(\gamma\tau) = J^k(\gamma, \tau)\, \rho_{ij}(\gamma)\, f_j(\tau)\,, \tag{2.20}$$

The matrix $\rho(\gamma)$ depends on the modular transformation γ. Furthermore, from Eq. (2.18), we know that $\rho(\gamma)$ defines a representation of the modular group, satisfying the group composition law

$$\rho(\gamma_1\gamma_2) = \rho(\gamma_1)\rho(\gamma_2)\,.$$

By comparing Eq. (2.20) with the modularity condition, we conclude that for any element $h \in \Gamma(N)$,

$$\rho(h) = \mathbf{1}\,.$$

Since $\rho(h) = \mathbf{1}$ for all $h \in \Gamma(N)$, the subgroup $\Gamma(N)$ lies in the kernel of ρ. As a result, the representation depends only on the equivalence classes $[\gamma] \in \Gamma/\Gamma(N)$, and thus it defines a representation of the finite modular group Γ'_N. From the defining relations of Γ'_N, the representation matrices fulfill

$$\rho(S)^4 = \rho(ST)^3 = \rho(T)^N = \mathbf{1}, \qquad \rho(R)\rho(T) = \rho(T)\rho(R).$$

Since every representation of a finite group can be decomposed into a direct sum of irreducible unitary representations, we can write ρ in block-diagonal form,

$$\rho \sim \rho_{\mathbf{r}_1} \oplus \rho_{\mathbf{r}_2} \oplus \cdots, \qquad \sum_i \dim \rho_{\mathbf{r}_i} = \dim \mathcal{M}_k(\Gamma(N)),$$

where $\rho_{\mathbf{r}_i}$ denotes an irreducible unitary representation of Γ'_N. In this way, it is always possible to choose a basis $Y_{\mathbf{r}}(\tau) \equiv (f_1(\tau), f_2(\tau), \ldots)^T$ in the space of modular forms such that $Y_{\mathbf{r}}$ transforms under the full modular group Γ as

$$Y_{\mathbf{r}}(\gamma\tau) = (c\tau + d)^k \rho_{\mathbf{r}}(\gamma) Y_{\mathbf{r}}(\tau), \qquad \gamma \in \Gamma. \tag{2.21}$$

In practice, it is sufficient to consider the generators S and T, so that

$$Y_{\mathbf{r}}(S\tau) = (-\tau)^k \rho_{\mathbf{r}}(S) Y_{\mathbf{r}}(\tau), \qquad Y_{\mathbf{r}}(T\tau) = \rho_{\mathbf{r}}(T) Y_{\mathbf{r}}(\tau).$$

Finally, taking $\gamma = R$ in Eq. (2.21), we find that in order for the modular forms $Y_{\mathbf{r}}(\tau)$ to be non-vanishing, it is required that

$$\rho_{\mathbf{r}}(R) = (-1)^k, \tag{2.22}$$

which equals to $+1$ for even k and -1 for odd k.

2.1.2 Even weight and inhomogeneous modular group

For instance, let us consider the case where the modular weight k is an even number. From (2.22), it is easy to see that

$$\rho_{\mathbf{r}}(R) = 1 .$$

In this case, the defining relations of the group Γ'_N reduce exactly to those of the group Γ_N, and consequently, modular forms of even weight can be organized into multiplets of Γ_N, up to, of course, an automorphy factor. This is particularly interesting, since we have seen that the inhomogeneous finite modular group exhibits isomorphisms with permutation groups, whose algebra is well-known.

There are no modular forms of negative weight, and modular forms of weight zero are constant. Moreover, modular forms of weight $2k$ and level N constitute a finite-dimensional vector space $\mathcal{M}_{2k}(\Gamma(N))$. Another important point is that the product of two modular forms of level N with weights $2k$ and $2k'$ is itself a modular form of level N and weight $2(k + k')$.

2.1.3 Constructing even weight modular forms of level 3

As an illustrative example, it is instructive to show how modular forms of even weight and level 3 are constructed. As a first observation, since $N = 3$, modular forms can be arranged into multiplets of the finite group A_4. The Dedekind eta function $\eta(\tau)$ is frequently used in the construction of modular forms. It is defined as

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) \ , \qquad q \equiv e^{2\pi i \tau} \ .$$

The eta function can also be expressed as an infinite series,

$$\eta(\tau) = q^{1/24} \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} \ .$$

Using the generators S and T defined previously, one finds that $\eta(\tau)$ transforms as

$$\eta(\tau + 1) = e^{i\pi/12} \eta(\tau) \ , \qquad \eta(-1/\tau) = \sqrt{-i\tau}\, \eta(\tau) \ .$$

In what follows, we adopt the method presented in [23] to construct explicitly a basis of weight 2 for $\Gamma(3)$. The dimension of this space can be computed using the formulas introduced earlier, yielding

$$\dim \mathcal{M}_2(\Gamma(3)) = 3 \ .$$

This implies that there exist three linearly independent modular forms of weight 2 at level 3. Let us begin by noting that if a modular form $f(\tau)$ transforms as

$$f(\tau) \longrightarrow e^{i\alpha} (c\tau + d)^k f(\tau) \ ,$$

then its logarithmic derivative transforms according to

$$\frac{d}{d\tau} \log f(\tau) \longrightarrow (c\tau + d)^2 \frac{d}{d\tau} \log f(\tau) + k\, c(c\tau + d) \ .$$

Furthermore, we can linearly combine several $f_i(\tau)$ with weights k_i as follow

$$\frac{d}{d\tau} \sum_i \alpha_i \log f_i(\tau) \longrightarrow (c\tau + d)^2 \frac{d}{d\tau} \sum_i \alpha_i \log f_i(\tau) + \left(\sum_i \alpha_i k_i \right) c(c\tau + d) \ ,$$

with

$$\sum_i \alpha_i k_i = 0 \ .$$

In this case, the inhomogeneous term vanishes, and consequently one obtains modular forms of weight 2. The seed functions for weight 2 modular forms at level 3 are given by [24]

$$f_1(\tau) = \eta\left(\frac{\tau}{3}\right) \ , \qquad f_2(\tau) = \eta\left(\frac{\tau + 1}{3}\right) \ ,$$

$$f_3(\tau) = \eta\left(\frac{\tau + 2}{3}\right) \ , \qquad f_4(\tau) = \eta(3\tau) \ .$$

These functions are closed under the action of the modular group, and each of them is mapped into itself under the action of $\Gamma(3)$. Under the action of the generator T, they transform as

$$f_1(\tau) \xrightarrow{T} f_2(\tau) \ , \qquad f_2(\tau) \xrightarrow{T} f_3(\tau) \ ,$$
$$f_3(\tau) \xrightarrow{T} e^{i\pi/12} f_1(\tau) \ , \qquad f_4(\tau) \xrightarrow{T} e^{i\pi/4} f_4(\tau) \ .$$

On the other hand, under the action of S they transform as

$$f_1(\tau) \xrightarrow{S} \sqrt{3}\sqrt{-i\tau}\, f_4(\tau) \ , \qquad f_2(\tau) \xrightarrow{S} e^{-i\pi/12}\sqrt{-i\tau}\, f_3(\tau) \ ,$$
$$f_3(\tau) \xrightarrow{S} e^{i\pi/12}\sqrt{-i\tau}\, f_2(\tau) \ , \qquad f_4(\tau) \xrightarrow{S} \frac{1}{\sqrt{3}}\sqrt{-i\tau}\, f_1(\tau) \ .$$

Hence the most general modular form of weight 2 and level 3 can be written as

$$\mathcal{F}(\alpha_1, \alpha_2, \alpha_3, \alpha_4|\tau) = \frac{d}{d\tau}\sum_{i=1}^{4} \alpha_i \log f_i(\tau) \ ,$$

with the constraint

$$\sum_{i=1}^{4} \alpha_i = 0 \ ,$$

which ensures the cancellation of the inhomogeneous term in the transformation. Under the action of the generators S and T, one finds

$$\mathcal{F}(\alpha_1, \alpha_2, \alpha_3, \alpha_4|\tau) \xrightarrow{S} \tau^2 \mathcal{F}(\alpha_4, \alpha_3, \alpha_2, \alpha_1|\tau) \ ,$$
$$\mathcal{F}(\alpha_1, \alpha_2, \alpha_3, \alpha_4|\tau) \xrightarrow{T} \mathcal{F}(\alpha_3, \alpha_1, \alpha_2, \alpha_4|\tau) \ .$$

As previously mentioned, it is always possible to choose a basis in the vector space of modular forms such that the basis vectors are arranged into multiplets of irreducible representations of the finite modular group. In the present case, there exist three independent modular forms $Y_i(\tau)$ which can be organized into an A_4 triplet (see the Appendix B for the details of the permutations groups),

$$Y^{(2)}_{\mathbf{3}} = \left(Y^{(2)}_{\mathbf{3},1}, Y^{(2)}_{\mathbf{3},2}, Y^{(2)}_{\mathbf{3},3}\right)^T \ ,$$

satisfying

$$Y^{(2)}_{\mathbf{3}}(-1/\tau) = \tau^2 \rho_{\mathbf{3}}(S) Y^{(2)}_{\mathbf{3}}(\tau) \ , \qquad Y^{(2)}_{\mathbf{3}}(\tau+1) = \rho_{\mathbf{3}}(T) Y^{(2)}_{\mathbf{3}}(\tau) \ .$$

Here $\rho_{\mathbf{3}}(S)$ and $\rho_{\mathbf{3}}(T)$ are the unitary representation in the triplet representation of A_4 [23],

$$\rho_{\mathbf{3}}(S) = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix} \ , \qquad \rho_{\mathbf{3}}(T) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} \ , \quad \text{with} \quad \omega = e^{2\pi i/3}.$$

Solving the above transformation properties, one obtains explicit expressions for $Y_i(\tau)$ as follows

$$\begin{aligned}
Y_1(\tau) &= \frac{3i}{2\pi}\,\mathcal{F}(1,1,1,-3 \mid \tau) && (2.23)\\
&= \frac{i}{2\pi}\left[\frac{\eta'(\tau/3)}{\eta(\tau/3)} + \frac{\eta'\left(\frac{\tau+1}{3}\right)}{\eta\left(\frac{\tau+1}{3}\right)} + \frac{\eta'\left(\frac{\tau+2}{3}\right)}{\eta\left(\frac{\tau+2}{3}\right)} - 27\,\frac{\eta'(3\tau)}{\eta(3\tau)}\right], && (2.24)
\end{aligned}$$

$$\begin{aligned}
Y_2(\tau) &= -\frac{3i}{\pi}\,\mathcal{F}(1,\omega^2,\omega,0 \mid \tau) && (2.25)\\
&= -\frac{i}{\pi}\left[\frac{\eta'(\tau/3)}{\eta(\tau/3)} + \omega^2\frac{\eta'\left(\frac{\tau+1}{3}\right)}{\eta\left(\frac{\tau+1}{3}\right)} + \omega\frac{\eta'\left(\frac{\tau+2}{3}\right)}{\eta\left(\frac{\tau+2}{3}\right)}\right], && (2.26)
\end{aligned}$$

$$\begin{aligned}
Y_3(\tau) &= -\frac{3i}{\pi}\,\mathcal{F}(1,\omega,\omega^2,0 \mid \tau) && (2.27)\\
&= -\frac{i}{\pi}\left[\frac{\eta'(\tau/3)}{\eta(\tau/3)} + \omega\frac{\eta'\left(\frac{\tau+1}{3}\right)}{\eta\left(\frac{\tau+1}{3}\right)} + \omega^2\frac{\eta'\left(\frac{\tau+2}{3}\right)}{\eta\left(\frac{\tau+2}{3}\right)}\right]. && (2.28)
\end{aligned}$$

For practical numerical computations, it is convenient to use the q-expansions of the modular forms, which are given by

$$\begin{aligned}
Y_1(\tau) &= 1 + 12q + 36q^2 + 12q^3 + 84q^4 + 72q^5 + 36q^6 + 96q^7 + 180q^8 + 129q^9 + 216q^{10} + \cdots\,,\\
Y_2(\tau) &= -6q^{1/3}\left(1 + 7q + 8q^2 + 18q^3 + 14q^4 + 31q^5 + 20q^6 + 36q^7 + 31q^8 + 56q^9 + 32q^{10} + \cdots\right),\\
Y_3(\tau) &= -18q^{2/3}\left(1 + 2q + 5q^2 + 4q^3 + 8q^4 + 6q^5 + 14q^6 + 8q^7 + 14q^8 + 10q^9 + 21q^{10} + \cdots\right).
\end{aligned}$$

From the above q-expansions we see that the modular forms $Y_i(\tau)$ satisfy the constraint

$$Y_2^2 + 2Y_1Y_3 = 0.$$

The whole ring of even weight modular forms of level 3 can be generated from the products of the weight 2 modular forms $Y_{1,2,3}(\tau)$. At weight 4, the tensor product of $Y_{\mathbf{3}}^{(2)}$ gives rise to three independent modular multiplets,

$$\begin{aligned}
Y_{1}^{(4)} &= (Y_{\mathbf{3}}^{(2)}Y_{\mathbf{3}}^{(2)})_{\mathbf{1}} = Y_1^2 + 2Y_2Y_3,\\
Y_{1'}^{(4)} &= (Y_{\mathbf{3}}^{(2)}Y_{\mathbf{3}}^{(2)})_{\mathbf{1}'} = Y_3^2 + 2Y_1Y_2,\\
Y_{\mathbf{3}}^{(4)} &= \frac{1}{2}(Y_{\mathbf{3}}^{(2)}Y_{\mathbf{3}}^{(2)})_{\mathbf{3}_S} = \begin{pmatrix} Y_1^2 - Y_2Y_3\\ Y_2^2 - Y_1Y_3\\ Y_3^2 - Y_1Y_2\end{pmatrix}.
\end{aligned}$$

There are seven linearly independent modular forms of level 3 and weight 6 and they decompose as $\mathbf{1} \oplus \mathbf{3} \oplus \mathbf{3}$ under A_4,

$$
\begin{aligned}
Y_{\mathbf{1}}^{(6)} &= (Y_{\mathbf{3}}^{(2)} Y_{\mathbf{3}}^{(4)})_{\mathbf{1}} = Y_1^3 + Y_2^3 + Y_3^3 - 3Y_1 Y_2 Y_3, \\
Y_{\mathbf{3}I}^{(6)} &= (Y_{\mathbf{3}}^{(2)} Y_{\mathbf{1}}^{(4)}) = (Y_1^2 + 2Y_2 Y_3) \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix}, \\
Y_{\mathbf{3}II}^{(6)} &= (Y_{\mathbf{3}}^{(2)} Y_{\mathbf{1}'}^{(4)}) = (Y_3^2 + 2Y_1 Y_2) \begin{pmatrix} Y_3 \\ Y_1 \\ Y_2 \end{pmatrix}.
\end{aligned}
$$

The weight 8 modular forms can be arranged into three singlets $\mathbf{1}$, $\mathbf{1}'$, $\mathbf{1}''$ and two triplets $\mathbf{3}$ of A_4,

$$
\begin{aligned}
Y_{\mathbf{1}}^{(8)} &= (Y_{\mathbf{3}}^{(2)} Y_{\mathbf{3}I}^{(6)})_{\mathbf{1}} = (Y_1^2 + 2Y_2 Y_3)^2, \\
Y_{\mathbf{1}'}^{(8)} &= (Y_{\mathbf{3}}^{(2)} Y_{\mathbf{3}II}^{(6)})_{\mathbf{1}'} = (Y_3^2 + 2Y_1 Y_2)^2, \\
Y_{\mathbf{1}''}^{(8)} &= (Y_{\mathbf{3}}^{(2)} Y_{\mathbf{3}I}^{(6)})_{\mathbf{1}''} = (Y_2^2 + 2Y_1 Y_3)^2, \\
Y_{\mathbf{3}I}^{(8)} &= (Y_{\mathbf{3}}^{(2)} Y_{\mathbf{1}}^{(6)}) = (Y_1^3 + Y_2^3 + Y_3^3 - 3Y_1 Y_2 Y_3) \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix}, \\
Y_{\mathbf{3}II}^{(8)} &= (Y_{\mathbf{3}}^{(2)} Y_{\mathbf{3}II}^{(6)})_{\mathbf{3}_A} = (Y_3^2 + 2Y_1 Y_2) \begin{pmatrix} Y_1^2 - Y_2 Y_3 \\ Y_2^2 - Y_1 Y_3 \\ Y_3^2 - Y_1 Y_2 \end{pmatrix}.
\end{aligned}
$$

We can repeat the same procedure for different levels, obtaining the corresponding modular forms in each case. An important question that remains is how such symmetries can be implemented in mass models, which was, after all, the original motivation for introducing modular symmetry. We leave this question for later, when its importance can be better appreciated. Now we want to explore what happens if we impose further conditions on our modular forms. So far, we have only required modular invariance together with holomorphy, which restricts the allowed modular to be positive. One may then ask what occurs if we relax the assumption of holomorphy. In that case, modular forms would no longer be required to be holomorphic functions, and negative modular weights could in principle be allowed.

2.2 Non-holomorphic modular forms

In the previous section, we mentioned the possibility of relaxing the condition of holomorphy and investigating the type of modular objects that arise in such a framework. To this end, we now assume that our "new" modular objects $Y(\tau)$ satisfy the following conditions [25]

$$Y(\gamma\tau) = J^k(\gamma,\tau)\, Y(\tau)\ , \qquad \gamma \in \Gamma(N), \tag{2.29}$$

$$\Delta_k Y(\tau) = 0\ . \tag{2.30}$$

The first condition corresponds to modularity, which we want to preserve. The second condition, known as the harmonic condition, allows us to move beyond the restriction of holomorphy. The operator Δ_k denotes the weight k (which is not restricted to be positive) hyperbolic Laplacian, defined as

$$\Delta_k = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + iky\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right), \tag{2.31}$$

where x and y are defined through $\tau = x + iy$. A modular function $Y(\tau)$ satisfying this condition is, in general, non-holomorphic in τ, although any holomorphic function automatically solves this differential equation. In addition, such modular functions must satisfy a suitable growth condition. In the present work, we restrict our attention to polyharmonic Maass forms, for which the growth condition is taken to be

$$Y(\tau) = \mathcal{O}(y^\alpha), \qquad \text{as } y \to +\infty, \quad \text{uniformly in } x,$$

for some real constant α. Using the fact that $T^N \in \Gamma(N)$, the modularity condition implies that $Y(\tau + N) = Y(\tau)$, thus introducing a notion of periodicity. Consequently, the modular function admits a Fourier expansion of the form

$$Y(\tau) = \sum_{n\in\frac{1}{N}\mathbb{Z}} a_n(y)\, q^n, \qquad q \equiv e^{2\pi i\tau}.$$

The harmonic condition imposes a differential equation on the coefficients $a_n(y)$, namely

$$\frac{d^2 a_n(y)}{dy^2} = \left(4\pi n - \frac{k}{y}\right)\frac{da_n(y)}{dy}. \tag{2.32}$$

Solving this equation, one finds that for $n \neq 0$ the solution takes the form

$$a_n(y) = c^+(n) + c^-(n)\,\Gamma(1-k, -4\pi n y), \qquad n \neq 0,$$

where $c^\pm(n)$ are integration constants. For the zero mode, the solution depends on the value of the weight k

$$a_0(y) = \begin{cases} c^+(0) + c^-(0)\, y^{1-k}, & k \neq 1, \\ c^+(0) + c^-(0)\, \ln y, & k = 1. \end{cases}$$

Here $\Gamma(1-k,-4\pi ny)$ denotes the incomplete gamma function. Using its recursive properties, one can explicitly compute its form for the first few integer values

$$\begin{aligned}
\Gamma(1,z) &= e^{-z},\\
\Gamma(2,z) &= (z+1)e^{-z},\\
\Gamma(3,z) &= (z^2+2z+2)e^{-z},\\
\Gamma(4,z) &= (z^3+3z^2+6z+6)e^{-z},\\
\Gamma(5,z) &= (z^4+4z^3+12z^2+24z+24)e^{-z}.
\end{aligned}$$

As a consequence, the Fourier expansion of $Y(\tau)$ can be written as

$$Y(\tau) = \sum_{n\in\frac{1}{N}\mathbb{Z}} c^+(n)\, q^n + c^-(0)\, y^{1-k} + \sum_{n\in\frac{1}{N}\mathbb{Z}\setminus\{0\}} c^-(n)\, \Gamma(1-k,-4\pi ny)\, q^n, \qquad q = e^{2\pi i\tau}. \tag{2.33}$$

In the special case $k = 1$, the term y^{1-k} must be replaced by $\ln y$. Furthermore, if $Y(\tau)$ is a polyharmonic Maass form, it satisfies

$$Y(\tau) = \mathcal{O}(y^\alpha) \quad \text{as } y\to\infty,$$

for some constant α. This growth condition implies that $c^-(n) = 0$ for $n < 0$. Therefore, the Fourier expansion reduces to

$$Y(\tau) = \sum_{\substack{n\in\frac{1}{N}\mathbb{Z}\\ n\geq 0}} c^+(n)\, q^n + c^-(0)\, y^{1-k} + \sum_{\substack{n\in\frac{1}{N}\mathbb{Z}\\ n<0}} c^-(n)\, \Gamma(1-k,-4\pi ny)\, q^n. \tag{2.34}$$

Note that the contributions proportional to $c^-(0)$ and $c^-(n)$ are non-holomorphic. In particular, they are absent in the case of purely holomorphic modular forms. Therefore, modular forms of level N and weight k (which are precisely the functions we saw in the previous section) constitute a special type of polyharmonic Maass forms. It is well known that the space of modular forms is closed under multiplication, the product of two modular forms of level N and respective weights k and k' yields another modular form of level N and weight $k+k'$. By contrast, this closure property does not generally extend to polyharmonic Maass forms. Indeed, the product of two such forms of level N and weights k and k' does not necessarily define a polyharmonic Maass form of weight $k+k'$, since the harmonicity condition may fail, particularly in situations where k or k' are negative.

Now, the interesting aspect of all this is that polyharmonic Maass forms also form vector spaces and satisfy properties analogous to those of modular forms. In particular, this means that polyharmonic Maass forms can also be arranged into multiplets that transform under irreducible representations of permutation groups. This is ultimately quite natural, since the whole construction is related to the group underlying modular forms, and at no point did we use the explicit expressions of the modular forms themselves.

2.2.1 Lifting modular forms to polyharmonic Maass forms

It is natural to ask whether polyharmonic Maass forms can be related in some way to modular forms. The answer is that they can indeed be related, and the connection is established through differential operators. These operators are defined as

$$D \equiv \frac{1}{2\pi i}\frac{\partial}{\partial \tau}\ , \qquad \xi_k \equiv 2iy^k\overline{\frac{\partial}{\partial \bar{\tau}}}\ .$$

More specifically, for any polyharmonic Maass form $Y(\tau)$ of level N and weight k, both $D^{1-k}Y(\tau)$ and $\xi_k Y(\tau)$ are modular forms of weight $2-k$ and level N. Meanwhile, for a multiplet $Y_{\mathbf{r}}^{(k)}(\tau)$ of polyharmonic Maass forms transforming in the irreducible representation $\mathbf{r}$ of Γ'_N (or Γ_N, depending on whether we consider even modular weights or arbitrary integer weights), it can be shown that both $\xi_k Y_{\mathbf{r}}^{(k)}$ and $D^{1-k}Y_{\mathbf{r}}^{(k)}$ are multiplets of modular forms of weight $2-k$ and level N, transforming in the representations $\mathbf{r}^*$ and $\mathbf{r}$, respectively. Therefore they must satisfy the modularity condition, which allows us to write

$$\left(\xi_k Y_{\mathbf{r}}^{(k)}\right)(\gamma\tau) = (c\tau+d)^{2-k}\rho_{\mathbf{r}^*}(\gamma)\left(\xi_k Y_{\mathbf{r}}^{(k)}\right)(\tau) \tag{2.35}$$

$$\left(D^{1-k} Y_{\mathbf{r}}^{(k)}\right)(\gamma\tau) = (c\tau+d)^{2-k}\rho_{\mathbf{r}}(\gamma)\left(D^{1-k} Y_{\mathbf{r}}^{(k)}\right)(\tau) \tag{2.36}$$

where

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is an element of Γ'_N (or Γ_N). Before proceeding further, it is important to clarify our goal. What we want is to connect polyharmonic Maass forms with modular forms. Since we already know how to construct modular forms, it may be possible to construct polyharmonic Maass forms if we understand the structure of the operators that connect these two objects. This is the main motivation for the calculation that follows. In particular, it is immediate to notice that in (2.36) the condition is automatically satisfied for $k=1$. Therefore, a weight 1 polyharmonic Maass form cannot be determined solely from the condition associated with the ξ_k operator. Regarding the conjugate representation $\rho_{\mathbf{r}^*}$, it is known to be equivalent to another representation of Γ'_N (or Γ_N) through a similarity transformation Ω,

$$\rho_{\mathbf{r}^*}(\gamma) = \Omega\rho_{\mathbf{r}'}(\gamma)\Omega^\dagger.$$

It is convenient to work in the basis in which the representations $\rho_{\mathbf{r}}(T)$ and $\rho_{\mathbf{r}'}(T)$ are diagonal, taking the form

$$\rho_r(T) = \mathrm{diag}\{1, e^{2\pi i r_2}, \cdots, e^{2\pi i r_d}\}, \qquad \rho_{r'}(T) = \mathrm{diag}\{1, e^{2\pi i r'_2}, \cdots, e^{2\pi i r'_d}\}, \qquad 0 \le r_i < 1.$$

with $r_i = 1 - r'_j$ for certain i and j. First, we use the modularity condition of $Y_{\mathbf{r}}^{(k)}(\tau)$, of level N and weight k, in order to obtain its Fourier expansion for the first component and for the remaining components:

$$Y_{r,1}^{(k)}(\tau) = c_1^+(0) + c_1^-(0)y^{1-k} + \sum_{n=1}^{\infty} c_1^+(n)q^n + c_1^-(-n)\Gamma(1-k, 4\pi ny)q^{-n}, \tag{2.37}$$

$$Y_{r,i}^{(k)}(\tau) = q^{r_i} \sum_{n=0}^{\infty} c_i^+(n)q^n + q^{r_i} \sum_{n=1}^{\infty} c_i^-(-n)\Gamma(1-k, 4\pi(n-r_i)y)\, q^{-n}, \qquad i > 1. \tag{2.38}$$

Next, we analyze $Y_{\mathbf{r}}^{(2-k)}$. The modularity condition implies that

$$Y_{\mathbf{r}}^{(2-k)}(\tau+1) = \rho_{\mathbf{r}}(T) Y_{\mathbf{r}}^{(2-k)}(\tau).$$

Using the fact that the chosen basis diagonalizes $\rho_{\mathbf{r}}(T)$, we can separate the first component of the multiplet from the remaining ones. For the first component we obtain

$$Y_{\mathbf{r},1}^{(2-k)}(\tau+1) = Y_{\mathbf{r},1}^{(2-k)}(\tau),$$

while for the remaining components we have

$$Y_{\mathbf{r},i}^{(2-k)}(\tau+1) = e^{2\pi i r_i} Y_{\mathbf{r},i}^{(2-k)}(\tau).$$

Therefore, the Fourier expansion of the first component of the multiplet is

$$Y_{\mathbf{r},1}^{(2-k)}(\tau) = \sum_{n=0}^{\infty} a_1(n)q^n.$$

For the remaining components we obtain

$$Y_{\mathbf{r},i}^{(2-k)}(\tau) = q^{r_i} \sum_{n=0}^{\infty} a_i(n)q^n, \qquad i > 1.$$

On the other hand, $\Omega Y_{\mathbf{r}}^{(2-k)}$ transforms under the dual representation $\rho_{\mathbf{r}}^*$, where

$$\rho_{\mathbf{r}}^*(T) = \mathrm{diag}\left(1, e^{-2\pi i r_i}\right).$$

Proceeding in exactly the same way, we obtain

$$(\Omega Y_{r'}^{(2-k)})_1(\tau) = \sum_{n=0}^{\infty} b_1(n)q^n,$$

$$(\Omega Y_{r'}^{(2-k)})_i(\tau) = q^{-r_i} \sum_{n=1}^{\infty} b_i(n)q^n, \qquad i > 1.$$

If there exist unique modular multiplets $Y^{(2-k)}_{\mathbf{r}'}$ and $Y^{(2-k)}_{\mathbf{r}}$ at level N, weight $2-k$ and representations $\mathbf{r}$ and $\mathbf{r}'$, then proportionality follows

$$\xi_k Y^{(k)}_{\mathbf{r}}(\tau) = \alpha \Omega Y^{(2-k)}_{\mathbf{r}'}(\tau) \;, \qquad D^{1-k} Y^{(k)}_{\mathbf{r}}(\tau) = \beta Y^{(2-k)}_{\mathbf{r}}(\tau) \tag{2.39}$$

where α and β are complex constants. Now, if we follow the derivation of [26], which essentially amounts to acting with ξ_k and D^{1-k} on the Fourier expansion of the polyharmonic Maass form, we obtain

$$\begin{aligned}
\xi_k Y^{(k)}_{r,1}(\tau) &= (1-k)\,\overline{c^-_1(0)} - \sum_{n=1}^{\infty}(4\pi n)^{1-k}\,\overline{c^-_1(-n)}\,q^n, \\
\xi_k Y^{(k)}_{r,i}(\tau) &= -q^{-r_i}\sum_{n=1}^{\infty}(4\pi(n-r_i))^{1-k}\,\overline{c^-_i(-n)}\,q^n, \\
D^{1-k} Y^{(k)}_{r,1}(\tau) &= (-4\pi)^{k-1}(1-k)!\,c^-_1(0) + \sum_{n=1}^{\infty} n^{1-k} c^+_1(n)\,q^n, \\
D^{1-k} Y^{(k)}_{r,i}(\tau) &= q^{r_i}\sum_{n=0}^{\infty}(n+r_i)^{1-k} c^+_i(n)\,q^n.
\end{aligned}$$

We can now compare the q-expansions on both sides of (2.39). From this comparison, we find that the Fourier coefficients of the polyharmonic Maass form multiplet $Y^{(k)}_{\mathbf{r}}(\tau)$ satisfy the following conditions.

$$\begin{aligned}
(1-k)\,\overline{c^-_1(0)} = \alpha b_1(0), &\qquad (-4\pi)^{k-1}(1-k)!\,c^-_1(0) = \beta a_1(0), \\
-(4\pi n)^{1-k}\,\overline{c^-_1(-n)} = \alpha b_1(n), &\qquad n^{1-k} c^+_1(n) = \beta a_1(n), \\
-(4\pi(n-r_i))^{1-k}\,\overline{c^-_i(-n)} = \alpha b_i(n), &\qquad (n+r_i)^{1-k} c^+_i(n) = \beta a_i(n).
\end{aligned}$$

Without loss of generality, we could choose the normalization factor $\alpha = 1$, then we have

$$\begin{aligned}
&\beta = (-4\pi)^{k-1}(-k)!\,\frac{\overline{b_1(0)}}{a_1(0)}, \qquad c^-_1(0) = \frac{\overline{b_1(0)}}{1-k}, \\
&c^-_1(-n) = -(4\pi n)^{k-1}\,\overline{b_1(n)}, \qquad c^+_1(n) = (-4\pi)^{k-1}(-k)!\,\overline{b_1(0)}\,\frac{a_1(n)}{a_1(0)}, \quad n \neq 0, \\
&c^-_i(-n) = -(4\pi(n-r_i))^{k-1}\,\overline{b_i(n)}, \qquad c^+_i(n) = (-4\pi(n+r_i))^{k-1}(-k)!\,\overline{b_1(0)}\,\frac{a_i(n)}{a_1(0)}.
\end{aligned}$$

Thus all the Fourier expansion coefficients except $c^+_1(0)$ of weight k polyharmonic Maass form multiplet $Y^{(k)}_r(\tau)$ at level N can be expressed in terms of those of the weight $2-k$ and level N modular forms. The value of $c^+_1(0)$ can be fixed from the moduluraty . In short, one can lift the level N and weight k modular forms to the level N and weight k polyharmonic Maass form multiplet via the following formula

$$Y_{r,1}^{(k)}(\tau) = c_1^+(0) + \frac{\overline{b_1(0)}}{1-k}\, y^{1-k} - \sum_{n=1}^{\infty}(4\pi n)^{k-1}\left[(-1)^k(-k)!\,\overline{b_1(0)}\frac{a_1(n)}{a_1(0)}q^n + \overline{b_1(n)}\Gamma(1-k,4\pi n y)q^{-n}\right], \tag{2.40}$$

$$\begin{aligned} Y_{r,i}^{(k)}(\tau) = q^{r_i}\sum_{n=0}^{\infty}(-4\pi(n+r_i))^{k-1}(-k)!\,\overline{b_1(0)}\frac{a_i(n)}{a_1(0)}q^n \\ - q^{r_i}\sum_{n=1}^{\infty}(4\pi(n-r_i))^{k-1}\,\overline{b_i(n)}\Gamma(1-k,4\pi(n-r_i)y)q^{-n}, \qquad i>1. \end{aligned} \tag{2.41}$$

Recall that the product of two polyharmonic Maass forms of weights k, k' at level N usually is not a polyharmonic Maass form of weight $k+k'$, since the Laplacian condition could be spoiled particularly if $k<0$ or $k'<0$. The explicit expressions of the multiplets may not appear particularly friendly at first sight, but their structure becomes much clearer once one does an example. At this point, it is useful to recall a question that was left open at the end of the previous section: why do we not construct mass models directly using this holomorphic modular symmetry? In other words, what motivates the search for non-holomorphic modular functions? The answer to this question arises naturally in the context of supersymmetric theories. Without going into too many technical details, supersymmetry typically requires that the relevant fields transform as holomorphic functions. Consequently, modular forms entering the superpotential must also be holomorphic. However, for several reasons, alternative scenarios beyond supersymmetry have been actively explored in recent years. In such a framework, there is no fundamental requirement enforcing holomorphicity. Therefore, it becomes reasonable to relax this condition and consider the possibility that the modular objects appearing in the Lagrangian are not necessarily holomorphic. Although we have not yet specified the precise transformation properties of the fields, it is already intuitive that if non-holomorphic modular forms are present in the theory, the fields themselves must transform in a correspondingly non-holomorphic way. In summary, the holomorphic nature of modular forms arises naturally within supersymmetric constructions. Since our goal is not to impose supersymmetry, we allow ourselves to relax the holomorphicity condition in the Lagrangian. With this in mind, we can now proceed to develop models based on non-holomorphic modular symmetry.

Chapter 3

Models with Modular Symmetry

3.1 Modular invariance

As mentioned before, modular flavor symmetry was originally formulated in a supersymmetric framework [24]. The approach we follow here opens the possibility of constructing a non-supersymmetric realization of modular flavor symmetry. The finite modular group remains unchanged. The generic matter fields are denoted by ψ_i and ψ_i^c. Their transformations under the modular group are specified by their modular weights $-k_\psi$ and $-k_{\psi^c}$, as well as by the irreducible representations ρ_ψ and ρ_{ψ^c} of Γ'_N (or Γ_N). Explicitly, their transformation properties are given by

$$\psi_i(x) \longrightarrow (c\tau + d)^{-k_{\psi_i}} \rho_\psi(\gamma)_{ij}\, \psi_j(x) \ , \tag{3.1}$$

$$\psi_i^c(x) \longrightarrow (c\tau + d)^{-k_{\psi_i^c}} \rho_{\psi^c}(\gamma)_{ij}\, \psi_j^c(x) \ . \tag{3.2}$$

where

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) \ .$$

We focus on the mass terms arising from the Yukawa interactions. The modular invariant Yukawa interaction can be written as

$$\mathcal{L}_Y^D = Y^{(k_Y)}(\tau)\, \psi^c \psi H + \text{h.c.} \ , \tag{3.3}$$

where we have adopted the two-component spinor notation, and ϕ refers to the Higgs field. The modular transformation of the Higgs field is given by

$$\phi(x) \longrightarrow (c\tau + d)^{-k_\phi} \rho_\phi(\gamma)\, \phi(x) \ . \tag{3.4}$$

Modular invariance requires $Y^{(k_Y)}(\tau)$ to be polyharmonic Maass forms of weight k_Y and level N, transforming in the representation ρ_Y of Γ_N, i.e.

$$Y^{(k_Y)}(\gamma\tau) = (c\tau + d)^{k_Y} \rho_Y(\gamma)\, Y^{(k_Y)}(\tau). \tag{3.5}$$

Each term in the Yukawa interactions $\mathcal{L}_Y^D$ has to be invariant under Γ_N (or Γ'_N), and its total modular weight must vanish. Hence k_Y must satisfy

$$k_Y = k_{\psi^c} + k_\psi + k_\phi, \tag{3.6}$$

and ρ_Y must satisfy

$$\rho_Y \otimes \rho_{\psi^c} \otimes \rho_\psi \otimes \rho_\phi \supset \mathbf{1}, \tag{3.7}$$

where $\mathbf{1}$ denotes the trivial singlet representation of Γ_N (or Γ'_N). On the other hand, for Majorana fermions, denoted by ψ^c, the corresponding mass terms is given by

$$\mathcal{L}_M = Y^{(k_Y)}(\tau)\, \psi^c \psi^c + \text{h.c.} \tag{3.8}$$

Analogously, the modular invariance requires

$$k_Y = 2k_{\psi^c}, \tag{3.9}$$

and

$$\rho_Y \otimes \rho_{\psi^c} \otimes \rho_{\psi^c} \supset \mathbf{1}. \tag{3.10}$$

Now that we know the conditions that must be met for modular invariance, we are ready to build models.

3.2 Type-I seesaw with non-holomorphic modular A_4

In this model [26], neutrino masses are generated through the type-I seesaw mechanism. For simplicity, we introduce only two right-handed neutrinos, $N_{1,2}$. The left-handed doublets are arranged into an A_4 triplet (which we denote by L), while the right-handed singlets transform as singlets of A_4. The charge assignments for the leptonic and scalar sectors under gauge group and the non-holomorphic modular A_4 symmetry can be seen in Table 3.1.

The most general Lagrangian for charged lepton sector invariant under the modular (and gauge) symmetry is

$$-\mathcal{L}_\ell = \alpha_1 \left(Y_{\mathbf{3}}^{(-4)}\overline{L}\right)_{\mathbf{1}'} \phi e_{1R} + \alpha_2 \left(Y_{\mathbf{3}}^{(-2)}\overline{L}\right)_{\mathbf{1}'} \phi e_{2R} + \alpha_3 \left(Y_{\mathbf{3}}^{(4)}\overline{L}\right)_{\mathbf{1}} \phi e_{3R} + h.c., \tag{3.11}$$

Table 3.1: Field content and quantum number assignments for the lepton and scalar sectors under the gauge group $SU(2)_L \times U(1)_Y$ and the non-holomorphic modular A_4 symmetry.

Fields	$SU(2)_L$	$U(1)_Y$	A_4	k
L_L	$\mathbf{2}$	-1	$\mathbf{3}$	-5
e_{1R}	$\mathbf{1}$	-2	$\mathbf{1}''$	1
e_{2R}	$\mathbf{1}$	-2	$\mathbf{1}''$	3
e_{3R}	$\mathbf{1}$	-2	$\mathbf{1}$	9
ϕ	$\mathbf{2}$	1	$\mathbf{1}$	0
N_1	$\mathbf{1}$	0	$\mathbf{1}''$	1
N_2	$\mathbf{1}$	0	$\mathbf{1}$	3

where the notation $\left(Y_{\mathbf{3}}^{(k)}\overline{L}\right)_{\mathbf{r}}$ refers to the $\mathbf{r}$ projection of the product $Y_{\mathbf{3}}^{(k)}\overline{L}$. In the same way, the most general Lagrangian for neutrino sector is given by

$$-\mathcal{L}_\nu = \beta_1 \left(Y_{\mathbf{3}}^{(-4)}\overline{L}\right)_{\mathbf{1}'} \tilde{\phi} N_1 + \beta_2 \left(Y_{\mathbf{3}}^{(-2)}\overline{L}\right)_{\mathbf{1}} \tilde{\phi} N_2 + \beta_3 Y_{\mathbf{1}'}^{(4)} \overline{N_1^c} N_2 + \frac{1}{2}\beta_4 Y_{\mathbf{1}}^{(6)} \overline{N_2^c} N_2 + h.c. \tag{3.12}$$

Now, we have to use the tensor products of A_4 in order to obtain singlet representation (the details of the group A_4 can be found in Appendix B)

$$\alpha_1 Y_{\mathbf{3}}^{(-4)}(\overline{L}\phi e_{1R})_{\mathbf{3}} = \alpha_1 Y_{\mathbf{3},1}^{(-4)}\overline{\ell_{2L}}\phi e_{1R} + \alpha_1 Y_{\mathbf{3},2}^{(-4)}\overline{\ell_{1L}}\phi e_{1R} + \alpha_1 Y_{\mathbf{3},3}^{(-4)}\overline{\ell_{3L}}\phi e_{1R}$$

$$\alpha_2 Y_{\mathbf{3}}^{(-2)}(\overline{L}\phi e_{2R})_{\mathbf{3}} = \alpha_2 Y_{\mathbf{3},1}^{(-2)}\overline{\ell_{2L}}\phi e_{2R} + \alpha_2 Y_{\mathbf{3},2}^{(-2)}\overline{\ell_{1L}}\phi e_{2R} + \alpha_2 Y_{\mathbf{3},3}^{(-2)}\overline{\ell_{3L}}\phi e_{2R}$$

$$\alpha_3 Y_{\mathbf{3}}^{(4)}(\overline{L}\phi e_{3R})_{\mathbf{3}} = \alpha_3 Y_{\mathbf{3},1}^{(4)}\overline{\ell_{1L}}\phi e_{3R} + \alpha_3 Y_{\mathbf{3},2}^{(4)}\overline{\ell_{3L}}\phi e_{3R} + \alpha_3 Y_{\mathbf{3},3}^{(4)}\overline{\ell_{2L}}\phi e_{3R}$$

Then, we can construct the mass matrix for the charged lepton sector

$$M_\ell = \frac{v}{\sqrt{2}} \begin{pmatrix} \alpha_1 Y_{\mathbf{3},2}^{(-4)} & \alpha_2 Y_{\mathbf{3},2}^{(-2)} & \alpha_3 Y_{\mathbf{3},1}^{(4)} \\ \alpha_1 Y_{\mathbf{3},1}^{(-4)} & \alpha_2 Y_{\mathbf{3},1}^{(-2)} & \alpha_3 Y_{\mathbf{3},3}^{(4)} \\ \alpha_1 Y_{\mathbf{3},3}^{(-4)} & \alpha_2 Y_{\mathbf{3},3}^{(-2)} & \alpha_3 Y_{\mathbf{3},2}^{(4)} \end{pmatrix} \tag{3.13}$$

For the neutrino sector we have to make the same calculation

$$\beta_1 Y_{\mathbf{3}}^{(-4)}(\overline{L}\tilde{\phi}N_1)_{\mathbf{3}} = \beta_1 Y_{\mathbf{3},1}^{(-4)}\overline{\ell_{2L}}\tilde{\phi}N_1 + \beta_1 Y_{\mathbf{3},2}^{(-4)}\overline{\ell_{1L}}\tilde{\phi}N_1 + \beta_1 Y_{\mathbf{3},3}^{(-4)}\overline{\ell_{3L}}\tilde{\phi}N_1$$

$$\beta_2 Y_{\mathbf{3}}^{(-2)}(\overline{L}\tilde{\phi}N_2)_{\mathbf{3}} = \beta_2 Y_{\mathbf{3},1}^{(-2)}\overline{\ell_{1L}}\tilde{\phi}N_2 + \beta_2 Y_{\mathbf{3},2}^{(-2)}\overline{\ell_{3L}}\tilde{\phi}N_2 + \beta_2 Y_{\mathbf{3},3}^{(-2)}\overline{\ell_{2L}}\tilde{\phi}N_2$$

Then, we can construct the Dirac block

$$m_D = \frac{v}{\sqrt{2}} \begin{pmatrix} \beta_1 Y_{\mathbf{3},2}^{(-4)} & \beta_2 Y_{\mathbf{3},1}^{(-2)} \\ \beta_1 Y_{\mathbf{3},1}^{(-4)} & \beta_2 Y_{\mathbf{3},3}^{(-2)} \\ \beta_1 Y_{\mathbf{3},3}^{(-4)} & \beta_2 Y_{\mathbf{3},2}^{(-2)} \end{pmatrix} \tag{3.14}$$

In the same way

$$M_R = \begin{pmatrix} 0 & \beta_3 Y_{\mathbf{1'}}^{(4)} \\ \beta_3 Y_{\mathbf{1'}}^{(4)} & \frac{1}{2}\beta_4 Y_{\mathbf{1}}^{(6)} \end{pmatrix} \tag{3.15}$$

Now we can use the typical seesaw formula to obtain the mass matrix for the neutrinos

$$M_\nu \simeq -m_D M_R^{-1} m_D^T$$

So

$$M_\nu \simeq -\frac{v^2}{2} \begin{pmatrix} \beta_1 Y_{\mathbf{3},2}^{(-4)} & \beta_2 Y_{\mathbf{3},1}^{(-2)} \\ \beta_1 Y_{\mathbf{3},1}^{(-4)} & \beta_2 Y_{\mathbf{3},3}^{(-2)} \\ \beta_1 Y_{\mathbf{3},3}^{(-4)} & \beta_2 Y_{\mathbf{3},2}^{(-2)} \end{pmatrix} \begin{pmatrix} 0 & \beta_3 Y_{\mathbf{1'}}^{(4)} \\ \beta_3 Y_{\mathbf{1'}}^{(4)} & \frac{1}{2}\beta_4 Y_{\mathbf{1}}^{(6)} \end{pmatrix}^{-1} \begin{pmatrix} \beta_1 Y_{\mathbf{3},2}^{(-4)} & \beta_1 Y_{\mathbf{3},1}^{(-4)} & \beta_1 Y_{\mathbf{3},3}^{(-4)} \\ \beta_2 Y_{\mathbf{3},1}^{(-2)} & \beta_2 Y_{\mathbf{3},3}^{(-2)} & \beta_2 Y_{\mathbf{3},2}^{(-2)} \end{pmatrix} \tag{3.16}$$

An aspect that was not mentioned previously is that the CP phase can also be explained through the incorporation of modular symmetry. Since modular forms are complex numbers, one can assume that the coupling constants α_i and β_i are real and the complex phases originate solely from the modular sector. This scenario is commonly referred to as purely geometric CP violation. As in the conventional seesaw framework, we now proceed to construct the corresponding mass matrices. In this way, we diagonalize the mass matrices of the charged leptonic sector and the neutrino sector by performing a χ^2 analysis. From this procedure, we obtain both the particle masses and the mixing parameters. The resulting observables are shown in Table 3.2, while the corresponding values for the parameters of the theory are presented in Table 3.3. In addition, we generate correlation plots by perturbing the best fit point. This allows us to observe that the solution does not correspond to an isolated point in parameter space, but rather that there exists a broader region predicting observables within 3σ, as illustrated in Figure 3.1.

In this way, we successfully reproduce the observables of the leptonic sector. It is important to note that the charged lepton masses are measured with very high precision, which makes their exact numerical fit computationally expensive. For this reason, we only require the predicted values to lie within approximately 5% of the experimental measurements. At this point, a more general question arises: is this the only possible model within this framework, or can other similar models be constructed? Here we do not refer to alternative mechanisms for generating neutrino masses, but rather to the possibility of obtaining different Yukawa structures through alternative assignments of modular weights or field representations. In principle, one can construct many different models by reassigning modular weights and representations. However, this does not guarantee the existence of a solution capable of reproducing the leptonic observables. Even small modifications in the representations of the fields under A_4 can lead to different Yukawa structures. As an illustration, we may consider the charge assignment shown in Table 3.4.

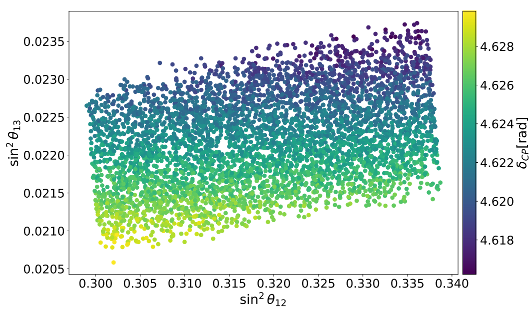
Obviously, the neutrino sector remains unchanged. However, let us examine what happens in

Table 3.2: Predicted values of the observables obtained from the model (considering NO).

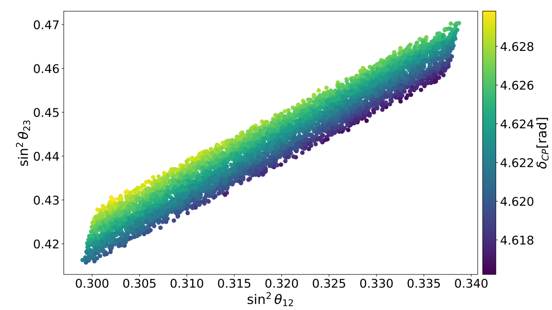
Observable	Value ($\chi^2 \simeq 0.5$)
m_e (eV)	5.1037×10^5
m_μ (eV)	1.0546×10^8
m_τ (eV)	1.7785×10^9
$\sin^2\theta_{12}$	0.31845
$\sin^2\theta_{13}$	0.02217
$\sin^2\theta_{23}$	0.44271
Δm^2_{21} (eV^2)	7.5113×10^{-5}
Δm^2_{32} (eV^2)	2.5094×10^{-3}
$\delta_{\rm CP}$ (rad)	4.62327

Table 3.3: Values for the parameters.

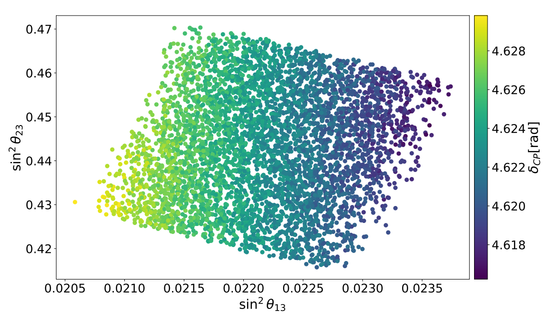
Parameter	Value
α_1	1.1962×10^{-5}
α_2	1.1843×10^{-2}
α_3	5.6073×10^{-4}
β_1	-2.0554×10^{-11}
β_2	-1.7476×10^{-13}
β_3	3.1267
β_4	-1.6758×10^{-1}
Re(τ)	8.1266×10^{-2}
Im(τ)	1.3042



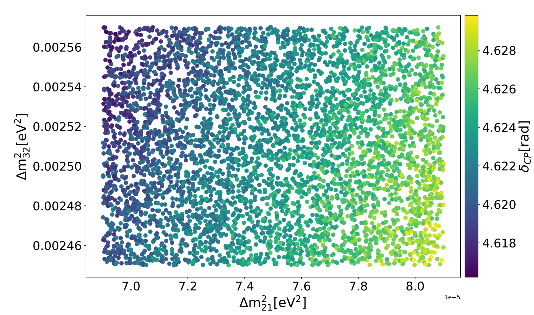
(a) Correlation for $\sin^2\theta_{12}$, $\sin^2\theta_{13}$ and δ_{CP}.



(b) Correlation for $\sin^2\theta_{12}$, $\sin^2\theta_{23}$ and δ_{CP}.



(c) Correlation for $\sin^2\theta_{13}$, $\sin^2\theta_{23}$ and δ_{CP}.



(d) Correlation for Δm^2_{21}, Δm^2_{32} and δ_{CP}.

Figure 3.1: Correlation plots for the observables of the leptonic sector.

the charged lepton sector. The most general Lagrangian that can be constructed is given by

$$-\mathcal{L}_\ell = \alpha_1 \left(Y_{\mathbf{3}}^{(-4)}\overline{L}\right)_{\mathbf{1}} \phi e_{1R} + \alpha_2 \left(Y_{\mathbf{3}}^{(-2)}\overline{L}\right)_{\mathbf{1}''} \phi e_{2R} + \alpha_3 \left(Y_{\mathbf{3}}^{(4)}\overline{L}\right)_{\mathbf{1}'} \phi e_{3R} \tag{3.17}$$

Table 3.4: Field content and quantum number assignments for the lepton and scalar sectors under the gauge group $SU(2)_L \times U(1)_Y$ and the non-holomorphic modular A_4 symmetry.

Fields	$SU(2)_L$	$U(1)_Y$	A_4	k
L_L	$\mathbf{2}$	-1	$\mathbf{3}$	-5
e_{1R}	$\mathbf{1}$	-2	$\mathbf{1}$	1
e_{2R}	$\mathbf{1}$	-2	$\mathbf{1}'$	3
e_{3R}	$\mathbf{1}$	-2	$\mathbf{1}''$	9
ϕ	$\mathbf{2}$	1	$\mathbf{1}$	0
N_1	$\mathbf{1}$	0	$\mathbf{1}''$	1
N_2	$\mathbf{1}$	0	$\mathbf{1}$	3

Then

$$\alpha_1 \left(Y_{\mathbf{3}}^{(-4)}\overline{L}\right)_{\mathbf{1}} \phi e_{1R} = \alpha_1 Y_{\mathbf{3},1}^{(-4)}\overline{\ell_{1L}}\phi e_{1R} + \alpha_1 Y_{\mathbf{3},2}^{(-4)}\overline{\ell_{3L}}\phi e_{1R} + \alpha_1 Y_{\mathbf{3},3}^{(-4)}\overline{\ell_{2L}}\phi e_{1R}$$

$$\alpha_2 \left(Y_{\mathbf{3}}^{(-2)}\overline{L}\right)_{\mathbf{1}''} \phi e_{2R} = \alpha_2 Y_{\mathbf{3},1}^{(-2)}\overline{\ell_{3L}}\phi e_{2R} + \alpha_2 Y_{\mathbf{3},2}^{(-2)}\overline{\ell_{2L}}\phi e_{2R} + \alpha_2 Y_{\mathbf{3},3}^{(-2)}\overline{\ell_{1L}}\phi e_{2R}$$

$$\alpha_3 \left(Y_{\mathbf{3}}^{(4)}\overline{L}\right)_{\mathbf{1}'} \phi e_{3R} = \alpha_3 Y_{\mathbf{3},1}^{(4)}\overline{\ell_{2L}}\phi e_{3R} + \alpha_3 Y_{\mathbf{3},2}^{(4)}\overline{\ell_{1L}}\phi e_{3R} + \alpha_3 Y_{\mathbf{3},3}^{(4)}\overline{\ell_{3L}}\phi e_{3R}$$

As in the previous model, we can construct the charged lepton mass matrix

$$M_\ell = \frac{v}{\sqrt{2}} \begin{pmatrix} \alpha_1 Y_{\mathbf{3},1}^{(-4)} & \alpha_2 Y_{\mathbf{3},3}^{(-2)} & \alpha_3 Y_{\mathbf{3},2}^{(4)} \\ \alpha_1 Y_{\mathbf{3},3}^{(-4)} & \alpha_2 Y_{\mathbf{3},2}^{(-2)} & \alpha_3 Y_{\mathbf{3},1}^{(4)} \\ \alpha_1 Y_{\mathbf{3},2}^{(-4)} & \alpha_2 Y_{\mathbf{3},1}^{(-2)} & \alpha_3 Y_{\mathbf{3},3}^{(4)} \end{pmatrix} \tag{3.18}$$

In Table 3.5 we present the observables predicted by the model. It can be seen that several of them lie outside the experimentally allowed 3σ ranges, indicating that this configuration fails to reproduce the leptonic observables. This result is particularly striking, since we only modified the singlet representations of the fields while keeping the modular weights unchanged, yet the solution disappears. In general, there is no clear criterion that allows one to determine in advance whether a given configuration of the model will admit a solution capable of reproducing the experimental observables. This makes the systematic construction of such models considerably more challenging. The situation becomes even more complicated due to an intrinsic ambiguity in the construction itself. In particular, there is no clear explanation for why the group A_4 is chosen, why specific representations of A_4 are assigned to the fields, or why particular modular weights are selected. While it is certainly appealing that modular symmetry may provide a framework capable of explaining several aspects of the flavor structure, the issue of parameter arbitrariness still persists. Ultimately, there is no clear indication of the fundamental origin of the modular symmetry A_4 itself. One argument that is often invoked is based on the idea of constructing economical models. In practice, this means working with relatively small groups that contain a limited number of irreducible representations, and avoiding very large modular weights, since higher weights generally lead to the appearance of more multiplets with the same weight. Such an economy argument may help to restrict the number of parameters in the model. However, it should be emphasized that this criterion is not fundamentally physical. Therefore, the question of the origin and the selection of the modular symmetry in these constructions remains an open problem.

Table 3.5: Predicted values of the observables obtained from the model (considering NO).

Observable	Value
m_e (eV)	5.2013×10^5
m_μ (eV)	1.0182×10^8
m_τ (eV)	1.7672×10^9
$\sin^2\theta_{12}$	0.07296
$\sin^2\theta_{13}$	0.02355
$\sin^2\theta_{23}$	0.36810
Δm^2_{21} (eV^2)	7.4273×10^{-5}
Δm^2_{32} (eV^2)	2.5009×10^{-3}
δ_{CP} (rad)	1.50082

3.3 Cocktail model with non-holomorphic modular S_3

As mentioned at the beginning, one possible alternative to the seesaw mechanisms is given by radiative mechanisms. In this class of models, neutrino masses are suppressed by loop factors. In this work, we focus on one particular realization, the so–called Cocktail Model, implemented with a non–holomorphic modular flavor symmetry S_3. The Cocktail model [21] is an extended Inert Doublet model where the active neutrino masses are generated at three-loop level. Its scalar sector includes the SM Higgs doublet ϕ, an inert scalar doublet η, one singly charged $\sigma^\pm$ and one doubly charged scalar field $S^{\pm\pm}$. In the theory under consideration, we assume that the S_3 modular flavor symmetry is spontaneously broken down to a preserved Z_2 symmetry, where the Z_2 charges of the scalar and fermionic particles of the model are given by $(-1)^k$ where k is the modular weight of the particle under consideration. The remnant Z_2 symmetry is crucial to guarantee the stability of the dark matter candidate. Unlike the Cocktail model [21], we do not need to add a preserved Z_2 symmetry, since it arises from the S_3 spontaneous symmetry breaking. The model particle content with their S_3 modular weights and transformations under the $SU(3)_C \times SU(2)_L \times U(1)_Y \times S_3$ symmetry are displayed in Table 3.6. Here $e_R = (e_{1R},\ e_{2R})^T$, $L_L = (l_{1L},\ l_{2L})^T$ denote S_3 leptonic doublets, and the indices refer to different families. The charge assignments under S_3 are chosen so that the model remains as minimal as possible. In principle, one could assume that the fields transform under higher weights and thereby fit all experimental observables. To avoid such arbitrariness, we simply assign modular weight zero to the leptonic fields. Moreover, polyharmonic Maass forms of weight zero admit both singlet and doublet representations, which naturally motivates assigning this weight to the leptonic sector. The Higgs doublet and the inert scalar doublet can be written

Table 3.6: Field content and quantum number assignments for the lepton and scalar sectors under the gauge group $SU(3)_C \times SU(2)_L \times U(1)_Y$ and the non-holomorphic modular S_3 symmetry.

Fields	$SU(3)_C$	$SU(2)_L$	$U(1)_Y$	S_3	k
L_L	$\mathbf{1}$	$\mathbf{2}$	-1	$\mathbf{2}$	0
l_{3L}	$\mathbf{1}$	$\mathbf{2}$	-1	$\mathbf{1}$	0
e_R	$\mathbf{1}$	$\mathbf{1}$	-2	$\mathbf{2}$	0
e_{3R}	$\mathbf{1}$	$\mathbf{1}$	-2	$\mathbf{1}$	0
ϕ	$\mathbf{1}$	$\mathbf{2}$	1	$\mathbf{1}$	0
η	$\mathbf{1}$	$\mathbf{2}$	1	$\mathbf{1}$	-1
$\sigma^{\pm}$	$\mathbf{1}$	$\mathbf{1}$	± 2	$\mathbf{1}$	-3
$S^{\pm\pm}$	$\mathbf{1}$	$\mathbf{1}$	± 4	$\mathbf{1}$	4

in the unitary gauge as follows

$$\phi = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ h \end{pmatrix} + \begin{pmatrix} 0 \\ v \end{pmatrix} \quad , \quad \eta = \frac{1}{\sqrt{2}} \begin{pmatrix} \Lambda^+ \\ H_0 + iA_0 \end{pmatrix} . \tag{3.19}$$

With the particle content and symmetries specified in Table 3.6, we display the following leptonic Yukawa interactions and the scalar potential compatible with the symmetries of the model

$$\begin{aligned} -\mathcal{L}_Y^{(l)} = {} & \alpha_1 Y_{\mathbf{1}}^{(0)} (\overline{L}_L \phi e_R)_{\mathbf{1}} + \alpha_2 Y_{\mathbf{2}}^{(0)} (\overline{L}_L \phi e_R)_{\mathbf{2}} + \alpha_3 Y_{\mathbf{2}}^{(0)} (\overline{L}_L \phi e_{3R})_{\mathbf{2}} \\ & + \alpha_4 Y_{\mathbf{2}}^{(0)} (\bar{l}_{3L} \phi e_R)_{\mathbf{2}} + \alpha_5 Y_{\mathbf{1}}^{(0)} (\bar{l}_{3L} \phi e_{3R})_{\mathbf{1}} \\ & + \beta_1 Y_{\mathbf{1}}^{(4)} (\overline{e_R^C} e_R S^{++})_{\mathbf{1}} + \beta_2 Y_{\mathbf{2}}^{(4)} (\overline{e_R^C} e_R S^{++})_{\mathbf{2}} \\ & + \beta_3 Y_{\mathbf{2}}^{(4)} (\overline{e_R^C} e_{3R} S^{++})_{\mathbf{2}} + \beta_4 Y_{\mathbf{1}}^{(4)} (\overline{e_{3R}^C} e_{3R} S^{++})_{\mathbf{1}} + \text{h.c.} \end{aligned} \tag{3.20}$$

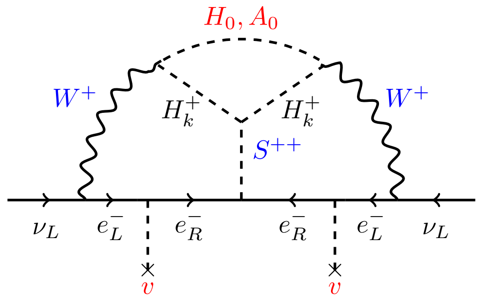


Figure 3.2: Representative loop diagram for radiative neutrino mass generation in the Cocktail model.

$$\begin{aligned}V = &-\mu_\phi^2\,(\phi^\dagger\phi) + \mu_\eta^2\,(\eta^\dagger\eta) - \mu_\sigma^2\,(\sigma^+\sigma^-) + \mu_S^2\,(S^{++}S^{--}) + \lambda_1\,(\phi^\dagger\phi)^2 + \lambda_2\,(\eta^\dagger\eta)^2 + \lambda_\sigma\,(\sigma^+\sigma^-)^2\\ &+ \lambda_S\,(S^{++}S^{--})^2 + \lambda_3\,(\phi^\dagger\phi)(\eta^\dagger\eta) + \lambda_4\,(\phi^\dagger\eta)(\eta^\dagger\phi) + \lambda_{\phi\sigma}(\phi^\dagger\phi)(\sigma^+\sigma^-) + \lambda_{\eta\sigma}(\eta^\dagger\eta)(\sigma^+\sigma^-)\\ &+ \lambda_{\phi S}(\phi^\dagger\phi)(S^{++}S^{--}) + \lambda_{\eta S}(\eta^\dagger\eta)(S^{++}S^{--}) + \lambda_{\sigma S}(\sigma^+\sigma^-)(S^{++}S^{--})\\ &+ \left[\lambda_5 Y_{\mathbf{1}}^{(-2)}(\phi^\dagger\eta)^2 + \xi\;\varepsilon_{ab}\phi^a\eta^b\sigma^+S^{--} + \kappa_1 Y_{\mathbf{1}}^{(-4)}\varepsilon_{ab}\phi^a\eta^b\sigma^- + \kappa_2 Y_{\mathbf{1}}^{(-2)}S^{--}\sigma^+\sigma^+ + h.c.\right],\end{aligned} \tag{3.21}$$

where $Y_{\mathbf{I}}^{(k)}$ are dimensionless holomorphic modular forms of the modulus τ, with k the modular weight and $\mathbf{I} = \mathbf{1}, \mathbf{2}$ labels the irreducible representations of S_3.

Due to the preserved Z_2 symmetry arising from the spontaneous breaking of the modular S_3 flavor group, the tiny active neutrino masses are generated at the three-loop level through the virtual exchange of singly and doubly charged scalar fields, as well as the CP-even and CP-odd neutral components of the inert scalar doublet η. The Feynman diagram contributing to the entries of the active neutrino mass matrix is shown in Fig. 3.2. The three-loop diagram responsible for active neutrino mass generation is governed by the scalar interaction parameters λ_5, κ_1, κ_2, and ξ.

After electroweak symmetry breaking we expand Eq. (3.21) around the vacuum and fix the modulus τ. The Maaß forms evaluated at τ become numerical factors and can be absorbed into effective couplings. We define $\widetilde{\kappa}_1 \equiv \kappa_1\, Y_{\mathbf{1}}^{(-4)}(\tau)$ and $\widetilde{\lambda}_5 \equiv \lambda_5\, Y_{\mathbf{1}}^{(-2)}(\tau)$. The inert doublet η contains a CP-even state H_0 and a CP-odd state A_0, whose relative splitting is controlled by $\widetilde{\lambda}_5$:

$$\Delta m_0^2 \equiv m_{A_0}^2 - m_{H_0}^2 = -2\,\mathrm{Re}\widetilde{\lambda}_5 v^2. \tag{3.22}$$

In the charged sector, the $\widetilde{\kappa}_1$ interaction induces mixing between Λ^+ and σ^+, leading to two physical charged scalars,

$$H_1^+ = \sin\beta\,\sigma^+ + \cos\beta\,\Lambda^+\ , \qquad H_2^+ = \cos\beta\,\sigma^+ - \sin\beta\,\Lambda^+\ , \tag{3.23}$$

with mixing angle

$$\tan(2\beta) = \frac{2\sqrt{2}\,\widetilde{\kappa}_1\,v}{2\mu_\eta^2 + 2\mu_\sigma^2 + v^2(\lambda_3 - \lambda_{\phi\sigma})}\,. \tag{3.24}$$

The corresponding charged splitting, $\Delta m_+^2 \equiv m_{H_2^+}^2 - m_{H_1^+}^2$, is given by

$$\Delta m_+^2 = \sqrt{2\,\widetilde{\kappa}_1^2 v^2 + \left[\frac{v^2}{2}\left(\lambda_3 - \lambda_{\phi\sigma}\right) + \mu_\eta^2 + \mu_\sigma^2\right]^2}\,. \tag{3.25}$$

The splittings Δm_0^2 and Δm_+^2 are key ingredients for the three-loop neutrino mass generation and for charged-lepton flavor-violating observables. In the next section we discuss the charged-lepton and neutrino mass matrices, and the resulting predictions for neutrino masses and mixing.

3.3.1 Lepton masses and mixings

From the leptonic Yukawa interactions, we find that the spontaneous breaking of the SM electroweak symmetry as well as of the modular S_3 flavor symmetry yield the following mass matrix for charged leptons:

$$M_\ell = \frac{v}{\sqrt{2}}\,Y, \tag{3.26}$$

and are obtained through the bi-unitary transformation:

$$U_L^\dagger M_\ell\, U_R = \mathrm{diag}(m_e, m_\mu, m_\tau). \tag{3.27}$$

The Yukawa matrix Y is fixed by the modular S_3 tensor product rules and reads

$$Y = \begin{pmatrix} \alpha_1 + \alpha_2 Y^{(0)}_{\mathbf{2},1} & -\alpha_2 Y^{(0)}_{\mathbf{2},2} & \alpha_3 Y^{(0)}_{\mathbf{2},1} \\ -\alpha_2 Y^{(0)}_{\mathbf{2},2} & \alpha_1 - \alpha_2 Y^{(0)}_{\mathbf{2},1} & \alpha_3 Y^{(0)}_{\mathbf{2},2} \\ \alpha_4 Y^{(0)}_{\mathbf{2},1} & \alpha_4 Y^{(0)}_{\mathbf{2},2} & \alpha_5 \end{pmatrix}, \tag{3.28}$$

where the modular forms $Y^{(0)}_{\mathbf{2}}$ of weight-zero are evaluated at the modulus τ. Eq. (3.28) therefore correlates the charged-lepton hierarchy with the restricted modular texture.

Active neutrino masses are generated at three loops and can be written in the weak basis as

$$M_\nu \simeq \frac{\sin(2\beta)}{(16\pi^2)^3}\left(\mathcal{A}_1\mathcal{I}_1 + \mathcal{A}_2\mathcal{I}_2\right)\, Y\, C\, Y^T, \tag{3.29}$$

where C collects the couplings of the doubly charged scalar S^{++} to right-handed charged leptons and is complex symmetric,

$$C = \begin{pmatrix} \beta_1 Y^{(4)}_{\mathbf{1}} + \beta_2 Y^{(4)}_{\mathbf{2},1} & -\beta_2 Y^{(4)}_{\mathbf{2},2} & \beta_3 Y^{(4)}_{\mathbf{2},1} \\ -\beta_2 Y^{(4)}_{\mathbf{2},2} & \beta_1 Y^{(4)}_{\mathbf{1}} - \beta_2 Y^{(4)}_{\mathbf{2},1} & \beta_3 Y^{(4)}_{\mathbf{2},2} \\ \beta_3 Y^{(4)}_{\mathbf{2},1} & \beta_3 Y^{(4)}_{\mathbf{2},2} & \beta_4 Y^{(4)}_{\mathbf{1}} \end{pmatrix}. \tag{3.30}$$

The coefficients $\mathcal{A}_{1,2}$ are

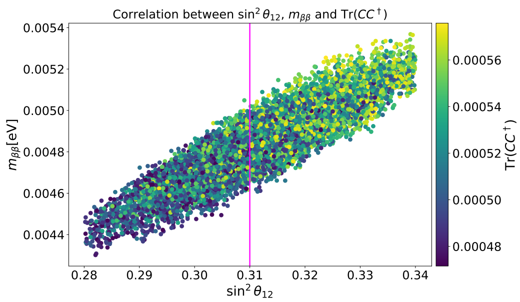
$$\begin{aligned}
\mathcal{A}_1 &\simeq \frac{5M_W^4}{v^6}\left(\Delta m_+^2\right)^2 \Delta m_0^2\,\frac{\kappa_2 Y_{\mathbf{1}}^{(-2)}\sin(2\beta)+\xi Y_{\mathbf{1}}^{(0)} v\cos(2\beta)}{m_{S^{++}}^{3/2} m_0^{1/2} m_+^2},\\
\mathcal{A}_2 &\simeq -\frac{29M_W^4}{v^6}\Delta m_+^2 \Delta m_0^2\,\frac{\xi Y_{\mathbf{1}}^{(0)} v}{m_{S^{++}}\, m_0^{1/2} m_+^{1/2}},
\end{aligned} \tag{3.31}$$

with the shorthand definitions $m_0^2 \equiv m_{A_0}^2 + m_{H_0}^2$ and $m_+^2 \equiv m_{H_2^+}^2 + m_{H_1^+}^2$. The loop integrals $\mathcal{I}_{1,2}$ are dimensionless $\mathcal{O}(1)$ functions of mass ratios. Finally, in the charged-lepton mass basis the neutrino matrix reads $m_\nu = U_L^T M_\nu U_L$, and the leptonic mixing is $U_{\text{PMNS}} = U_L^\dagger U_\nu$, with $U_\nu^T m_\nu U_\nu = \text{diag}(m_1, m_2, m_3)$. A representative benchmark satisfying the constraints arising from oscillation data and electroweak precision observables is selected to illustrate the phenomenological implications of the model. By performing random variation of the parameters around the their best fit point values, we generate correlation plots that illustrate the region of parameter space that are consistent with the leptonic observables, as shown in the figures below (see Figure 3.3). Although the benchmark point is obtained from the minimization of the χ^2 function, in the correlation plots we allow variations within a 5% range for the charged lepton masses, since their experimental uncertainties are extremely small. For the remaining observables, the standard 3σ constraints are imposed. In addition, the real and imaginary parts of the modulus τ are kept fixed in the correlation plots, since even variations at the 1% do not generate a parameter space consistent with the leptonic observables. This indicates that the model is extremely sensitive to variations of the modulus τ, which is not a surprise since the texture of Y and C are completely determinated by this parameter.

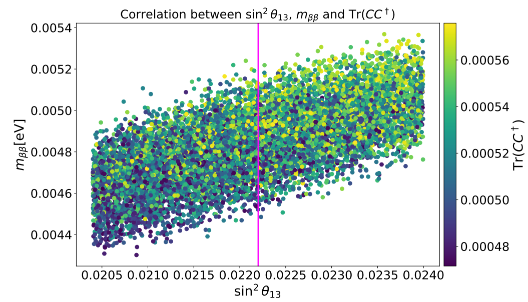
In principle, additional correlation plots can be obtained; however, the Figures presented here focus exclusively on the variables that exhibit the strongest correlations among themselves. Figures 2.a and 2.b show the allowed values for $\sin^2\theta_{12}$ and $\sin^2\theta_{13}$ with respect to $m_{\beta\beta}$ and the CP phase. The effective Majorana mass is completely a prediction of the model. Its maximum value is found to be $m_{\beta\beta}^{\max} \simeq 0.0056\,[\text{eV}]$, which lies well below the current upper bound. Figure 2.c illustrates the correlation among $\sin^2\theta_{12}$, $\sin^2\theta_{13}$ and the CP phase. Finally, the Figure 2.d shows the correlation between the sum of neutrino masses, Δm_{32}^2 and the CP phase. Once again, the sum of neutrino masses is a prediction of the model, with maximum value $\Sigma m_\nu^{\max} \simeq 0.063\,[\text{eV}]$, which is below the upper limit of $0.072\,[\text{eV}]$ reported by Planck+DESI.

3.3.2 Constraints for the full solution

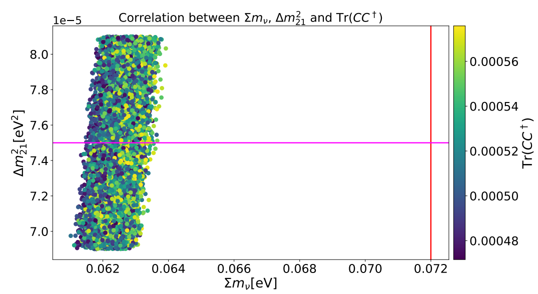
The decay $\mu \to e\gamma$ is one of the most actively searched processes of charged lepton flavor violation (CLFV). Since the model contains a doubly charged scalar, the process $\mu \to e\gamma$ arises at the one-



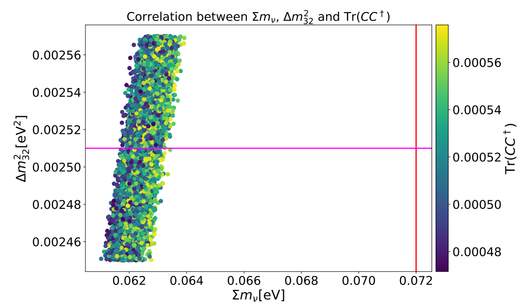
(a) Correlation for $\sin^2\theta_{12}$, $m_{\beta\beta}$ and $\mathrm{Tr}(CC^\dagger)$



(b) Correlation for $\sin^2\theta_{13}$, $m_{\beta\beta}$ and $\mathrm{Tr}(CC^\dagger)$



(c) Correlation for $\sum m_\nu$, Δm^2_{21} and $\mathrm{Tr}(CC^\dagger)$



(d) Correlation for $\sum m_\nu$, Δm^2_{32} and $\mathrm{Tr}(CC^\dagger)$

Figure 3.3: Correlation plots between the observables of the leptonic sector. The magenta lines represent the current experimental values, while the red line denotes a theoretical upper limit. Panels (a)-(d) show the correlations among: (a) $\sin^2\theta_{12}$, $m_{\beta\beta}$ and $\mathrm{Tr}(CC^\dagger)$; (b) $\sin^2\theta_{13}$, $m_{\beta\beta}$ and $\mathrm{Tr}(CC^\dagger)$; (c) Σm_ν, Δm^2_{21} and $\mathrm{Tr}(CC^\dagger)$; (d) Σm_ν, Δm^2_{32} and $\mathrm{Tr}(CC^\dagger)$.

loop level and is mediated by this scalar. We perform a scan over the parameter space of the model to obtain its predictions for the branching ratio of this process. The results are shown in Figure 3.4. However, not everything is as promising as it initially appears. Although the model predicts values for $\mathrm{Br}(\mu \to e\gamma)$ within the allowed range, there are other CLFV processes whose predicted rates exceed the current experimental bounds. The coupling matrix C is precisely the matrix that encodes these processes, and therefore its entries must satisfy the corresponding experimental bounds. The solutions we found do not satisfy all the CLFV constraints. In fact, we have not found any solution that simultaneously fulfills all the relevant conditions. It is also worth noting that we have not even addressed the constraints arising from the scalar potential in detail. Beyond obtaining the expressions for the mass splittings, the boundedness from below conditions represent an additional and independent set of constraints that must be imposed on the solutions of the model. Even at a more basic level, perturbativity of the couplings must also be required. In particular, the couplings should remain below 4π (at least in the scalar sector). We have not found solutions that satisfy the perturbativity condition either. In summary, although we find solutions that reproduce the leptonic

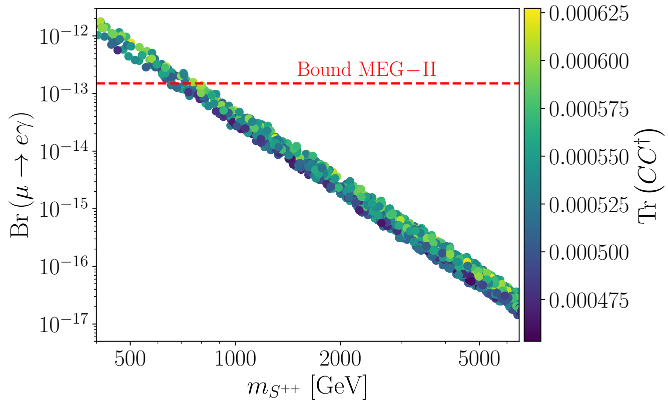


Figure 3.4: Predicted branching ratio $\mathrm{Br}(\mu \to e\gamma)$ as a function of the doubly charged scalar mass $m_{S^{++}}$. The red dashed line indicates the current MEG-II upper bound. The color bar represents $\mathrm{Tr}(CC^\dagger)$.

observables, these solutions fail once the remaining theoretical and experimental constraints are imposed. Our approach may be somewhat ambitious, since many works in the literature tend, to some extent, to relax or ignore some of the conditions mentioned above. In any case, there is no guarantee that fully consistent solutions exist within this type of modular model.

Chapter 4

Conclusion

In summary, we have identified a symmetry that allows us to describe the mixing patterns that appear in mass models. This symmetry establishes that the coupling constants are no longer arbitrary parameters, but instead correspond to polyharmonic Maass forms. This provides a possible explanation for the origin of the CP violating phase. In addition, we constructed and fitted a Type-I seesaw model with modular A_4 symmetry, finding that it is capable of reproducing the observables of the leptonic sector within 3σ. We also obtained correlation plots for the model and observed that there exists a region in parameter space that successfully predicts these observables. This is an important result, since it indicates that the solution we found does not correspond to an isolated point in parameter space. We also discussed that the assignment of modular weights and representations for the fields under the modular symmetry is, in principle, arbitrary. Although one could argue in favor of more economical theories, there is currently no clear physical principle that constrains these assignments. In fact, we observed that not all models constructed under these symmetries are able to fit the experimental data, and we do not yet have a general criterion that allows us to determine whether a given model will have viable solutions, in the sense of reproducing the observables of the leptonic sector. We then analyzed the Cocktail Model with modular S_3 symmetry, which provides a radiative alternative to the seesaw mechanism. In this case, the goal is to explain simultaneously the origin of neutrino masses and to provide a candidate for dark matter. Once again, the modular symmetry allows us to find solutions that reproduce the observables of the leptonic sector. However, when more stringent conditions are imposed, such as constraints from charged lepton flavor violation (CLFV) processes and stability conditions of the scalar potential, we encounter difficulties in obtaining complete solutions, since the modular symmetry strongly restricts these sectors. In principle, there could exist some combination of representations and modular weights capable of satisfying all these conditions simultaneously. Nevertheless, since there is currently no systematic criterion indicating how to identify such a configuration, finding it becomes extremely difficult. In summary, the models studied are able to reproduce the observables of the leptonic sector and therefore provide a possible explanation for the observed mixing patterns. At the same time, however, these models remain too broad, since there are no clear physical constraints that determine the representations and modular weights under which the fields transform.

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Appendix A

Chirality

As is well known, in order to define the Dirac equation it is necessary to introduce the four gamma matrices (or Dirac matrices). From these matrices one can define the so-called chirality matrix γ^5, whose expression is given by

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 \, .$$

The chirality matrix is Hermitian and therefore it can be diagonalized by means of a unitary transformation,

$$U\gamma^5 U^\dagger = \gamma^5_{\text{diag}} \, ,$$

where U is a unitary matrix. Moreover, it satisfies $(\gamma^5)^2 = \mathbf{1}$, from which it follows that its eigenvalues are ± 1. Indeed, in the chiral representation,

$$\gamma^0_C = \begin{pmatrix} 0 & -\mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix} \, , \qquad \gamma^i_C = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \, ,$$

the chirality matrix is diagonal,

$$\gamma^5_C = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} \, . \tag{A.1}$$

Let us denote by ψ_R and ψ_L the fields that are eigenstates of γ^5 with eigenvalues $+1$ and -1, respectively:

$$\gamma^5\psi_R = +\psi_R \, ,$$
$$\gamma^5\psi_L = -\psi_L \, .$$

The chiral fields ψ_R and ψ_L are referred to as *right-handed* and *left-handed*, respectively. Any generic spinor field ψ can always be decomposed into its chiral components,

$$\psi = \psi_R + \psi_L \, , \tag{A.2}$$

with

$$\psi_R = \frac{1+\gamma^5}{2}\psi \ ,$$

$$\psi_L = \frac{1-\gamma^5}{2}\psi \ .$$

It is convenient to define the chiral projection operators

$$P_R \equiv \frac{1+\gamma^5}{2} \ ,$$

$$P_L \equiv \frac{1-\gamma^5}{2} \ ,$$

which satisfy the properties

$$P_R + P_L = \mathbf{1} \ ,$$

$$(P_R)^2 = P_R \ ,$$

$$(P_L)^2 = P_L \ ,$$

$$P_R P_L = P_L P_R = 0 \, .$$

Appendix B

Permutation groups

B.1 N=2

The polyharmonic Maaß forms of level $N = 2$ can be arranged into irreducible multiplets of $\Gamma_2 \cong S_3$. The S_3 group can be generated by two generators S and T satisfying the relations

$$S_3 = \{S, T \mid S^2 = T^2 = (ST)^3 = 1\}. \tag{B.1}$$

The S_3 group has a doublet representation $\mathbf{2}$ and two singlet representations $\mathbf{1}, \mathbf{1}'$, and the modular generators S and T are represented by

$$\begin{aligned}
\mathbf{1}:& \quad S = 1, \qquad T = 1, \\
\mathbf{1}':& \quad S = -1, \qquad T = -1, \\
\mathbf{2}:& \quad S = \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\end{aligned}$$

The product of two doublets is given by $\mathbf{2} \times \mathbf{2} = \mathbf{1} \oplus \mathbf{1}' \oplus \mathbf{2}$, and the contraction rules are given by

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}_{\mathbf{2}} \otimes \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}_{\mathbf{2}} = (\alpha_1\beta_1 + \alpha_2\beta_2)_{\mathbf{1}} \oplus (\alpha_1\beta_2 - \alpha_2\beta_1)_{\mathbf{1}'} \oplus \begin{pmatrix} \alpha_1\beta_1 - \alpha_2\beta_2 \\ -\alpha_1\beta_2 - \alpha_2\beta_1 \end{pmatrix}_{\mathbf{2}}. \tag{B.2}$$

There are two linearly independent weight 2 modular forms at level 2 [27]:

$$Y_1(\tau) = \frac{2i}{\pi}\left[\frac{\eta'(\tau/2)}{\eta(\tau/2)} + \frac{\eta'((\tau+1)/2)}{\eta((\tau+1)/2)} - 8\frac{\eta'(2\tau)}{\eta(2\tau)}\right], \tag{B.3}$$

$$Y_2(\tau) = \frac{2\sqrt{3}i}{\pi}\left[\frac{\eta'(\tau/2)}{\eta(\tau/2)} - \frac{\eta'((\tau+1)/2)}{\eta((\tau+1)/2)}\right]. \tag{B.4}$$

The weight 2 modular forms of level 2 forms a doublet **2** of S_3 [27] and the q-expansion is

$$Y_{\mathbf{2}}^{(2)} = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} 1 + 24q + 24q^2 + 96q^3 + 24q^4 + 144q^5 + \cdots \\ 8\sqrt{3}\, q^{1/2} \left(1 + 4q + 6q^2 + 8q^3 + 13q^4 + 12q^5 + \cdots\right) \end{pmatrix}. \tag{B.5}$$

Using the method of chapter 2, one can obtain the explicit expressions of polyharmonic Maass forms from the modular forms at level 2. The weight $k = 0$ polyharmonic Maass forms can be arranged into a doublet $Y_{\mathbf{2}}^{(0)}$ and a trivial singlet $Y_{\mathbf{1}}^{(0)}$ of S_3. Note that $Y_{\mathbf{1}}^{(0)}$ is a constant and without loss of generality we can take $Y_{\mathbf{1}}^{(0)} = 1$. The master equation for the weight 0 polyharmonic Maass form $Y_{\mathbf{2}}^{(0)}$ of level 2 is of the following

$$\xi_0\big(Y_{\mathbf{2}}^{(0)}\big) = Y_{\mathbf{2}}^{(2)}, \qquad D\big(Y_{\mathbf{2}}^{(0)}\big) = -\frac{1}{4\pi} Y_{\mathbf{2}}^{(2)}. \tag{B.6}$$

where we have chosen the normalization factor $\alpha = 1$, and thus $\beta = -\frac{1}{4\pi}$. Then, we can obtain the q-expansion of $Y_{\mathbf{2}}^{(0)} = (Y_{\mathbf{2},1}^{(0)}, Y_{\mathbf{2},2}^{(0)})^T$ as

$$\begin{aligned} Y_{\mathbf{2},1}^{(0)} &= y - \frac{6e^{-4\pi y}}{\pi q} - \frac{3e^{-8\pi y}}{\pi q^2} - \frac{8e^{-12\pi y}}{\pi q^3} - \frac{3e^{-16\pi y}}{2\pi q^4} + \cdots \\ &\quad - \frac{4\log 2}{\pi} - \frac{6q}{\pi} - \frac{3q^2}{\pi} - \frac{8q^3}{\pi} - \frac{3q^4}{2\pi} - \frac{36q^5}{5\pi} + \cdots, \end{aligned} \tag{B.7}$$

$$\begin{aligned} Y_{\mathbf{2},2}^{(0)} &= \frac{4\sqrt{3}\, q^{1/2}}{\pi} \left(e^{-2\pi y} + 4\frac{e^{-6\pi y}}{3q^2} + 6\frac{e^{-10\pi y}}{5q^3} + 8\frac{e^{-14\pi y}}{7q^4} + 13\frac{e^{-18\pi y}}{9q^5} + \cdots \right) \\ &\quad - \frac{4\sqrt{3}\, q^{1/2}}{\pi} \left(1 + \frac{4}{3}q + \frac{6}{4}q^2 + \frac{8}{7}q^3 + \frac{13}{9}q^4 + \frac{12}{11}q^5 + \cdots \right). \end{aligned} \tag{B.8}$$

The exact expressions of $Y_{\mathbf{2},1}^{(0)}$ and $Y_{\mathbf{2},2}^{(0)}$ are

$$Y_{\mathbf{2},1}^{(0)} = y - \frac{4\log 2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\sigma_1(2n)}{2n}\left(q^n + \frac{e^{-4\pi n y}}{q^n}\right) - \frac{\sigma_1(n)}{n}\left(q^{2n} + \frac{e^{-8\pi n y}}{q^{2n}}\right), \tag{B.9}$$

$$Y_{\mathbf{2},2}^{(0)} = \frac{4\sqrt{3}}{\pi}\sum_{n=0}^{\infty} \frac{\sigma_1(2n+1)}{2n+1}\left[q^{(2n+1)/2} + \frac{e^{-2\pi(2n+1)y}}{q^{(2n+1)/2}}\right]. \tag{B.10}$$

The weight 4 modular forms of level 2 are $Y_{\mathbf{1}}^{(4)}$ and $Y_{\mathbf{2}}^{(4)}$ with

$$Y_{\mathbf{1}}^{(4)} = Y_1^2 + Y_2^2, \qquad Y_{\mathbf{2}}^{(4)} = \begin{pmatrix} Y_1^2 - Y_2^2 \\ -2Y_1Y_2 \end{pmatrix}. \tag{B.11}$$

They can be lifted to weight -2 polyharmonic Maaß forms

$$\xi_{-2}\big(Y_{\mathbf{1}}^{(-2)}\big) = Y_{\mathbf{1}}^{(4)}, \qquad D^3\big(Y_{\mathbf{1}}^{(-2)}\big) = -\frac{2}{(4\pi)^3} Y_{\mathbf{1}}^{(4)}. \tag{B.12}$$

Using the general result of (2.41), we find the weight $k=-2$ polyharmonic Maaß forms of level 2 as

$$\begin{aligned} Y_1^{(-2)} = \frac{y^3}{3} - \frac{15\,\Gamma(3,4\pi y)}{4\pi^3 q} - \frac{135\,\Gamma(3,8\pi y)}{32\pi^3 q^2} - \frac{35\,\Gamma(3,12\pi y)}{9\pi^3 q^3} + \cdots \\ - \frac{\pi\,\zeta(3)}{12\,\zeta(4)} - \frac{15q}{2\pi^3} - \frac{135q^2}{16\pi^3} - \frac{70q^3}{9\pi^3} - \frac{1095q^4}{128\pi^3} - \frac{189q^5}{25\pi^3} - \frac{35q^6}{4\pi^3} + \cdots \,, \end{aligned} \tag{B.13}$$

$$\begin{aligned} Y_{2,1}^{(-2)} = \frac{y^3}{3} + \frac{9\,\Gamma(3,4\pi y)}{4\pi^3 q} + \frac{57\,\Gamma(3,8\pi y)}{32\pi^3 q^2} + \frac{7\,\Gamma(3,12\pi y)}{3\pi^3 q^3} + \frac{441\,\Gamma(3,16\pi y)}{256\pi^3 q^4} + \cdots \\ + \frac{\pi\,\zeta(3)}{30\,\zeta(4)} + \frac{9q}{2\pi^3} + \frac{57q^2}{16\pi^3} + \frac{14q^3}{3\pi^3} + \frac{441q^4}{128\pi^3} + \frac{567q^5}{125\pi^3} + \cdots \,, \end{aligned} \tag{B.14}$$

$$\begin{aligned} Y_{2,2}^{(-2)} = \frac{4\sqrt{3}\,q^{1/2}}{\pi^3}\left(\frac{\Gamma(3,2\pi y)}{2q} + \frac{14\,\Gamma(3,6\pi y)}{27q^2} + \frac{63\,\Gamma(3,10\pi y)}{125q^3} + \frac{172\,\Gamma(3,14\pi y)}{343q^4} + \cdots\right) \\ + \frac{4\sqrt{3}\,q^{1/2}}{\pi^3}\left(1 + \frac{28q}{27} + \frac{126q^2}{125} + \frac{344q^3}{343} + \frac{757q^4}{729} + \cdots\right). \end{aligned} \tag{B.15}$$

In principle, one can continue computing polyharmonic Maass forms for higher weights. However, this is not necessary for the construction of models.

B.2 N=3

The finite inhomogeneous modular group Γ_3 is isomorphic to A_4, which corresponds to the symmetry group of the regular tetrahedron. This group can be generated by the elements S and T, which satisfy the defining relation

$$A_4 = \{S,T \mid S^2 = T^3 = (ST)^3 = 1\}. \tag{B.16}$$

The group has four irreducible representations. To see this, let us recall a basic result from the theory of finite groups,

$$|G| = \sum_i n_i^2, \tag{B.17}$$

where $|G|$ denotes the order of the group and n_i are the dimensions of its irreducible representations. Furthermore, the number of irreducible representations is equal to the number of conjugacy classes of the group.

In the case of A_4, there are four conjugacy classes and the order of the group is 12. Therefore, the only possibility consistent with the above relation is that the group contains three singlet irreducible representations and one triplet irreducible representation.

The generators S and T in the irreducible representations of A_4 are given by

$$\mathbf{1}: \qquad S = 1\,, \qquad T = 1,$$

$$\mathbf{1}': \qquad S = 1\,, \qquad T = \omega,$$

$$\mathbf{1}'': \qquad S = 1\,, \qquad T = \omega^2,$$

$$\mathbf{3}: \qquad S = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix},$$

where $\omega = e^{2\pi i/3}$. The triplet representation $\mathbf{3}$ is a real representation, meaning that its complex conjugate is equivalent to itself. In other words, the conjugate representation is related to the original one through a similarity transformation,

$$\rho^*_{\mathbf{3}}(T) = \Omega\, \rho_{\mathbf{3}}(T)\, \Omega^\dagger, \qquad \rho^*_{\mathbf{3}}(S) = \Omega\, \rho_{\mathbf{3}}(S)\, \Omega^\dagger, \tag{B.18}$$

where Ω is the similarity transformation matrix given by

$$\Omega = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{B.19}$$

For two generic A_4 triplets $\alpha = (\alpha_1, \alpha_2, \alpha_3)^T$ and $\beta = (\beta_1, \beta_2, \beta_3)^T$, the tensor product decomposition is

$$\mathbf{3} \otimes \mathbf{3} = \mathbf{1} \oplus \mathbf{1}' \oplus \mathbf{1}'' \oplus \mathbf{3}_S \oplus \mathbf{3}_A. \tag{B.20}$$

The corresponding contraction rules are given by

$$\begin{aligned} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}_{\mathbf{3}} \otimes \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}_{\mathbf{3}} &= (\alpha_1\beta_1 + \alpha_2\beta_3 + \alpha_3\beta_2)_{\mathbf{1}} \oplus (\alpha_3\beta_3 + \alpha_1\beta_2 + \alpha_2\beta_1)_{\mathbf{1}'} \oplus (\alpha_2\beta_2 + \alpha_1\beta_3 + \alpha_3\beta_1)_{\mathbf{1}''} \\ &\oplus \begin{pmatrix} 2\alpha_1\beta_1 - \alpha_2\beta_3 - \alpha_3\beta_2 \\ 2\alpha_3\beta_3 - \alpha_1\beta_2 - \alpha_2\beta_1 \\ 2\alpha_2\beta_2 - \alpha_1\beta_3 - \alpha_3\beta_1 \end{pmatrix}_{\mathbf{3}_S} \oplus \begin{pmatrix} \alpha_2\beta_3 - \alpha_3\beta_2 \\ \alpha_1\beta_2 - \alpha_2\beta_1 \\ \alpha_3\beta_1 - \alpha_1\beta_3 \end{pmatrix}_{\mathbf{3}_A}. \end{aligned} \tag{B.21}$$

Another important aspect to consider is the tensor product between the different singlet representations, since these products are used to construct invariant terms in the Lagrangian. The products are given by

$$\mathbf{1} \otimes \mathbf{1} = \mathbf{1}\ ,\ \mathbf{1} \otimes \mathbf{1}' = \mathbf{1}' \otimes \mathbf{1} = \mathbf{1}'\ ,\ \mathbf{1} \otimes \mathbf{1}'' = \mathbf{1}'' \otimes \mathbf{1} = \mathbf{1}''$$

$$\mathbf{1}' \otimes \mathbf{1}' = \mathbf{1}'' \ , \ \mathbf{1}' \otimes \mathbf{1}'' = \mathbf{1}'' \otimes \mathbf{1}' = \mathbf{1}$$
$$\mathbf{1}'' \otimes \mathbf{1}'' = \mathbf{1}'.$$

On the other hand, it can be shown that there are three modular forms of weight 2 and level 3, which can be organized into a triplet representation of A_4 as follows:

$$Y_{\mathbf{3}}^{(2)} = \begin{pmatrix} Y_1(\tau) \\ Y_2(\tau) \\ Y_3(\tau) \end{pmatrix} \ , \tag{B.22}$$

con la q-expansion

$$\begin{aligned} Y_1(\tau) &= (1 + 12q + 36q^2 + 12q^3 + ...) \ , \\ Y_2(\tau) &= -6q^{1/3}(1 + 7q + 8q^2 + 18q^3 + ...) \ , \\ Y_3(\tau) &= -18q^{2/3}(1 + 2q + 5q^2 + 4q^3 + ...) \ . \end{aligned}$$

One can obtain five independent modular forms of weight 4 by taking the tensor product of $Y_{\mathbf{3}}^{(2)}$ with itself, resulting in

$$\begin{aligned} Y_{\mathbf{3}}^{(4)} &= \left(Y_1^2 - Y_2Y_3, Y_3^2 - Y_1Y_2, Y_2^2 - Y_1Y_3\right)^T \ , \\ Y_{\mathbf{1}}^{(4)} &= Y_1^2 + 2Y_2Y_3 \ , \\ Y_{\mathbf{1}'}^{(4)} &= Y_3^2 + 2Y_1Y_2 \ . \end{aligned}$$

To compute the modular forms of weight 6, we follow essentially the same procedure used for the weight 4 case, except that the tensor product structure changes. For integer weights $k > 2$, polyharmonic Maass forms coincide with the modular forms of level 3. Moreover, the modular form $Y_{\mathbf{3}}^{(2)}$ implies the existence of a polyharmonic Maass form of weight 0, namely $Y_{\mathbf{3}}^{(0)}$. Using the expression in Eq. (2.41), we find that the q-expansion of $Y_{\mathbf{3}}^{(0)}$ is given by

$$\begin{aligned} Y_{3,1}^{(0)} = y &- \frac{3e^{-4\pi y}}{\pi q} - \frac{9e^{-8\pi y}}{2\pi q^2} - \frac{e^{-12\pi y}}{\pi q^3} - \frac{21e^{-16\pi y}}{4\pi q^4} - \frac{18e^{-20\pi y}}{5\pi q^5} - \frac{3e^{-24\pi y}}{2\pi q^6} + \cdots \\ &- \frac{9\log 3}{4\pi} - \frac{3q}{\pi} - \frac{9q^2}{2\pi} - \frac{q^3}{\pi} - \frac{21q^4}{4\pi} - \frac{18q^5}{5\pi} - \frac{3q^6}{2\pi} + \cdots , \end{aligned}$$

$$\begin{aligned} Y_{3,2}^{(0)} = &\frac{27q^{1/3}e^{\pi y/3}}{\pi} \left(\frac{e^{-3\pi y}}{4q} + \frac{e^{-7\pi y}}{5q^2} + \frac{5e^{-11\pi y}}{16q^3} + \frac{2e^{-15\pi y}}{11q^4} + \frac{2e^{-19\pi y}}{7q^5} + \frac{4e^{-23\pi y}}{17q^6} + \cdots \right) \\ &+ \frac{9q^{1/3}}{2\pi} \left(1 + \frac{7q}{4} + \frac{8q^2}{7} + \frac{9q^3}{5} + \frac{14q^4}{13} + \frac{31q^5}{16} + \frac{20q^6}{19} + \cdots \right) , \end{aligned}$$

$$\begin{aligned} Y_{3,3}^{(0)} = &\frac{9q^{2/3}e^{2\pi y/3}}{2\pi} \left(\frac{e^{-2\pi y}}{q} + \frac{7e^{-6\pi y}}{4q^2} + \frac{8e^{-10\pi y}}{7q^3} + \frac{9e^{-14\pi y}}{5q^4} + \frac{14e^{-18\pi y}}{13q^5} + \frac{31e^{-22\pi y}}{16q^6} + \cdots \right) \\ &+ \frac{27q^{2/3}}{\pi} \left(\frac{1}{4} + \frac{q}{5} + \frac{5q^2}{16} + \frac{2q^3}{11} + \frac{2q^4}{7} + \frac{9q^5}{17} + \frac{21q^6}{20} + \cdots \right) . \end{aligned}$$

In the same way, the polyharmonic Maass form of weight -2 can be obtained from the modular forms of weight 4. The components of the triplet are given by

$$
\begin{aligned}
Y_{3,1}^{(-2)}(\tau) = {} & \frac{y^3}{3} + \frac{21\,\Gamma(3,4\pi y)}{16\pi^3 q} + \frac{189\,\Gamma(3,8\pi y)}{128\pi^3 q^2} + \frac{169\,\Gamma(3,12\pi y)}{144\pi^3 q^3} + \frac{1533\,\Gamma(3,16\pi y)}{1024\pi^3 q^4} + \cdots \\
& + \frac{\pi}{40}\frac{\zeta(3)}{\zeta(4)} + \frac{21q}{8\pi^3} + \frac{189q^2}{64\pi^3} + \frac{169q^3}{72\pi^3} + \frac{1533q^4}{512\pi^3} + \frac{1323q^5}{500\pi^3} + \frac{169q^6}{64\pi^3} + \cdots\,,
\end{aligned}
$$

$$
\begin{aligned}
Y_{3,2}^{(-2)}(\tau) = {} & -\frac{729q^{1/3}}{16\pi^3}\left(\frac{\Gamma(3,8\pi y/3)}{16q} + \frac{7\Gamma(3,20\pi y/3)}{125q^2} + \frac{65\Gamma(3,32\pi y/3)}{1024q^3} + \frac{74\Gamma(3,44\pi y/3)}{1331q^4} + \cdots\right) \\
& - \frac{81q^{1/3}}{16\pi^3}\left(1 + \frac{73q}{64} + \frac{344q^2}{343} + \frac{567q^3}{500} + \frac{20198q^4}{2197} + \frac{4681q^5}{4096} + \cdots\right),
\end{aligned}
$$

$$
\begin{aligned}
Y_{3,3}^{(-2)}(\tau) = {} & -\frac{81q^{2/3}}{32\pi^3}\left(\frac{\Gamma(3,4\pi y/3)}{q} + \frac{73\Gamma(3,16\pi y/3)}{64q^2} + \frac{344\Gamma(3,28\pi y/3)}{343q^3} + \frac{567\Gamma(3,40\pi y/3)}{500q^4} + \cdots\right) \\
& - \frac{729q^{2/3}}{8\pi^3}\left(\frac{1}{16} + \frac{7q}{125} + \frac{65q^2}{1024} + \frac{74q^3}{1331} + \cdots\right).
\end{aligned}
$$

Finally, we obtain the polyharmonic Maass forms of weight -4.

$$
\begin{aligned}
Y_{3,1}^{(-4)} = {} & \frac{y^5}{5} - \frac{549}{3328\pi^5}\left(\frac{\Gamma(5,4\pi y)}{q} + \frac{33\Gamma(5,8\pi y)}{32q^2} + \frac{14641\Gamma(5,12\pi y)}{14823q^3} + \frac{1057\Gamma(5,16\pi y)}{1024q^4} + \cdots\right) \\
& - \frac{3\pi}{728}\frac{\zeta(5)}{\zeta(6)} - \frac{1647}{416\pi^5}\left(q + \frac{33q^2}{32} + \frac{14641q^3}{14823} + \frac{1057q^4}{1024} + \frac{3126q^5}{3125} + \cdots\right),
\end{aligned}
$$

$$
\begin{aligned}
Y_{3,2}^{(-4)} = {} & \frac{72171q^{1/3}}{212992\pi^5}\left(\frac{\Gamma(5,8\pi y/3)}{q} + \frac{33344\Gamma(5,20\pi y/3)}{34375q^2} + \frac{1025\Gamma(5,32\pi y/3)}{1024q^3} + \cdots\right) \\
& + \frac{6561q^{1/3}}{832\pi^5}\left(1 + \frac{1057q}{1024} + \frac{16808q^2}{16807} + \frac{51579q^3}{50000} + \frac{371294q^4}{371293} + \cdots\right),
\end{aligned}
$$

$$
\begin{aligned}
Y_{3,3}^{(-4)} = {} & \frac{2187q^{2/3}}{6656\pi^5}\left(\frac{\Gamma(5,4\pi y/3)}{q} + \frac{1057\Gamma(5,16\pi y/3)}{1024q^2} + \frac{16808\Gamma(5,28\pi y/3)}{16807q^3} + \cdots\right) \\
& + \frac{216513q^{2/3}}{26624\pi^5}\left(1 + \frac{33344q}{34375} + \frac{1025q^2}{1024} + \frac{1717888q^3}{1771561} + \frac{16808q^4}{16807} + \cdots\right).
\end{aligned}
$$

Evidently, one can continue computing higher polyharmonic Maass forms. However, the ones obtained here are already sufficient for constructing models and exploring the implications of the modular symmetry.