UNIVERSIDAD TÉCNICA FEDERICO SANTA MARÍA

DEPARTAMENTO DE ELECTRÓNICA
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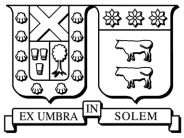

UNIVERSIDAD TECNICA
FEDERICO SANTA MARIA

IDENTIFICATION AND CONTROL METHODS UTILIZING RANK AND CARDINALITY OPTIMIZATION APPROACH

Thesis presented by

Gabriel Andrés Urrutia Bustos

in partial fulfillment of the requirements for the professional title of

Ingeniero Civil Electrónico

and the degree of

Magíster en Ciencias de la Ingeniería Electrónica

Principal supervisor
Dr. Juan Carlos Agüero Vásquez

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This THESIS is presented in partial fulfillment of the requirements for the professional title of Ingeniero Civil Electrónico and the degree of Magíster en Ciencias de la Ingeniería Electrónica at Universidad Técnica Federico Santa María, Valparaíso, Chile.

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Valparaíso, January 2017.

A mi Mamá, Papá y Hermano,
por su constante dedicación,
apoyo incondocional
e infinito amor.

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En primer lugar, quisiera agradecer el infinito amor y cariño que me ha brindado mi familia a lo largo de estos años. Especialmente, agredecer a mis padres por su constante guía y apoyo, los que sin duda hicieron posible y alivianaron en gran medida el camino que he recorrido. No cabe la menor duda de que gran parte de lo que soy se lo debo a ellos, y por lo mismo les estaré eternamente agradecido.

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ABSTRACT

This Thesis addresses a class of optimization problems that can be found in several areas, such as system identification and control. Particularly, these problems are formulated by using rank and cardinality constraints in order to obtain low rank matrices or induce sparsity of the solution.

Rank-constrained optimization problems are found in control and system identification. Low-order controller design problems are well known examples where the formulation utilizes Linear Matrix Inequalities (LMIs) and rank constraints over matrices for bounding the controller's order and closed loop stability degree.

Promotion of sparsity in identification and control problems can bring many practical advantages in the final solution. In model selection, by formulating the identification problem with a cardinality (ℓ_0-norm) constraint over the parameter vector, a simplified or specific structure of the model can be obtained. In control applications sparsity can be induced on the solution of an optimal control problem, thus limiting the number of active actuators at each time step.

Although low-rank and sparsity are desirable characteristics in the solution of many problems of interest, solving these type of problems poses computational difficulties. Many approaches that rely on approximations and specific tailored solutions are available in the literature in order to overcome the inherent complexity of the problem. However,

in this work a novel rank-constraint representation is used which, aims to solve (not an approximation but) a problem that is equivalent to the original in the sense that they both have the same global optimum. The resulting problem can also be solved using standard nonlinear programming tools.

The work hereby presented is divided in three main parts. First, an overview of state-of-the art techniques for solving cardinality and rank-constrained problems is shown.

The second part of the thesis presents optimization problems with cardinality constraints in the field of model selection, parameter estimation and optimal control.

The third part of the thesis addresses a rank-constrained optimization problem when designing a low-order controller with prescribed degree of stability. The formulation of this problem includes LMI and rank constraints.

Keywords – Rank-constrained optimization, cardinality-constrained optimization, sparsity, system identification, optimal control, low-order controller design.

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Chapter 1

INTRODUCTION

The study and solution of optimization problems that induce sparsity (solution vector with many entries equal to zero) and have rank constraints on matrices has recently attracted huge attention. The main reason of this interest is their inherent practical application, which can found in the formulations of problems in many fields, such as control theory, signal processing, machine learning, computer vision, among others [1].

A problem inducing sparsity can be formulated by using the ℓ_0-(pseudo)-norm (defined as the number of non-zero entries of a vector) as a constraint on the solution vector, or as a penalization term in the cost function. In addition, the ℓ_0-(pseudo)-norm greatly increases the complexity of the problem due to the ℓ_0-(pseudo)-norm being nonconvex and nonlinear. Moreover, it also adds a combinatorial behavior to the problem.

Due to the high complexity of these type of problems, alternative forms of solution have been developed in order to avoid the direct treatment of the ℓ_0-(pseudo)-norm. One of the most used techniques is the convex relaxation of the problem, which is achieved through the use of the ℓ_1-norm (LASSO method) [2,3]. Although fast in giving a solution, this approach does not solve the original problem. Thus the solution, in general, can be suboptimal. Another issue related to these type of approximations of the ℓ_0-(pseudo)-norm is that they often add additional variables to the problem in order to regulate the trade-off between sparsity and other desirable features, such as the goodness of the approximation of the ℓ_0-(pseudo)-norm. The correct choice of the value of this variables has the difficulty of requiring additional knowledge of the solution. Other approximations of the ℓ_0-(pseudo)-norm include the use of the ℓ_q-(pseudo)-norm and the use of exponential approximations [4], which aim to improve the approximation given by the ℓ_1-norm while compromising computational complexity.

When dealing with rank constraints on matrices, iterative algorithms have been developed in order to find a matrix whose rank complies with a certain bound [5, 6]. The problem with these algorithms is that in general they treat the problem locally, which leads to local optima. Approximation of rank constraints are also found in the literature, such as the use of the nuclear norm instead of the rank constraint [7]. This heuristic

method poses a convex relaxation to the rank constraint, just as the ℓ_1-norm is to the ℓ_0-(pseudo)-norm, turning the original problem into a semidefinite optimization problem.

The use of approximations of the ℓ_0-(pseudo)-norm and rank constraints often leads to unsatisfactory results, which has motivated the study of alternative forms of representation of the constraints [1, 8–11]. In this work an equivalent representation of the rank constraint [8] is used to reformulate rank constrained and cardinality constrained problems. The alternative formulation has the same global optimum as the original problem.

As stated before, optimization problems involving rank or cardinality constraints appear in many areas of interest. In system identification, the use of information criteria such as Akaike's Information Criterion (AIC) or Bayesian Information Criterion (BIC) are commonly used when measuring the goodness of fit and complexity of a set of models. The ℓ_0-(pseudo)-norm can be used to state a more general problem by addressing the cardinality of the parameter vector, and thus the complexity of the selected model.

In the field of optimal control, cardinality constraints can be imposed over the input vector in a Model Predictive Control (MPC) optimization scheme. By restricting the cardinality of the input vector at each time step, the number of active inputs of the control scheme is limited to a certain bound. Finally, rank-constrained optimization schemes are found in controller design problems. A particular case of this problem is the low-order controller design with a prescribed order of closed loop stability degree, which states a feasibility problem involving Linear Matrix Inequalities (LMI) and rank constraints over matrices.

1.1 Overview of Thesis Content

A short description of the topics treated in each chapter of the Thesis is given below.

Chapter 2 gives a short introduction on the state of the art methods used for sparsity and rank-constrained problems.

Chapter 3 addresses the problem of sparse logistic regression using cardinality constraints and information criteria, such as AIC. The cardinality of the parameter vector is accounted by using the ℓ_0-(pseudo)-norm. In this problem, an equivalent representation for rank constraints on matrices [8] is used to reformulate the problem.

Chapter 4 studies the problem of model selection by approximating the ℓ_0-(pseudo)-norm with exponential terms in order to reduce the numerical complexity of the problem. A problem involving *hidden variables* in system identification is also studied and solved using the Expectation-Maximization (EM) algorithm with the exponential approximation of the cardinality constraint.

Chapter 5 investigates the stability of an optimal feedback control problem with cardinality constraints on the input vector. Sufficient conditions are provided for guaranteeing practical stability of the closed-loop. Solution of the formulated problem is done by addressing the combinatorial nature of the problem through a differnt framework. The reformulation is carried out by using an equivalent representation of the cardinality constraint.

Chapter 6 addresses the problem of low-order controller design with prescribed order of stability. An optimization problem stated using linear matrix inequalities with rank constraint is formulated and solved using an equivalent representation for rank constraints. Comparison with state of the art methods is also shown.

Chapter 7 summarizes the conclusions of the thesis and proposes possible future research work.

1.2 Summary of the Contributions of the Thesis

The main contribution of the thesis are the following:

1. The thesis provides an introduction into the field of optimization problems with cardinality and rank constraints. State of the art approximating methods are presented together with their benefits and drawbacks. Equivalent representations of rank constraints proposed by different authors are also studied, and their application in problems of interest is analyzed.

2. A classification problem using logistic regression with cardinality constraint over the number of parameters is presented. Treatment of the ℓ_0-(pseudo)-norm as a

constraint and a penalization term is done using and equivalent representation. A comparison among standard Maximum Likelihood (ML) estimation, cardinality constrained ML and the use of Information Criteria, such as AIC, is performed. This type of problem is of great interest in different areas such as gene selection, machine learning, among others.

3. Study of an optimization approach for model selection using AIC by treating the ℓ_0-(pseudo)-norm formulation through an approximation utilizing exponential terms. This approximation reduces, in general, the computational load. Performance of the proposal is tested through simulation, which gives similar results when utilizing the original AIC scheme (testing all possible models and then selecting the best based on AIC). A similar procedure is performed in the case of *hidden variables*, where model selection is done using the EM algorithm. In this case, the proposed method outperforms the original AIC scheme.

4. A problem of feedback control with constrained number of active inputs is investigated. A novel quadratic model predictive control strategy is presented, where sparsity and stability of the solution is assured. Sparsity of the solution is treated by constraining the ℓ_0-(pseudo)-norm of the solution vector at each control horizon instant. Additional bounding constraints are also imposed over the system state and control input. Alternative solutions are provided for the case of convex and non-convex bounding, by using a mixed integer linear programming tool (CPLEX) and non-linear programming tool (BARON) respectively. The computational complexity corresponding to the inherent combinatorial behavior of the resulting optimization problem is reduced, in general, by using an equivalent representation of the ℓ_0-(pseudo)-norm, and solved using standard nonlinear programming tools.

5. A low-order control design with a given closed loop stability degree is studied. Formulation of the problem includes the use of Linear Matrix Inequalities (LMI) and rank constraints in order to limit the controller's order. Solution is carried out utilizing an equivalent representation, and comparison with state of the art algorithms is carried out through simulations. The proposed approach outperforms two algorithms available in the literature, obtaining faster closed loop systems. The proposed approach makes the formulation and solution of similar control design problems possible.

1.3 Publications by the Author

1. Referred journal

   (a) R.P. Aguilera, G. Urrutia, R.A. Delgado, J.C. Agüero and D. Dolz, "Stability Analysis of Quadratic Model Predictive Control with ℓ_0-input constraint", Submitted to journal.

2. Referred conferences

   (a) G. Urrutia, R.A. Delgado, R. Carvajal, D. Katselis and J.C. Agüero, "Sparse Logistic Regression Utilizing Cardinality Constraints and Information Criteria", In 2016 IEEE Conference on Control Applications (CCA).

   (b) G. Urrutia, R.A. Delgado and J.C. Agüero, "Low-order Control Design Using a Novel Rank-constrained Optimization Approach", In 2016 Australian Control Conference (AUCC).

   (c) R. Carvajal, G. Urrutia, and J.C. Agüero, "An Optimization-based Algorithm for Model Selection using an approximation of Akaike's Information Criterion", In 2016 Australian Control Conference (AUCC).

1.4 Notation

Let $\mathbb{R}$ and $\mathbb{R}_{\geq 0}$ denote the real and non-negative real number sets. Uppercase letters matrices and bold lowercase letters denote vectors. Calligraphic letters denote sets. $\text{rank}(A)$ denotes the rank of a matrix and $\text{trace}(A)$ denotes its trace. I denotes an identity matrix of appropriate dimension. For a vector $\boldsymbol{x}$, $\text{diag}(\boldsymbol{x})$ denotes the diagonal matrix with diagonal entries the elements of vector $\boldsymbol{x}$ and $|\boldsymbol{x}|$ denotes the vector with entries the absolute values of the entries of $\boldsymbol{x}$. $\mathbb{R}^n$ is the set of $n \times 1$ real vectors and $\mathbb{R}^{m\times n}$ is the set of $m \times n$ real matrices. $\mathbb{S}^n$ stands for the set of $n \times n$ symmetric matrices and $\mathbb{S}^n_+$ denotes the set of $n \times n$ positive-semidefinite matrices. For the $n \times n$ matrices A, $B \in \mathbb{S}^n$, $A \geq B$ denotes the Löwner partial ordering, i.e., $A - B \in \mathbb{S}^n_+$. We represent the transpose of a given matrix A and a vector $\boldsymbol{x}$ via $(A\boldsymbol{x})' = \boldsymbol{x}'A'$.

Special notation is used in chapter 5 due to additional definitions that help to better illustrate the problem studied.

Chapter 2

OVERVIEW ON RANK-CONSTRAINED OPTIMIZATION METHODS

2.1 Introduction

In this chapter, a short introduction to state of the art methods for solving optimization problems with rank constraints and cardinality constraints is given. These methods mainly consist of tailored algorithms, approximations or alternative forms of representing a rank-constraint, all of which aim to reduce the computational complexity of the problem and improve the goodness of the solution. A small discussion on the benefits and drawbacks of each method is also presented.

This chapter begins with a short description of the general rank-constrained optimization problem with the special case of cardinality constrained optimization. In order to contextualize the general problem approach, some applications of these formulations are mentioned.

In the second part of the chapter, methods for solving rank constrained optimization problems are shown. A distinction is made between approximations and equivalent representations. Methods for the particular case of cardinality constrained problems are also shown.

Finally, a novel representation of rank constraints is shown [8]. This representation was used for solving system identification and control problems presented in the following chapters of the thesis.

2.2 Problem Description

Several problems involving rank constraints can be found in a very wide variety of fields. Some applications of these type of problems include model reduction, controller design,

matrix completion, factor analysis and machine learning, among others.

A general formulation of a Rank constrained optimization problem is as follows

$$\begin{aligned} \mathcal{P}_{rco}: \min_{x\in\mathbb{R}^p} \quad & f(x) \\ s.t. \quad & x \in \Omega \\ & \operatorname{rank}(G(x)) \leq r \end{aligned} \tag{2.1}$$

where $r \in \mathbb{R}$, $\Omega \subset \mathbb{R}^n$ and $G(x) \in \mathbb{R}^{m\times l}$ are given.

Similarly, another class of problems where the rank of a matrix is to be minimized can be formulated as

$$\begin{aligned} \mathcal{P}_{rco}: \min_{x\in\mathbb{R}^p, r\in\mathbb{R}} \quad & r \\ s.t. \quad & x \in \Omega \\ & \operatorname{rank}(G(x)) \leq r \end{aligned} \tag{2.2}$$

Rank-constrained optimization problems face the difficulties of being non-linear, non-convex and having a highly combinatorial nature. These drawbacks tend to increase the computational load of the problem, specially the fact of having a combinatorial behavior, which can make a problem with a high number of variables practically unsolvable. Rank-constrained optimization problems are classified as NP-hard problems, which means there isn't a known polynomial time algorithm that would solve all instances of the problem.

Rank-minimization problems may encounter extra difficulties when discerning if a solution is optimal or not. This drawback of the formulation is due to a wide range of solutions complying with a certain rank. To help avoiding this behavior, additional stopping criteria can be added to the solution algorithm together with the addition of extra terms to the cost functional.

A particular case of rank constrained optimization is cardinality constrained optimization, see (2.3). Cardinality constrained problems can be written in this form by utilizing a function $G(x)$ with diagonal form. Note that the cardinality constrained problems tend to promote sparsity over the entries of the solution vector. Cardinality of a vector is usually represented through the use of the ℓ_0-(pseudo)-norm, which corresponds to the number of non-zero valued entries of a given vector.

$$\begin{aligned} \mathcal{P}_{cco}: \min_{x\in\mathbb{R}^p} \quad & f(x) \\ s.t. \quad & ||x||_0 \leq r \end{aligned} \tag{2.3}$$

where r is a given constant.

2.3 Previous Work on Rank-constrained Optimization and rank-minimization

Several attempts on reducing the complexity of rank-constrained problems have been done in the past.

A result worth mentioning is the one published by Eckart and Young in 1936 [12]. In that paper, they showed an analytic solution to the problem of approximating a given matrix by one of lower rank. The problem in question is the following

$$\begin{aligned} \min_{X \in \mathbb{R}^{m \times n}} \quad & ||X - Z||_F^2 \\ s.t. \quad & \text{rank}(X) \leq r \end{aligned} \tag{2.4}$$

for a given $Z \in \mathbb{R}^{m \times n}$, where $|| \cdot ||_F$ is the frobenius norm, which is defined as

$$||A||_F = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2} = \sqrt{\sum_{i=1}^{min\{m,n\}} \sigma_i(A)^2} \tag{2.5}$$

where $\sigma_i(A)$ denotes de ith singular value of A.

Problem (2.4) has analytic solution (shown in [12]), which corresponds to be a truncated version of the Singular Value Decomposition (SVD) of matrix Z. If Z has the SVD $Z = U[diag(\sigma(Z))]V'$, then the solution corresponds to only consider the r singular values with highest absolute value, *i.e.* $\hat{X} = U_{1:r}[diag(\sigma_{1:r}(Z))]V'_{1:r}$. The solution is unique, unless $\sigma_r(Z) = \sigma_{r+1}(Z)$ (here σ denotes singular values order in nonincreasingly order).

Note that the minimum value of (2.4) corresponds to the sum of the squares of the singular values that were not taken into account when constructing the approximating matrix. This fact can be also used to solve the following rank minimization problem

$$\begin{aligned} \min_{X \in \mathbb{R}^{m \times n}} \quad & \text{rank}(X) \\ s.t. \quad & ||X - Z||_F^2 \leq \epsilon \end{aligned} \tag{2.6}$$

where ϵ is a certain bound for the approximation error.

The result presented in [12], although useful for the case with no additional constraints on the solution, represents a motivation for other solution methods.

As shown, rank-constrained optimization is closely related to rank minimization problems. Rank minimization problems with a convex set as constraint are usually solved by utilizing the trace heuristic or the nuclear norm and Log-det heuristics. An important result in this matter is given in [7] where it is proven that the nuclear norm (sum of

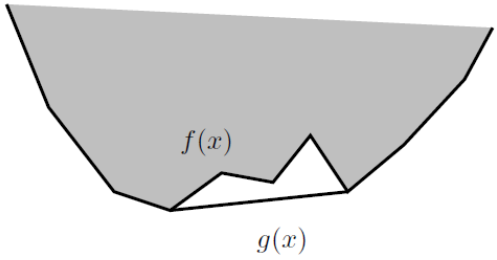

Fig. 2.1: Convex envelope given by $g(x)$ of a function $f(x)$.

the singular values, denoted by $||\cdot||_*$) corresponds to the convex envelope of the rank function, and therefore it is a convex relaxation of the rank constraint.

The convex envelope of a certain set X corresponds to the smallest convex set that contains the set X (an illustration of this matter can be found in Fig 2.1, which is available in [7]). The biggest advantage of doing a convex-relaxation is the decrease in computational load of the problem, which often allows the use of algorithms that found a solution in polynomial time for its solution. On the down side, the relaxed problem doesn't have (in general) the same optimal solution as the original problem. Nevertheless, the solution obtained from relaxed formulations can be used as a starting points or as additional information for more complex algorithms. As an example, the use of the nuclear norm heuristic will give a solution that corresponds to a lower bound for the true solution.

On the other hand, a solution to the relaxation of a rank minimization problem over symmetric and positive semidefinite matrices was previously given where by using the trace heuristic, which although efficient, was constrained to a particular type of matrices. The use of the nuclear norm generalizes the trace heuristic, which allows for the attainment of the solution of a much wider class of problems. The general problem is as follows (for more details see [7]):

$$\begin{aligned} \min_{X\in\mathbb{R}^{m\times n}, t\in\mathbb{R}} \quad & t \\ s.t. \quad & \text{rank}(X) \leq t \\ & X \in \mathcal{C} \end{aligned} \tag{2.7}$$

By introducing the nuclear norm, a convex relaxation of the problem is achieved:

$$\begin{aligned} \min_{X\in\mathbb{R}^{m\times n}} \quad & t \\ s.t. \quad & ||X||_* \leq t \\ & X \in \mathcal{C} \end{aligned} \tag{2.8}$$

Following [7, Lemma 2]), (2.8) can be rewritten as follows

$$\begin{aligned} \min_{X\in\mathbb{R}^{m\times n}, Y\in\mathbb{R}^{m\times m} and Z\in\mathbb{R}^{n\times n}} \quad & \operatorname{trace}(Y)+\operatorname{trace}(Z) \\ s.t. \quad & \begin{bmatrix} Y & X \\ X' & Z \end{bmatrix} \geq 0 \\ & X \in \mathcal{C} \end{aligned} \tag{2.9}$$

where $\mathcal{C}$ denotes a convex set, matrices $Y = Y'$ and $Z = Z'$ are slack variables. If the convex set $\mathcal{C}$ can be written as an LMI, problem (2.9) can be solved utilizing standard semi-definite programming (SDP) tools.

Problems regarding LMI are of great interest in engineering and science. Solution to this type of problems have been studied using several approaches, including alternating projection methods [13, 14], Newton-like algorithms [5] and iterative rank minimization [6] among others.

Another generalization of the trace heuristic for solving rank constrained optimization problems can be found in [9]. In this latter, an iteration of convex problems sequence is defined in order to find low-rank matrices solutions.

The problem of finding low-rank matrices on a convex set can be defined as a feasibility problem as follows

$$\begin{aligned} \operatorname*{find}_{G\in\mathbb{S}^n} \quad & G \\ s.t. \quad & G \in \mathcal{C} \\ & G \geq 0 \\ & \operatorname{rank}(G) \leq r \end{aligned}$$

where n is a given positive constant, and $\mathcal{C} \subseteq \mathbb{R}^n$

As shown in [9] this problem can be equivalently expressed as an iterative convex procedure, which will have the same global optimum:

$$\begin{aligned} \min_{G\in\mathbb{S}^n} \quad & \langle G, W\rangle \\ s.t. \quad & G \in \mathcal{C} \\ & G \geq 0 \end{aligned} \tag{2.10}$$

where the slack matrix W corresponds to a *direction vector*, which is the solution of the following semidefinte program (SDP) for $0 \leq r \leq n-1$

$$\sum_{i=r+1}^{n} \lambda(G^\star)_i = \min_{W \in \mathbb{S}^n} \ \langle G^\star, W \rangle \tag{2.11}$$
$$s.t. \ \ 0 \leq W \leq I$$
$$\text{trace}(W) = N - r$$

where $G^\star$ is the solution matrix of problem (2.10) for a given direction matrix W. The idea of this algorithm is to iteratively solve problems (2.10) and (2.11) until global convergence is achieved, which is reached when the sum of the $n - r$ non-increasingly ordered eigenvalues λ of $G^\star$ are zero ($\sum_{i=r+1}^{n} \lambda(G^\star)_i = 0$).

Although a condition for optimality is defined, the attainment of the global optimum is not assured, while local convergence is not guaranteed. This is mainly due to the additional local minima that the iterative algorithm adds to the resulting problem.

2.4 Previous Work on Cardinality-constrained Optimization

In cardinality constrained (or penalized) optimization problems, the goal is to promote sparsity on the solution vector, which translates into setting a certain number of entries from the solution vector to zero. A direct application of this approach is model fitting, where by reducing the size of the model (number of variables used in the model), better and more accurate models can be obtained (avoiding overfitting).

As it was noted before, formulating the problem utilizing the ℓ_0-(pseudo)-norm introduces a very high computational load because of its combinatorial behavior. In order to avoid this and reduce runtimes, several authors have replaced the ℓ_0-(pseudo)-norm with other regularization terms of the form shown below in (2.12). The idea of the regularization term is to approximate in some sense the ℓ_0-(pseudo)-norm, simplifying the problem and reducing its complexity.

$$\mathcal{P}_{rp}: \ \min_{\theta \in \mathbb{R}^p} \ f(\theta) + \beta R(\theta) \tag{2.12}$$
$$s.t. \ \ \theta \in \Omega$$

where $\Omega \subseteq \mathbb{R}^p$

In the general expression shown in (2.12), $f(\theta)$ corresponds to the cost function, β is a trade-off term between the cost function and the regularization term $R(\theta)$.

Setting the value of the regularization parameter is a key part of this type of methods. The value of the trade-off directly affects the performance of these methods when adjusting the number of variables used in the final solution. In the cardinality constrained version of these approximations (2.13), this issue is present when choosing the constant bounding the constraint, t:

$$\mathcal{P}_{rc}: \min_{\theta \in \mathbb{R}^p} \; f(\theta) \qquad (2.13)$$
$$s.t. \;\; R(\theta) \leq t$$
$$\theta \in \Omega$$

Some common regularization terms include the use of the ℓ_1-norm (LASSO), approximation by the ℓ_q-(pseudo)-norm (with $0 < q < 1$) or the use of exponential terms [4].

2.4.1 The LASSO

One of the most used techniques for avoiding the use of ℓ_0-(pseudo)-norm on the formulation of these problems is the LASSO (Least Absolute Shrinkage and Selection Operator) method. The LASSO consists of a convex relaxation of the cardinality constrained problem by utilizing the ℓ_1-norm as the regularization term ($R(\theta) := ||\theta||_1$). The use of the ℓ_1-norm is based on the fact that it is considered the best convex approximation of the ℓ_0-(pseudo)-norm [2, 3, 15]. Although the regularization term introduced in the LASSO is not strictly convex nor differentiable, several techniques from convex analysis can be used to solve the resulting optimization problem. Such techniques include proximal methods [16], subgradient methods [17], semi-smooth Newton [18], among others.

The LASSO technique is used in many research areas when tackling sparsity inducing problems. As an example, optimal control schemes promoting sparsity by using the ℓ_1-norm can be found in [19–22].

Although by using the ℓ_1-norm a convexified version of the original problem is obtained, and thus reducing computation times, in general, the solution will differ from the solution obtained when solving the original cardinality-constrained problem. This is due to the ℓ_1-norm being just an approximation of the ℓ_0-(pseudo)-norm. Another difficulty of the LASSO can be found when using it in a cardinality constrained problems, where the bounding constant of the constraint must be set although it does not represent cardinality. Nevertheless, the benefits of using LASSO are widely known and have motivated the development of variants which aim to solve some limitation and to improve results of the technique. These variants are the elastic net [23], group lasso [24–26] and fused-lasso [27].

2.4.2 Approximation using the ℓ_q-(pseudo)-norm

Another popular regularization term used in reducing the complexity of the cardinality constrained problem is the use of ℓ_q-(pseudo)-norm, which is defined as $R(\theta) := ||\theta||_q$ with $0 < q < 1$ [28–32]. Note that this penalty term is non-convex, thus increases

the computational complexity of the resulting optimization problem in comparison to the LASSO. However, the advantage of this approach is the possibility of selecting the "*quality*" of approximation of the ℓ_0-(pseudo)-norm: by choosing $q \to 0$ a much better approximation of the original problem is achieved, on the cost of increasing the complexity of the problem. On the other hand, while selecting $q \to 1$, the quality of the approximation worsens while decreasing the computational load. Sparsity of the solution is improved and the bias related to the use ℓ_1-norm as a regularization term is reduced.

The ℓ_q-(pseudo)-norm with $q = \frac{1}{2}$ is considered the best choice between quality of the approximation of the ℓ_0-(pseudo)-norm and computational load [29, 33, 34].

Because the resulting problem is non-convex, the attainment of the global optimum of the optimization problem is a relatively hard task. A particular class of algorithms developed to provide a solution to problems having non-convex penalty terms are coordinate descend methods [28, 32]. The drawback of this algorithms is that convergence to the global optimum is not guaranteed and only a special kind of local convergence is proven.

2.4.3 Approximation using exponential terms

Another approximation of the ℓ_0-(pseudo)-norm used in optimization is the use of smooth non-convex exponential terms [4].

$$\|\boldsymbol{\theta}\|_0 \approx L - \sum_{l=1}^{L} e^{-a|\theta_l|}, \tag{2.14}$$

where L corresponds to the length of the solution vector, and a is a parameter that adjust the accuracy of the approximation (as the value of a increases, the quality of the approximation also increases).

From (2.14) it can be seen that the concave exponential term will penalize large entries from the solution vector θ. This penalization term however, will not exceed the value L which is the length of the vector, thus being of great interest for cardinality constraint problems because of the direct relation between the bounding constant and cardinality.

Solution of this type of problems can be achieved by linear programming iterative methods [4], which have fast solving times but local convergence.

2.5 Main Result on Rank-Constrained Optimization

In this section we include the following results [8] for clarity and completeness of the thesis work.

2.5.1 Rank-Constrained Optimization

The following theorem presents an equivalent representation of rank constraints.

Theorem 1. *Let $G \in \mathbb{R}^{m\times n}$ then the following expressions are equivalent:*

1. *rank$(G) \leq r$*
2. *$\exists W_R \in \Phi_{n,r}$, such that $GW_R = 0_{m\times n}$*
3. *$\exists W_L \in \Phi_{m,r}$, such that $W_L G = 0_{m\times n}$*

   where

$$\Phi_{n,r} = \{W \in \mathbb{S}^n, 0 \leq W \leq I, trace(W) = n - r\} \tag{2.15}$$

Proof: For a detailed proof see [8]. □

Consider the following general rank-constrained optimization problem over a convex set

$$\begin{aligned} \mathcal{P}_{rco}: \min_{\boldsymbol{x}\in\mathbb{R}^p} \quad & f(\boldsymbol{x}) \\ s.t. \quad & \boldsymbol{x} \in \Omega \\ & \text{rank}(G(\boldsymbol{x})) \leq r \end{aligned} \tag{2.16}$$

Now consider the following optimization problem that incorporates bilinear constraints

$$\begin{aligned} \mathcal{P}_{rco,equiv}: \min_{\boldsymbol{x}\in\mathbb{R}^p, W\in\mathbb{S}^n} \quad & f(\boldsymbol{x}) \\ s.t. \quad & \boldsymbol{x} \in \Omega \\ & G(\boldsymbol{x})W = 0_{m\times n} \\ & 0 \leq W \leq I_n \\ & \text{trace}(W) = n - r \end{aligned} \tag{2.17}$$

where $\Omega \subset \mathbb{R}^p$ is a constraining set, the cost function $f : \mathbb{R}^p \to \mathbb{R}$ is the objective function and $G : \mathbb{R}^p \to \mathbb{R}^{m\times n}$.

Considering Theorem 1, the two optimization problems $\mathcal{P}_{rco}$ and $\mathcal{P}_{rco,equiv}$ stated before are equivalent in the sense that both have the same global optimum.

Theorem 2. *A vector $\boldsymbol{x}^\star \in \mathbb{R}^p$ is a global solution of $\mathcal{P}_{rco}$ if and only if there exists a $W^\star$ such that the pair $(\boldsymbol{x}^\star, W^\star)$ is a global solution of $\mathcal{P}_{rco,equiv}$.*

Proof: For a detailed proof, see [8]. □

A difference with other approaches [1,9] lies in the possibility of this equivalent representation to be used in a wider class of rank constrained and rank penalized optimization problems (it is not limited to positive semidefinite matrices).

2.5.2 Cardinality-Constrained Optimization

The problem $\mathcal{P}_{rco}$ can also be used to solve ℓ_0-(pseudo)-norm constrained problems (i.e cardinality-constrained optimization problems) instead of those involving the rank of a matrix. The cardinality problem can be stated from $\mathcal{P}_{rco}$ by considering $G(x) = diag(\boldsymbol{x})$. By this definition, the cardinality of x is expressed by means of the rank of a matrix, $||\boldsymbol{x}||_0 = \text{rank}(G(\boldsymbol{x}))$.

This particular case arises the following corollary:

Corollary 1. *[8] Let $\boldsymbol{x} \in \mathbb{R}^n$, then the cardinality of x is $||\boldsymbol{x}||_0 \leq r$ if and only if there exist a $\boldsymbol{w} \in \{\boldsymbol{w} \in \mathbb{R}^n | 0 \leq w_i \leq 1, i = 1, ..., n; \sum_{i=1}^n w_i = n - r\}$, such that $x_i w_i = 0$, $\forall i = 1, .., n$.*

Next we consider the following ℓ_0-(pseudo)-norm constrained optimization problem

$$\begin{aligned} \mathcal{P}_{\ell_0 co}: \min_{\boldsymbol{x} \in \mathbb{R}^n} \quad & f(\boldsymbol{x}) \\ s.t. \quad & \boldsymbol{x} \in \Omega \\ & ||x||_0 \leq r \end{aligned} \tag{2.18}$$

From Corollary 1, problem $\mathcal{P}_{\ell_0 co}$ can be reformulated as an optimization problem subject to bilinear constraints as follows

$$\begin{aligned} \mathcal{P}_{\ell_0 co, equiv}: \min_{\boldsymbol{x} \in \mathbb{R}^n, \boldsymbol{w} \in \mathbb{R}^n} \quad & f(\boldsymbol{x}) \\ s.t. \quad & w_i x_i = 0, \ i = 1, ..., n \\ & 0 \leq w_i \leq 1 \ i = 1, ..., n \\ & \sum_{i=1}^n w_i = n - r \\ & \boldsymbol{x} \in \Omega \end{aligned} \tag{2.19}$$

2.5.3 Cost function involving ℓ_0-(pseudo)-norm

Theorem 1 can also be used in problems that incorporate the ℓ_0-(pseudo)-norm in its cost functional. This is done by using the epigraph notation [35] as follows

$$\begin{aligned} \mathcal{P}_{rm}: \min_{\boldsymbol{\theta} \in \mathbb{R}^p} \quad & r \\ s.t. \quad & \boldsymbol{\theta} \in \Omega \\ & \text{rank}(G(\boldsymbol{\theta})) \leq r \end{aligned} \tag{2.20}$$

where $G(\boldsymbol{x}) = diag(\boldsymbol{x})$. Problem $\mathcal{P}_{rm}$ is equivalent to $\mathcal{P}_{rm,eq}$

$$\begin{aligned} \mathcal{P}_{rm,eq}: \quad \min_{\boldsymbol{\theta}\in\mathbb{R}^p, W\in\mathbb{S}^n} \quad & n - \text{trace}(W) \\ s.t. \quad & \boldsymbol{\theta} \in \Omega \\ & G(\boldsymbol{\theta})W = 0_{m\times n} \\ & 0 \leq W \leq I \end{aligned} \tag{2.21}$$

Chapter 3

SPARSE LOGISTIC REGRESSION UTILIZING CARDINALITY CONSTRAINTS AND INFORMATION CRITERIA

In this chapter we address the problem of estimating a sparse parameter vector that defines a logistic regression. The problem is then solved using two approaches: i) inequality constrained Maximum Likelihood estimation and ii) penalized Maximum Likelihood which is closely related to Information Criteria such as AIC. For the promotion of sparsity, we utilize a nonlinear constraint based on the ℓ_0-(pseudo)-norm of the parameter vector. The corresponding optimization problem is solved using an equivalent representation of the problem that is simpler to solve. We illustrate the benefits of our proposal with an example that is inspired by a gene selection problem in DNA microarrays.

3.1 Introduction

Sparse estimation problems are of great interest in the scientific community. Several applications that consider sparse estimation are approached by incorporating a regularization/penalty term as a mean of inducing sparsity, such as the ℓ_1-norm in the classical Lasso [3] and an $\ell_1 - \ell_2$ norm combination in the Elastic Net [23]. However, it is usually difficult to give those penalties a physical meaning. On the other hand, the ℓ_0-(pseudo)-norm can also be used to promote (induce) sparsity, see e.g [8, 36, 37], and its interpretation is based on understanding that the ℓ_0-(pseudo)-norm is a cardinality function that outputs the number of *active elements*, i.e. the non-zero ones. Additionally, sparse estimation problems can be defined as an inequality constrained optimization problem, see e.g. [38]. A traditional approach to solve ℓ_0-constrained optimization problems is to

utilize the so-called Greedy Algorithms [39], which provide a suboptimal solution to the optimization problem. The solution is computed in several iterations, where, on each iteration, the element of the parameter vector that has the most impact on the objective function is included in the solution. The iterations are repeated until the ℓ_0-(pseudo)-norm constraint is satisfied.

Classification algorithms such as logistic regression and support vector machine are tools to perform fault detection (see e.g. [40, 41]). Logistic regression is applied when the output is discrete valued and can only take a finite number of values, such as $\{1, 0\}$ or "on–off" [42]. Logistic regressions are based on the *logistic function*, which is defined by a parameter vector that, in turn, defines the *boundaries* of the regions in the space associated with the discrete output values [42, 43].

In this chapter we focus on the estimation of sparse parameter vector using an inequality constrained Maximum Likelihood (ML) approach based on the ℓ_0-(pseudo)-norm in a logistic regression, in contrast to the more common practice of inducing sparsity by introducing a penalty term. In [28], a sparse logistic regression problem is solved by the use of $\ell_{1/2}$-norm penalization. The $\ell_{1/2}$-norm can be thought as a balance between the ℓ_1-norm and the ℓ_q-(pseudo)-norm with q close to 0. In that sense, the $\ell_{1/2}$-norm solution is better than the ℓ_1-norm solution in terms of sparseness, while it is also better than the ℓ_q-(pseudo)norm (q close to 0) in terms of convergence [29]. A similar approach was considered in [44], where a Group Lasso penalized logistic regression model was considered. The Group Lasso [24] has the advantage of doing variable selection on grouped variables in linear regression models. This approach is capable to induce sparsity in the solution of high dimensional problems. In order to obtain sparse estimates, it is common to choose a *bound* or *threshold* to turn into zero the elements of vector $\boldsymbol{\theta}$ whose estimates exhibit an absolute value that is less than the *bound*. On the other hand, for $0 < q \leq 1$, it is common to express the ℓ_q-(pseudo)-norm of the parameter θ as $\lambda||\boldsymbol{\theta}||_q^q$. However, the choice of the hyperparameter λ and q are of great importance, since they define the maximum number of zero entries of $\boldsymbol{\theta}$ that can be identified [45].

Here, we consider the recently published approach for solving a general class of problems [8], where ℓ_0 inequality constrained optimization problems lie in. The approach in [8] accounts for a reinterpretation of the original problem, obtaining an equivalent optimization that is simpler to solve than the original problem. The main goal of our proposal is the attainment of low complexity models, which can be achieved utilizing the ℓ_0-(pseudo)-norm. On the other hand, a popular method model slection is Akaike's information Criterion (AIC) [46], which we also consider for comparison purposes in our numerical examples.

3.2 Maximum Likelihood Estimation of Logistic Regression Models

3.2.1 Logistic Regression

Logistic regression is a classification algorithm that models the output of a categorical dependent variable, that is, the output is discrete valued (taking, in general, a limited and fixed number of possible values) and can express the presence or absence of a given attribute or characteristic (known as class). The model considered in logistic regressions is based on the *sigmoid* or *logistic function*

$$c(z) = \frac{1}{1 + e^{-z}}, \tag{3.1}$$

where $z \in \mathbb{R}$ is the independent variable used for predictions in the logistic regression model [42]. Notice that $\forall z \in \mathbb{R}$, $0 < c(z) < 1$, which allows for the *logistic function* to be interpreted as an estimation of the probability that a given attribute is present or not. In general, the variable z is defined in terms of a regression and a parameter vector $\boldsymbol{\theta}$. In turn, *training data* (measurements, surveys, etc.) is used to estimate $\boldsymbol{\theta}$. In particular, for a linear classifier, the variable z is modelled as

$$z(\boldsymbol{x}) = \boldsymbol{a}'\boldsymbol{x} + b, \tag{3.2}$$

where $\boldsymbol{a} \in \mathbb{R}^M$, $\boldsymbol{x} \in \mathbb{R}^M$, $b \in \mathbb{R}$. Thus, $\boldsymbol{\theta} = [\boldsymbol{a}'\, b]'$. The definition of $z(\boldsymbol{x})$ in (3.2) maps an M dimensional space to the probability of a given class or feature [47].

The resulting model that describes the probability of a given set of attributes belonging to a certain class is given by

$$p(\boldsymbol{Y} = y_i | \boldsymbol{X} = \boldsymbol{x}_i) = c(\boldsymbol{x}_i; \boldsymbol{\theta}) = \frac{1}{1 + e^{-(\boldsymbol{x}_i'\boldsymbol{a} + b)}}, \tag{3.3}$$

where $\boldsymbol{X}$ is the set of characteristics or attributes that are related to a given class through the probability of $\boldsymbol{Y}$ for a given $\boldsymbol{x}_i$. For example, if the model considers an "on-off" (or "present-absent") characteristic, then $\boldsymbol{Y} = \{y_1, y_2\}$ represents the set of classes and $\boldsymbol{X}$ is the set of grouped attributes. Then $p(\boldsymbol{Y} = y_1 | \boldsymbol{X} = \boldsymbol{x}_i) = 1 - p(\boldsymbol{Y} = y_2 | \boldsymbol{X} = \boldsymbol{x}_i)$.

3.2.2 Logistic Regression ML Estimation

For a logistic regression, the *likelihood function* is defined in terms of the individual probabilities of each possible value of the output variable. Thus, for a collection of N samples, we have

$$p(\boldsymbol{y}|\boldsymbol{\theta}) = \prod_{i=1}^{N} p(\boldsymbol{Y} = y_i | \boldsymbol{X} = \boldsymbol{x}_i), \tag{3.4}$$

where $\boldsymbol{y} = [y_1, ..., y_N]'$. Hence, the *log-likelihood* function is defined as

$$\ell(\boldsymbol{\theta}) = \sum_{i=1}^{N} \log p(\boldsymbol{Y} = y_i | \boldsymbol{X} = \boldsymbol{x}_i). \tag{3.5}$$

For a binomial logistic regression, with data set $\{z_i, y_i\}_1^N$, where $y_i \in \{0, 1\}$ and $z_i = z(\boldsymbol{x}_i)$, the *likelihood function* is defined as [47]

$$p(\boldsymbol{y}|\boldsymbol{\theta}) = \prod_{i=1}^{N} \kappa_i^{z_i} (1 - \kappa_i)^{1-z_i}, \tag{3.6}$$

with

$$\kappa_i = p(\boldsymbol{Y} = y_i | \boldsymbol{X} = \boldsymbol{x}_i) = \frac{1}{1 + e^{-(\boldsymbol{x}_i' \boldsymbol{a} + b)}}.$$

In case of indexing the classes as $y_i \in \{-1, 1\}$, after a mathematical arrangement, the probability of the pair $(\boldsymbol{x}_i, y_i)$ can be written in the following general form

$$P(\boldsymbol{Y} = y_i | \boldsymbol{X} = \boldsymbol{x}_i) = \frac{1}{1 + e^{-y_i(\boldsymbol{x}' \boldsymbol{a} + b)}} \tag{3.7}$$

Using (3.5), the negative log-likelihood can be written as follows

$$\ell(\boldsymbol{a}, b | \mathcal{S}) = \sum_{i=1}^{N} lse(0, -y_i(\boldsymbol{x}_i' \boldsymbol{a} + b) \tag{3.8}$$

where $lse(\boldsymbol{w}) = log(\sum_{i=1}^{n} e^{w_i})$ is the LogSumExp function.

3.3 Sparse Parameter Estimation in Logistic Regression Systems

3.3.1 Constrained ML Estimation in Logistic Regressions

A sparse estimate for the parameter vector $\boldsymbol{\theta}$ can be obtained by solving the following cardinality-constrained optimization problem:

$$\begin{aligned} \mathcal{P}: \min_{\boldsymbol{\theta}} \ & \ell(\boldsymbol{y}|\boldsymbol{\theta}) \\ s.t. \ & ||\boldsymbol{\theta}||_0 \leq r \end{aligned} \tag{3.9}$$

where the cardinality of $\boldsymbol{\theta}$ is constrained by $||\boldsymbol{\theta}||_0 \leq r$, limiting the complexity of the model. In general, the optimization problem (3.9) involves a high computational cost. On the other hand, the wrong choice of r might result in an increase of the bias in the estimator (3.9). To avoid this problem, we consider the alternative and equivalent (has

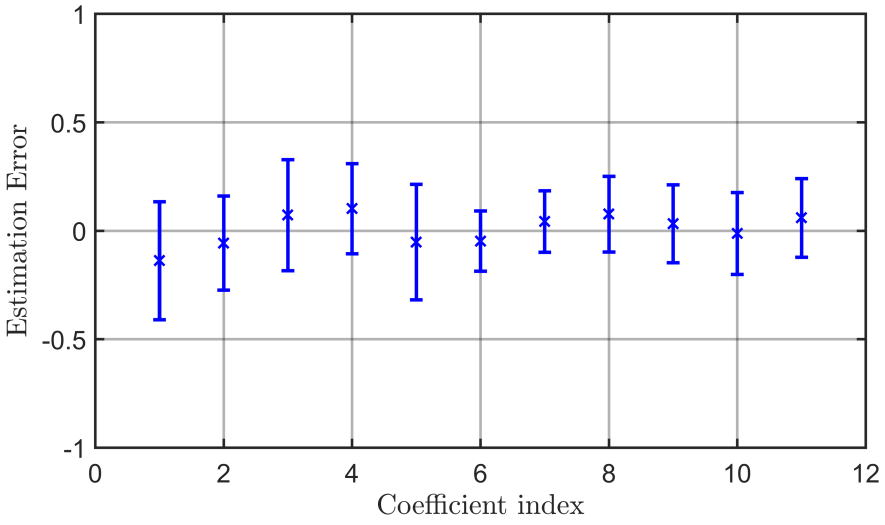

Fig. 3.1: Parameter estimation via Maximum Likelihood.

the same global optimum) optimization problem (see Apendix B, Corollary 1)

$$\begin{aligned} \mathcal{P}_{\text{eq}}: \min_{\boldsymbol{\theta},\boldsymbol{w}} \;& \ell(\boldsymbol{y}|\boldsymbol{\theta}) \\ s.t. \;& \theta_i w_i = 0 \; i = 1, ..., M+1 \\ & 0 \leq w_i \leq 1, \; i = 1, ..., M+1 \\ & \sum_{i=1}^{M+1} w_i = M + 1 - r \end{aligned} \tag{3.10}$$

where $\boldsymbol{w} = [w_1, ..., w_{M+1}]'$ is a *latent variable* that allows for this representation of (3.9) based on the incorporation of a linear and a bilinear constraint. Here, $\ell(\boldsymbol{y}|\boldsymbol{\theta})$ denotes a negative log-likelihood function.

3.3.2 AIC Applied to Logistic Regressions

Another way to induce sparsity in a model description is by considering the complexity and the goodness of a model (i.e. the number of elements) in the cost function, as in Akaike's Information Criterion (AIC) and the Bayesian Information Criterion (BIC) [46, 48]. In particular, AIC considers the number of parameters and the likelihood of these parameters fitting the model, and gives a measure of balance between them. The smaller the AIC number, the better the choice of parameters. For a system model defined by a parameter vector $\boldsymbol{\theta}$, AIC is given by

$$AIC(\boldsymbol{\theta}) = 2r - 2\ell(\boldsymbol{y}|\boldsymbol{\theta}) \tag{3.11}$$

where r is the number of parameters used and $\ell(\boldsymbol{y}|\boldsymbol{\theta})$ is the log-likelihood function of the model. For different model structures, the selection of the best model is carried out simultaneously by the following minimization problem [49]

$$\mathcal{P}_{\text{AIC}}: \ \arg\min_{\boldsymbol{\theta}} \ \ 2(||(\boldsymbol{\theta})||_0) + 2\ell(\boldsymbol{y}|\boldsymbol{\theta}) \tag{3.12}$$

where the cardinality of $\boldsymbol{\theta}$ is expressed in terms of its ℓ_0-(pseudo) norm. *Remark:* Note that problems (3.9) and (3.12) are similar. However in (3.9) cardinality of the parameter vector is imposed as a hard constraint, whereas in (3.12) the penalty term promotes sparsity. In the same way, traditional sparsity problems can be rewritten as in (3.9).

In a similar way to $\mathcal{P}_{\text{eq}}$ in (3.10), $\mathcal{P}_{\text{AIC}}$ can be reformulated as (see Apendix C):

$$\begin{aligned} \mathcal{P}_{\text{AIC,eq}}: \ \min_{\boldsymbol{\theta},\boldsymbol{w}\in\mathbb{R}^{M+1}} \ \ & 2(M+1-\sum_{i=1}^{M+1} w_i) + 2\ell(\boldsymbol{\theta}|\mathcal{S}) \\ s.t. \ \ & \theta_i w_i = 0 \ i = 1,...,M+1 \\ & 0 \le w_i \le 1, \ i = 1,...,M+1, \end{aligned} \tag{3.13}$$

where the term $M+1-\sum_{i=1}^{M+1} w_i$ is an upper bound of $||\boldsymbol{\theta}||_0$.

The optimization problems $\mathcal{P}_{eq}$ and $\mathcal{P}_{AIC,eq}$ defined in (3.10) and (3.13) are equivalent to the problems $\mathcal{P}$ and $\mathcal{P}_{AIC}$ in (3.9) and (3.12) respectively in the sense that they have the same global optima. However, the equivalent problems in (3.10) and (3.13) have more local minima than the original ones [8, 50, 51].

The main benefit of these equivalent representations is that they might help to reduce the computational load corresponding to the combinatorial nature of the original problems and have recently been analyzed in [52]. In addition, these problems can be solved by a standard nonlinear programming software, such as BARON [53, 54].

Note that the problems (3.9) and (3.12) have non-convex functionals, which implies that traditional optimization toolboxes such as CPLEX and CVX cannot be directly applied.

3.4 Numerical Example

In this section we investigate the performance of the proposed strategy to the solution of the sparse logistic regression. Our motivation comes from a specific application that has attracted a lot of interest in recent years in the Bioinformatics community, namely the gene selection based on DNA microarrays for the diagnosis of cancer. The main goal is to identify the gene biomarkers so that different types of cancer are easily classified and predicted with high accuracy. The corresponding mathematical problem for gene selection involves an appropriate regularization step to adequately deal with the high

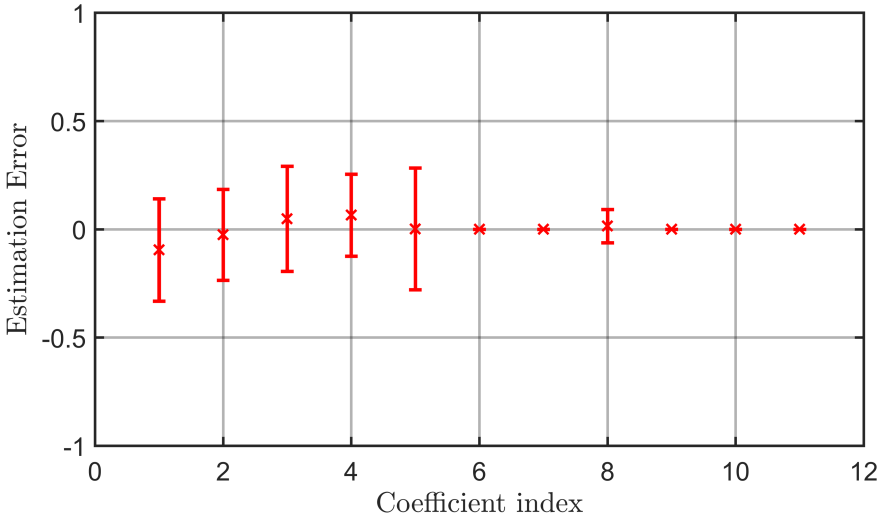

Fig. 3.2: Parameter estimation via ℓ_0-(pseudo)-norm constrained Maximum Likelihood.

dimensionality and ill-conditioning of the selection process. This is due to the fact that from a biological perspective, only a small subset of genes is strongly indicative of a targeted disease and most of the genes are irrelevant to the classification of different types of cancer [28]. Irrelevant selected genes may reduce the accuracy and distort the process of classification.

For our problem, let us suppose that we have N samples $\mathcal{S} = \{(\boldsymbol{x}_1, y_1), (\boldsymbol{x}_2, y_2), ..., (\boldsymbol{x}_N, y_N)\}$, where $\boldsymbol{x}_i = [x_{i1}, x_{i2}, ..., x_{iM}]'$ is the ith input pattern, denoting the M features/genes and $y_i \in \{-1, 1\}$ indicates the class of the corresponding sample with respect to a specific cancer type.

The classifier for any input/output pair $(\boldsymbol{x}_i, y_i)$ is selected to be the logistic regression model shown in (3.3). Therefore, for any gene set $\boldsymbol{x}_i$, the classifier $c(\boldsymbol{x}_i; \boldsymbol{\theta})$ predicts the corresponding class of cancer type y_i. Thus the probability of cancer type $\boldsymbol{Y} = 1$ for a given set of attributes can be written as:

$$P(\boldsymbol{Y} = 1 | \boldsymbol{X} = \boldsymbol{x}_i) = c(\boldsymbol{x}_i; \boldsymbol{a}, b). \tag{3.14}$$

which corresponds to the probability that a given gene pattern belongs to a class of a specific cancer type, given a certain linear combination of the predictors (genes).

The simulation setup is as follows:

i) a set of vectors $\boldsymbol{f}_{i0}, \boldsymbol{f}_{i1}, ..., \boldsymbol{f}_{ip}$, $i = 1, 2, ..., N$ is drawn according to the standard normal distribution $\mathcal{N}(\boldsymbol{0}, I)$,

ii) given a correlation coefficient $\rho \in [0, 1]$, the entries of the regressors are generated

according to the relationship $x_{ij} = f_{ij}\sqrt{1-\rho} + f_{i0}\sqrt{\rho}$, $j = 1, 2, ..., M$ [55],

iii) the simulated data is generated according to the logistic model $z(\boldsymbol{x}) = \boldsymbol{x}'\boldsymbol{a} + b$,

iv) additive noise is included in order to account for unobserved differences that may correspond to modelling error or measurement noise, obtaining

$$\tilde{c}(\boldsymbol{x}_i; \boldsymbol{\theta}) = \frac{1}{1 + e^{-(\boldsymbol{a}'\boldsymbol{x}_i + b + \nu_i)}}, \tag{3.15}$$

with $\nu_i \sim \mathcal{N}(0, \sigma^2)$, $\sigma = 0.2$.

For the simulations, $M = 10$, $N = 200$, $b = 0$, $\boldsymbol{a} = [1\,1\,-1\,-1\,1\,0\,0\,0\,0\,0]'$, and the cardinality of the solution (r) is set to be equal to the number of parameters used in (3.15), that is $||\boldsymbol{a}||_0 = 5$ (since $b = 0$). We also consider 30 Monte Carlo simulations. Note that $\boldsymbol{\theta} = (\boldsymbol{a}, b)$.

In order to compare the results, we calculate the average error and standard deviation of the estimates for each coefficient. The average error $\bar{\epsilon}_i$ for an estimated coefficient is taken as follows

$$\bar{\epsilon}_i = \frac{1}{N_{exp}} \sum_{j=1}^{N_{exp}} \theta_i^0 - \theta_i^{[j]} \tag{3.16}$$

where N_{exp} is the number of Monte Carlo simulations, θ_i^0 is the true parameter and $\theta_i^{[j]}$ is the i-th parameter estimate corresponding to the j-th experiment.

The standard deviation σ_i for each estimate is calculated as follows

$$\sigma_i = \sqrt{\frac{1}{N_{exp}} \sum_{j=1}^{N_{exp}} (\theta_i^{[j]} - \bar{\epsilon}_i)^2} \tag{3.17}$$

The results of the numerical examples are shown in Figs. 3.1, 3.2, and 3.3, where the average error and standard deviation of the estimates are plotted. In Fig. 3.1 we show the estimation error and standard deviation for ML estimation. The average number of non-zero elements used in the solutions is 11, thus the estimates are clearly not sparse. In Fig. 3.2 the estimation error and deviation of the estimates of problem $\mathcal{P}_{\text{eq}}$ are shown. Clearly, the estimation error is smaller than the one obtained by using ML, and the standard deviation is even much smaller for the coefficients that are zero in the original model. Finally, in Fig. 3.3 we show the corresponding error estimates and deviation of the estimates for AIC. The estimation error is bigger compared to the ones obtained from $\mathcal{P}_{\text{eq}}$, but smaller in comparison with the ML approach. The standard deviations of the last six coefficients are also smaller than the ML approach, showing that the most plausible model only included the first 5 nonzero elements of $\boldsymbol{a}$ and $b = 0$.

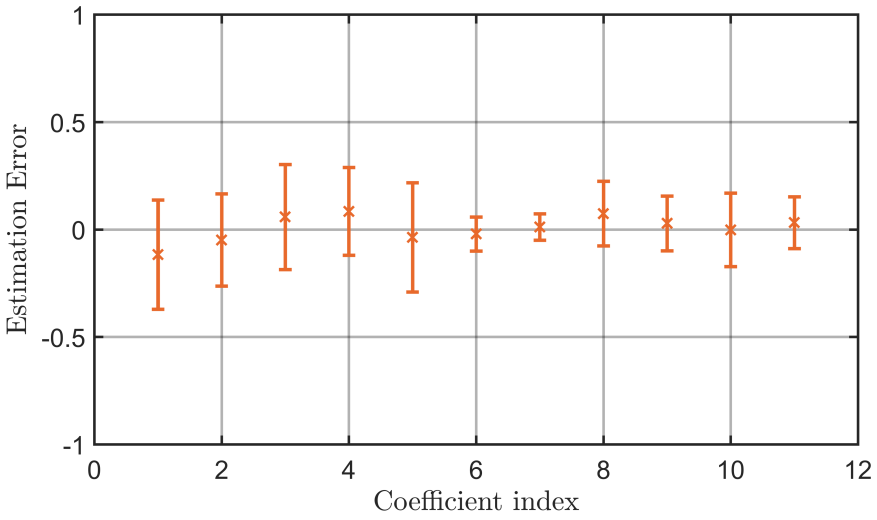

Fig. 3.3: Parameter estimation via Akaike's Information Criteria (AIC).

3.5 Conclusion

In this chapter we addressed a sparse logistic regression estimation problem. We have studied two approaches: i) penalized Maximum Likelihood via AIC and ii) constrained ML. The inequality constraint is obtained by utilizing the ℓ_0-(pseudo) norm of the parameter vector, which accounts for its cardinality. The corresponding optimization problem is then rewritten in an equivalent form, yielding a less computationally demanding problem. We compared our solutions against standard Maximum Likelihood estimation estimates. The simulations show that constrained ML performs better than AIC and ML when the number of features is known.

Chapter 4

AN OPTIMIZATION-BASED ALGORITHM FOR MODEL SELECTION USING AN APPROXIMATION OF AKAIKE'S INFORMATION CRITERIA

In this chapter, we consider an optimization approach for model selection using Akaike's Information Criterion (AIC) by incorporating the ℓ_0-(pseudo)-norm as a penalty function to the log-likelihood function. In order to reduce the numerical complexity of the optimization problem, we propose to approximate the ℓ_0-(pseudo)-norm by an exponential term. We focus on problems with hidden variables— i.e. where there are random variables that we cannot measure, and the Expectation-Maximization (EM) algorithm. We illustrate the benefits of our proposal via numerical simulations.

4.1 Introduction

The utilization of Information Criteria is perhaps one of the most common ways to solve model selection problems [56–58]. In particular, Akaike's Information Criterion (AIC) [59] and the (asymptotic) Bayesian Information Criterion (BIC) [48] are based upon the formulation and the evaluation of the likelihood function and on a penalty term that involves the number of parameters of the chosen system model. Thus, the incorporation of the penalty promotes the attainment of low complexity models. However, the implementation of AIC and BIC usually involves high computational loads, particularly when the candidate models are of high dimension. This is due to the fact that AIC and BIC are usually solved in a combinatorial fashion, evaluating the corre-

sponding cost function for every parameter combination. To overcome this difficulty, the ℓ_0-(pseudo)-norm can be introduced as a penalty for the likelihood function, accounting for the cardinality of the parameter vector. On the other hand, the ℓ_0-(pseudo)-norm also introduces a high computational load, which is reduced by replacing it by one of the less demanding, more "*friendly*" approximations. Those approximations include the ℓ_q-(pseudo)-norm [30–32] ($0 < q < 1$) and the ℓ_1-norm [3, 60], traditionally associated to sparsity identification problems.

In this chapter, we consider the utilization of another approximation of the ℓ_0-(pseudo)-norm [4]. This approximation has been considered for classification and cardinality constrained problems, see e.g. [61, 62] and the references therein. Our departure from the works mentioned here corresponds to the application of the ℓ_0-(pseudo)-norm and its approximation (as in [4]) to perform model selection in dynamical systems with *hidden variables*. Those *hidden variables* correspond to variables for which there are no available measurements. However, if their behaviour can be modelled, they can be estimated. In particular, a popular tool for system identification with *hidden variables* is the Expectation-Maximization (EM) algorithm [60, 63, 64]. In this context, the ℓ_0-(pseudo)-norm is applied to the corresponding E-step.

4.2 Akaike's Information Criterion

Without loss of generality, and for simplicity of the presentation, as a mechnism for performing parameter estimation we focus on AIC. However, our results directly apply to (asymptotic) BIC as well.

Akaike's information criterion establishes a relationship between Maximum Likelihood (ML) estimation and the Kullback-Leibler information [57]. This relationship can be expressed as:

$$\text{AIC} = -\log p(\boldsymbol{y}|\boldsymbol{\theta}) + L, \tag{4.1}$$

where $p(\boldsymbol{y}|\boldsymbol{\theta})$ is the likelihood function, $\boldsymbol{y}$ is the measurement data, $\boldsymbol{\theta}$ the parameters to be estimated, and L is the number of estimated parameters. In AIC, the goal is to obtain a model that minimizes the expression in (4.1), not only providing a good fit but also with the minimum of parameters possible, penalizing the dimension of the parameters.

4.2.1 AIC via the ℓ_0-(pseudo)-norm

The ℓ_0-(pseudo)-norm accounts for the cardinality of a vector. That is, if the dimension of a vector $\boldsymbol{v}$ is L, and there are no zero elements in $\boldsymbol{v}$, then $\|\boldsymbol{v}\|_0 = L$. In the same way, $\|[\boldsymbol{v}', 0, 0, 0, 0, 0]'\|_0 = L$. This behaviour of the ℓ_0-(pseudo)-norm allows for a

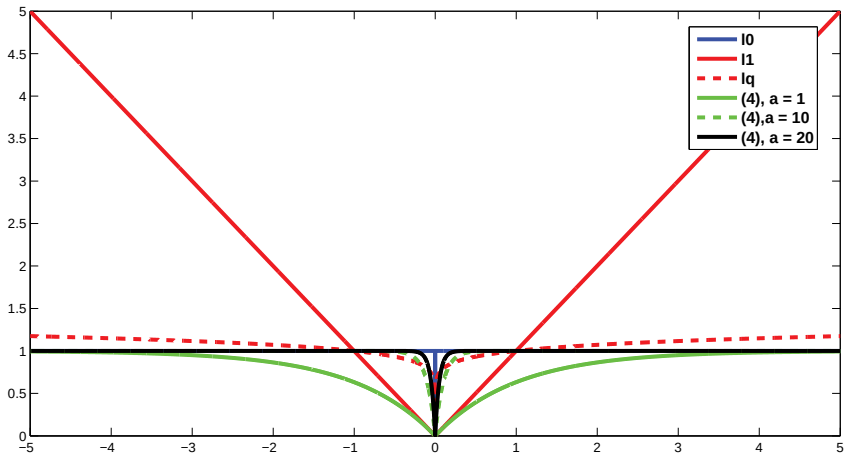

Fig. 4.1: Approximations of the ℓ_0-(pseudo)-norm.

reformulation of AIC as:

$$\text{AIC} = -\log p(\boldsymbol{y}|\boldsymbol{\theta}) + \|\boldsymbol{\theta}\|_0, \tag{4.2}$$

and the model is selected as

$$\hat{\boldsymbol{\theta}} = \arg\min_{\boldsymbol{\theta}} -\log p(\boldsymbol{y}|\boldsymbol{\theta}) + \|\boldsymbol{\theta}\|_0. \tag{4.3}$$

The problem in (4.3) can be solved in different ways see e.g. [8, 65] and the references therein. However, there is a high computational cost associated to those solutions. Nevertheless, the computational load can be reduced by approximating $\|\boldsymbol{\theta}\|_0$ by a suitable expression.

4.2.2 Approximations of the ℓ_0-(pseudo)-norm

Among others, the ℓ_q-(pseudo)-norm [30–32] ($0 < q < 1$), $\lambda\|\boldsymbol{\theta}\|_q^q$, and the ℓ_1-norm [3,60], $\lambda\|\boldsymbol{\theta}\|_1$, are common choices for approximating $\|\boldsymbol{\theta}\|_0$. Another approximation, that has been utilized in the area of optimzation (and is not very popular in the areas of system identification and signal processing) is [4]

$$\|\boldsymbol{\theta}\|_0 \approx L - \sum_{l=1}^{L} e^{-a|\theta_l|}, \tag{4.4}$$

where $a > 0$. The behaviour of the approximation in (4.4) is illustrated in Fig. 4.1.

Clearly, from Fig. 4.1, the approximation that behaves the closest to $\|\boldsymbol{\theta}\|_0$ is given by (4.4), with $a >> 0$. In fact, the greater a, the closer the approximation in (4.4) is to the actual ℓ_0-(pseudo)-norm.

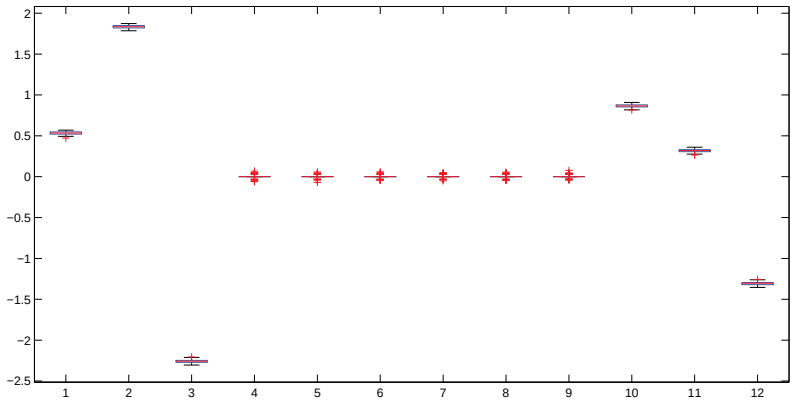

Fig. 4.2: Boxplot of model selection for a linear regression system using AIC.

4.3 Numerical Examples

4.3.1 Linear regression system

To illustrate the performance of the proposed method, we first consider the following linear system:

$$y_k = \boldsymbol{u}'_{k:k+L-1}\boldsymbol{\theta} + w_k, \tag{4.5}$$

where $\boldsymbol{\theta} = [0.54, 1.83, -2.26, 0, 0, 0, 0, 0, 0, 0.86, 0.32, -1.31]$ is the unknown parameter vector, $\boldsymbol{u}_{k:k+L-1} = [u_k, u_{k+1}, ..., u_{k+L-1}]'$ is a known input signal, and $w_k \sim \mathcal{N}(0, \sigma_w^2)$ is the measurement noise. The estimation and model selection procedure is carried out via AIC (4.1), evaluating all the possible combinations of parameters for $\boldsymbol{\theta}$, and via the ℓ_0-(pseudo)-norm using (4.4), with $a = 100$. For this example, we have considered $N = 200$ measurement points and 150 Monte Carlo simulations. The initialization of the algorithm is given by the least squares estimate of $\boldsymbol{\theta}$. The results are shown in Fig. 4.2. The behaviour that is obtained by applying the approximation $\|\boldsymbol{\theta}\|_0 \approx L - \sum_{l=1}^{L} e^{-a|\theta_l|}$

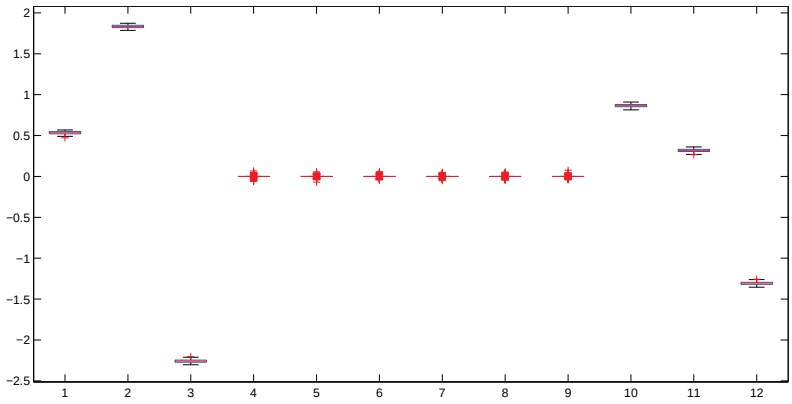

Fig. 4.3: Boxplot of model selection for a linear regression system using the proposed ℓ_0 approximation.

is very similar to the one obtained by applying AIC directly. In fact, most of the zeros (from every Monte Carlo simulation) were found by both AIC and our proposal, with the approximation showing a nearly identical performance as AIC, as shown in Fig. 4.2.

4.3.2 Nonlinear system with hidden variables

Here we have considered the following nonlinear system:

$$\begin{aligned} z_{k+1} &= 0.9z_k + \xi_k, \\ y_k &= \sin\left(\frac{\omega(k-1)}{N} + z_k\right) \boldsymbol{u}'_{k:k+p-1}\boldsymbol{\theta} + \eta_k, \end{aligned} \tag{4.6}$$

where $\boldsymbol{u}_{k:k+L-1} = [u_k, u_{k+1}, ..., u_{k+L-1}]'$ is a known input, $\theta = [-0.77, -1.55, 0, 0, 0, 0, 0.46]'$ is the unknown parameter vector, $N = 200$ is the number of measurement points, $\omega = 5$ is a known constant parameter, $\xi_k \sim \mathcal{N}(0, 0.1)$, and $\eta_k \sim \mathcal{N}(0, 0.1)$. For the estimation of $\boldsymbol{\theta}$, the log-likelihood function in (4.3) is replaced by the corresponding auxiliary function $\mathcal{Q}(\boldsymbol{\theta}, \hat{\boldsymbol{\theta}}^{(i)})$ from the E-step in the EM algorithm. That is, the problem to solve (iteratively) is given by:

E-step: Compute

$$\mathcal{Q}\left(\boldsymbol{\theta}, \hat{\boldsymbol{\theta}}^{(i)}\right) = E\left[\log p(\boldsymbol{z}, \boldsymbol{y}|\boldsymbol{\theta})|\boldsymbol{y}, \hat{\boldsymbol{\theta}}^{(i)}\right].$$

M-step: Solve

$$\hat{\boldsymbol{\theta}}^{(i+1)} = \arg\max_{\boldsymbol{\theta}} \mathcal{Q}\left(\boldsymbol{\theta}, \hat{\boldsymbol{\theta}}^{(i)}\right).$$

For more details on the EM algorithm see e.g. [30, 60, 63, 64, 66] and the references therein. Notice that the attainment of the auxiliary function $\mathcal{Q}\left(\boldsymbol{\theta}, \hat{\boldsymbol{\theta}}^{(i)}\right)$ depends upon the attainment of $p(\boldsymbol{z}|\boldsymbol{y})$. In this example, we have considered the utilization of Particle Filtering and Particle Smoothing [67] for that purpose. On the other hand, for the attainment of the estimates utilizing AIC, the auxiliary cost function $\mathcal{Q}\left(\boldsymbol{\theta}, \hat{\boldsymbol{\theta}}^{(i)}\right)$ must be solved for every possible system model structure, i.e. every parameter combination. As explained before, to overcome this difficulty, we utilize the approximation $\|\boldsymbol{\theta}\|_0$ in (4.4), which implies that the auxiliary cost function $\mathcal{Q}\left(\boldsymbol{\theta}, \hat{\boldsymbol{\theta}}^{(i)}\right)$ in E-step is modified as follows:

$$\mathcal{Q}\left(\boldsymbol{\theta}, \hat{\boldsymbol{\theta}}^{(i)}\right) = E\left[\log p(\boldsymbol{z}, \boldsymbol{y}|\boldsymbol{\theta})|\boldsymbol{y}, \hat{\boldsymbol{\theta}}^{(i)}\right] - L + \sum_{l=1}^{L} e^{-a|\theta_l|}. \tag{4.7}$$

Note that $\mathcal{Q}\left(\boldsymbol{\theta}, \hat{\boldsymbol{\theta}}^{(i)}\right)$ in (7) is now non-convex. We compare the performance in obtaining the correct model structure of two methods for applying AIC based on the EM algorithm:

i) when AIC is directly utilized (we optimize for all the models candidates). This is an optimization problem that is combinatorial in nature.

ii) when replacing the ℓ_0-(pseudo)-norm with the approximation in (4.4). We propose to minimize the cost function by using BARON [54], a toolbox for global optimization based on the optimization technique known as branch-and-cut [53].

We have also considered 55 Monte Carlo simulations. The initialization of the algorithm is given by least squares assuming that there is no hidden variable. The results are shown in Fig. 4.4. In this figure, the estimates exhibit a similar behaviour as for the linear example presented in the previous section. Here the zeros of $\boldsymbol{\theta}$ were correctly estimated in most of the Monte Carlo simulations. Additionally, the performance of the estimator in (4.4) is better than directly applying AIC (4.1), which is shown in Fig. 4.5. This improved performance is due to the fact that the EM algorithm allows for the attainment of a local optima. Thus, when evaluating all the parameter combinations, the spaces in which the local optima are obtained are different to each other, and hence the filtering processes as well, yielding in many situations an incorrect estimate. In contrast, the incorporation of the regularization term (the approximation of the ℓ_0-(pseudo)-norm) allows for performing the filtering and optimization processes in the correct space, and thus the attainment of the correct model is more achievable.

4.4 CONCLUSIONS

In this chapter we have shown an optimization approach for model selection using AIC based on an approximation of the ℓ_0-(pseudo)-norm. Our analysis has been performed on a simulation basis, resulting in a good performance of our proposal, obtaining a very similar result to what is obtained when using the original AIC cost function when there are no *hidden variables* and with a considerable reduction of the computational load. On the other hand, in the case with hidden variables, our approach not only reduced the computational load, but also outperformed the method of directly optimizing the auxiliary function with an ℓ_0-(pseudo)-norm penalization term. The reason of the attainment of a different solution is that the EM algorithm is a local optimization algorithm. This implies that incorporating the penalization term in the optimization of the auxiliary function improves the capacity of finding the global optima when utilizing the EM algorithm.

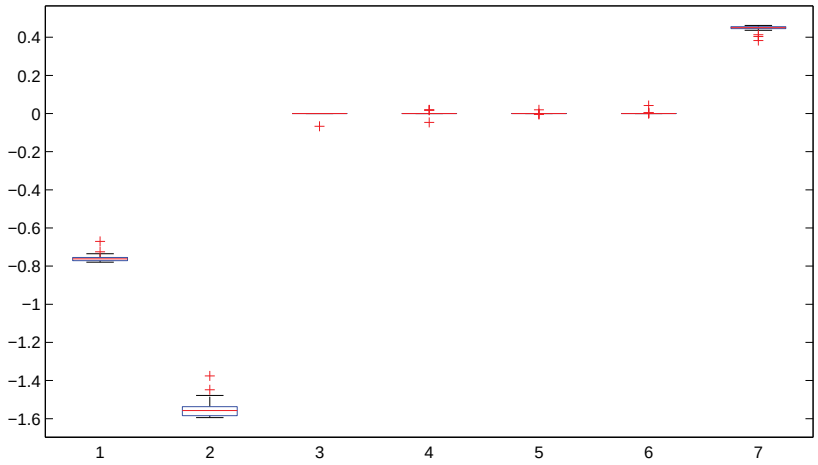

Fig. 4.4: Boxplot of model selection for the nonlinear system in (4.6) with approximate ℓ_0-(pseudo)-norm and $a = 100$.

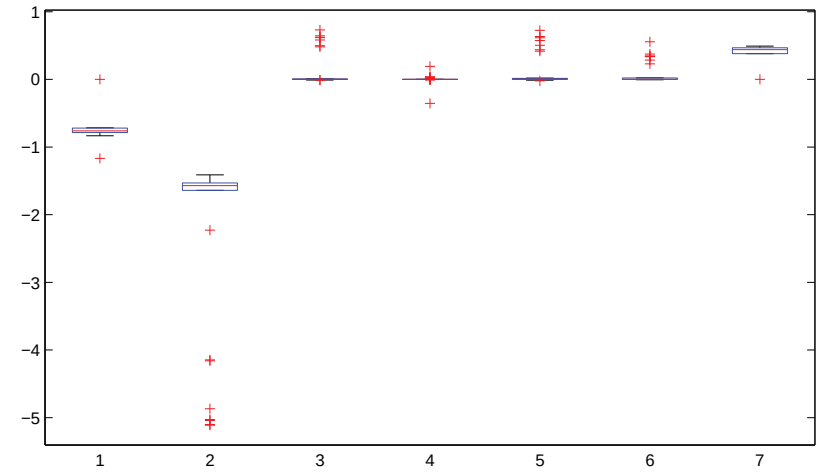

Fig. 4.5: Boxplot of model selection for the nonlinear system in (4.6) using AIC as shown in (4.3).

Chapter 5

STABILITY ANALYSIS OF QUADRATIC MODEL PREDICTIVE CONTROL WITH ℓ_0-INPUT CONSTRAINT

We investigate feedback control with a constrained number of active inputs. This problem is known as sparse control. Specifically, we describe a novel quadratic model predictive control strategy that guarantees sparsity by bounding directly the ℓ_0-(pseudo)-norm of the control input vector at each control horizon instant. Besides this sparse constraint, bounded constraints are also imposed to both control input and system state. Under this scenario, we provide sufficient conditions for guaranteeing practical stability of the closed-loop. Without introducing any relaxation in the sparsity constraints, we turn the combinatorial optimization problem into an equivalent optimization problem that can be solved utilizing standard non-linear programming toolboxes that returns the global optimum input control sequence.

5.1 Introduction

Classical control theory considers the full control action vector to govern a process [68]. However, in the latest years the control community has been attracted to study the so-called sparse control [69], where one intend to control a process employing a reduced number of inputs, see e.g. [70, 71]. Decreasing the number of active control inputs can benefit the operation of control systems. For instance, sparse control has been proposed in [72] to alleviate the traffic information when dealing with limited communication resources. This can also be useful to reduce the power budget when transmitting through self-powered devices [73]. Considering the spacecraft rendezvous problem, sparse control

was deployed in [20] to minimize propellant ejection and to accommodate the minimum impulse constraint.

Other research fields have also incorporated sparse vectors, which have gained an increasing number of interesting applications in system identification [60, 74], state estimation [75], compressive sampling [76, 77], power networks [78], and over-actuated control systems [19] among others.

The inherent characterization of sparsity is through the ℓ_0-(pseudo)-norm (number of non-zero elements of a vector). However, explicitly including ℓ_0-(pseudo)-norm constraints in the control decision problem leads to an NP-hard combinatorial problem [79]. Mainly, three approaches have been proposed in optimal control problems to avoid the involved computational burden: i) a greedy algorithm known as Orthogonal Matching Pursuit (OMP) algorithm [72], ii) a ℓ_1-norm relaxation [21, 70] and more recently iii) an algorithm based in coordinate descent type methods [80].

OMP algorithms [38] rely on computing suboptimal solutions satisfying ℓ_0 constraints. Even if it is computationally inexpensive, adding further constraints into the optimization problem (as states and control inputs belonging to convex sets) is not a simple task. However, approaches based on a ℓ_1-norm relaxation do offer enough flexibility to introduce these kind of constraints. Despite the fact that in [74] the authors proposed an approach to choose the regularization parameter of a modified ℓ_1-norm regularization algorithm, the ℓ_1-norm has no clear meaning in most application (as it just represent the sum of the absolute values). On the other hand, coordinate descent type methods [81], where one decision variable is update at each iteration by means of some selection rule, handle the ℓ_0-(pseudo)-norm but provide only local minima [80].

Works such as [20,21,71] and [72] have introduced sparsity constraints on the control inputs when dealing with model predictive control (MPC) problems. While in [20,21,71] the authors also included extra convex constraints in the optimization procedure, in [72] this issue is not clarified. Still, these approaches evade ℓ_0-(pseudo)-norm restrictions and lack of limiting the number of active control actions at each control horizon instant.

In the current work, we develop a receding horizon technique for quadratic MPC controllers with explicitly ℓ_0-constraints on each control horizon instant. The contribution of the current work is twofold: i) we establish sufficient conditions to guarantee asymptotic and practical stability of the closed loop system considering that the input sequence staisfies a combination of a non-convex (but closed) and a cardinality constraints, and ii) we re-write the corresponding optimization problem into an equivalent form that, in the simulation study in section 5.6 performs better than alternative formulations available in the existent literature [62]. Also, as another novelty, we address the chattering phenomenon (infinite frequency of switching of active control inputs [82]) when the system

is close to the origin by using a dual-MPC strategy. This work extends the work in [36] by: i) including extra bounded constraints (on the states and control actions) in the optimization problem, ii) guaranteeing practical stability instead of exponential stability (due to the additional constraints), iii) considering the problem of chattering and iv) improving the ℓ_0 optimization algorithm to obtain a global optimum instead of a local one.

The remainder of this chapter has the following structure. Section 5.2 introduces some preliminary definitions on practical stability. In Section 5.3 we describe the MPC control problem with ℓ_0 constraints. The ℓ_0 optimization algorithm is discussed in Section 5.4 and in Section 5.5 we address the stability issues. Numerical studies are included in Section 5.6 and Section 5.7 draws conclusions.

Special Notation for Chapter 5

The difference between two given sets $\mathcal{A} \subseteq \mathbb{R}^n$ and $\mathcal{B} \subseteq \mathbb{R}^n$ is denoted by $\mathcal{A} \backslash \mathcal{B} = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x} \in \mathcal{A}, \boldsymbol{x} \notin \mathcal{B}\}$. The Euclidean norm is denoted via $|\cdot|$ while the weighted Euclidean norm (squared) is denoted by $|\boldsymbol{x}|_P^2 = \boldsymbol{x}' P \boldsymbol{x}$. Additionally, the induced norm of a given matrix A is its largest singular value. The maximum and minimum eigenvalues of a given matrix A are represented via $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ respectively. $\vec{\mathbf{0}}_m$, and $\vec{\mathbf{1}}_m$ denote vectors with only zero or one entries respectively. Vector $\boldsymbol{e}_i$ represents the i-th column of the identity matrix, and it is used jointly with a given vector to represent its i-th entry.

Definition 1. *A function σ: $\mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be a $\mathcal{K}$-function if it is continuous, strictly increasing and $\sigma(0) = 0$*

Definition 2. *σ is a $\mathcal{K}_\infty$ function if it is a $\mathcal{K}$-function and unbounded ($\sigma(s) \to \infty$ as $s \to \infty$)*

Definition 3. *A function β: $\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a $\mathcal{KL}$-function if it is continuous and if, for each $t \geq 0$, the function $\beta(\cdot, t)$ is a $\mathcal{K}$-function and for each $s \geq 0$ the function $\beta(s, \cdot)$ is non-increasing and satisfies $\beta(s, t) \to 0$ as $t \to \infty$.*

5.2 Preliminaries: Practical Stability

In this section, the main aspects on practical stability for discrete-time systems are given. These concepts are based on the *regional* input-to-state *practical* stability framework presented in [83,84]. Here, the term *regional* is related to the fact that stability properties hold only in a specific region, which is often the case when system constraints are present,

see [85]. The term *practical* is used to emphasize that, in some cases, only stability of a neighbourhood of the origin can be guaranteed, see e.g., [86].

Consider a discrete-time system described by:

$$\boldsymbol{x}_{k+1} = f(\boldsymbol{x}_k), \quad f(0) = 0, \tag{5.1}$$

where $\boldsymbol{x}_k \in \mathbb{R}^n$ is the system state and $f(\cdot)$ is not necessarily continuous.

Definition 4 (Positively Invariant Set)**.** *A set $\mathcal{A} \subseteq \mathbb{R}^n$ is said to be a* positively invariant *(PI) set for the system* (5.1) *if $f(\boldsymbol{x}) \in \mathcal{A}$, for all $\boldsymbol{x} \in \mathcal{A}$.*

Definition 5 (UpAS)**.** *The system* (5.1) *is said to be* Uniformly practically Asymptotically Stable *(UpAS) in $\mathcal{A} \subseteq \mathbb{R}^n$ if $\mathcal{A}$ is a PI set for* (5.1) *and if there exist a $\mathcal{KL}$-function β, and a nonnegative constant $\delta \geq 0$ such that*

$$|\boldsymbol{x}_k| \leq \beta(|\boldsymbol{x}_0|, k) + \delta, \quad \forall \boldsymbol{x}_0 \in \mathcal{A}, k \in \mathbb{N}.$$

Particularly, if $\delta = 0$ then, system (5.1) *is said to be UAS. If $\mathcal{A} \triangleq \mathbb{R}^n$ then, system* (5.1) *is said to be globally UpAS.*

Definition 6 (Practical-Lyapunov function)**.** *A (not necessarily continuous) function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is said to be a* practical-Lyapunov function *in $\mathcal{A}$ for the system* (5.1) *if $\mathcal{A}$ is a PI set and if there exist a compact set, $\Omega \subseteq \mathcal{A}$, neighbourhood of the origin, $\boldsymbol{x} = 0$, some $\mathcal{K}_\infty$-functions α_1, α_2, and α_3, and a constants $d \geq 0$, such that*

$$V(|\boldsymbol{x}|) \geq \alpha_1(|\boldsymbol{x}|), \qquad \forall \boldsymbol{x} \in \mathcal{A}, \tag{5.2}$$

$$V(|\boldsymbol{x}|) \leq \alpha_2(|\boldsymbol{x}|), \quad \forall \boldsymbol{x} \in \Omega, \tag{5.3}$$

$$V(f(\boldsymbol{x})) - V(\boldsymbol{x}) \leq -\alpha_3(|\boldsymbol{x}|) + d, \quad \forall \boldsymbol{x} \in \mathcal{A}. \tag{5.4}$$

If $\mathcal{A} \triangleq \mathbb{R}^n$ then, the function, V, is said to be a global practical-Lyapunov function.

Theorem 3 ([84])**.** *If* (5.1) *admits a practical-Lyapunov function in $\mathcal{A}$, then it is UpAS in $\mathcal{A}$.*

5.3 Problem Description

Consider the following discrete-time linear time-invariant system:

$$\boldsymbol{x}_{k+1} = A\boldsymbol{x}_k + B\boldsymbol{u}_k, \tag{5.5}$$

where $\boldsymbol{x}_k \in \mathbb{X} \subseteq \mathbb{R}^n$ is the system state, $\boldsymbol{u}_k \in \mathbb{U} \subseteq \mathbb{R}^m$ is the control input vector. Here, both $\mathbb{X}$ and $\mathbb{U}$ are assumed to be compact sets which contain the origin in its

interior. Moreover, convexity is only assumed for $\mathbb{X}$. The pair (A, B) is assumed to be stabilizable where the matrix A is not necessarily Schur stable. In this case we seek to control system (5.5), if possible, by using an MPC with a reduced number of active inputs γ, i.e., $\gamma \in \{0, \ldots, m\}$. To this end, one needs to design a controller which can provide the best possible actuation considering only γ active inputs while the remaining $m-\gamma$ inactive inputs will take a null value. For this problem, we denote by $\boldsymbol{\sigma} \in \mathbb{R}^m$ the binary vector which indicates the active and inactive inputs, i.e., the $i-th$ component of $\boldsymbol{\sigma}$ is given by:

$$\boldsymbol{e}_i'\boldsymbol{\sigma}_k = \begin{cases} 1 & \text{if } \boldsymbol{e}_i'\boldsymbol{u}_k \text{ is active,} \\ 0 & \text{otherwise } (\boldsymbol{e}_i'\boldsymbol{u}_k = 0), \end{cases}$$

for all $i \in \{1, \ldots, m\}$, where $\boldsymbol{e}_i$ is the i-th column of the identity matrix. Thus, the number of non-zero elements of vector $\boldsymbol{\sigma}_k$ (ℓ_0-(pseudo)-norm) is $|\boldsymbol{\sigma}_k|_0 = \gamma$. To formulate the MPC optimal problem, we first consider the following cost function

$$V_N(\boldsymbol{x}, \vec{\boldsymbol{u}}) = \sum_{j=0}^{N-1} \ell(\hat{\boldsymbol{x}}_j, \hat{\boldsymbol{u}}_j) + V_f(\hat{\boldsymbol{x}}_N), \tag{5.6}$$

where N is the prediction horizon, and $\ell(\hat{\boldsymbol{x}}, \hat{\boldsymbol{u}}) = |\hat{\boldsymbol{x}}|_Q^2 + |\hat{\boldsymbol{u}}|_R^2$ is the stage cost with Q and R positive definite matrices, while the term $V_f(\hat{\boldsymbol{x}}) = |\hat{\boldsymbol{x}}|_P^2$, in which P is positive definite, represents the terminal cost. The vector $\vec{\boldsymbol{u}}$ in (5.6) contains the tentative control actions over the prediction horizon, i.e.,

$$\vec{\boldsymbol{u}} = \left[\hat{\boldsymbol{u}}_0', \ldots, \hat{\boldsymbol{u}}_{N-1}'\right]' \in \mathbb{R}^{Nm}.$$

The MPC optimization of interest for the current state, $\boldsymbol{x}_k = \boldsymbol{x}$, is given as

$$\mathbb{P}_N(\boldsymbol{x}): \; V_N^{op}(\boldsymbol{x}) = \min_{\vec{\boldsymbol{u}}} \{V_N(\boldsymbol{x}, \vec{\boldsymbol{u}})\}, \tag{5.7}$$

subject to:

$$\hat{\boldsymbol{x}}_{j+1} = A\hat{\boldsymbol{x}}_j + \hat{\boldsymbol{u}}_j, \tag{5.8}$$

$$\hat{\boldsymbol{u}}_j \in \mathbb{U}, \tag{5.9}$$

$$\hat{\boldsymbol{x}}_j \in \mathbb{X}, \tag{5.10}$$

$$|\hat{\boldsymbol{u}}_j|_0 \leq \gamma, \tag{5.11}$$

$$\hat{\boldsymbol{x}}_N \in \mathbb{X}_f \subseteq \mathbb{X}, \tag{5.12}$$

for all $j \in \{0, \ldots, N-1\}$, where $\hat{\boldsymbol{x}}_0 = \boldsymbol{x}_k$ and $\gamma \leq m$.

Here, (5.9) and (5.10) take into account the system bounded constraints, where $\mathbb{U}$ is not necessarily convex. Constraint (5.11) encompasses the number of active inputs (sparse) constraint over the prediction horizon. Constraint (5.12) is the, so-called, terminal constraint. Similarly to convex MPC formulations, the terminal region $\mathbb{X}_f$ and

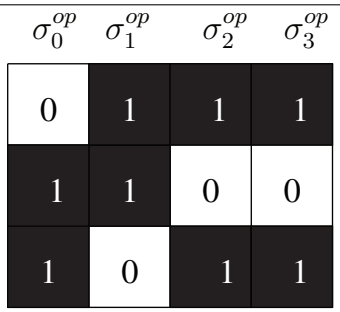

Fig. 5.1: Illustration of an optimal active input sequence, $\vec{\boldsymbol{\sigma}}^{op}$, for a prediction horizon $N = 4$.

matrix P can be designed to guaranty stability of the resulting closed-loop [87]. Their design will be considered in the stability analysis presented in Section 5.5. We define the set $\mathcal{U}(\boldsymbol{x})$ to represent all the input sequences sequences, $\vec{\boldsymbol{u}}$, which satisfy constraints (5.9)-(5.12).

Consequently, the optimal input sequence, $\vec{\boldsymbol{u}}^{op}(\boldsymbol{x})$, is the one which minimizes the cost function, i.e.,

$$\vec{\boldsymbol{u}}^{op}(\boldsymbol{x}) \triangleq \arg\left\{\min_{\vec{\boldsymbol{u}}\in\mathcal{U}(\boldsymbol{x})} V_N(\boldsymbol{x},\vec{\boldsymbol{u}})\right\}.$$

Thus, the resulting optimal solution is the, so-called, optimal input control sequence

$$\vec{\boldsymbol{u}}^{op}(\boldsymbol{x}) = \left[(\hat{\boldsymbol{u}}_0^{op})', \ldots, (\hat{\boldsymbol{u}}_{N-1}^{op})'\right]', \tag{5.13}$$

while the resulting optimal state sequence is:

$$\vec{\boldsymbol{x}}^{op}(\boldsymbol{x}) = \left[\boldsymbol{x}', (\hat{\boldsymbol{x}}_1^{op})', \ldots, (\hat{\boldsymbol{x}}_N^{op})'\right]'.$$

Additionally, for this particular problem, we also obtain the resulting optimal active input sequence, given by:

$$\vec{\boldsymbol{\sigma}}^{op}(\boldsymbol{x}) = \left[(\boldsymbol{\sigma}_0^{op})', \ldots, (\boldsymbol{\sigma}_{N-1}^{op})'\right]'.$$

Notice that the elements of $\vec{\boldsymbol{\sigma}}^{op}(\boldsymbol{x})$ may differ from each other. However, $|\boldsymbol{\sigma}_j^{op}| \leq \gamma$ for all $j \in \{0, \ldots, N-1\}$. For example, if $N = 4$, $m = 3$ and $\gamma = 2$ a possible $\vec{\boldsymbol{\sigma}}^{op}(\boldsymbol{x})$ is shown in Fig. 5.1.

We also denote the domain of attraction of the cost function, $V_N(\boldsymbol{x})$, via

$$\mathrm{X}_N \triangleq \{\boldsymbol{x} \in \mathbb{X} : \mathcal{U}(\boldsymbol{x}) \neq \emptyset\}.$$

Therefore, X_N contains all $\boldsymbol{x} \in \mathbb{X}$ such that there exists a control sequence $\vec{\boldsymbol{u}} \in \mathcal{U}(\boldsymbol{x})$ satisfying conditions (5.9)-(5.12).

Finally, we use a *receding horizon* technique, i.e., only the first element of $\vec{\boldsymbol{u}}^{op}(\boldsymbol{x})$ is applied to the system at each sampling instant (see, e.g. [87]). The solution of the optimal problem, $\mathbb{P}_N(\boldsymbol{x})$ in (5.7), yields the sparse MPC law, $\kappa_N(\cdot) : \mathrm{X}_N \to \mathbb{U}$,

$$\boldsymbol{\kappa}_N(\boldsymbol{x}) \triangleq \hat{\boldsymbol{u}}_0^{op}. \tag{5.14}$$

Consequently, the resulting sparse MPC loop can be represented via

$$\boldsymbol{x}_{k+1} = A\boldsymbol{x}_k + B\kappa_N(\boldsymbol{x}_k). \tag{5.15}$$

In the following section, we will present a general method to solve an optimization problem subject to ℓ_0-norm constraints. This solution is then used to solve the quadratic MPC problem in (5.7)-(5.12).

5.4 ℓ_0-constrained based solution

Consider the following ℓ_0-constrained optimization problem

$$\mathbb{P}_0 : \min_{\boldsymbol{x}\in\mathbb{R}^p} f(\boldsymbol{x}), \tag{5.16}$$

subject to:
$$\boldsymbol{x} \in \Omega$$
$$|\boldsymbol{x}_k|_0 \leq \gamma$$

A way of handling cardinality constraints is through the following mixed-integer programming formulation [62]:

$$\mathbb{P}_{0,MIP} : \min_{\boldsymbol{x}\in\mathbb{R}^p, \boldsymbol{z}\in\{0,1\}^p} f(\boldsymbol{x}), \tag{5.17}$$

subject to:
$$\boldsymbol{x} \in \Omega$$
$$-L\boldsymbol{e}_i'\boldsymbol{z} \leq \boldsymbol{e}_i'\boldsymbol{x} \leq L\boldsymbol{e}_i'\boldsymbol{z} \tag{5.18}$$
$$\vec{\mathbf{1}}_m'\boldsymbol{z}_k = \gamma \tag{5.19}$$

where each entry of vector $\boldsymbol{z}$ is set to be binary, and $\boldsymbol{e}_i$ is the i-th column of the identity matrix. By means of constraint (5.18), a semi-continuous behavior is induced on variable $\boldsymbol{x}_i$.

To solve the problem $\mathbb{P}_{0,MIP}$, standard Mixed-Integer Quadratic Programming (MIQP) solvers such as CPLEX or BARON [53] can be used. However, in this work the input is restricted to belong to a compact set that may be non-convex. Therefore, it cannot be handled by CPLEX [62].

Also, consider the following optimization problem involving bilinear constraints

$$\mathbb{P}_{0equiv}: \min_{\boldsymbol{x},\boldsymbol{w}\in\mathbb{R}^p} f(\boldsymbol{x}), \tag{5.20}$$

subject to:

$$\boldsymbol{x} \in \Omega$$

$$(\boldsymbol{e}_i'\boldsymbol{x})(\boldsymbol{e}_i'\boldsymbol{w}) = 0 \tag{5.21}$$

$$\vec{\mathbf{0}}_p \leq \boldsymbol{w}_k \leq \vec{\mathbf{1}}_p \tag{5.22}$$

$$\vec{\mathbf{1}}_m'\boldsymbol{w}_k = p - \gamma \tag{5.23}$$

where $\Omega \subset \mathbb{R}^p$ is a constraining set, $f : \mathbb{R}^p \to \mathbb{R}$ is the objective function, and $\boldsymbol{e}_i$ is the i-th column of the identity matrix. The following result, that was independently obtained in [8,50], shows that problems $\mathbb{P}_0$ in (5.16) and $\mathbb{P}_{0equiv}$ in (5.20) are equivalent.

Theorem 4 ([8,37,50,52])**.** *A vector $\boldsymbol{x}^* \in \mathbb{R}^p$ is a global solution of $\mathbb{P}_0$ if and only if there exists a vector $\boldsymbol{w}^* \in \mathbb{R}^p$ such that the pair $(\boldsymbol{x}^*, \boldsymbol{w}^*)$ is a global solution of $\mathbb{P}_{0equiv}$.*

Results in [50] and [37] are similar. However, [37] have been obtained in a more general framework where constraints on the rank of a matrix are utilized.

A key observation is that $\mathbb{P}_{0equiv}$ can be solved by using standard tools of nonlinear programming. In particular, we obtain a global solution of $\mathbb{P}_{0equiv}$ by using the optimization software BARON [53].

In problem $\mathbb{P}_{0equiv}$, the auxiliary variable $\boldsymbol{w}$ in (5.21)-(5.23), at the optimum is a binary variable taking value 1 for those entries of the state with value 0. Additional constraints over $\boldsymbol{w}$ can be included in the optimization problem to manage how the zero and non-zero elements of $\boldsymbol{x}$ interact. These interactions are difficult to handle by relaxation methods such as the ℓ_1-norm heuristic. In addition, our approach obtains a solution in less time than the corresponding binary non-linear programming (i.e., $w_i \in \{0,1\}$) for the simulation study in Section 5.6.

Remark 1. *The proposed approach can easily handle ℓ_0-(pseudo)-norm constraints over a selection in the vector, i.e. $|diag_m(\boldsymbol{a}_i)\vec{\boldsymbol{u}}|_0 \leq \gamma$, where a_i is a given vector with entries $\{0,1\}$. We use this approach latter in the chapter to solve problem* (5.7)-(5.12)*, where ℓ_0-(pseudo)-norm constraints are imposed on several selections of vector $\vec{\boldsymbol{u}}$. In addition, we can also minimize the ℓ_0-(pseudo)-norm of the whole optimal input vector, i.e., $|\vec{\boldsymbol{u}}|_0 \leq \gamma$.*

Therefore, a comparison between both approaches is done using the optimization software BARON.

Application to Quadratic MPC

Next, we apply the ideas presented above to quadratic MPC. The quadratic MPC with ℓ_0-input constraint described by (5.7)-(5.12) can be equivalently formulated as the following optimization problem

$$\mathbb{P}_{equiv,N}(\boldsymbol{x}):\ V^{op}_{equiv,N}(\boldsymbol{x}) = \min_{\vec{\boldsymbol{u}},\vec{\boldsymbol{w}}}\{V_N(\boldsymbol{x},\vec{\boldsymbol{u}})\}, \tag{5.24}$$

subject to:

$$\hat{\boldsymbol{x}}_{j+1} = A\hat{\boldsymbol{x}}_j + \hat{\boldsymbol{u}}_j, \tag{5.25}$$
$$\hat{\boldsymbol{u}}_j \in \mathbb{U}, \tag{5.26}$$
$$\hat{\boldsymbol{x}}_j \in \mathbb{X}, \tag{5.27}$$
$$\mathrm{diag}_m(w_j)\boldsymbol{u}_j = \vec{\mathbf{0}}_m, \tag{5.28}$$
$$\vec{\mathbf{0}}_m \leq \boldsymbol{w}_j \leq \vec{\mathbf{1}}_m, \tag{5.29}$$
$$w_j'\vec{\mathbf{1}}_m = m - \gamma, \tag{5.30}$$
$$\hat{\boldsymbol{x}}_N \in \mathbb{X}_f \subseteq \mathbb{X}, \tag{5.31}$$

for all $j \in \{0,\ldots,N-1\}$, where $\hat{\boldsymbol{x}}_0 = \boldsymbol{x}_k$ and $\gamma \leq m$. Note that in this case the set $\mathcal{U}(\boldsymbol{x})$ represents all the input sequences, $\vec{\boldsymbol{u}}$, that satisfy constraints (5.26)-(5.31).

The ℓ_0-(pseudo)-norm constraint (5.11), in problem $\mathbb{P}_N(\boldsymbol{x})$ in (5.7), is substituted by (5.28)-(5.30) in problem $\mathbb{P}_{equiv,N}(\boldsymbol{x})$ as per (5.24). This substitution allows us to obtain a global solution of $\mathbb{P}_N(\boldsymbol{x})$ by using standard tools in nonlinear programming over $\mathbb{P}_{equiv,N}(\boldsymbol{x})$. Note that the equivalence between $\mathbb{P}_N(\boldsymbol{x})$ and $\mathbb{P}_{equiv,N}(\boldsymbol{x})$ holds in the global optimum (see [8, 52]).

5.5 Stability Analysis

In this section, sufficient conditions to guarantee stability of the sparse MPC loop in (5.15) are established.

Firstly, we define the predicted state sequence as

$$\vec{\boldsymbol{x}}_{[1:N]} = [\hat{\boldsymbol{x}}_1', \ldots, \hat{\boldsymbol{x}}_N']'.$$

Considering an initial system state $\hat{\boldsymbol{x}}_0 = \boldsymbol{x}$, from (5.8), we obtain

$$\vec{\boldsymbol{x}}_{[1:N]} = \Lambda x + \Phi\vec{\boldsymbol{u}},$$

where

$$\Lambda \triangleq \begin{bmatrix} A \\ A^2 \\ \vdots \\ A^N \end{bmatrix}, \quad \Phi \triangleq \begin{bmatrix} B & 0 & \cdots & 0 & 0 \\ AB & B & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ A^{N-1}B & A^{N-2}B & \cdots & AB & B \end{bmatrix}.$$

Thus, the cost function (5.6) can be re-written as

$$V_N(\boldsymbol{x}, \vec{\boldsymbol{u}}) = \nu(\boldsymbol{x}) + \vec{\boldsymbol{u}}' H_N \vec{\boldsymbol{u}} + 2\vec{\boldsymbol{u}}' F_N \boldsymbol{x},$$

where the term $\nu(\boldsymbol{x})$ is independent of $\vec{\boldsymbol{u}}$ and $H_N \triangleq \Phi' Q_N \Phi + R_N \in \mathbb{R}^{Nm \times Nm}$, $F_N \triangleq \Phi' Q_N \Lambda \in \mathbb{R}^{Nm \times n}$, with $Q_N \triangleq \operatorname{diag}\{Q, \ldots, Q, P\} \in \mathbb{R}^{Nn \times Nn}$, $R_N \triangleq \operatorname{diag}\{R, \ldots, R\} \in \mathbb{R}^{Nm \times Nm}$. Notice that, since Q, R, and P are positive definite, so is H_N. Based on this representation, the following unconstrained optimal input, $\vec{\boldsymbol{u}}^{op}_{uc}(\boldsymbol{x})$, can be defined, see [87].

Lemma 1 (Unconstrained Solution)**.** *If for the optimal problem $\mathbb{P}_N(\boldsymbol{x})$ in (5.7), constraints (5.9)-(5.12) are not taken into account, i.e., $\mathbb{U} \triangleq \mathbb{R}^m$, $\mathbb{X} = \mathbb{X}_f \triangleq \mathbb{R}^n$, and $\gamma = m$, then $V_N(\boldsymbol{x}, \vec{\boldsymbol{u}})$ is minimized when*

$$\vec{\boldsymbol{u}}^{op}_{uc}(\boldsymbol{x}) \triangleq \arg\left\{ \min_{\vec{\boldsymbol{u}} \in \mathbb{R}^{Nm}} V_N(\boldsymbol{x}, \vec{\boldsymbol{u}}) \right\} \triangleq -H_N^{-1} F_N \boldsymbol{x}. \tag{5.32}$$

for all $\boldsymbol{x} \in \mathbb{R}^n$.

5.5.1 Sparse Local Controller

We propose to prove stability of ℓ_0-input constrained MPC loop in (5.15) by examining properties of a feasible local controller based on the optimal nominal solution presented in (5.32) with prediction horizon $N = 1$; cf. [86]. To take into account the ℓ_o-input constraint, for a given $\gamma = |\boldsymbol{\sigma}|$, we consider the following sparse matrix

$$L_\sigma = \operatorname{diag}_m\{\boldsymbol{\sigma}\} \in \mathbb{R}^{m \times m}.$$

Thus, the proposed feasible local controller is given by

$$\boldsymbol{\kappa}_f(\boldsymbol{x}) = K_\sigma \boldsymbol{x} = (K + \Delta_\sigma)\boldsymbol{x}, \tag{5.33}$$

where

$$\Delta_\sigma = (L_\sigma - I)K,$$

with

$$K = -H_1^{-1} F_1 = (B'PB + R)^{-1} B'PA.$$

Thus, based on the nominal local controller, we chose the terminal region in (5.12) as:

$$\mathbb{X}_f \triangleq \left\{ \boldsymbol{x}' P \boldsymbol{x} \leq \varphi_x : \boldsymbol{x} \in \mathbb{X}, K_\sigma \boldsymbol{x} \in \mathbb{U} \right\}. \tag{5.34}$$

Here, $\varphi_x \in \mathbb{R}_{\geq 0}$ is designed to obtain the largest ellipsoid which guarantees that for all $\boldsymbol{x} \in \mathbb{X}, K_\sigma \boldsymbol{x} \in \mathbb{U}$. Notice that since the origin belongs to $\mathbb{X}$ and $\mathbb{U}$ then, $\mathbb{X}_f \neq \emptyset$.

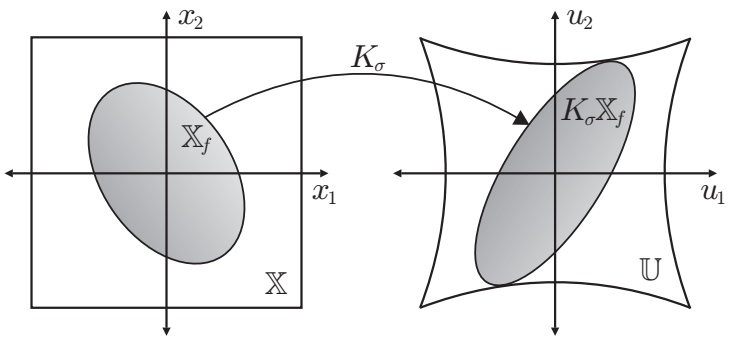

Fig. 5.2: Illustration of the terminal set $\mathbb{X}_f \subset \mathbb{X} \subset \mathbb{R}^2$ for the case when $\mathbb{U} \subset \mathbb{R}^2$ is non-convex.

Therefore, both the proposed local controller and terminal region provide that for all $\boldsymbol{x} \in \mathbb{X}_f$, $\boldsymbol{\kappa}_f(\boldsymbol{x}) \in \mathbb{U}$. It is important to emphasize that in this work the compact set $\mathbb{U}$ is not restricted to be convex. However, the local controller, $\boldsymbol{\kappa}_f(\boldsymbol{x})$, maps the states in $\mathbb{X}_f$ to the convex set $K_\sigma\mathbb{X}_f$, which is contained by $\mathbb{U}$, i.e., $K_\sigma\mathbb{X}_f \subseteq \mathbb{U}$. This is illustrated in Fig. 5.2.

On the other hand, the closed-loop expression for system (5.5) governed by the local controller (5.33) is given by

$$\boldsymbol{x}_{k+1} = A_{K_\sigma}\boldsymbol{x} = (A_K + B\Delta_\sigma)\boldsymbol{x}_k, \quad \forall \boldsymbol{x}_k \in \mathbb{X}_f, \tag{5.35}$$

where $A_K = A + BK$, and the term $B\Delta_\sigma$ represents the sparse control effect on the "nominal system", $\boldsymbol{x}_{k+1} = A_K\boldsymbol{x}_k$.

Theorem 5. *Suppose that the terminal cost, $V_f(\boldsymbol{x})$, in* (5.6) *is designed such that the matrix P is chosen to be the solution to the algebraic Riccati equation*

$$A_K' P A_K - P + Q^* = 0, \quad Q^* \triangleq Q + K'RK. \tag{5.36}$$

If γ in (5.11) *is chosen such that*

$$Q^* - \Psi_\sigma \succ 0, \tag{5.37}$$

where

$$\Psi_\sigma = (2A_K + B\Delta_\sigma)'PB\Delta_\sigma.$$

Then, $\boldsymbol{\kappa}_f$ in (5.33) *is a uniformly exponentially stable sparse local controller in $\mathbb{X}_f$ for the system* (5.5).

Proof. We first consider the terminal cost, $V_f(\boldsymbol{x})$ in (5.6), as a candidate Lyapunov function. Therefore, we apply Theorem 3 with $\alpha_1(s) = a_1 s^2$ and $\alpha_2(s) = a_2 s^2$, where

$a_1 \triangleq \lambda_{\min}(P), \quad a_2 \triangleq \lambda_{\max}(P)$. Direct calculations give that:

$$\begin{aligned} V_f(A\boldsymbol{x} + B\boldsymbol{\kappa}_f(\boldsymbol{x})) - V_f(\boldsymbol{x}) \\ =&\boldsymbol{x}'\big(A'_{K_\sigma} P A_{K_\sigma} - P\big)\boldsymbol{x} \\ =&\boldsymbol{x}'\big(A'_K P A_K - P + 2A'_K P B\Delta_\sigma + B'\Delta'_\sigma P \Delta_\sigma B\big)\boldsymbol{x}. \end{aligned}$$

Since matrix P is chosen according to (5.36), it follows that

$$V_f(A\boldsymbol{x} + B\boldsymbol{\kappa}_f(\boldsymbol{x})) - V_f(\boldsymbol{x}) = -\boldsymbol{x}'(Q^* - (2A_K + \Delta_\sigma)' P \Delta_\sigma)\boldsymbol{x}.$$

Then, considering the proposed stabilizing condition (5.37), property (5.4) holds with $\alpha_3(s) = a_3 s^2$, where $a_3 = \lambda_{\min}(Q^* - \Psi) > 0$. Therefore, it follows that

$$\Delta V_f(\boldsymbol{x}_k) = V_f(\boldsymbol{x}_{k+1}) - V_f(\boldsymbol{x}_k) \leq -a_3|\boldsymbol{x}_k|^2, \tag{5.38}$$

for all $\boldsymbol{x}_k \in \mathbb{X}_f$. This allows us to establish the following relationship

$$V_f(\boldsymbol{x}_{k+1}) \leq \rho V_f(\boldsymbol{x}_k), \quad \forall \boldsymbol{x}_k \in \mathbb{X}_f. \tag{5.39}$$

Taking into account inequality (5.38), it follows that

$$0 \leq V_f(\boldsymbol{x}_{k+1}) \leq V_f(\boldsymbol{x}_k) - a_3|\boldsymbol{x}_k|^2 \leq (a_2 - a_3)|\boldsymbol{x}_k|^2.$$

Hence, $0 < a_3 \leq a_2$, which implies that $\rho = 1 - \frac{a_3}{a_2} \in [0, 1)$.

Therefore, considering (5.34) and (5.39), we have that $\mathbb{X}_f$ is a PI set for (5.35). Moreover, for all $\boldsymbol{x} \in \mathbb{X}_f$, $\boldsymbol{\kappa}_f(x) \in \mathbb{U}$. By iterating (5.39), it is possible to exponentially bound the system state evolution via:

$$|\boldsymbol{x}_k|^2 \leq \frac{a_2}{a_1}\rho^k |\boldsymbol{x}_0|^2, \quad \forall k > 0, \boldsymbol{x}_0 \in \mathbb{X}_f.$$

Thus, $\limsup_{k\to\infty} |\boldsymbol{x}_k| = 0$, provided that $\boldsymbol{x}_0 \in \mathbb{X}_f$.

Consequently, the proposed sparse local controller, $\boldsymbol{\kappa}_f(x)$ in (5.33), is a stabilizing controller for (5.5) for all $\boldsymbol{x} \in \mathbb{X}_f$. More precisely, the local sparse MPC loop (5.35) is uniformly exponentially stable. □

Remark 2. *Since $Q, R \succ 0$ and the pair (A, B) is stabilizable then, there exists a unique solution of the discrete algebraic Ricatti equation* (5.36)*, i.e., $P > 0$. Moreover, A_K in* (5.35) *is Schur stable; see [88].*

5.5.2 Multi-Step Sparse MPC Stability Analysis

Based on the proposed stabilizing sparse local controller, $\boldsymbol{\kappa}_f(\boldsymbol{x})$, we next establish sufficient conditions for practical stability for the ℓ_0-input constrained multi-step MPC loop in (5.15).

Theorem 6. *Consider the positive constants* $c_1 = \lambda_{\min}(P)$, $c_2 = \lambda_{\max}(P + W_\sigma)$, *and* $c_3 = \lambda_{\min}(Q)$, *where*

$$\begin{aligned} W_\sigma &= F' H_N^{-1} (\mathcal{L}_\sigma - I) H_N (\mathcal{L}_\sigma - I) H_N^{-1} F \in \mathbb{R}^{n \times n}, \\ \mathcal{L}_\sigma &= diag\{L_\sigma, \ldots, L_\sigma\} \in \mathbb{R}^{Nm \times Nm}. \end{aligned}$$

Suppose that $\boldsymbol{x}_0 \in X_N$ *and matrix* P, *in* $V_f(\boldsymbol{x})$, *is designed as per* (5.36). *If the proposed sparse local controller,* $\boldsymbol{\kappa}_f(\boldsymbol{x})$, *in* (5.33) *satisfies both* (5.37) *and*

$$|G_\sigma| < a_1 \left(\frac{c_3}{c_2} \right), \tag{5.40}$$

where

$$G_\sigma = \Delta_\sigma' H_1 \Delta_\sigma.$$

Then, the MPC closed-loop system (5.15) *is UpAS for all* $\boldsymbol{x} \in X_N$, *with*

$$\mathcal{D}_\delta \triangleq \left\{ \boldsymbol{x} \in \mathbb{X}_f : \boldsymbol{x}' P \boldsymbol{x} \leq \delta = \frac{c_3 \varphi_x}{c_2 a_1} |G_\sigma| \right\} \tag{5.41}$$

as an ultimately invariant set.

Proof. To prove this theorem, we verify conditions presented in Definition 6. Since matrix P in (5.6) satisfies (5.36), the unconstrained solution, $\vec{\boldsymbol{u}}_{uc}^{op}(\boldsymbol{x})$ in (5.32) can be expressed via:

$$\vec{\boldsymbol{u}}_{uc}^{op}(\boldsymbol{x}) = \begin{bmatrix} (K\hat{\boldsymbol{x}})' & (K\hat{\boldsymbol{x}}_1)' & \ldots & (K\hat{\boldsymbol{x}}_{N-1})' \end{bmatrix}'$$

Now, the optimal cost function, $V_N^{op}(\boldsymbol{x})$, with $\boldsymbol{x}_k = \boldsymbol{x}$, can be rewritten as:

$$\begin{aligned} V_N^{op}(\boldsymbol{x}) &= V_N^{op}(\boldsymbol{x}, \vec{\boldsymbol{u}}^{op}(\boldsymbol{x})) \\ &= \boldsymbol{x}' P \boldsymbol{x} + (\vec{\boldsymbol{u}}^{op}(\boldsymbol{x}) - \vec{\boldsymbol{u}}_{uc}^{op}(\boldsymbol{x}))' H_N (\vec{\boldsymbol{u}}^{op}(\boldsymbol{x}) - \vec{\boldsymbol{u}}_{uc}^{op}(\boldsymbol{x})). \end{aligned} \tag{5.42}$$

Notice that when constraints (5.9)-(5.12) are not taken into account, i.e., $\mathbb{X} = \mathbb{R}^n$, $\mathbb{U} = \mathbb{R}^m$, $\mathbb{X}_f = \mathbb{R}^n$, and $\gamma = m$, we have that $\vec{\boldsymbol{u}}^{op}(\boldsymbol{x}) = \vec{\boldsymbol{u}}_{uc}^{op}(\boldsymbol{x})$. Hence, $V_{uc}^{op}(\boldsymbol{x}) = \boldsymbol{x}' P \boldsymbol{x}$. Therefore, it follows that

$$V_N^{op}(\boldsymbol{x}) \geq c_1 |\boldsymbol{x}|^2, \quad \forall \boldsymbol{x} \in \mathrm{X}_N,$$

Thus, property (5.2) holds with $\alpha_1(s) = c_1 s^2$.

Then, we obtain an upper bound for the cost function for the case when $\gamma \leq m$. To do this, we use the following suboptimal solution[1] based on the proposed sparse local controller, $\boldsymbol{\kappa}_f(\boldsymbol{x})$ in (5.33):

$$\tilde{\boldsymbol{u}} = \mathcal{L}_\sigma \vec{\boldsymbol{u}}_{uc}^{op}(x) = \begin{bmatrix} (L_\sigma K \hat{\boldsymbol{x}})' & (L_\sigma K \hat{\boldsymbol{x}}_1)' & \dots & (L_\sigma K \hat{\boldsymbol{x}}_{N-1})' \end{bmatrix}',$$

for all $\boldsymbol{x} \in \mathbb{X}_f$. Thus, the optimal cost function satisfies that:

$$\begin{aligned} V_N^{op}(\boldsymbol{x}) &\leq V_N(\boldsymbol{x}, \tilde{u}) \\ &= \boldsymbol{x}' P \boldsymbol{x} + \vec{\boldsymbol{u}}_{uc}^{op}(\boldsymbol{x})' (\mathcal{L}_\sigma - I)' H_N (\mathcal{L}_\sigma - I) \vec{\boldsymbol{u}}_{uc}^{op}(\boldsymbol{x}) \end{aligned}$$

Then, we obtain that

$$V_N^{op}(\boldsymbol{x}) \leq c_2 |\boldsymbol{x}|^2, \quad \forall \boldsymbol{x} \in \mathbb{X}_f. \tag{5.43}$$

Thus, property (5.3) holds with $a_2(s) = c_2 s^2$.

Taking into account the proposed stabilizing sparse local controller, $\kappa_f(x)$, we adopt the shifted sequence approach, based on (5.13) (see [87]), and use the following feasible input sequence

$$\tilde{\boldsymbol{u}}(\boldsymbol{x}_{k+1}) = [(\hat{\boldsymbol{u}}_1^{op})', \dots, (\hat{\boldsymbol{u}}_{N-1}^{op})', \kappa_f(\hat{\boldsymbol{x}}_N)']'.$$

This generates the following state sequence:

$$\tilde{\boldsymbol{x}}(\boldsymbol{x}_{k+1}) = [(\hat{\boldsymbol{x}}_1^{op})', \dots, (\hat{\boldsymbol{x}}_N^{op})', (\hat{\boldsymbol{x}}_{N+1})']'.$$

Notice that by constraint (5.12), $\hat{\boldsymbol{x}}_N \in \mathbb{X}_f$. Therefore, since $\boldsymbol{\kappa}_f(\boldsymbol{x})$ satisfies (5.37), we have that $\hat{\boldsymbol{x}}_{N+1} = A_{K\sigma} \hat{\boldsymbol{x}}_N \in \mathbb{X}_f$.

By optimality, we obtain the bound

$$V_N^{op}(\boldsymbol{x}_{k+1}) \leq V_N^{op}(\boldsymbol{x}_{k+1}, \tilde{u}(\boldsymbol{x}_{k+1}))$$

Comparing (5.42) with (5.43), we obtain that

$$\begin{aligned} \Delta V_N^{op}(\boldsymbol{x}) &= V_N^{op}(\boldsymbol{x}_{k+1}) - V_N^{op}(\boldsymbol{x}_k) \\ &\leq -|\boldsymbol{x}|_Q^2 + |\hat{\boldsymbol{x}}_{N+1}|_P^2 - |\hat{\boldsymbol{x}}_N|_P^2 + |\hat{\boldsymbol{x}}_N|_Q^2 + |\kappa_f(\hat{\boldsymbol{x}}_N)|_R^2 \\ &= -|\boldsymbol{x}|_Q^2 + |(A_K + B\Delta_\sigma)\hat{\boldsymbol{x}}_N|_P^2 - |\hat{\boldsymbol{x}}_N|_P^2 + |\hat{\boldsymbol{x}}_N|_Q^2 \\ &\quad + |(K + \Delta_\sigma)\hat{\boldsymbol{x}}_N|_R^2 \\ &= -|\boldsymbol{x}|_Q^2 + \hat{\boldsymbol{x}}_N' \Delta_\sigma' (B'PB + R) \Delta_\sigma \hat{\boldsymbol{x}}_N \\ &\quad + \hat{\boldsymbol{x}}_N' \left(A_K' P A_K - P + Q^* + 2(A_K' P B + K' R) \Delta_\sigma \right) \hat{\boldsymbol{x}}_N \end{aligned}$$

[1] It is important to highlight that this suboptimal input sequence is only used to facilitate the stability analysis. The actual optimal input sequence, $\vec{\boldsymbol{u}}^{op}(\boldsymbol{x})$, may present sparse elements which might differ from each other, see e.g. Fig. 5.1.

Since matrix P is chosen according to (5.36), we have that

$$A'_K PB + K'R = A'PB + K'(B'PB + R) = 0.$$

Then, we obtain that

$$\Delta V_N^{op}(\boldsymbol{x}) \leq -|\boldsymbol{x}|_Q^2 + \hat{\boldsymbol{x}}'_N G_\sigma \hat{\boldsymbol{x}}_N.$$

Taking into account that $\hat{\boldsymbol{x}}_N \in \mathbb{X}_f$, and considering from (5.34) that

$$a_1|\boldsymbol{x}|^2 \leq \boldsymbol{x}'P\boldsymbol{x} \leq \varphi_x, \quad \forall \boldsymbol{x} \in \mathbb{X}_f,$$

it follows that

$$\Delta V_N^{op}(\boldsymbol{x}) \leq -c_3|\boldsymbol{x}|^2 + d, \quad \forall \boldsymbol{x} \in \mathrm{X}_N, \tag{5.44}$$

Therefore, condition (5.4) holds with $\alpha_3(s) = c_3 s^2$ and $d = \frac{\varphi_x}{a_1}|G_\sigma|$.

Now, suppose that for an instant $t > 0$, $\boldsymbol{x} \in \mathbb{X}_f$. Then, using (5.43) and (5.44), it is possible to establish that

$$\begin{aligned} V_N^{op}(\boldsymbol{x}_{k+1}) &\leq V_N^{op}(\boldsymbol{x}_k) - c_3|\boldsymbol{x}_k|^2 + d \\ &\leq V_N^{op}(\boldsymbol{x}_k) - \frac{c_3}{c_2} V_N^{op}(\boldsymbol{x}_k) + d \\ &\leq \rho_n V_N^{op}(\boldsymbol{x}_k) + d \end{aligned} \tag{5.45}$$

where $\rho_n = 1 - \frac{c_3}{c_2} \in [0, 1)$. Therefore, by iterating (5.45), the optimal cost function will be exponentially bounded by

$$V_N^{op}(\boldsymbol{x}_k) \leq \rho_n^k V_N^{op}(\boldsymbol{x}_t) + \left(\frac{1-\rho_n^k}{1-\rho_n}\right) d, \quad \forall k \geq t, \boldsymbol{x} \in \mathbb{X}_f.$$

From (5.42), we have that $|\boldsymbol{x}|_P^2 = x'Px \leq V_N^{op}(\boldsymbol{x})$. Consequently, considering (5.43), the system state evolution will be exponentially bounded via

$$|\boldsymbol{x}_k|_P^2 \leq c_2 \rho^k |\boldsymbol{x}_t| + \left(\frac{1-\rho_n^k}{1-\rho_n}\right) d, \quad \forall k \geq t, \boldsymbol{x} \in \mathbb{X}_f,$$

Finally, the system state will be ultimately bounded by

$$\limsup_{k\to\infty} |\boldsymbol{x}_k|_P^2 \leq \frac{d}{1-\rho_n} = \delta.$$

By taking into account the condition (5.40), we obtain that $\delta < \varphi_x$. Therefore, comparing (5.34) with (5.41), we have that $\mathcal{D}_\delta \subset \mathbb{X}_f$.

Consequently, by Theorem 3, the multi-step sparse MPC loop (5.15) is UpAS for all $\boldsymbol{x}_0 \in \mathrm{X}_N \backslash \mathbb{X}_f$ and practically exponentially stable for $k > t$. □

Theorem 6 establishes that for all $\boldsymbol{x}_0 \in \mathrm{X}_N$, the system state will be steered by the multi-step sparse predictive controller, $\boldsymbol{\kappa}_N(\boldsymbol{x})$ in (5.14), towards the terminal region $\mathbb{X}_f \subset \mathrm{X}_N$ and then (with the same controller) into the ultimately bounded set $\mathcal{D}_\delta \subset \mathbb{X}_f$.

Remark 3. *Notice that decay rate ρ_n in the previous Theorem depends on the binary variable $\boldsymbol{\sigma}$. Thus, one can use the results of this theorem to reduce the number of input to guarantee stability of the closed-loop while obtaining a desired performance in terms of the decay rate ρ_n.*

Dual-Mode Sparse MPC Formulation

By using the local sparse controller in (5.33), it is possible to define a dual-mode sparse MPC strategy as follows:

$$\boldsymbol{\kappa}_{DM}(\boldsymbol{x}) = \begin{cases} \boldsymbol{\kappa}_N(\boldsymbol{x}), & \boldsymbol{x} \in X_N \backslash \mathbb{X}_f \\ \boldsymbol{\kappa}_f(\boldsymbol{x}), & \boldsymbol{x} \in \mathbb{X}_f \end{cases}$$

Thus, the resulting dual-mode sparse MPC loop can be expressed via:

$$\boldsymbol{x}_{k+1} = A\boldsymbol{x}_k + B\boldsymbol{\kappa}_{DM}(\boldsymbol{x}_k), \quad \forall \boldsymbol{x}_k \in X_N. \tag{5.46}$$

Theorem 7 (Stability of Dual-Mode Sparse MPC)**.** *Suppose that the matrix P in the terminal cost, $V_f(\boldsymbol{x})$, satisfies* (5.36)*, and the proposed sparse local controller, $\boldsymbol{\kappa}_f(\boldsymbol{x})$ in* (5.33)*, satisfies both* (5.37) *and* (5.40)*, then* (5.46) *is UAS, i.e.,* $\limsup_{k\to\infty} |\boldsymbol{x}_k| = 0$ *for all $\boldsymbol{x}_0 \in X_N$.*

Proof. The proof can be derived based on the proofs of Lemma 1 and Theorem 6. □

The proposed dual-mode sparse MPC, $\boldsymbol{\kappa}_{DM}(\boldsymbol{x})$, allows the system state to achieve the origin by relying on the local sparse controller, $\boldsymbol{\kappa}_f(\boldsymbol{x})$. Thus, infinite number of switches of the control signal on a finite-time interval, i.e. chattering effects (see [82] for further details), can be avoided. Note that chattering is a harmful phenomenon because it leads to undesirable vibrations of mechanical elements and overheating of electronic devices.

5.6 Simulation study

Here, we illustrate the benefits of the proposed sparse MPC strategy. Consider the system (5.5) with

$$A = \begin{bmatrix} 0.0721 & 0.6583 & -0.4689 & 0.2238 \\ -0.1881 & 0.5344 & 0.2543 & -0.6755 \\ 0.6522 & 0.3096 & 0.5503 & 0.1500 \\ -0.4926 & 0.1645 & 0.5091 & 1.0681 \end{bmatrix} \tag{5.47}$$

$$B = \begin{bmatrix} 0.2138 & 0.3385 & -0.1888 \\ 0.4112 & -0.0666 & -0.2024 \\ 0.6095 & 0.1967 & 0.2353 \\ -0.2627 & -0.0707 & 0.3762 \end{bmatrix} \tag{5.48}$$

where $\boldsymbol{x}_k \in \mathbb{R}^4$, $\boldsymbol{u}_k \in \mathbb{R}^3$. Matrix A has 2 unstable eigenvalues, and the pair (A, B) is controllable. The sparse constraint over the input is set as $|\boldsymbol{u}_k|_0 \leq \gamma$, with $\gamma = 2$. Additionally, a convex constraint is imposed to the system as $|\boldsymbol{x}_k| \leq \delta_x = 3$.

The sparse MPC strategy (5.15) was implemented with parameters $N = 4$, $Q = \mathcal{I}_{4\times 4}$, and $R = diag([3\ \ 3\ \ 3])$. The terminal cost, $V_f = |\boldsymbol{x}|_P^2$, is chosen in order to satisfy condition (5.36), yielding:

$$P = \begin{bmatrix} 2.5820 & -0.2209 & -0.3899 & -0.8719 \\ -0.2209 & 3.1023 & 1.2104 & 1.0563 \\ -0.3899 & 1.2104 & 5.2290 & 2.7497 \\ -0.8719 & 1.0563 & 2.7497 & 6.3644 \end{bmatrix}.$$

In order to illustrate the benefit of the proposed approach, we introduce a non-convex constraint over each vector input $\hat{\boldsymbol{u}}_j$ of the prediction horizon. These constraints are as follows

$$\hat{\boldsymbol{u}}_j' Q_1 \hat{\boldsymbol{u}}_j \leq 1, \tag{5.49}$$

$$\hat{\boldsymbol{u}}_j' Q_2 \hat{\boldsymbol{u}}_j + f_2 \hat{\boldsymbol{u}}_j + \rho_2 \geq 1, \tag{5.50}$$

where

$$Q_1 = \begin{bmatrix} 0.3472 & 0 & 0 \\ 0 & 0.3472 & 0 \\ 0 & 0 & 0.3472 \end{bmatrix}, \tag{5.51}$$

$$Q_2 = \begin{bmatrix} 3.1250 & 0 & 0 \\ 0 & 3.1250 & 0 \\ 0 & 0 & 3.1250 \end{bmatrix}, \tag{5.52}$$

$$f_2 = \begin{bmatrix} -10.6066 & 0 & 0 \end{bmatrix}, \tag{5.53}$$

$$\rho_2 = 9. \tag{5.54}$$

Based on the proposed design, the terminal region is chosen as:

$$\mathbb{X}_f \triangleq \left\{ \boldsymbol{x}_f' P \boldsymbol{x}_f \leq \varphi_x = 4.1223 \right\}.$$

This value of φ_x assures that the terminal region satisfies that $\mathbb{X}_f \subset \mathbb{X}$, and that $K_\sigma \boldsymbol{x} \in \mathbb{U}$ (definition of terminal region in (5.34)).

For this example, a vector of active inputs $\sigma = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}'$ satisfies condition (5.37). Therefore, by Theorem 5, system (5.5), with (5.47) and (5.48), governed by the proposed sparse MPC is UpAS.

Starting from the initial state $x_0 = \begin{bmatrix} 1 & -1.5 & -1 & 2 \end{bmatrix}'$, the proposed sparse MPC strategy implemented using the solver BARON [53].

An exhaustive search method (i.e., evaluating all possibilities and then selecting the optimal one) was implemented using BARON by fixing zeros in the standard MPC problem and solving the resulting quadratic programming (QP) problem. This approach proved to be impractical for this particular example due to the big amount of time required for some solutions. This is due to a resulting complex optimization problem when forcing some variables to be zero valued.

Solution of the resulting MPC problem using the proposed approach ($\mathbb{P}_{equiv,N}$) is obtained using BARON optimization software. For comparison purposes, the same ℓ_0-input constrained MPC problem is formulated using a mixed-integer approach and solved also by utilizing BARON.

We also apply the sparse local controller strategy shown in subsection 5.5.1. This control strategy is applied after simulation step 13, once the state vector is near the origin.

The results of the simulations of the two different approaches are shown in Figs. 5.3 and 5.4. Here, $u\_\texttt{bilinear}$ and $u\_\texttt{mixed\_integer}$ represent the optimal sparse input obtained by the proposed sparse MPC strategy and the mixed-integer approach. These

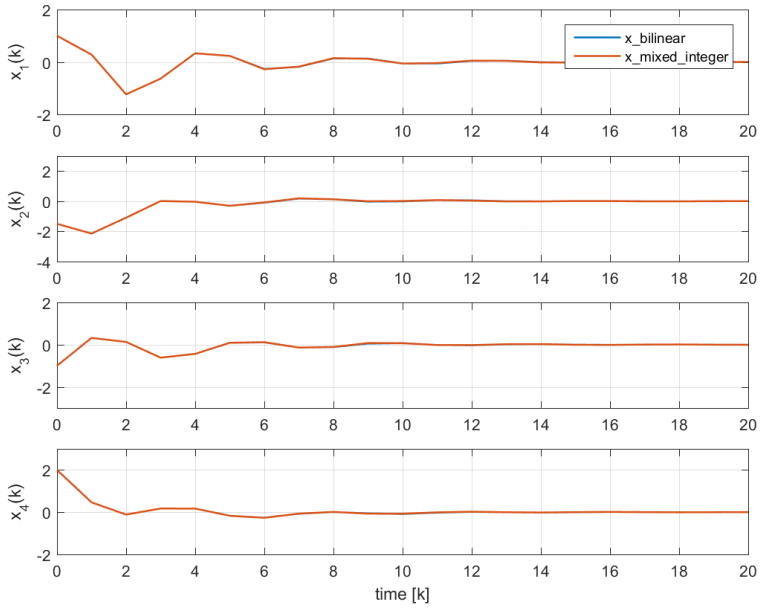

Fig. 5.3: System state trajectory at each time step.

inputs lead to the corresponding system state trajectories denoted by $x\_\texttt{bilinear}$ and $x\_\texttt{mixed\_integer}$ respectively. From Figs. 5.3, 5.4 and 5.5, it can be noticed that the system constraints are satisfied, and that the system is led to the origin by using only 2 active inputs. Moreover, in Figs. 5.4 and 5.5 we note that the optimal inputs applied obtained with the two strategies are practically the same. Only a slightly differences arises when the state is near the origin, which could be due to numerical problems. Some chattering can be observed before we commute to the local controller (specially in input u_2).

Finally, an important matter to analyze is the execution time carried out for each optimization approach. The computing time of the proposed approach was 46.2 seconds, while the mixed-integer formulation took 83.6 seconds, thus being slower. However, a more comprehensive study is needed for the general case, in order to derive further conclusions.

5.7 Conclusion

In this chapter we have addressed the problem of sparse feedback control utilizing a quadratic MPC technique for deterministic time-invariant linear systems written in state-space form. The proposed control strategy considers only some of the available inputs as “active” at each control horizon instant. This condition is imposed by utilizing an

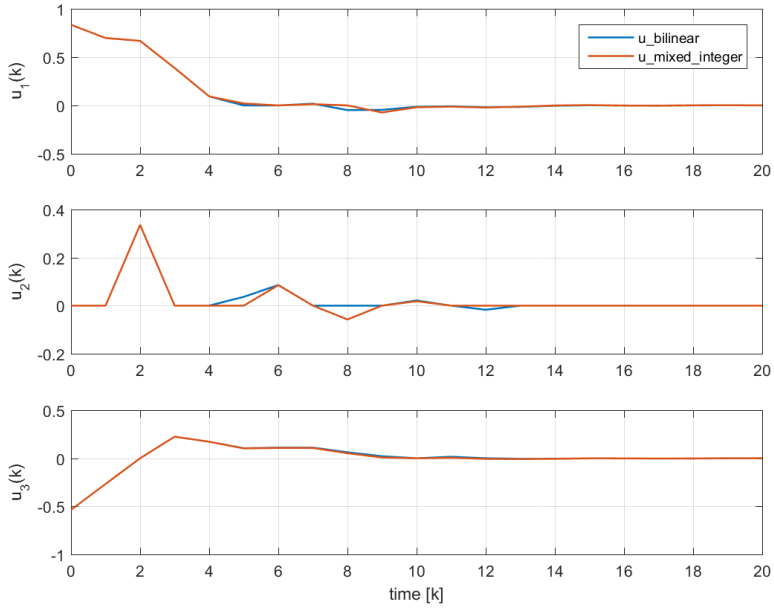

Fig. 5.4: Sparse optimal input at each time step.

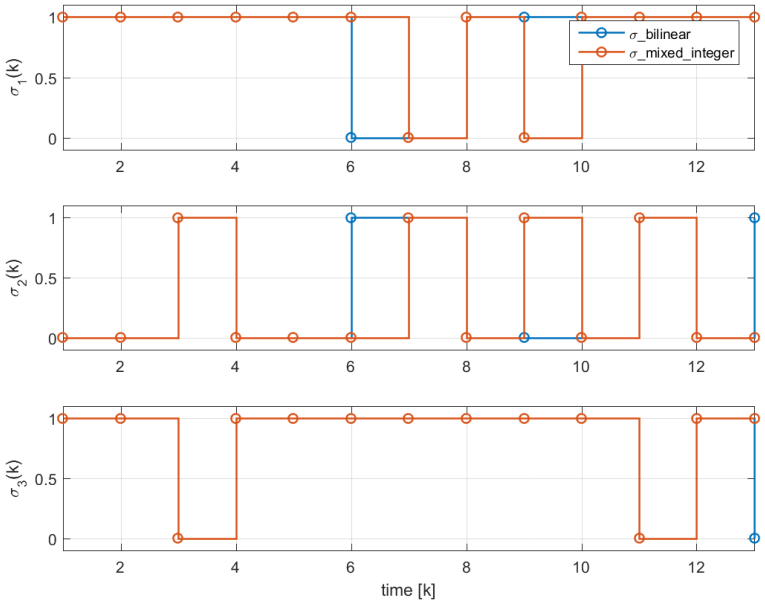

Fig. 5.5: Optimal active input sequence $\vec{\boldsymbol{\sigma}}^{op}$ for each simulation time k.

ℓ_0-(pseudo)-norm constraint. The resulting optimization problem was then rewritten into an equivalent optimization problem which can be solved utilizing a non-linear pro-

gramming optimization toolbox (e.g. BARON). Additional conditions were proposed in order to assure stability of the feedback system. Finally, we proposed a solution for the potential chattering effect that might happen when the state approaches the origin.

Chapter 6

LOW-ORDER CONTROL DESIGN USING A NOVEL RANK-CONSTRAINED OPTIMIZATION APPROACH

In this chapter, a recent equivalent representation of rank constraints is used to design a low-order controller with prescribed degree of stability. We solve an optimization problem involving linear matrix inequalities and rank constraints. We illustrate the potential of the proposed approach by comparing with similar approaches available in the literature.

6.1 Introduction

Rank-constrained optimization has gained increased attention in the last decades. Recent advances in convex optimization and the development of easy-to-use optimization software have helped to increase the usage of such software within the systems and control community. The success of nuclear norm, log-det and trace heuristics, see e.g. [7,89], in some problems have motivated several researchers to formulate a large number of engineering problems in terms of optimization problems that include rank constraints. A classic example of such engineering problems arises in system identification, where the order of a rational system is equal to the rank of an infinite dimensional Hankel matrix [90]. Another example is Factor Analysis (see e.g. [37]), where the number of latent factors is equal to the rank of a covariance matrix.

Although heuristics such as the nuclear norm provide a convenient way to address rank constraints in optimization problems, there is an inherent loss of performance in the use of this heuristic [91]. Moreover, most heuristics consider the rank constraint as a soft constraint, i.e the obtained solution may violate the rank constraint. This

approach to deal with rank constraints may be unsatisfactory in some applications. This has motivated the development of methods that consider the rank constraint as a hard constraint, see e.g. [8, 92, 93]. These methods are based on the notion of equivalent representations of a rank constraint. These equivalent representations are aimed at overcoming some of the undesirable features of the rank function, namely, non-linearity, non-smoothness and non-convexity of the rank function. In this chapter, we focus on the rank-constraint representation described in [8] that allows to solve rank-constrained optimization problems within the framework of nonlinear programming.

In the rank-constraint representation in [8] the value of the rank in the rank constraint, say r, is controlled by the value of the trace of an auxiliary matrix. These presents a major advantage over other rank-constraints representations where the value of r is controlled by the size of auxiliary matrices, see e.g. [94] or by including a constraint that is non-convex on r, see e.g. [92], [93].This feature may be useful in problems where the value of the rank to be constrained is unknown *a priori*.

In this chapter we apply the rank-constraint representation described in [8] to a Reduced Order Output Feedback stabilization problem and address variants of the same problem. We then perform a numerical comparison of the proposed approach against state-of-the-art alternative methods.

6.2 Reduced Order Output Feedback

To illustrate the features of the rank-constraint representation in [8] we apply it to a rank-constrained optimization problem. In particular, we focus in the problem of Reduced order output feedback stabilization. In this section, we described the approach proposed in [14] that uses Linear Matrix Inequalities (LMI) to find a reduced order controller for the output feedback stabilization problem. The benefit of this formulation is that it allows us to define an optimization framework to solve the problem of interest.

Consider a continuous time, linear time invariant (LTI) system

$$\dot{\boldsymbol{x}}(\boldsymbol{t}) = A\boldsymbol{x}(\boldsymbol{t}) + B\boldsymbol{u}(\boldsymbol{t}) \tag{6.1}$$

$$\boldsymbol{y}(\boldsymbol{t}) = C\boldsymbol{x}(\boldsymbol{t}) \tag{6.2}$$

where $\boldsymbol{x} \in \mathbb{R}^n$ is the system state, $\boldsymbol{u} \in \mathbb{R}^m$ is the control signal and $\boldsymbol{y} \in \mathbb{R}^p$ is the measured output. The controller is given by

$$\begin{bmatrix} \dot{\boldsymbol{x}}_{\boldsymbol{c}}(\boldsymbol{t}) \\ \boldsymbol{u}(\boldsymbol{t}) \end{bmatrix} = K \begin{bmatrix} \boldsymbol{x}_{\boldsymbol{c}}(\boldsymbol{t}) \\ \boldsymbol{y}(\boldsymbol{t}) \end{bmatrix} \tag{6.3}$$

where $K \in \mathbb{R}^{(n_c+m)\times(n_c+p)}$ and $\boldsymbol{x}_{\boldsymbol{c}} \in \mathbb{R}^{n_c}$ is the controller state.

Define $\tilde{A} = \begin{bmatrix} A & 0 \\ 0 & 0_{n_c} \end{bmatrix}$, $\tilde{B} = \begin{bmatrix} 0 & B \\ I_{n_c} & 0 \end{bmatrix}$ and $\tilde{C} = \begin{bmatrix} 0 & I_{n_c} \\ C & 0 \end{bmatrix}$. The following lemma establishes necessary and sufficient conditions to make sure that the real part of the closed loop system ($\dot{\tilde{\boldsymbol{x}}} = \mathcal{A}\tilde{\boldsymbol{x}} = (\tilde{A} + \tilde{B}K\tilde{C})\tilde{\boldsymbol{x}}$) poles are on the left of $s = -\alpha$.

Lemma 2. *(see [14]) Let $\mathcal{A}$ be a given square matrix and α be a given positive scalar. Then the following statements are equivalent:*

1. *The system $\dot{\tilde{x}} = \mathcal{A}\tilde{x}$ is α-stable (with prescribed degree of stability α).*
2. *There exists a matrix $\mathcal{Y} \succ 0$ such that $(\mathcal{A} + \alpha I)'\mathcal{Y} + \mathcal{Y}(\mathcal{A} + \alpha I) \prec 0$.*

Note that statement 2 of Lemma 2 involves a bilinear form of the two unknown matrices $\mathcal{Y}$ and K (since the closed loop matrix $\mathcal{A}$ depends on K). In [14], the unknown controller terms are eliminated from the bilinear form and necessary and sufficient conditions for the existence of an α-stabilizing controller of order n_c are found. In [14] the existence of a α-stabilizing controller of order n_c for a given $\alpha > 0$ can be tested by solving a set of LMI subject to rank constraint as described below.

Consider a system defined as in (6.1)-(6.2) and a given scalar $\alpha > 0$. Solving the following feasibility problem for $X \succ 0$ and $Y \succ 0$ assures that an α-stabilizing controller of order n_c exists (see [14]).

$$
\begin{aligned}
\mathcal{P}_0 : \qquad & \text{Find} \;\; X, Y \in \mathcal{S}^n \\
& \text{s.t.} \;\; -B^{\perp}(AX + XA' + 2\alpha X)B^{\perp'} \succ 0 \\
& \qquad\;\; -C'^{\perp}(YA + A'Y + 2\alpha Y)C^{\perp} \succ 0 \\
& \qquad\;\; \begin{bmatrix} X & I \\ I & Y \end{bmatrix} \succeq 0 \\
& \qquad\;\; \operatorname{rank} \begin{bmatrix} X & I \\ I & Y \end{bmatrix} \leq n + n_c
\end{aligned}
$$

where B' is a matrix of maximal rank such that its rows are orthogonal and $B'B = 0$. Similar conditions hold for $C'^{\perp}$.

Solution matrices X and Y of $\mathcal{P}_0$ are related in the following way to statement 2 of Lemma 2 (see [14] for details):

$$
\mathcal{Y} = \begin{bmatrix} Y & Y_{12} \\ Y_{12}' & Y_{22} \end{bmatrix} \quad \text{and} \quad \mathcal{Y}^{-1} = \begin{bmatrix} X & X_{12} \\ X_{12}' & X_{22} \end{bmatrix}
$$

where $X_{22}, Y_{22} \in \mathbb{R}^{n_c \times n_c}$. Thus, by solving $\mathcal{P}_0$ the unknown α-stabilizing controller K can be found by solving the LMI found in statement 2 of Lemma 2 as described below.

Consider the Matrix Inversion Lemma, and take the eigenvalue decomposition $X - Y^{-1} = V\Lambda V'$, where Λ is a diagonal matrix whose entries are the eigenvalues ordered in decreasing order. Define $R = V(:, 1 : \lambda_{n_c})diag(\lambda_1^{1/2}, ..., \lambda_{n_c}^{1/2})$ and $\tilde{X} = \begin{bmatrix} X & R \\ R' & I \end{bmatrix}$. A controller K that fulfills statement 2 of Lemma 2 is found by solving the following optimization problem:

$$\begin{aligned} \mathcal{P}_K : \max_{\gamma \in \mathbb{R}, K} \quad & \gamma \\ s.t. \quad & (\tilde{A} + \tilde{B}K\tilde{C})\tilde{X} + \tilde{X}(\tilde{A} + \tilde{B}K\tilde{C})' + 2\gamma\tilde{X} \leq 0 \end{aligned} \tag{6.4}$$

Solution γ of problem $\mathcal{P}_K$ represents a lower bound for the stability degree of the closed loop system $\dot{\tilde{\boldsymbol{x}}} = \mathcal{A}\tilde{\boldsymbol{x}}$ [5].

A Newton-like method to solve problems involving rank constrained linear matrix inequalities (LMI) is presented in [5]. In particular they use it to solve problem $\mathcal{P}_0$. It is important to note the cited approach locally solves the problem. The algorithm is implemented in the LMIRank solver that is freely distributed by the authors.

Rank Minimization Approach

In this section we describe the method presented in [6] to solve a similar problem. This is mentioned for comparison purposes only. In [6] an iterative rank minimization procedure is presented and used for (locally) solving the similar problem of finding a stabilizing controller of a certain order (stability degree is not a constraint). The algorithm in [6] reduces the rank of a matrix constraint in a convex set. In [6], the problem of determining the existence of a low order controller is treated (for a system described as in (6.1)-(6.2)).

Lemma 3. *(see [6], [95]) There exists a stabilizing output feedback law of order k if and only if the global minimum of the rank minimization problem is less than $n + k$.*

$$\begin{aligned} \mathcal{P}_{rk} : \min_{W_1, W_2, \sigma} \quad & rank \begin{bmatrix} W_1 & I \\ I & W_2 \end{bmatrix} \\ s.t. \quad & AW_1 + W_1A' \prec \sigma BB' \\ & A'W_2 + W_2A \prec \sigma C'C \\ & \begin{bmatrix} W_1 & I \\ I & W_2 \end{bmatrix} \succeq 0 \end{aligned} \tag{6.5}$$

$$\sigma > 0 \tag{6.6}$$

where $W_1, W_2 \in \mathbb{S}^n$ and $\sigma \in \mathbb{R}^+$.

Note that problem $\mathcal{P}_{rk}$ represents a particular case of $\mathcal{P}_1$. This can be seen by taking $\lim \alpha \to 0^+$ and by considering that $B^{\perp}B = 0$ (for further insight, see section 2.6 of [96]).

Note also that problem $\mathcal{P}_{rk}$ must incorporate a stop condition. This is due to the existence of an infinite class of controllers satisfying the stated conditions. In the iterative rank minimization algorithm proposed in [6], the program is stopped once a desired order, i.e. rank, is achieved.

In this chapter we implement the cited approach to solve problem $\mathcal{P}_0$ and compare its performance with other methods.

6.3 Equivalence by Rank Constraint Representation

In this section we use the approach presented in [8] to find a equivalent representation of problem $\mathcal{P}_0$. The need of equivalent representations for rank constraints arise because the rank function has several features that are undesirable in optimization problems. In particular, the rank function is non-smooth, non-linear and non-convex. In the optimization literature, smoothness and convexity are widely exploited, and the lack of such features in the rank function limits the tools that can be used in the to solve the optimization problem. Thus, equivalent representations aim at overcoming at least one of these undesirables features of the rank function. Recently, equivalent representations have been utilized to avoid the direct treatment of rank constraints ([8, 94], page 241 of [9]).

The following result describes the equivalent representation of a rank constraint presented in [8].

Lemma 4. *Let $G \in \mathbb{R}^{m\times n}$, then the following expressions are equivalent*

(i) $\operatorname{rank}(G) \leq r$

(ii) $\exists W \in \Phi_{n,r}$, *such that* $GW = 0_{m\times n}$

where

$$\Phi_{n,r} = \{W \in \mathbb{S}^n,\ 0 \preceq W \preceq I, \mathit{trace}(W) = n - r\} \tag{6.7}$$

Proof. See [8]. □

The result from Lemma 4 can be seen as a generalization of the one provided in [9] (the former can be used on rank constraints over real matrices of all sizes). An advantage of Lemma 4 is that it represents a rank-constraint in a form that can be used for optimization purposes. The constraints imposed by the set $\Phi_{n,r}$ can be handled by Semidefinite Programming. However, the bilinear condition, $GW = 0$, need to be

addressed in the context of nonlinear programming. Optimization software as BARON [54] [53] allows us to solve problems with this type of bilinear constraints and to obtain a global solution.

This approach to deal with rank constraints has been applied in several framework including: Model Predictive control [36], Factor Analysis [37] and to nonlinear system identification [65].

The following result presents an equivalent representation of problem $\mathcal{P}_0$.

Theorem 8. *Let $W \in \mathbb{S}^{2n}$, $0 \preceq W \preceq I$, trace$(W) = n - n_c$, then feasibility problem $\mathcal{P}_0$ is equivalent (in the sense that has same global optimum) to the following problem*

$$
\begin{aligned}
\mathcal{P}_1 : \qquad & \text{Find} \;\; X, Y \in \mathcal{S}^n, W \in \mathcal{S}^{2n} \\
& s.t. \;\; - B^{\perp}(AX + XA' + 2\alpha X)B^{\perp'} \succeq 0 \\
& \quad - C'^{\perp}(YA + A'Y + 2\alpha Y)C^{\perp} \succeq 0 \\
& \quad \begin{bmatrix} X & I \\ I & Y \end{bmatrix} \succeq 0 \\
& \quad \begin{bmatrix} X & I \\ I & Y \end{bmatrix} W = 0 \\
& \quad \text{trace}(W) = 2n - (n + n_c) \\
& \quad 0 \preceq W \preceq I
\end{aligned}
$$

Proof. Problems $\mathcal{P}_0$ and $\mathcal{P}_1$ both have the same feasible set with respect to X and Y. Moreover, these consditions do not depend on W. Hence, to prove the equivalence between $\mathcal{P}_0$ and $\mathcal{P}_1$ it suffices with using Lemma 4, which proves the validity of the rank-constraint representation. □

As stated in Theorem 8, problem $\mathcal{P}_0$ can be transformed into a problem, $\mathcal{P}_1$, that does not explicitly include the rank constraint, but has same optimum as the original problem. This new formulation of the problem can be solved by standard tools of nonlinear programming such as those provided by software BARON [53, 54].

6.4 Further Extensions of the Approach

Similar to Theorem 8 we can also formulate other problems of interest.

6.4.1 Minimization of Controller's Order

Lemma 4 relates the upper bound of a rank constraint with the trace of an auxiliary matrix W. This allows the formulation of a rank minimization problem by maximizing the trace of W. Thus, the problem of finding the α-stabilizing controller with minimum order can be formulated as follows (for a given α)

$$
\begin{aligned}
\mathcal{P}_2 : \quad & \min_{X,Y\in\mathcal{S}^n, W\in\mathcal{S}^{2n}} 2n - \operatorname{trace}(W) \\
\text{s.t.} \quad & -B^{\perp}(AX + XA' + 2\alpha X)B^{\perp'} \succ 0 \\
& -C'^{\perp}(YA + A'Y + 2\alpha Y)C^{\perp} \succ 0 \\
& \begin{bmatrix} X & I \\ I & Y \end{bmatrix} \succeq 0 \\
& \begin{bmatrix} X & I \\ I & Y \end{bmatrix} W = 0 \\
& 0 \preceq W \preceq I
\end{aligned}
$$

Similar rank representations can be found in the literature. In [1], a representation involving two auxiliary matrices whose dimensions depend on the rank constraint's bound r is proposed. Due to the dependence of the auxiliary matrices dimensions with the rank bound, it is not plausible to use the representation in [1] to state problem $\mathcal{P}_2$.

In [6], the reduced order controller is obtained by an iterative approach that minimizes the rank of a matrix. This approach however only assures local convergence, while the representation applied in this work has the same global optimum as the original problem and can be obtain utilizing nonlinear programming techniques.

Remark 4. *It is important to note that when implementing problem $\mathcal{P}_2$ a stop mechanism must be incorporated. Suppose that for a given system, the minimum order possible for a stabilizing controller is n_c. Problem $\mathcal{P}_2$ is defined as a minimization problem, thus it will not stop once found a controller of order n_c, in fact it will continue searching in the infinite set of controllers, trying to find one of even smaller order. To fix this issue, we add a time constraint while solving $\mathcal{P}_2$ with the global optimization software BARON. Other authors such as [6] set a lower bound for the achieved controller's order which can be compared in each step of their iterative rank minimization approach.*

6.4.2 Optimizing for controller order and α-stabilizing degree

Feasibility problem $\mathcal{P}_0$ stated before can be extended into a maximization problem, where the biggest value for parameter α is to be found (α is treated as a variable). Thus by

maximizing α and maintaining the rank constraint, the resulting pair of matrices (X, Y) can be used to obtain the fastest stabilizing controller of order n_c. This leads into the following optimization problem

$$
\begin{aligned}
\mathcal{P}_3 : \quad & \max_{\alpha\in\mathbb{R}_{>0}, X, Y\in\mathbb{S}^n, W\in\mathcal{S}^{2n}} \alpha \\
\text{s.t.} \quad & -B^{\perp}(AX + XA' + 2\alpha X)B^{\perp'} \succ 0 \\
& -C'^{\perp}(YA + A'Y + 2\alpha Y)C^{\perp} \succ 0 \\
& \begin{bmatrix} X & I \\ I & Y \end{bmatrix} \succeq 0 \\
& \begin{bmatrix} X & I \\ I & Y \end{bmatrix} W = 0 \\
& \operatorname{trace}(W) = 2n - (n + n_c) \\
& 0 \preceq W \preceq I
\end{aligned}
$$

The flexibility of the approach presented can be also used to another where the stability degree and controller order are optimize at the same time by imposing a trade-off between them:

$$
\begin{aligned}
\mathcal{P}_4 : \quad & \min_{\alpha\in\mathbb{R}_{>0}, X, Y\in\mathbb{S}^n, W\in\mathcal{S}^{2n}} \operatorname{trace}(W) - \alpha \\
\text{s.t.} \quad & -B^{\perp}(AX + XA' + 2\alpha X)B^{\perp'} \succ 0 \\
& -C'^{\perp}(YA + A'Y + 2\alpha Y)C^{\perp} \succ 0 \\
& \begin{bmatrix} X & I \\ I & Y \end{bmatrix} \succeq 0 \\
& \begin{bmatrix} X & I \\ I & Y \end{bmatrix} W = 0 \\
& 0 \preceq W \preceq I
\end{aligned}
$$

This problems are currently been studied in order to understand their solution space.

6.5 Numerical Comparison

In this section we carry numerical examples in order to compare the performance and effectiveness of the approach.

We consider the reduced order feedback control problem used in [13] and [5]. The

system has the following state-space matrices

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}'$$

6.5.1 α-stabilizing Controller

First, we search for α-stabilizing controllers of order $n_c = 2$ for given values of α (which imposes a lower bound for the stabilization degree). In Table 6.1, the resulting stability degree $\hat{\alpha}$ of the closed loop system is shown. Note that that since we are solving a feasibility problem it is possible to obtain a closed loop with greater stabilizing degree. We use three different approaches to solve $\mathcal{P}_0$: i) the approach presented in Orsi *et. al.* [5], ii) the approach in Sun *et. al.* [6] and iii) formulating an alternative problem utilizing an equivalent representation of rank constraints. We constrain the computation-time and maximum number of iterations to compare the different approaches. Orsi's approach was limited to 20000 iterations while Sun's and our approach was limited to a 500$[s]$ computation time.

Table 6.1: Achieved Closed Loop α-stability solving $\mathcal{P}_1$

	Orsi *et. al.*		Sun *et. al.*		Equivalent Representation	
α	$\hat{\alpha}$	$T[s]$	$\hat{\alpha}$	$T[s]$	$\hat{\alpha}$	$T[s]$
0.2	0.203	1.4	0.2	2.4	0.305	151.2
0.42	0.420	3.0	-	-	0.506	440.8
0.46	0.467	172.2	-	-	0.521	380.3
0.5	-	-	-	-	0.500	125.4
0.502	-	-	-	-	0.529	451.7

Although our approach takes more time than the one shown in [5], we are able of finding controllers that result in a better closed loop stability degree $\hat{\alpha}$, or that others could not find. This is due to the nonlinear programming technique that the solver utilizes. Given that the goal is to find a static controller K, the solution time of the approach is not of much relevance depending on the solver used, a suboptimal solution could be obtain if time is a constraint. Note that the closed loop stability degree obtained is not necessarily the same as the required, and in some cases is far greater. This shows

that the solver doesn't work in an iterative way (improving some parameter at each step), which might lead to local minimum.

Although the equivalent representation for rank constraints allows us to use nonlinear programming techniques, the problem is still computationally demanding. Complexity and computational load might increase for some particular problems. This is seen for example when solving $\mathcal{P}_0$ for higher closed loop stability degree (i.e. increasing α).

6.5.2 Reducing controller order

Next we solve problem $\mathcal{P}_2$, where the objective is to minimize the controller's order for a user specified stabilizing degree α. Considering Remark 4 we add a time constraint into the solver of $t_{max} = 300[s]$. In Table 6.2 are shown the results such as achieved controller order $\hat{n_c}$ and correspondent closed loop stabilizing degree.

Table 6.2: Controller's order n_c for given stabilizing degree α, solution time $t_{max} = 300[s]$

α	$\hat{n_c}$	$\hat{\alpha}$
0.1	2	0.105
0.2	3	0.221
0.5	3	0.786
0.7	3	0.786
1	3	1.378

We note that in relatively the same time used for examples for problem $\mathcal{P}_1$ we have achieved a better closed loop stabilizing degree.

6.6 Conclusions and future work

In this chapter we addressed the problem of designing a reduced order feedback control. We incorporated rank constraints in order to restrict the order of the unknown controller through an optimization problem. The resulting optimization framework gives us the possibility of formulating additional rank constrained problems for control design. We have applied an equivalent rank constraint representation to reformulate the problem into another one that is equivalent in a global optimum sense. The resulting (global optimum-equivalent) problem can be solved by using nonlinear programming techniques. We also formulated two additional extensions of the original reduced order control problem: 1) maximization of the stability degree, given a controller's order, 2) given a certain stability

degree, minimize the order of the controller. This shows the versatility of the rank constraint representation to solve different control design problems. Finally numerical examples were shown in order to illustrate the performance of the proposed method.

Chapter 7

CONCLUSIONS

7.1 Overview

Rank constrained and rank penalized optimization problems are a class of optimization formulations with great potential in the scientific world. Application of this type of formulation can be seen and foreseen in a wide variety of areas of interest, including system identification and control.

The handling of the rank constraint or penalization term is a very important step when attempting to solve these type of problems. As it has been shown in the introducing chapter, several algorithms and approaches exist. These algorithm normally relay on approximations (convex or non-convex) in order to formulate a similar easy-to-solve problem with (in general) a different optimum. Other procedures formulate iterative search algorithms where, although condition for identifying convergence to a global optimum are presented, only local convergence is assured.

Solving these problem by utilizing an equivalent representation that has the same global optimum and allows the attainment of this solution by using standard non-linear programming tools is thus a great breakthrough.

The thesis work here presented starts by introducing the general formulation of rank constrained and rank penalized problems while also detailing their possible applications and the difficulties that they pose. State of the art solution methods are shown with their benefits and drawback, including a novel equivalent representation which is used in the solution of the system identification and control problems presented.

The remaining chapters of the thesis present different problems of great interest for the community of system identification and control. These problems include cardinality constrained problems (identification and optimal control) and one rank constrained problem related to low-order controller design. Comparisons with existing methods have been shown and a discussion of the solutions obtained was presented for each case.

7.2 Summary of contributions by chapters

The main contribution of the thesis are the following:

- Chapter 2 introduces rank constrained optimization together with its promising applications and algorithms commonly used for their solution.

- In chapter 3 a classification problem using logistic regression with cardinality constraint over the number of parameters is presented. Key features of model and sample set are identified by promoting sparsity in the solution vector. A comparison between standard Maximum Likelihood (ML) estimation, cardinality constrained ML and the use of Information Criteria such as AIC is performed, by identifying a given model with a sparse parameter vector.

  Simulation show that the formulation involving cardinality constrained ML performs better than the other alternatives, but requires a prior knowledge or estimation of the number of parameters to be estimated. On the other hand, penalized ML or AIC give an excellent a very good estimate of the parameter values and of the number of parameters by providing sparse solution vector for each experiment. This type of problem has big interest in areas of gene selection, machine learning, among others.

- Chapter 4 presents the study of an cardinality penalized optimization approach for model selection. Solution of the problem is done by approximating the cardinality constraint with smooth non-convex exponential terms. The use of this approximation intends to reduce the time spent in obtaining the solution while also performing a good approximation over the ℓ_0-norm.

  It is found that when estimating the parameters of a linear regression with AIC, the result of the proposed "approximating" approach does not differ with the results obtained when solving the original AIC scheme (testing all possible models and then selecting the best based on AIC).

  A similar procedure is performed in an identification scheme involving *hidden variables*, where model selection is done using Expectation Maximization (EM). Surprisingly, in this case by using EM with the proposed method (cardinality-approximation penalty), better estimates of the model parameter are obtained than when performing EM using the original AIC scheme (cardinality penalty). In a simple way: performing an approximation gives better results than solving the true problem. An explanation to this seemingly awkward behavior can reside on the iterative nature of the EM algorithm, where iterative solution of the cardinality penalized formulation (AIC) will result in a more erratic overall behavior.

- In chapter 5 a problem of feedback control with constrained number of active inputs is investigated. A novel quadratic model predictive control strategy is presented, where sparsity and stability of the solution is assured. Sparsity of the solution is treated by constraining the solution vector utilizing the ℓ_0-norm at each control horizon instant. Additional bounding constraints are also imposed over the system state and control input.

  Although the formulated problem contains difficult constraints such as non-convex closed sets and cardinality constraints, sufficient conditions are stated in order to guarantee asymptotic and practical stability of the closed-loop control solution.

  Alternative solutions are provided for the case of convex and non-convex bounding, by respectively using a mixed integer linear programming tool (CPLEX) and non-linear programming tool (BARON). The inherent combinatorial behavior of the resulting optimization problem is addressed by using an equivalent representation of the ℓ_0-norm, and solved using standard nonlinear programming tools.

  In order to avoid a possible chattering behavior due to continuously changing control input entries between active and non-active, a dual control strategy is developed when the system's state is near the origin.

- Finally in chapter 6, a low-order control design with given closed loop stability degree is studied. Formulation of the problem includes the use of Linear Matrix Inequalities (LMI) and rank constraints in order to limit the controller's order. Solution is carried out utilizing an equivalent representation, and comparison with state of the art algorithms is done through simulations.

  The proposed approach outperforms two algorithms available in the literature, by obtaining faster closed loop systems.

  Also, additional controller design formulation are possible to be defined and also their solution is attempted by using the equivalent rank constraint representation. Other possible problems are: minimization of controller's order, maximization of the stabilizing degree of the controller, and optimizing both parameter at the same time. It is found that when optimizing the stabilizing degree of the controller, additional stopping conditions must be added to the problem formulation.

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