Exploring Ultraviolet Complete Extensions of a Vector Dark Matter Model via the $\mathfrak{so}(4)$ algebra

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Abstract

In previous works it has been shown that a minimalist model for an isotriplet vector dark matter surfers from loss of unitarity because of the coupling between the vector field and the Higgs boson [1]. Nevertheless, algebraic manipulation of the vector sector by unifying the weak interaction fields with the vector dark matter fields reveals a possible path towards an ultraviolet completion. It will be shown in this work that the resulting vector sector constructed by linear combinations of these fields reproduces the gauge sector of a Yang-Mills theory over a group with an $\mathfrak{so}(4)$ algebra. The algebra isomorphism $\mathfrak{so}(4) \simeq \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ implies that an ultraviolet completion can be constructed by using the SO(4) cover group $SU(2) \times SU(2)$ to formulate the desired Yang mills theory. We have managed to construct such Yang-Mills theory which reproduces the desired gauge sector and we are currently studying the constraints on the parameter space coming from current experimental bounds related to dark matter and standard model mass spectrum.

Dedication and acknowledgements

First of all, I deeply thank my family for their unwavering support. I thank my sister for being a wonderful companion when we set our differences aside. I am grateful to my aunt for her cheerful presence and unconditional love. I also thank my godfather for his ability to listen, his emotional intelligence, and his wise advice. But above all, I am especially grateful to my mother, whose constant affection and care have always played a crucial role in difficult and stressful times. Her kindness has helped me overcome challenges such as this thesis, and for that, I will always be thankful.

For me, family extends beyond relatives, so it is only natural to express my gratitude to my closest circle of friends. Martín, Cheche, and Tammy are the first names that come to mind when I reflect on where the paths of friendship and family intertwine. To them, whom I consider more than friends, I express my deepest appreciation for choosing to stand by my side, both in moments of joy and in times of sorrow. I also want to thank my other friends who have supported me along the way, particularly Benja and Sergio, who have made my daily routine a pleasant journey.

On a more formal note, I extend my sincere gratitude to the evaluation committee for taking the time to review this thesis. I know it is longer than many (certainly longer than I expected), so your dedication is truly appreciated. In particular, I thank Professors Carolina and Alfonso, my advisors, for their patience in answering my questions (especially the oddly specific ones). I am also deeply grateful for their willingness to listen, even to my most unconventional ideas, as that encouragement has been a great motivation to keep learning.

I would also like to give special acknowledgment to Yerko A. Muñoz. He may not know it, but his undergraduate thesis served as a fundamental guideline for structuring my introduction.

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Chapter 1

Introduction to the Dark Matter problem

The problem of dark matter is one of the predominant mysteries of modern physics and has persisted since at least the 1930s. Defined by what we do not observe rather than by the detection of some phenomena, it arises from astronomical observations that reveal a discrepancy: *the visible matter in different systems is insufficient to account for their observed behavior*. To address this, two main hypotheses emerge: either our understanding of gravity is incomplete, requiring modifications to gravitational theory, or there exists a form of matter undetectable by current instruments, effectively rendering it invisible. This hypothetical form of matter is what we referred to as *dark matter*.

In particle physics, we address the problem by adopting the latter hypothesis, which claims that the issue arises from a type of matter (i.e. a set of particles) that remains undetected. Based on this assumption, many expected properties of dark matter can be calculated, providing a foundational framework for constructing models. This thesis builds upon that framework, with the objective of developing an *ultraviolet completion* of an existing dark matter quantum field model discussed in chapter 3. We call the extension of this model the *dark−electroweak* model, as it unifies both sectors under one unified sector and will be introduced in chapter 4.

To this end, this first chapter introduces the reader to the fundamental lines of evidence for dark matter and what they reveal about its nature. Special attention will be given to WIMPs at the end of this chapter, as they are the dark matter candidates of the models this thesis explores.

1.1 Astrophysical evidence

Dark matter is known only indirectly, as its presence is necessary to explain the behavior of many astronomical systems. Undetected matter provides the simplest explanation for why most large scale gravitationally bound systems exhibit the properties they do. In this sense, the only observed interaction between dark matter and *baryonic matter* (i.e., the ordinary matter that makes up planets, stars, and galaxies) is through gravity. Therefore, we begin by reviewing some of the most significant observations that reveal these interactions.

1.1.1 Galaxy clusters

Although the idea of an invisible type of matter being a major constituent of the universe was not new at the time [2–4], the first well-registered study that made a solid case for such a claim was carried out by Fritz Zwicky in 1932 [5]. By applying the Virial theorem to the Coma galaxy cluster (a system with distance order of Megaparcecs), he noticed that the gravitational effects of visible galaxies were too small for the fast orbits they had, which meant that unseen matter was needed to hold the galaxy cluster together. To see this, an analysis of the cluster under Newtonian gravity will suffice [6][1]. Let us consider the acceleration of the ith galaxy in the cluster as

$$\ddot{\mathbf{x}}_i = G \sum_{j \neq i} m_j \frac{\mathbf{x}_j - \mathbf{x}_i}{\left|\mathbf{x}_j - \mathbf{x}_i\right|^3} \tag{1.1}$$

such that the potential energy of the system takes the form

$$U = -\frac{G}{2} \sum_{\substack{i,j \\ i \neq j}} \frac{m_i m_j}{\left|\mathbf{x}_j - \mathbf{x}_i\right|}. \tag{1.2}$$

It can be shown that one can rewrite this expression as

$$U = -\alpha \frac{GM^2}{r_h} \tag{1.3}$$

with $M := \sum m_i$ the total mass, r_h the *half-mass radius*[2] of the cluster and α a dimensionless numerical factor that relates to the density profile of the cluster. Empirically we find that these systems have $\alpha \approx 0.4$. We also know that the kinetic energy can be written

$$K = \frac{1}{2} M \left\langle v^2 \right\rangle \tag{1.4}$$

with the mean square velocity being

$$\left\langle v^2 \right\rangle := \frac{1}{M} \sum_i m_i |\dot{\mathbf{x}}_i|. \tag{1.5}$$

[1]All numerical values for this analysis have also been taken from this reference.

[2]This is the radius of a sphere centered in the cluster's center of mass that encloses half of its total mass M.

From here steady-state virial theorem can be applied such that

$$U + 2K = 0 \implies K = -\frac{U}{2}. \tag{1.6}$$

Replacing K and U gives us

$$\frac{1}{2}M\langle v^2\rangle = \frac{\alpha G M^2}{2r_h} \tag{1.7}$$

which implies that the virial theorem can be used to estimate the mass of a cluster:

$$M = \frac{\langle v^2\rangle r_h}{\alpha G}. \tag{1.8}$$

Obtaining representative values for $\langle v^2\rangle$ and r_h is no easy task, but one can make an estimate of $\langle v^2\rangle$ by studying the redshift of galaxies in the cluster and approximate r_h by assuming that the mass-to-light ratio is constant with radius with the cluster being roughly spherical. Under this assumptions we can arrive at an estimation of $\langle v^2\rangle \approx 2.32\times10^{12}\,\mathrm{m^2s^{-2}}$ and $r_h \approx 1.5\,\mathrm{Mpc} \approx 4.6\times10^{22}\,\mathrm{m}$, which yields a total mass of

$$M = \frac{\langle v^2\rangle r_h}{\alpha G} \approx \frac{(2.32\times10^{12}\,\mathrm{m^2\,s^{-2}})(4.6\times10^{22}\mathrm{m})}{(0.4)(6.7\times10^{-11}\,\mathrm{m^3s^{-2}kg^{-1}})} \approx 2\times10^{15}M_\odot \tag{1.9}$$

If it were only from measuring luminosity of galaxies and x-ray emissions of intracluster gas, it would lead to an estimate of $M \approx 2.3\times10^{14}M_\odot$, which is about ten percent of the mass we calculated based on the mean square velocity. Zwicky coined a term for this missing matter, calling it *dunkle materie* (translated into English as dark matter), a name still used today.

1.1.2 Galaxy rotation and dark matter halos

Further evidence of mass-to-light ratio anomalies came from measurements of galaxy rotation curves (smaller systems than clusters, on the scale of kiloparsecs). In 1939, H.W. Babcock reported the rotation curve of the Andromeda galaxy, suggesting that the mass-to-luminosity ratio increases with radial distance. This analysis was followed by numerous similar observations, which confirmed that the outer regions of galaxies rotate faster than expected based on visible matter alone [7, 8]. This led to the hypothesis that an unseen matter distribution must be present in these regions.

One of the most significant observations supporting this idea was conducted by Vera Rubin and Kent Ford, who studied the rotation curves $v(r)$ of different spiral galaxies with remarkable accuracy [9, 10]. They measured that these curves tended to flatten as the radial distance increased, rather than exhibiting the expected behavior $v(r) \propto r^{-1/2}$ as shown in Figure 1.1.

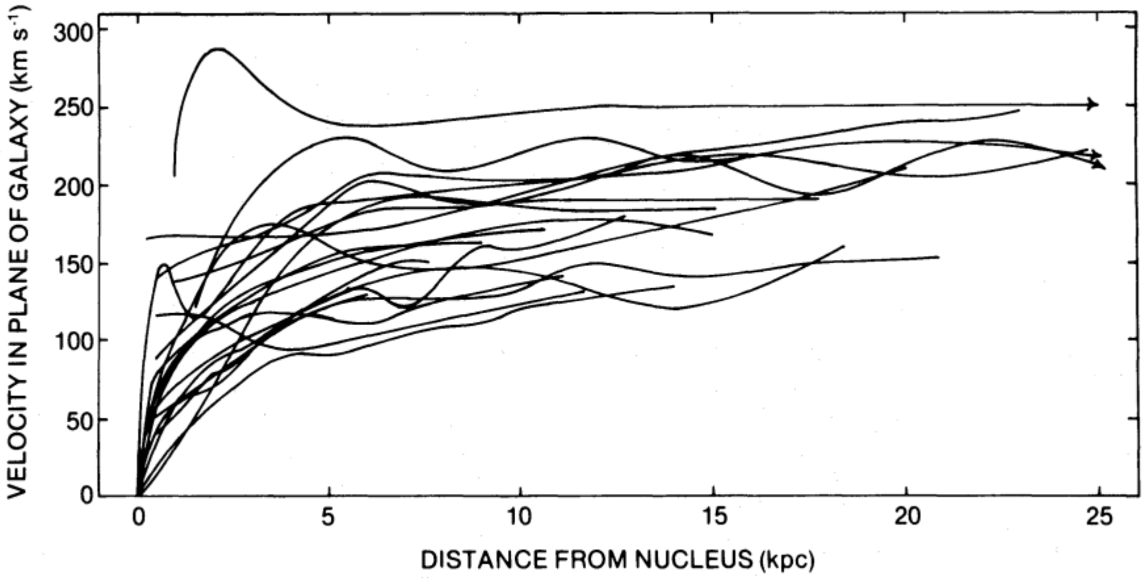

Figure 1.1: Galaxy rotation curves from the original paper from Rubin *et al.* [10]. Galaxies present a flattening of the circular velocity in outer regions instead of decreasing.

Once more, our expectations on the behavior of $v(r)$ come from a Newtonian treatment [6, 11], justified by the characteristics of the system. Let us enclose the galactic disk within a Gaussian sphere of radius r. According to measurements of the visible matter distribution, most of the galaxy's mass M is contained within this sphere. Thus, for $r \geq r_0$ a characteristic radius of the disk, the mass distribution is approximately $m(r) \approx M$. Under these assumptions, the circular velocity in the outer regions of the galaxy can be expressed as

$$v(r) = \sqrt{\frac{Gm(r)}{r}} \approx \sqrt{\frac{GM}{r}}, \tag{1.10}$$

Since M is constant, $v(r)$ should decrease in the outer regions as $v(r) \propto r^{-1/2}$. However, observations show that $v(r) \approx$ const., which can only be true if $m(r) \sim r$ for $r \geq r_0$. Meaning that there exists an additional mass contribution that remained undetected.

There are multiple reasons to think that dark matter distributions do not collapse into disks like baryonic matter does. The most obvious reason is their lack of radiation emission (which, if present, would have already been observed), implying that dark matter does not interact[3] electromagnetically (i.e. a dissipation channel). This, suggests that dark matter is distributed in *halos* surrounding galaxies and clusters. Let us do an order of magnitude estimate for the size of our galaxy's halo. In order to do this, we consider the rough assumption that the halo is a spherical distribution[4]. We also assume a homogeneous distribution taking into account a

[3] Or, if it does, its coupling to the photon is heavily suppressed.

[4] There is a general consensus that the actual shape of dark matter halos resembles more a flattened ellipsoid [12–14].

dark matter local density of $\rho_{dm} \sim 0.3\,\mathrm{GeV/cm^3}$ [14], and a mass upper bound given by the virial mass[5] $M_{halo} \sim 10^{12}\,\mathrm{M_\odot}$ [15]. This leads to an expression

$$M_{halo} \sim 4\pi \int_0^{R_{halo}} \rho_{dm} r^2 dr \implies R_{halo} \sim 100\,\mathrm{kpc}. \tag{1.11}$$

As you can see from Figure 1.1, galaxy disks extend around the order of $R_{disk} \sim 10\,\mathrm{kpc}$, which is also true for the Milky way. Clearly, this means that that the dark matter halo extends around an order of magnitude further away from where most baryonic matter is concentrated. This has profound implications, particularly, it implies that dark matter should be nonrelativistic (i.e. move at velocities much lower than the speed of light). To see that this is the case, let us the same expression used for the velocity of stars (1.10) for the motion of dark matter particles at the edge of our approximated halo:

$$v(R_{halo}) = \sqrt{\frac{GM_{halo}}{R_{halo}}} \sim 200\,\mathrm{km/s} \ll c. \tag{1.12}$$

We call constructions of dark matter that work under this consideration models of *cold* dark matter[6]. The standard model of cosmology called ΛCDM works under this consideration and so will this thesis.

1.1.3 Gravitational lensing

Until now, only systems that are considered to be classical have been discussed. Thus, one may question if the issue of dark matter is merely a problem regarding systems that are treated under the scope of Newtonian gravity, but this is not the case. The problem extends all the way to the regime of general relativity. For example, it is well known that if a celestial body if massive enough, it can bend the trajectory of light creating what is called a *gravitational lens* (see Figure 1.2). Since this effect is created by the curvature of space-time due to the objects mass, one can directly link the observed deflection angle to it. According to general relativity [6, 16] , the deflection angle $\Delta\phi$ of a light ray passing close to an spherically symmetric object of mass M is given by[7]

$$\Delta\phi \approx \frac{4GM}{c^2 R}, \tag{1.13}$$

where R is the radius of the massive body. The fact that celestial bodies can deflect light means they can act like lenses. In particular, if the light of a distant star travels near a concentrations a *MACHO*[8] (i.e. a concentration of dark matter), the deflection produces an image of the star,

[5] Mass enclosed within a sphere with a virial radius, a distance within which the virial theorem applies.

[6] In contrast to *hot* dark matter models, where it is assumed assume that particles move at ultrarelativistic speeds.

[7] Assuming the light ray passes very close to the massive object, making the impact parameter approximately equal to the radius of the massive object.

[8] *Massive compact halo objects*.

which is distorted and amplified. If this concentration of dark matter is aligned with line of sight of a distant observer, the image produced forms a ring of an angular radius

$$\theta_E = \sqrt{\frac{4GM}{c^2}\frac{y_0 - x_0}{y_0 x_0}}. \tag{1.14}$$

where y_0 corresponds to the distance the distance from the earth to the source and x_0 the distance from the earth to the lensing body[9] (see Figure 1.3). This angular measure is called the Einstein radius.

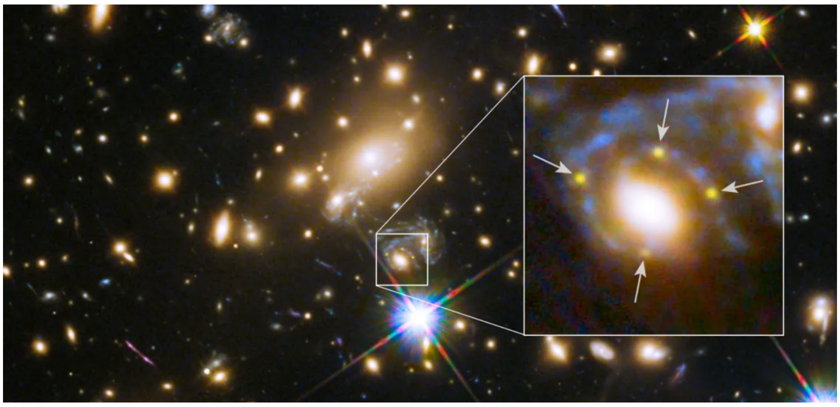

Figure 1.2: Actual image of gravitational lensing taken by the Hubble telescope [17]. The image shows a galaxy from a cluster producing multiple images of one distant supernova behind it.

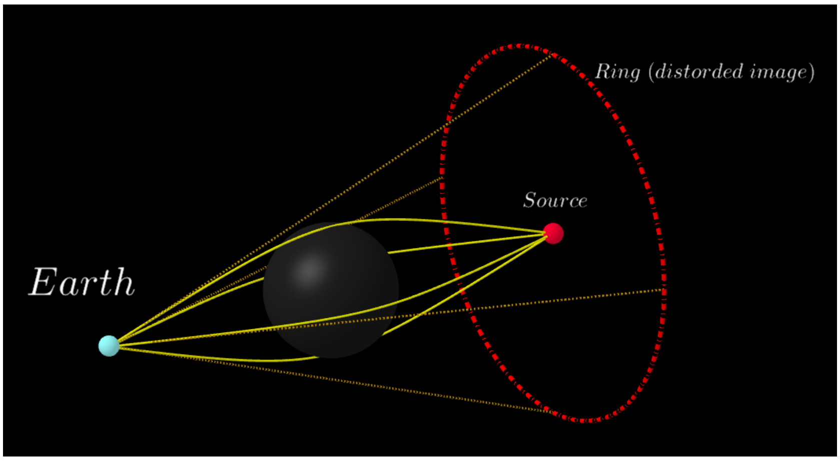

Figure 1.3: Pictorial portrait of the phenomenon of gravitational lensing. Light rays (yellow lines) emitted by a source (red sphere) travel close to a MACHO (black sphere) of radius R. we expect to observe a distorted ring−like image (red dotted circumference) of the source around it. In (1.14), y_0 corresponds to the distance between the earth (Cyan sphere) and the center of the spherical MACHO. In the other hand, x_0 corresponds to the distance between the center of this dark matter distribution.

[9]We treat each object as point−like.

Indeed, one of the strongest evidences for the existence of dark matter is the observation of these gravitational lenses in the Bullet cluster, firs carried out by Douglas Clowe *et al.* [18]. The bullet cluster is a system conformed by the collision of two individual galaxy clusters. Due to this collision, the dark matter (non−interacting) component of the cluster and the irradiating plasma (dissipative due to interactions) are spatially segregated. Meaning that, as the clusters merge, the dark matter distributions should “pass right through each other” while most baryionic content is left behind due to interaction. The regions where this dark matter should reside, do in fact present a concentration of image distortions due to unseen mass (See fig 1.4). The fact clusters like this[10] presents gravitational lensing also implies that dark matter should be concentrated in halos and not disk−like structures.

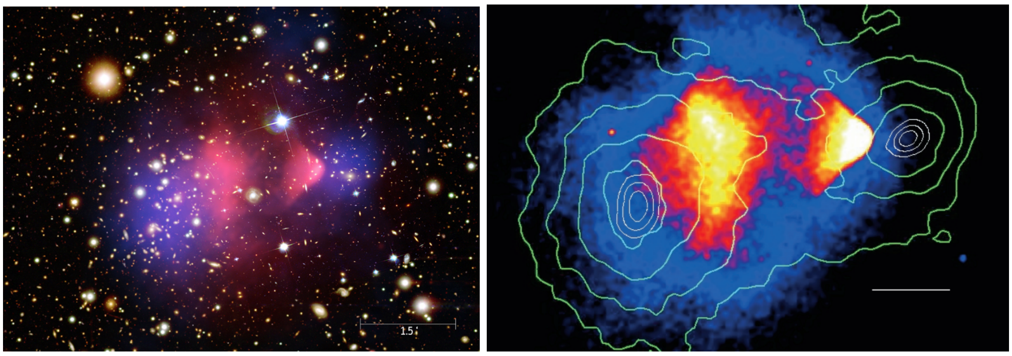

Figure 1.4: The left panel shows the Chandra X-ray image [19] of hot gas (pink), which contains most of the normal matter in the cluster. The majority of mass (blue) is mapped via gravitational lensing coming from unseen matter sources (i.e. dark matter). The Right panel shows an image from the original article [18]. It shows the contour mapping of this lensing distortion over the thermal image of the cluster.

1.2 Cosmic microwave background and Relic density

Until now, we have reviewed several lines of evidence supporting the existence of dark matter, yet significant challenges remain in determining its particle nature. However, one crucial piece of evidence allows us to narrow down the range of possibilities significantly: the *Cosmic Microwave Background* (CMB).

The cosmic microwave background is a remnant of low energy electromagnetic radiation originating from an early stage of the universe, known as the *time of last scattering*. This marks the last instance when photons from the early universe interacted with free electrons before baryonic matter transitioned from an ionized plasma to neutral atoms (see Figure 1.5). At this point, the expansion of the universe had progressed to the extent that the rate at which photons scatter

[10]Another example is the cluster MACS J0025.4-1222.

from electrons became smaller than the *Hubble parameter* (which tells us the rate at which the universe expands)[11].

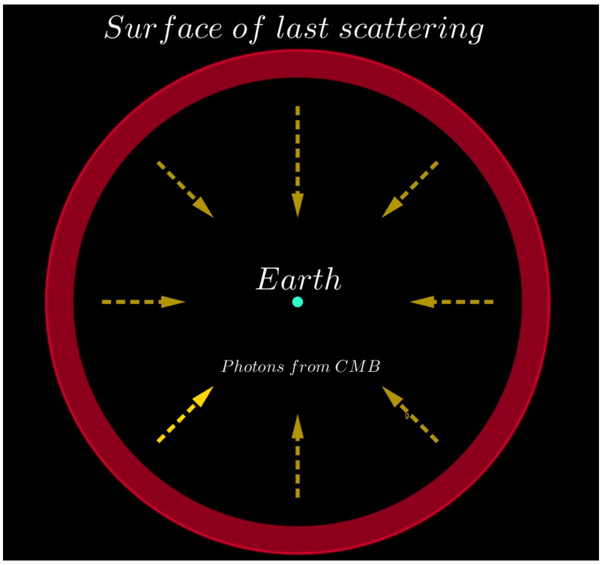

Figure 1.5: A pictorial representation of the surface of last scattering. As the universe expands, a spherical surface (red) continuously drifts away from each point in space. This surface marks the region where photons (yellow arrows) traveling toward the center (e.g. the Earth) last scattered.

The detection of this radiation from all directions enables the construction of the temperature map $T(\phi,\theta)$, illustrated in Figure 1.7. While many aspects of these measurements are of interest, we focus on their anisotropies

$$\frac{\delta T}{T}(\phi,\theta) = \frac{T(\phi,\theta) - \langle T \rangle}{\langle T \rangle}, \tag{1.15}$$

where $\langle T \rangle \approx 2.725$ K represents the mean CMB temperature. Deviations associated with the anisotropies at the time of last scattering can be explained by the gravitational effect of primordial density fluctuations in the distribution of nonbaryonic dark matter. Before the decoupling of photons, these dark matter concentrations would have continuously pulled the plasma inwards[12] while rising pressure due to compression pushed the plasma outwards. We call this phenomenon *baryonic acoustic oscillations*[13].

The connection between temperature anisotropies and fluctuations in the gravitational potential allows for the determination of the dark matter relic density, through cosmological model fits to the CMB power spectrum[14] [6, 20, 21]. The relic density is a crucial quantity for the creation

[11] For a more detailed discussion, see [6].

[12] Only through gravitational interaction, since dark matter does not seem to interact significantly in other ways.

[13] For a *very* good illustration of baryonic acoustic oscillations and their relation to the CMB, see the visualizations by Professor Adam Hincks: `https://adh-sj.info/bao_cmb.php`.

[14] Defined as the sequence of $C_l := \left\langle |a_{lm}|^2 \right\rangle$, where a_{lm} are the expansion coefficients given by $\frac{\delta T}{T}(\phi,\theta) = \sum a_{lm} Y_{lm}(\phi,\theta)$.

of quantum field theories that seek to describe it, as it allows to place bounds on the parameter space of models. It corresponds to the abundance of dark matter in the present, determined by its interactions in the early universe. To date, measurements of the CMB yield a relic density for cold dark matter [22] of

$$\Omega_{DM} h^2 = 0.120 \pm 0.001, \tag{1.16}$$

where h is the dimensionless Hubble constant. This measurement can be directly linked to quantum field models of dark matter through statistical mechanics and thermodynamics of the early universe. This abundance is defined as

$$\Omega_{DM} := \frac{\rho_{DM}}{\rho_c} \quad \Big/ \rho_c := \frac{3H_0^2}{8\pi G}. \tag{1.17}$$

Here ρ_{DM} is the present energy density of dark matter, and H_0 the present value of the Hubble parameter[15]. Thus, we seek a way to obtain ρ_{DM} in terms of parameter of the model. What will follow is an outline of the process of how this is done.

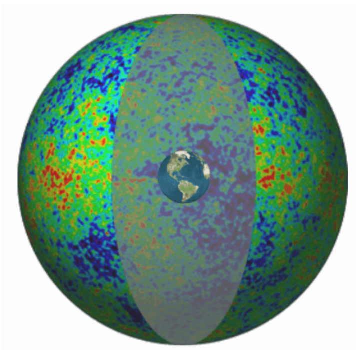

Figure 1.6: Thermal map of the CMB. The scan carried out from earth is done in terms of the spherical coordinates (ϕ, θ). This yields a spherical shell that is latter flattened for better visualization. Image extracted from [23].

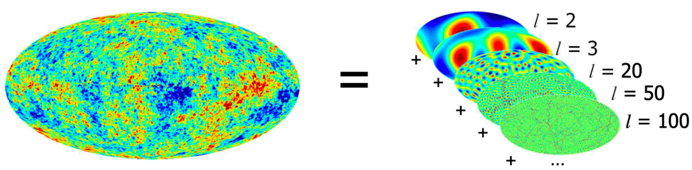

Figure 1.7: A flattened thermal map of the CMB, with the power spectrum decomposition on the left. The sequence of the spectrum is used to analyze anisotropies in the temperature map. Image extracted from [24].

[15] Related to the dimensionless Hubble constant as $H_0 \equiv h \times 100$ Km s^{-1} Mpc^{-1}.

To describe the evolution of a particle specie from the early universe to the present, a relativistic formulation of the Boltzmann transport equation is used [25–27]:

$$p^{\alpha}\frac{\partial f_{\chi}}{\partial x^{\alpha}}-\Gamma^{\alpha}_{\beta\gamma}p^{\beta}p^{\gamma}\frac{\partial f_{\chi}}{\partial p^{\alpha}}=C\left[f_{\chi}\right], \tag{1.18}$$

where $f_{\chi}(\vec{p},t)$ is the particle distribution function for dark matter χ dependent on time and 3−momentum[16]. To describe the universe's expansion, the *Robertson-Walker* interval

$$ds^2=dt^2-a^2(t)\left(\frac{dr^2}{1-kr^2}+r^2d\theta+r^2\sin^2\theta d\phi\right) \tag{1.19}$$

is used. Besides the usual spacetime comoving spherical coordinates, $a(t)$ corresponds to the scale factor [6, 16]. Under this metric, (1.18) gives the simplified structure

$$E\frac{\partial f_{\chi}}{\partial t}-H|\vec{p}|^2\frac{\partial f_{\chi}}{\partial E}=C\left[f_{\chi}\right], \tag{1.20}$$

with $H:=\frac{\dot{a}}{a}$ the definition of the previously mentioned Hubble parameter. The collision operator $C[f_{\chi}]$ is given by the interactions between the particle content as the universe evolves (i.e. the rate of interaction). Thus, it is a function of the matrix elements of the process

$$|X\rangle=\left|(\chi,p),\left(A_1,p_1^{(X)}\right),\ldots\right\rangle\longrightarrow|Y\rangle=\left|\left(B_1,p_1^{(Y)}\right),\left(B_2,p_2^{(Y)}\right),\ldots\right\rangle. \tag{1.21}$$

Here, A_i ($i=1,...,N_X$) and B_j ($j=1,...,N_Y$) are generic particles of the theory each with its associated momenta $p_i^{(X)}$ and $p_j^{(Y)}$ respectively. It can be written explicitly [20, 28, 29] as

$$C\left[f_{\chi}\right]=\sum_{X,Y}\int(2\pi)^4\delta^4(p_X-p_Y)\left(|\mathcal{M}|^2_{X\to Y}f_{\chi}f_Xf_{\to Y}-|\mathcal{M}|^2_{Y\to X}f_Y(1\pm f_{\chi})f_{\to X}\right)d\Pi_Xd\Pi_Y. \tag{1.22}$$

Where p_X is the total initial momentum, p_Y the total final momentum and

$$d\Pi_Z:=\prod_{k=1}^{N_Z}\frac{d^3\vec{p}_k^{(Z)}}{(2\pi)^3}\frac{g_k^{(Z)}}{2E_k^{(Z)}}\quad\wedge\quad f_Z=\prod_{k=1}^{N_Z}f_k^{(Z)}\quad\wedge\quad f_{\to Z}=\prod_{k=1}^{N_Z}\left(1\pm f_k^{(Z)}\right)\ \Big/ Z=X,Y. \tag{1.23}$$

the Lorentz invariant measures together with the products of particle distribution functions for both states state. In these expressions, the choice between ± is dependent on each particle (including dark matter) being either a boson (+) or fermion (−). The factor g_k also accounts for particle spin degeneracy.

Having all elements of the equation, we seek the abundance of dark matter by solving the Boltzmann equation for the number density

$$n_{\chi}=\frac{g_{\chi}}{(2\pi)^3}\int f_{\chi}(\vec{p},t)d^3\vec{p}. \tag{1.24}$$

[16] Classically, the function is also dependent on spatial position.

This is done by integrating (1.20) over its phase space measure $\int(\,\cdot\,)d\Pi_\chi$. Doing so yields the equation

$$\frac{1}{a^3}\frac{d(n_\chi a^3)}{dt} = \frac{g_\chi}{(2\pi)^3}\int \frac{C\left[f_\chi\right]}{E}d^3\vec{p}. \tag{1.25}$$

Generally, the distribution functions would be given by either the Bose-Einstein (+) or Fermi-Dirac (−) distributions

$$f_\pm = \frac{1}{e^{(E-\mu)/T}\mp 1} \tag{1.26}$$

depending on the nature of each particle. Nevertheless, the equation is solved in the high energy limit $E-\mu \gg T$ in which all particles behave under Boltzmann statistics [30]

$$f_\pm \to e^{-(E-\mu)/T}. \tag{1.27}$$

In this regime, $f_k^{(Z)} \ll 1$, meaning the product $f_{\to Z} \to 1$. This simplifies the collision operator to

$$C\left[f_\chi\right] = \sum_{X,Y}\int (2\pi)^4\delta^4(p_X-p_Y)|\mathcal{M}|^2\left(e^{-\frac{1}{T}\left(E-\mu+\Sigma\left(E_k^{(X)}-\mu_k^{(X)}\right)\right)} - e^{-\frac{1}{T}\Sigma\left(E_k^{(Y)}-\mu_k^{(Y)}\right)}\right)d\Pi_X d\Pi_Y. \tag{1.28}$$

where it has also been assumed that $|\mathcal{M}|^2_{X\to Y} = |\mathcal{M}|^2_{Y\to X} \equiv |\mathcal{M}|^2$. Conservation of energy enforces that

$$e^{-\frac{1}{T}\left(E-\mu+\Sigma\left(E_k^{(X)}-\mu_k^{(X)}\right)\right)} - e^{-\frac{1}{T}\Sigma\left(E_k^{(Y)}-\mu_k^{(Y)}\right)} = e^{-\frac{E_X}{T}}\left(e^{\frac{1}{T}\Sigma\mu_k^{(X)}} - e^{\frac{1}{T}\Sigma\mu_k^{(Y)}}\right), \tag{1.29}$$

with the total energy of the process

$$E_X := E + \sum_{k=1}^{N_X} E_k^{(X)} = \sum_{k=1}^{N_Y} E_k^{(Y)}. \tag{1.30}$$

Further simplifications can be done by expressing this collision operator in terms of the number density of each particle[17]

$$n_k^{(Z)} = \frac{g_k^{(Z)}}{(2\pi)^3}e^{\frac{1}{T}\mu_k^{(Z)}}\int e^{-\frac{1}{T}E_k^{(Z)}}d^3\vec{p}_k^{(Z)}, \tag{1.31}$$

the equilibrium or closed system number density

$$\bar{n}_k^{(Z)} = \frac{g_k^{(Z)}}{(2\pi)^3}\int e^{-\frac{1}{T}E_k^{(Z)}}d^3\vec{p}_k^{(Z)}, \tag{1.32}$$

and the *thermally averaged cross section*

$$\langle\sigma_{XY}v\rangle := \frac{1}{\prod \bar{n}_k^{(X)}}\int (2\pi)^4\delta^4(p_X-p_Y)|\mathcal{M}|^2 e^{-\frac{E_X}{T}}d\Pi_\chi d\Pi_X d\Pi_Y. \tag{1.33}$$

[17] Now considering a Boltzmann distribution.

Under these quantities, (1.25) is rewritten as

$$\frac{1}{a^3}\frac{d(n_\chi a^3)}{dt} = \sum_{X,Y} \langle \sigma_{XY} v\rangle \, \bar{n}_\chi \prod_{k=1}^{N_X} \bar{n}_k^{(X)} \left(\prod_{k=1}^{N_Y} \frac{n_k^{(Y)}}{\bar{n}_k^{(Y)}} - \frac{n_\chi}{\bar{n}_\chi} \prod_{k=1}^{N_X} \frac{n_k^{(X)}}{\bar{n}_k^{(X)}} \right). \tag{1.34}$$

This differential equation implicitly describes the evolution of the energy content of dark matter, since n_χ is a function of $E = E(t)$ and $\mu = \mu(t)$. Nevertheless, obtaining the *exact* dynamics for this equation also depend on a whole system of differential equations for the number density of each particle of the theory (standard model and beyond). Clearly, this approach is futile, so assumptions which yield simplifications are used in order to obtain an effective description of the system. This way, one seeks to obtain a value for the present number density, reached when the system reaches thermodynamic equilibrium (i.e. its behavior for $t \to \infty$). This quantity can then be related to the energy density under a proportionality factor

$$\rho_{DM} = \langle \varepsilon \rangle_0 \, n_\chi^{(0)}, \tag{1.35}$$

where $\langle \varepsilon \rangle_0$ is the present average energy per particle[18] and $n_\chi^{(0)}$ the present dark matter number density.

1.3 Weakly Interacting Massive Particles

Within the cold dark matter framework, numerous candidates have been proposed, including *axions*, *sterile neutrinos*, and *primordial black holes* [31]. Nevertheless, one class of particles gained particular attention in the 1980s when it was realized that a stable, weakly interacting particle with a mass around the electroweak scale would naturally produce a relic abundance close to the observed dark matter density [32]. These particles, referred to as *Weakly Interacting Massive Particles* (WIMPs), have since become one of the most extensively studied dark matter candidates. Over the years, various attempt to introduce them as extensions of the standard model have been done. Experimental efforts continue to search for them through direct and indirect detection, as well as collider experiments.

This thesis explores and develops a WIMP dark matter model, aiming to provide a consistent theoretical framework that matches observational constraints. To set the stage for this analysis, we now turn to the study of the Boltzmann equation (1.34), under the scope of a generic WIMP scenario such as the one outlined in Chapter 3.

Typically, WIMPs have an annihilation channel[19] as the process that binds them to the rest of the standard model. This means that two heavy dark matter particles χ can annihilate producing

[18]Formally defined as $\langle \varepsilon \rangle := \dfrac{\int f_\chi E d^3\vec{p}}{\int f_\chi d^3\vec{p}}$

[19]A typical example is the one of *Higgs portals*. Meaning processes of the type $\chi\chi \to hh$, where h is the higgs boson.

two light standard model particles ψ [30]. Considering this to be the only relevant process for the evolution of the number density n_χ simplifies (1.34) substantially

$$\frac{1}{a^3}\frac{d(n_\chi a^3)}{dt} = \langle \sigma v \rangle \bar{n}_\chi^2 \left(\frac{n_\psi^2}{\bar{n}_\psi^2} - \frac{n_\chi^2}{\bar{n}_\chi^2} \right). \tag{1.36}$$

Further simplification comes from the assumption standard model particles are in equilibrium with the rest of the Baryonic plasma. This implies that their number density behaves as the number density of a closed system (i.e. $n_\psi = \bar{n}_\psi$)

$$\frac{1}{a^3}\frac{d(n_\chi a^3)}{dt} = \langle \sigma v \rangle \left(\bar{n}_\chi^2 - n_\chi^2 \right). \tag{1.37}$$

From here, our analysis relies on the fact that temperatures T evolve inversely proportional to the scale factor a. To prove that this is in fact the case, let us consider the first law of thermodynamics for our system:

$$dU = TdS - PdV \implies dS = \frac{dU}{T} + \frac{P}{T}dV. \tag{1.38}$$

For U internal energy of our system and P the associated pressure. U is given by the total energy density ρ such that

$$dS = \left(\frac{\rho + P}{T} \right) dV. \tag{1.39}$$

As the universe expands, the overall volume element is given by a^3, so

$$dS = \left(\frac{\rho + P}{T} \right) d(a^3). \tag{1.40}$$

Invoking the second law of thermodynamics by assuming the universe to be a closed system ($dS = 0$) yields

$$\frac{\rho + P}{T} a^3 = S_0, \tag{1.41}$$

where S_0 is the entropy of the universe. Since most of the energy density of the system comes from the photons within the plasma, we can use the fact that in photon gases $\rho = 3P$ and

$$\rho(t) = \rho_0 a(t)^{-4}. \tag{1.42}$$

Such that ρ_0 can be set to be the present total energy density. This result that can be obtained from Einsteins field equations using the Robertson-Walker metric [20, 30]. These expressions yield the relation

$$T = \frac{4\rho_0}{3S_0} a^{-1}. \tag{1.43}$$

Meaning that $T \propto a^{-1}$. This argument matters because it allows to define a new dynamic variable

$$Y := \frac{n_\chi}{T^3}, \tag{1.44}$$

such that the quantity a^3T^3 remains constant. Under this new variable (1.37) turns into

$$\frac{dY}{dt} = T^3\left(\bar{Y}^2 - Y^2\right), \tag{1.45}$$

where $\bar{Y} := \bar{n}_\chi/T^3$. It is also convenient to introduce a new parametrization for the evolution of the system. This is done by defining

$$x := \frac{m_\chi}{T} \tag{1.46}$$

for m_χ the mas of the WIMP. This quantity is appropriate to describe the system's evolution since it monotonically increases with time

$$\frac{dx}{dt} = m_\chi \frac{3S_0}{4\rho_0}\frac{da}{dt} = \frac{m_\chi}{aT}\frac{da}{dt} \equiv xH. \tag{1.47}$$

Since both of these parameters (x, H) are always positive, then x increases as time does. This leaves the differential equation as

$$\frac{dY}{dx} = \frac{m_\chi^3}{x^4 H}\left(\bar{Y}^2 - Y^2\right). \tag{1.48}$$

A last change of variable that we can do comes from the implicit dependence of the Hubble parameter on temperature and x as a consequence. Notice that

$$H = \frac{1}{a}\frac{da}{dt} = -\frac{1}{T}\frac{dT}{dt}. \tag{1.49}$$

We claim that the time derivative of the temperature is proportional to the inverse of itself

$$\frac{dT}{dt} \propto \frac{1}{T} \tag{1.50}$$

Making $H \propto T^{-2} \propto x^2$. For the sake of our main discussion, we will not dive into the reasons why this is the case, but it is related to the *relativistic degrees do freedom* of the system [20]. Assuming this, the temperature−independent quantity

$$H_\chi := Hx^2 \tag{1.51}$$

allows to express all x−dependence to appear explicitly in the dynamic equation:

$$\frac{dY}{dx} = -\frac{\lambda}{x^2}\left(\bar{Y}^2 - Y^2\right) \quad \Big/ \lambda := \frac{m_\chi^3 \langle\sigma v\rangle}{H_\chi}. \tag{1.52}$$

[20] For a in depth explanation, see chapter *Standard cosmology* of [20].

This equation can be solved numerically without much trouble. Figure 1.8 shows $Y(x)$ for different orders of magnitude of λ, which is the parameter that contains the thermally averaged cross section. Notice that all of these curves clearly converge towards a fixed value Y_∞. This change between the curves being in a stage of equilibrium and the rapid convergence towards a fixed value is what we call *freeze−out*. From here, further approximations can be done to obtain an estimate of the energy density. A fairly simple estimation results to be [30]:

$$\rho_{DM} \approx \frac{m_\chi}{30} n_\chi^{(0)} \equiv \frac{m_\chi}{30} Y_\infty T_0^3, \tag{1.53}$$

where T_0 is the temperature at present. Having found ρ_{DM} as a function of $Y_\infty = Y_\infty(\langle\sigma v\rangle)$ is what links the observations of the CMB with the WIMP model.

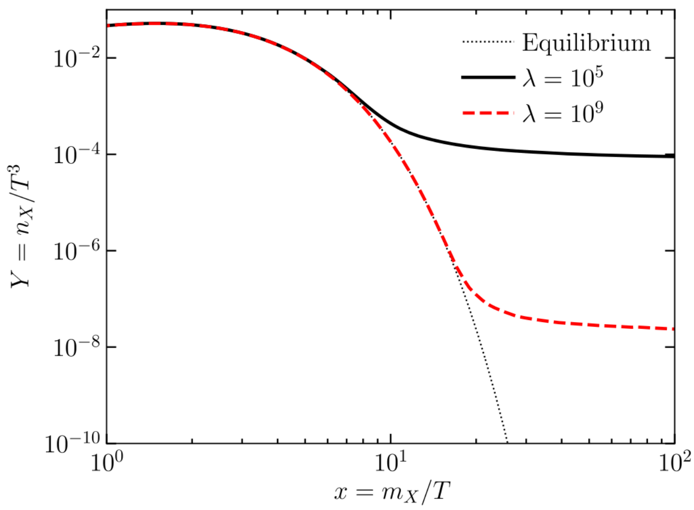

Figure 1.8: $Y(x)$ for $\lambda = 10^5$ and $\lambda = 10^9$. $\bar{Y}(x)$ is also plotted next to them. For smaller orders of magnitude of λ, the feeze−out happens sooner and Y converges to higher values. Image extracted from [30].

CHAPTER 2

YANG-MILLS THEORIES AND THE $\mathfrak{so}(4)$ ALGEBRA

The theory of groups and algebras is a mathematical framework that has taken over modern theoretical physics, specially those areas that make use of quantum field theories such as particle physics. It is a language that allows to express the symmetries of systems via the different representations that actions applied to a system can take. Particularly, Lie groups and their algebras are the center piece of *gauge theories*, which are constructions based on actions of spacetime−dependent group representations over the Lagrangian. A fundamental property of quantum gauge theories is that have been proven to be both renormalizable and unitary though the use of mechanisms such as the *BRST symmetry*.

An intrinsic problem of gauge theories, however, is that they alone do not provide a mechanism for generating mass for the *gauge fields* they introduce. Fortunately, one of the most remarkable results of late 20th−century theoretical physics is that the inclusion of *spontaneous symmetry breaking* allows these and other fields to acquire mass without violating the original gauge symmetry. This approach to the construction of gauge theories made possible the formulation of the Standard Model of particle physics and has proven to be an essential framework for building ultraviolet complete models. Particularly, it is within this framework that the dark−electroweak model is built.

For this last reasons, this chapter aims to review the most important aspects of the algebra from which we will formulate a gauge theory: $\mathfrak{so}(4)$. We will also provide a general overview of *Yang-Mills* theories (i.e. non−abelian gauge theories) as well as the process of spontaneous symmetry breaking in gauge theories, using the Standard Model as an example. Lastly, we will include a historical revision on the subjects of unitarity and renormalization of gauge theories, without going into the details of the mathematical machinery.

The objective of this chapter is thus to review the key aspects of *how* our formalism works[1]. This chapter will assume the reader is familiarized at a basic level with the theory of Lie groups and their representations. Nevertheless, appendix A aims to cover in a general manner the key aspects necessary to understand this chapter in order to keep this thesis as self–contained as possible.

2.1 The $\mathfrak{so}(4)$ agebra

Many times refereed as the algebra of "rotations in four dimensions", $\mathfrak{so}(4)$ is defined by having a set of generators T_k, $k = 1,...,6$ that can be divided in two subsets:

$$T_k = \begin{cases} J_k \, , \, k = 1,\ldots,3 \\ K_{k-3} \, , \, k = 4,\ldots,6. \end{cases} \tag{2.1}$$

These satisfy the commutation relations

$$[J_a, J_b] = i\epsilon_{abc} J_c \quad \wedge \quad [K_a, K_b] = i\epsilon_{abc} J_c \quad \wedge \quad [J_a, K_b] = i\epsilon_{abc} K_c, \tag{2.2}$$

for $a, b = 1,2,3$. Notice that the first commutation relation defines a subalgebra $\mathfrak{su}(2) \subset \mathfrak{so}(4)$, which will prove to be essential for understanding the gauge sector of the model we will construct. Another important observation is that, in contrast to other algebras, $\mathfrak{so}(4)$ is not defined by a single commutation relation. This is because the algebra is not simple, but rather *semisimple*. Therefore, it can be written as direct sum of simple algebras. To see that this is the case, let us define new generators that serve as an alternative basis for the algebra:

$$J_a^+ = \frac{1}{2}(J_a + K_a) \quad \wedge \quad J_a^- = \frac{1}{2}(J_a - K_a). \tag{2.3}$$

By simply using the commutation relations in (2.2), we obtain new relations for the algebra. These are

$$\left[J_a^+, J_b^+\right] = i\epsilon_{abc} J_c^+ \quad \wedge \quad \left[J_a^-, J_b^-\right] = i\epsilon_{abc} J_c^- \quad \wedge \quad \left[J_a^+, J_b^-\right] = 0, \tag{2.4}$$

which are the commutation relations corresponding to $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$. This leads to conclude that $\mathfrak{so}(4) = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$, since vector spaces are basis independent[2]. Thus, representations of the whole algebra can be constructed by talking Kronecker the sum of representations of each subalgebra. We can label representations built this way as the pair (n_1, n_2) with n_1 being the dimension of the representation of the first copy of $\mathfrak{su}(2)$ and n_2 the dimension of the second one. As a consequence, representations of $\mathfrak{so}(4)$ generators can be constructed as

$$J_a = J_a^+ \oplus_K J_a^- \quad \wedge \quad K_a = J_a^+ \ominus_K J_a^-, \tag{2.5}$$

[1]For those interested in a comprehensive treatment of the fundamental properties that make gauge theories successful quantum field theories, as well as *why* they exhibit these properties, see [33–35].

[2]Notice that $\mathfrak{su}(2)$ is a simple algebra.

with $J_k^{\pm}$ generators of the different copies of $\mathfrak{su}(2)$[3]. In particular, the $(2,2)$ representation constructed using Pauli matrices:

$$J_a^+ = \frac{\sigma_a}{2} \qquad \wedge \qquad J_a^- = \frac{\sigma_a}{2} \tag{2.6}$$

corresponds to the fundamental representation of the algebra, with the "usual" form of the $\mathfrak{so}(4)$ generators[4] being obtained by the similarity transformation

$$\tilde{J}_a = M^{-1} J_a M \qquad \wedge \qquad \tilde{K}_a = M^{-1} K_a M \quad \Big/ M = \begin{bmatrix} 0 & 1 & -i & 0 \\ i & 0 & 0 & -1 \\ -i & 0 & 0 & -1 \\ 0 & -1 & -i & 0 \end{bmatrix}. \tag{2.7}$$

During the construction of the model, we will concern ourselves with representations $(2,1)$, $(1,2)$ and $(2^*,2)$[5]. Each of this representation for the generators of $\mathfrak{su}(2)$ can be found in the following table:

Representation	Generator ($J_a^{\pm}$)
1	0
2	$\frac{1}{2}\sigma_a$
2^*	$-\frac{1}{2}\sigma_a^*$

Notice that these representations of $\mathfrak{so}(4)$ will not yield the group SO(4) when exponentiated, but rather its covering group SU(2) × SU(2).

When it comes to group actions obtained for a given algebra representation, the transformations U can always be written as

$$\begin{aligned} U &= \exp\left[i\left(\theta_a J_a + \eta_b K_b\right)\right] = \exp\left[i\left(\alpha_a^+ J_a^+ \oplus_K \alpha_b^- J_b^-\right)\right] = \exp\left[i\alpha_a^+ J_a^+\right] \otimes_K \exp\left[i\alpha_b^- J_b^-\right] \\ U &\equiv U_+ \otimes_K U_-, \end{aligned} \tag{2.8}$$

where $U_{\pm}$ are the individual SU(2) transformations. For $(2,1)$ and $(1,2)$, the group actions reduce to SU(2) transformations in their fundamental representations, since the other SU(2) copy yields the trivial action:

$$U_{(2,1)} = \exp\left[i\alpha_a^+ \frac{\sigma_a}{2}\right] \qquad \wedge \qquad U_{(1,2)} = \exp\left[i\alpha_b^- \frac{\sigma_b}{2}\right]. \tag{2.9}$$

In contrast, to build $(2^*,2)$ representation, notice that group transformations in the 2−dimensional and 2^*−dimensional representations are related by conjugation

$$\exp\left[i\alpha_a \frac{\sigma_a}{2}\right] = \left(\exp\left[-i\alpha_a \frac{\sigma_a^*}{2}\right]\right)^*. \tag{2.10}$$

[3]We also adopt the notation $A \ominus B \equiv A \oplus -B$.

[4]Representation obtained from differentiating the manifold of real 4×4 unit determinant orthogonal matrices.

[5]We adopt the notation 2^* for the anti-fundamental representation of $\mathfrak{su}(2)$.

This allows to write the $(2^*,2)$ representation also in terms of SU(2) fundamental representations

$$U_{(2^*,2)} = U^*_{(2,1)} \otimes_K U_{(1,2)}. \tag{2.11}$$

Clearly, $U_{(2^*,2)}$ is a linear transformation that acts on $\mathbb{C}^4$. However, it is common work in the isomorphic $\mathscr{M}_{2\times 2}(\mathbb{C})$ space instead. To do this, one considers the matrix $\Phi \in \mathscr{M}_{2\times 2}(\mathbb{C})$ for which the vector $\varphi \in \mathbb{C}^4$ is its vectorization:

$$\varphi = \mathrm{Vec}(\Phi). \tag{2.12}$$

Given that $U_{(2^*,2)} = U^*_{(2,1)} \otimes_K U_{(1,2)}$, the transformation rule for this object becomes:

$$\begin{aligned} \varphi' &= \left(U^*_{(2,1)} \otimes_K U_{(1,2)}\right)\varphi, \\ \mathrm{Vec}(\Phi') &= \left(U^*_{(2,1)} \otimes_K U_{(1,2)}\right)\mathrm{Vec}(\Phi), \\ \mathrm{Vec}(\Phi') &= \mathrm{Vec}\left(U_{(1,2)}\,\Phi\, U^\dagger_{(2,1)}\right), \end{aligned} \tag{2.13}$$

where we have used the vectorization property of matrix products:

$$\mathrm{Vec}(ABC) = (C^T \otimes_K A)\mathrm{Vec}(B). \tag{2.14}$$

Thus, the bidoublet Φ transforms as

$$\Phi' = U_{(1,2)}\,\Phi\, U^\dagger_{(2,1)}. \tag{2.15}$$

Using this form, all invariants built with φ can also be constructed under a bidoublet Φ formulation via identities such as (2.14) and

$$\mathrm{Tr}\left(A^\dagger B\right) = \mathrm{Vec}(A)^\dagger \mathrm{Vec}(B). \tag{2.16}$$

As we will see in Chapter 4, this method of working with $(2^*,2)$ representations often simplifies computations and the construction of group invariants.

2.2 Yang-Mills theories

First introduced by C. N. Yang and R. L. Mills [36], a Yang-Mills theory is a gauge theory which is built from fields that transform under different representations of a non−abelian group[6]. As with any gauge theory, group transformations are consider *local*, which means that the group parameters are functions that depend on physical spacetime position. It is by operations between these fields that group invariant terms can be constructed, allowing one to build a Lagrangian for the system.

[6] Although Yang-Mills theories are constructed with non−abelian groups in mind, abelian gauge theories can be derived as a particular case of these.

To start this construction, let $\psi_i \in V_i$ be a generic field of the theory in a vector space V_i, with $i = 1,...,N_\psi$ an index for these different fields. Also, let G be the non-Abelian Lie group with a Lie algebra $\mathfrak{g}$. Group objects can be represented by parameter-dependent linear operators acting acting on each on a different vector space[7]:

$$\begin{aligned} U_i : \mathbb{R}^N \times V_i &\to V_i \\ (\theta_1,...,\theta_N,\, \psi_i) &\to U_i(\theta_1,...,\theta_N)\psi_i, \quad \text{(No sum over } i) \end{aligned} \tag{2.17}$$

where θ_a ($a = 1,...,N$) corresponds a to a group parameter, meaning N is the number of group generators. In short, this means that all fields will transform as

$$\psi'_i = U_i\psi_i. \tag{2.18}$$

From here, let us consider the group parameters to be spacetime−dependent functions

$$\begin{aligned} \theta_a : \mathbb{R}^4 &\to \mathbb{R} \\ x &\to \theta_a(x), \end{aligned} \tag{2.19}$$

which means that all group objects can be understood as a spacetime functions. The group structure obtained from promoting these parameters to functions is what we call a *gauge group*. A direct consequence of this is that the set of transformations of G constitutes a manifold over spacetime. Thus, we write these local transformations as

$$U_i(x) = \exp\left[i\theta_a(x)T_a^{(i)}\right], \tag{2.20}$$

with the different $T_a^{(i)}$ generators of G in a representation that acts over V_i. Since the group actions now represent a manifold, a covariant derivative is required to properly describe the field dynamics. Its necessity arises from the fact that norms of field differences are not invariant under local group transformations

$$\left|\psi_i(y) - \psi_i(x)\right| \neq \left|U_i(y)\psi_i(y) - U_i(x)\psi_i(x)\right|. \tag{2.21}$$

This implies that usual spacetime derivatives over fields do not have a well define magnitude, rendering them unable to properly describe field variations. To build a measure of change along the manifold with a well defined norm and transformation rule, one can define a position dependent operator $\mathrm{W}_i(x,y)$ called *Wilson line* that links two different points of the manifold. This operator is characterized by its transformation rule

$$\mathrm{W}'_i(x,y) = U_i(x)\mathrm{W}_i(x,y)U_i^{-1}(y), \tag{2.22}$$

[7]We will assume this operators to be unitary, such that vector norms are group invariant.

and its equivalence with the identity when evaluated in the same position

$$W(x,x) = \mathbb{1}. \tag{2.23}$$

With this, we can see that the quantity

$$\left(\mathrm{W}_i(x,y)\psi_i(y) - \psi_i(x)\right)' = U(x)\left(\mathrm{W}_i(x,y)\psi_i(y) - \psi_i(x)\right) \tag{2.24}$$

transforms as a vector of V_i, meaning that its norm is invariant. This allows to define a manifold (covariant) derivative as

$$D_\mu \psi_i(x) := \lim_{\Delta x^\mu \to 0} \frac{\mathrm{W}_i(x, x+\Delta x)\psi_i(x+\Delta x) - \psi_i(x)}{\Delta x^\mu}. \tag{2.25}$$

An explicit and more useful expression can be obtained by expanding

$$\mathrm{W}_i(x, x+\Delta x) = \mathbb{1}_i + igA_\mu^{(i)}(x)\Delta x^\mu + \mathcal{O}(\Delta x^2), \tag{2.26}$$

with $A_\mu^{(i)}(x)$ an operator field proportional to the spacetime derivative of the Wilson line and $g \in \mathbb{R}$ a fixed number named *coupling constant*. Replacing this expansion on (2.25) and taking the limit yields

$$D_\mu \psi_i = \partial_\mu \psi_i - igA_\mu^{(i)}\psi_i. \tag{2.27}$$

From equation (2.24) we know that this covariant derivative satisfies $\left(D_\mu\psi_i\right)' = U_i(x)\left(D_\mu\psi_i\right)$, meaning that

$$\begin{aligned}\left(\partial_\mu\psi_i' - ig\left(A_\mu^{(i)}\right)'\psi_i'\right) &= U_i\left(\partial_\mu\psi_i - igA_\mu^{(i)}\right) \\ \left(\left(\partial_\mu U_i\right)\psi_i + U_i\partial_\mu\psi_i - ig\left(A_\mu^{(i)}\right)'U_i\psi_i\right) &= \\ U_i\left(U_i^{-1}\left(\partial_\mu U_i\right)\psi_i + \partial_\mu\psi_i - igU_i^{-1}\left(A_\mu^{(i)}\right)'U_i\psi_i\right) &=\end{aligned} \tag{2.28}$$

Comparing both sides from this equation yields the transformation rule for the field $A_\mu^{(i)}$

$$A_\mu^{(i)} = U_i^{-1}\left(A_\mu^{(i)}\right)'U_i - \frac{i}{g}U_i^{-1}\left(\partial_\mu U_i\right) \tag{2.29}$$

or equivalently

$$\left(A_\mu^{(i)}\right)' = U_iA_\mu^{(i)}U_i^{-1} - \frac{i}{g}\left(\partial_\mu U_i\right)U_i^{-1} \tag{2.30}$$

Obtaining an explicit expression in terms of the group parameters is not straightforward, since in general

$$\frac{\partial}{\partial\theta_b}\exp\left[i\theta_aT_a^{(i)}\right] \neq iT_b^{(i)}\exp\left[i\theta_aT_a^{(i)}\right] \tag{2.31}$$

even if the definition of the generators (A.7) suggests so. Instead, the actual formula for the derivative is given by

$$\frac{\partial}{\partial \theta_b} \exp\left[i\theta_a T_a^{(i)}\right] = i \int_0^1 \exp\left[is\theta_a T_a^{(i)}\right] T_b \exp\left[i(s-1)\theta_a T_a^{(i)}\right] ds. \tag{2.32}$$

Meaning that

$$-\frac{i}{g}\left(\partial_\mu U_i\right) U_i^{-1} = \int_0^1 \exp\left[is\theta_a T_a^{(i)}\right] \left(\partial_\mu \theta_b T_b\right) \exp\left[-is\theta_a T_a^{(i)}\right] ds, \tag{2.33}$$

where we have used the property

$$\exp\left[i\lambda_1\theta_a T_a^{(i)}\right] \exp\left[i\lambda_2\theta_a T_a^{(i)}\right] = \exp\left[i(\lambda_1+\lambda_2)\theta_a T_a^{(i)}\right] \tag{2.34}$$

for $\lambda_1, \lambda_2 \in \mathbb{R}$. Since the argument of (2.33) corresponds to group conjugation of an algebra element $\partial_\mu \theta_b T_b$ and Lie algebras are closed under this action, the whole expression will correspond to an element of the algebra

$$-\frac{i}{g}\left(\partial_\mu U_i\right) U_i^{-1} = \left(\partial_\mu \theta_b\right) \eta_{bc} T_c^{(i)} \tag{2.35}$$

For some set of coefficients $\eta_{bc} \in \mathbb{R}$ determined by computing the integral. Thus, in terms of these auxiliary coefficients we write

$$\left(A_\mu^{(i)}\right)' = \exp\left[i\theta_a T_a^{(i)}\right] A_\mu^{(i)} \exp\left[-i\theta_a T_a^{(i)}\right] + \left(\partial_\mu \theta_b\right) \eta_{bc} T_c^{(i)}. \tag{2.36}$$

This transformation is also built from group conjugation $UA_\mu U^{-1}$ and an inhomogeneous term given by the addition of the $\mathfrak{g}$−valuated function $\partial_\mu \theta_b \eta_{bc} T_c$. Conjugation and addition are both automorphisms over $\mathfrak{g}$, so we can assume $A_\mu \in \mathfrak{g}$ in order to ensure group transformations do not represent a mapping to a different vector space. This consideration implies that these fields can be written as

$$A_\mu(x) = A_\mu^a(x) T_a, \tag{2.37}$$

with A_μ^a a real−valued field. We have also dropped the index i, since A_μ will correspond to a field that will be treated abstractly, and can take representations that do not act in any particular vector space of the theory. We can also see that allowing the group parameters to be functions, gives the field A_μ an additional dependence on the group parameters (or rather, their derivatives) that does not come from the action of a group representation (linear operation). It turns out that the choice for an explicit form of $\theta_a(x)$ constitutes an additional degree of freedom that does not alter the dynamics of the system (i.e. the Lagrangian). In fact, multiple instance of the same group transformation *do not* need to correspond to *one* same arbitrary function: consecutive group actions can have different associated functions as group parameters. This freedom of choice in the group parameter functions is what we call *gauge freedom* and imposing particular conditions on this set of functions is what we call *gauge fixing*. It is after these concepts that the algebra−valued object A_μ recives its name: a *gauge field*.

We have seen that the covariant derivative of fields in V_i is itself an object of the same space, but it is not the only object with this feature relevant for the construction of a Yang-Mills theory. While the dynamics of the different fields ψ_i and the way they communicate within the manifold are determined by the covariant derivative, the dynamics of the gauge fields are determined by the commutator of these derivatives. It can be easily verified that said commutator acting over a field ψ_i is an object of V_i since it transforms as

$$\left([D_\mu, D_\nu]\,\psi_i\right)' = U_i\,[D_\mu, D_\nu]\,\psi_i \tag{2.38}$$

For this object, the following explicit form can be obtained

$$[D_\mu, D_\nu] = -ig\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) - g^2\,[A_\mu, A_\nu] \tag{2.39}$$

An important observation is that this object is in $\mathfrak{g}$, since it is built from derivatives and a commutator of A_μ. This motivates a definition for a new algebra−valued field

$$F_{\mu\nu} := \frac{i}{g}\,[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu - ig\,[A_\mu, A_\nu]\,, \tag{2.40}$$

such that $F^{(i)}_{\mu\nu}\psi_i \in V_i$. From this definition, two observations follow. First, notice that the transformation rule for this object is given by group conjugation

$$F'_{\mu\nu} = U_i F_{\mu\nu} U_i^{-1}. \tag{2.41}$$

And secondly, as any other element of $\mathfrak{g}$, the field can be written as the linear combination

$$F_{\mu\nu} = F^a_{\mu\nu} T_a, \tag{2.42}$$

with the coefficients given by

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - ig f_{abc} A^b_\mu A^c_\nu, \tag{2.43}$$

such that f_{abc} is the algebra structure constant. We call this object the *field strength tensor*.

As stated before, both the covariant derivative and the field strength tensor account for the dynamics of all fields. This can be realized through the construction of the Lagrangian invariants depending on the type of object:

- Scalar field ψ_i: $\left(D_\mu\psi_i\right)^\dagger\left(D^\mu\psi_i\right)$.

- Spinor field ψ_i: $i\bar{\psi}_i\gamma^\mu D_\mu\psi_i$.

- Gauge field A_μ: $-\frac{1}{2}\mathrm{Tr}\left\{F_{\mu\nu}F^{\mu\nu}\right\}$.

As an additional note on group invariant terms, notice that this is not possible to build invariant mass terms for gauge fields. It is straightforward to verify that

$$M^2\mathrm{Tr}\{A_\mu A^\mu\} = \frac{1}{2}M^2 A^a_\mu A^{\mu a} \tag{2.44}$$

is not group invariant. For this reason, plain Yang-Mills without a mechanism to introduce mass for the gauge field components are sometimes said to be *massless* gauge theories.

2.2.1 Gauge fields of semisimple algebras

An special but relevant case worth discussing is that of models built from semisimple algebras such as $\mathfrak{so}(4)\oplus\mathfrak{u}(1)$. These are algebras can be written as direct sums of simple algebras. This means that a gauge field corresponant to the original algebra can always be decomposed into subalgebra−associated gauge fields. To see that this is the case, let ($\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ be a semisimple algebra, this means that an arbitrary object of $x \in \mathfrak{g}$ can be written as

$$x = \alpha_1 x_1 \oplus \alpha_2 x_2, \tag{2.45}$$

with $x_k \in \mathfrak{g}_k$ and $\alpha_k \in \mathbb{R}$ ($k = 1,2$)[8]. In particular, the gauge field associated with $\mathfrak{g}$ can be written as the normalized linear combination

$$A_\mu = \cos(\alpha)A^{(1)}_\mu \oplus \sin(\alpha)A^{(2)}_\mu, \tag{2.46}$$

where $\alpha \in \mathbb{R}$ and $A^{(k)}_\mu \in \mathfrak{g}_k$ are the subalgebra-valued gauge fields[9]. Since the covariant derivative is constructed from actions of the whole group, it is written as

$$D_\mu = \partial_\mu - igA_\mu. \tag{2.47}$$

Nevertheless, it can be expressed in terms of the subalgebras using equation (2.46), leading to

$$D_\mu = \partial_\mu - i\left(g_1 A^{(1)}_\mu \oplus g_2 A^{(2)}_\mu\right). \tag{2.48}$$

Where we defined the subgroup coupling constants as

$$g_1 := g\cos(\alpha) \quad\wedge\quad g_2 := g\sin(\alpha) \tag{2.49}$$

The reader can verify that, if $T^{(k)}_a \in \mathfrak{g}_k$ are the generators of the algebras such that $U_k = \exp\left[\theta^{(k)}_a T^{(k)}_a\right]$ is the correspondent group transformation, then the transformation rule for A_μ

[8]Here $\oplus$ stands for the abstract direct sum and not necessarily its representation as a Kronecker sum.

[9]Do not confuse (k) with the previous (i) superscript used for indexing vector spaces. k is a index for subalgebras completely unrelated to the latter.

will be given by

$$A'_\mu = UA_\mu U^{-1} - \frac{i}{g}(\partial_\mu U)U^{-1} \tag{2.50}$$
$$= \cos(\alpha)\left(U_1 A_\mu^{(1)} U_1^{-1} - \frac{i}{g_1}(\partial_\mu U_1)U_1^{-1}\right) \oplus \sin(\alpha)\left(U_2 A_\mu^{(2)} U_2^{-1} - \frac{i}{g_2}(\partial_\mu U_2)U_2^{-1}\right)$$
$$\equiv \cos(\alpha)\left(A_\mu^{(1)}\right)' \oplus \sin(\alpha)\left(A_\mu^{(2)}\right)',$$

with the full group transformation

$$U = \exp\left[\theta_a^{(1)} T_a^{(1)} \oplus \theta_b^{(2)} T_b^{(2)}\right]. \tag{2.51}$$

This construction allows to associate different intensities to the interactions between the gauge fields and the rest. In the general case, if $\mathfrak{g}$ is a semisimple algebra such that

$$\mathfrak{g} = \bigoplus_{k=1}^{N} \mathfrak{g}_k \tag{2.52}$$

We can write its normalized gauge field as the direct sum linear combination

$$A_\mu = \bigoplus_{k=1}^{N} \alpha_k A_\mu^{(k)} \tag{2.53}$$

where the coefficients satisfy

$$\sum_{k=1}^{N} \alpha_k^2 = 1. \tag{2.54}$$

When it comes to the dynamic term for these types of gauge fields, one considers the "decoupled" field strength tensor for each subalgebra. These can be defined as

$$F_{\mu\nu}^{(k)} := \frac{i}{g_k}\left[D_\mu^{(k)}, D_\nu^{(k)}\right] = \partial_\mu A_\nu^{(k)} - \partial_\nu A_\mu^{(k)} - ig_k\left[A_\mu^{(k)}, A_\nu^{(k)}\right], \tag{2.55}$$

with the $k-$subalgebra "partial" covariannt derivative

$$D_\mu^{(k)} := \left(D_\mu\right)_{A_\mu^{(j)}=0} \quad \Big/ \forall j \neq k \tag{2.56}$$

We now define the $\mathfrak{g}$ strength field tensor for as the $\mathfrak{g}-$valued object

$$F_{\mu\nu} := \bigoplus_{k=1}^{N} F_{\mu\nu}^{(k)} \tag{2.57}$$

such that the kinetic term for the whole gauge field is the sum of individual kinetic terms

$$-\frac{1}{2}\mathrm{Tr}\left\{F_{\mu\nu}F^{\mu\nu}\right\} = -\frac{1}{2}\sum_{k=1}^{N}\mathrm{Tr}\left\{F_{\mu\nu}^{(k)}F^{(k)\mu\nu}\right\}, \tag{2.58}$$

because of the orthogonality between the generators T_a ($a = 1, 2, \ldots, \dim(\mathfrak{g})$):

$$\mathrm{Tr}\left\{T_a T_b\right\} \propto \delta_{ab}. \tag{2.59}$$

This allows each subalgebra gauge field to evolve independently, implicitly dictating the dynamics of the original gauge field A_μ.

2.3 Higgs mechanism: the Standard model

Spontaneous symmetry breaking occurs when a system invariant under a symmetry group G is expressed in terms of dynamical variables that do not transform homogeneously under the full group G, but rather under a subgroup $H \subsetneq G$ (e.g. a stabilizer). This process “hides” the original symmetry of the Lagrangian because the new fields do not transform under a representation of G. When this happens, we say that the original symmetry has been *spontaneously broken*.

The most common way to achieve this effect is to introduce a scalar−dependent potential whose minima are not invariant under the actions of G. By expanding the scalar field around one of these minima (called the vacuum states), we define new dynamical variables that transform under a representation of the subgroup H but not of G, thereby breaking the group symmetry spontaneously. If the broken symmetry corresponds to a continuous symmetry (local or global), a number $n \leq \dim(\mathfrak{g})$ of fields associated with the internal degrees of freedom of the scalar field will emerge. In particular, a subset of them[10] will correspond to massless fields called *Goldstone bosons*, as they behave as such once the theory is quantized [37–41].

In gauge theories, one of the most important consequences of spontaneous symmetry breaking is the ability to introduce mass terms for the components of the gauge field in the Lagrangian without violating the underlying symmetry. Moreover, it can be shown that all Goldstone bosons can be absorbed by these gauge fields through a *unitary gauge*, rendering them nonphysical. This process of generating mass for gauge fields via spontaneous symmetry breaking is known as the *Higgs mechanism* [42–44].

In this section, we will show how the Higgs mechanism operates in the context of the Electroweak sector of the Standard Model, as a similar approach is taken for the model we seek to formulate. Additionally, Appendix B illustrates different types of spontaneous symmetry breaking using simpler examples, which may serve as an introductory reference for the following discussion if needed.

2.3.1 Gauge sector

The electroweak sector of the standard model is formulated as a Yang-Mills theory over the group $G = \mathrm{SU}(2)_L \times U(1)_Y$. This means that group actions over the fields come in the form of

$$U = \exp\left[i\theta_a(x)J_a \oplus i\theta_Y(x)T_Y\right]. \tag{2.60}$$

[10]All of them in the context of models with global symmetries. This result is calles the *Goldstone theorem*.

For J_a the $SU(2)_L$ generators and T_Y the $U(1)_Y$ generator. Therefore, it is a theory equipped with a gauge field written as the linear combination

$$A_\mu = \cos(\theta_W) A_\mu^{\mathfrak{su}(2)} \oplus \sin(\theta_W) A_\mu^{\mathfrak{u}(1)}, \tag{2.61}$$

where θ_W is called the Weinberg angle. The first subalgebra gauge fields is given by

$$A_\mu^{\mathfrak{su}(2)} := W_\mu \equiv W_\mu^a J_a \tag{2.62}$$

with J_a a representation of the SU(2) generators. The other field corresponds to

$$A_\mu^{\mathfrak{u}(1)} := B_\mu T_Y. \tag{2.63}$$

With T_Y a representation of the $U(1)_Y$ generator. These fields define the covariant derivative

$$D_\mu = \partial_\mu - igA_\mu \equiv \partial_\mu - i\left(g_L W_\mu^a J_a \oplus g_Y B_\mu T_Y\right), \tag{2.64}$$

with $g_L := g\cos\theta_W$ and $g_Y := g\sin\theta_W$. For fields that do not transform trivially, we consider the direct sum to be represented by the Kronecker sum:

$$g_L W_\mu^a J_a \oplus g_Y B_\mu = g_L W_\mu^a J_a \otimes \mathbb{1} + \mathbb{1} \otimes g_Y B_\mu T_Y. \tag{2.65}$$

We also consider the generators to be given by

$$J_a = \frac{\sigma_a}{2} \ \wedge \ T_Y = \frac{Y}{2}, \tag{2.66}$$

where $Y \in \mathbb{Z}$ labels different one−dimensional representations. We call this number *hypercharge*. Therefore, the covariant derivative takes the form

$$D_\mu = \partial_\mu - ig_L W_\mu^a J_a - ig_Y B_\mu \frac{Y}{2}\mathbb{1}. \tag{2.67}$$

From here, we can obtain the dynamic terms for the gauge fields by considering a different representation of the direct sum in (2.64). Let us consider the direct sum of matrix representations such that

$$g_L W_\mu^a J_a \oplus g_Y B_\mu T_Y = g_L W_\mu^a J_a + g_Y B_\mu T_Y = \begin{bmatrix} g_L W_\mu^a \frac{\sigma_a}{2} & \mathbf{0} \\ \mathbf{0} & g_Y B_\mu \frac{1}{\sqrt{2}} \end{bmatrix}, \tag{2.68}$$

with

$$J_a = \begin{bmatrix} \frac{\sigma_a}{2} & \mathbf{0} \\ \mathbf{0} & 0 \end{bmatrix} \ \wedge \ T_Y = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{1}{\sqrt{2}} \end{bmatrix}. \tag{2.69}$$

This representation ensures the correct normalization for the traces of strength field tensor products[11]. Under this representation, we can define the field strength tensor

$$F_{\mu\nu} = W_{\mu\nu} + B_{\mu\nu} T_Y, \tag{2.70}$$

[11]This representation staisfies $\mathrm{Tr}\{T_a T_b\} = \frac{1}{2}\delta_{ab}$.

with the $\mathfrak{su}(2)$ and $\mathfrak{u}(1)$ field strength tensors

$$W_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu - g_L \left[W_\mu, W_\nu\right] \quad \wedge \quad B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu \tag{2.71}$$

Which directly yields the dynamic term

$$-\frac{1}{2}\mathrm{Tr}\left\{F_{\mu\nu}F^{\mu\nu}\right\} = -\frac{1}{2}\mathrm{Tr}\left\{W_{\mu\nu}W^{\mu\nu}\right\} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu}. \tag{2.72}$$

2.3.2 Scalar sector and stabilizer subgroup

The scalar sector is built from a scalar potential and a dynamic term:

$$\mathscr{L}_s := \left(D_\mu\phi\right)^\dagger \left(D^\mu\phi\right) - m^2\phi^\dagger\phi - \frac{\lambda}{4}(\phi^\dagger\phi)^2. \tag{2.73}$$

Where ϕ is a $\mathbb{C}^2$ doublet. The potential present vacua in all points within the G orbit of

$$\langle\phi\rangle = \sqrt{\frac{2m^2}{\lambda}} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \equiv \begin{bmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{bmatrix}. \tag{2.74}$$

Although not evident at first sight, $\langle\phi\rangle$ implicitly defines a stabilizer subgroup of G: $\mathrm{U}(1)_{em}$. To see that this is the case, let us write the representation for the group elements that act on the scalar doublet. We can go back to (2.60) and consider the representation (2.66). Taking the Kronecker sum of these representations yields

$$U = \exp\left[i\left(\theta_a\frac{\sigma_a}{2} + \theta_Y\frac{Y}{2}\mathbb{1}\right)\right]. \tag{2.75}$$

Consider the following change of basis for the $\mathfrak{g} = \mathfrak{su}(2)_L \oplus \mathfrak{u}(1)_Y$ algebra:

$$\left\{\frac{\sigma_1}{2}, \frac{\sigma_2}{2}, \frac{\sigma_3}{2}, \frac{Y}{2}\mathbb{1}\right\} \to \left\{\frac{\sigma_1}{2}, \frac{\sigma_2}{2}, Q^+, Q^-\right\}, \tag{2.76}$$

where the *charge generators* are defined as

$$Q^\pm := \frac{\sigma_3}{2} \pm \frac{Y}{2}\mathbb{1}. \tag{2.77}$$

Under this new basis, we can rewrite (2.75) as[12]

$$\exp\left[i\left(\theta_1\frac{\sigma_1}{2} + \theta_2\frac{\sigma_2}{2} + \theta_+Q^+ + \theta_-Q^-\right)\right] = \exp\left[i\left(\tilde\theta_1\frac{\sigma_1}{2} + \tilde\theta_2\frac{\sigma_2}{2}\right)\right]\exp\left[i\tilde\theta_{em}Q^+\right]\exp\left[i\tilde\theta_3Q^-\right], \tag{2.78}$$

with the new parameters

$$\theta_\pm := \frac{1}{2}(\theta_3 \pm \theta_Y). \tag{2.79}$$

[12]This decomposition is validated by the Baker–Campbell–Hausdorff formula (A.16) since $\left[Q^+, Q^-\right] = 0$, $\left[\frac{\sigma_1}{2}, Q^\pm\right] \propto \frac{\sigma_2}{2}$ and Vice versa.

Notice that both Q^+ and Q^- are diagonal matrices with real entries. This makes them both reducible representations of the $\mathfrak{u}(1)$ generator. In particular, setting the $U(1)_Y$ representation to $Y = 1$ for the scalar leaves $\langle\phi\rangle \in \mathrm{Ker}(Q^+)$:

$$Q^+ \langle\phi\rangle = 0. \tag{2.80}$$

Thus, we identify the existence of a stabilizer U(1) subgroup defined by this generator (independent of any individual representation D):

$$U(1)_{em} := \{ g \in G \, / D(g)\langle\phi\rangle \equiv U_{em}\langle\phi\rangle = \langle\phi\rangle \}. \tag{2.81}$$

Clearly, this means that the generic $U(1)_{em}$ transformations are given by

$$U_{em} = \exp\left[i\tilde{\theta}_{em}(x) Q^+ \right]. \tag{2.82}$$

with Q^+ given abstractly by

$$Q^+ = J_3 + T_Y. \tag{2.83}$$

2.3.3 Unitary gauge

Knowing that expressing ϕ in terms of $\langle\phi\rangle$ will spontaneously break G to $U(1)_{em}$, we discuss the role of gauge fixing. We now that a full parametrization of a $\mathbb{C}^2$scalar doublet can be obtained by writing

$$\phi(x) = \exp\left[i\chi\pi_a(x)\frac{\sigma_a}{2} \right] \begin{bmatrix} 0 \\ \dfrac{h(x)+v}{\sqrt{2}} \end{bmatrix}, \tag{2.84}$$

with χ a normalization factor. By substitution of this expression on its dynamic term one finds the fields $\pi_a(x)$ to be the massless Goldstone bosons. As stated before, we can eliminate these by working in a particular gauge. To do this in an illustrative way, let us consider an *overparametrized* expression obtain by adding a global phase $\rho Y/2 \in \mathbb{R}$

$$\phi(x) = \exp\left[i\chi\pi_a(x)\frac{\sigma_a}{2} \right] \begin{bmatrix} 0 \\ \dfrac{h(x)+v}{\sqrt{2}} e^{i\rho Y/2} \end{bmatrix} = \exp\left[i\chi\pi_a(x)\frac{\sigma_a}{2} + i\rho\frac{Y}{2}\mathbb{1} \right] \begin{bmatrix} 0 \\ \dfrac{h(x)+v}{\sqrt{2}} \end{bmatrix}. \tag{2.85}$$

We can rearrange this expression in the following way:

$$\begin{aligned} \phi &= \exp\left[i\pi_1\frac{\sigma_1}{2} + i\pi_2\frac{\sigma_2}{2} + i\frac{\pi_3+\rho}{2}Q^+ + i\frac{\pi_3-\rho}{2}Q^- \right] \begin{bmatrix} 0 \\ \dfrac{h+v}{\sqrt{2}} \end{bmatrix} \\ &= \exp\left[i\chi\pi_1'\frac{\sigma_1}{2} + i\chi\pi_2'\frac{\sigma_2}{2} \right] \exp\left[i\chi\pi_3' Q^- \right] \begin{bmatrix} 0 \\ \dfrac{h+v}{\sqrt{2}} \end{bmatrix}. \end{aligned} \tag{2.86}$$

Where we have used the fact that $\exp[i\xi Q^+][0\ 1]^T = [0\ 1]^T$, $\forall \xi \in \mathbb{R}$. Lets write ϕ as the result of a G transformation acting over another scalar doublet $\tilde{\phi}$ in the following way

$$\phi = U\tilde{\phi} \tag{2.87}$$

$$\exp\left[i\chi\pi'_1\frac{\sigma_1}{2} + i\chi\pi'_2\frac{\sigma_2}{2}\right]\exp\left[i\chi\pi'_3 Q^-\right]\begin{bmatrix} 0 \\ \frac{h+v}{\sqrt{2}} \end{bmatrix} = \exp\left[i\tilde{\theta}_1\frac{\sigma_1}{2} + i\tilde{\theta}_2\frac{\sigma_2}{2}\right]\exp[i\tilde{\theta}_3 Q^-]\exp[i\tilde{\theta}_{em}Q^+]\tilde{\phi}.$$

We can now fix the gauge such that $\tilde{\phi} = \frac{1}{\sqrt{2}}(h+v)[0\ 1]^T$, which means enforcing $\tilde{\theta}_a = \chi\pi'_a$. Once this is done, the gauged transformation rules of all fields will ensure no π'_a fields appear in the Lagrangian. This choice also places no constrains over $\theta_{em}(x)$, leaving $\mathrm{U}(1)_{em}$ as a unbroken gauge group. The reader can verify that the process of gauge fixing just carried out is equivalent to setting θ_a in (2.75) equal to $\chi\pi_a$ in (2.84), as it also manages to eliminate the Goldstone fields from the Lagrangian[13]. Nevertheless, this later approach is less illustrative of the remanence of Q^+ as the generator of the unbroken subgroup.

2.3.4 Mass for the bosons

By construction, the scalar will spontaneously break the symmetry down to its stabilizer. Consequently, the gauge field component associated with this remaining symmetry will be the only one retaining a direct role as a gauge field. This becomes even more apparent upon fixing the gauge, as it leaves only this field as the remaining gauge degree of freedom. Naturally, this stabilizer gauge field will be a linear combination of W^3_μ and B_μ. To see that this is the case, we write the gauge field (2.61) by considering a change of basis similar to (2.76)

$$\begin{aligned} A_\mu &= \cos(\theta_W)W^b_\mu\frac{\sigma_b}{2} + \cos(\theta_W)W^3_\mu\frac{\sigma_3}{2} + \sin(\theta_W)B_\mu\frac{Y}{2}\mathbb{1} \\ &= \cos(\theta_W)W^b_\mu\frac{\sigma_b}{2} + \frac{1}{2}\left(\cos(\theta_W)W^3_\mu + \sin(\theta_W)B_\mu\right)Q^+ + \frac{1}{2}\left(\cos(\theta_W)W^3_\mu - \sin(\theta_W)B_\mu\right)Q^- \\ &\equiv \cos(\theta_W)\frac{1}{\sqrt{2}}\left(W^+_\mu\sigma^+ + W^-_\mu\sigma^-\right) + \frac{1}{2}A^\gamma_\mu Q^+ + \frac{1}{2}Z_\mu Q^-. \end{aligned} \tag{2.88}$$

Here we have defined the new vector fields

$$\begin{aligned} W^\pm &:= \frac{1}{\sqrt{2}}\left(W^1_\mu \mp iW^2_\mu\right) \\ A^\gamma_\mu &:= \cos(\theta_W)W^3_\mu + \sin(\theta_W)B_\mu \\ Z_\mu &:= \cos(\theta_W)W^3_\mu - \sin(\theta_W)B_\mu \end{aligned} \tag{2.89}$$

[13]In this case, θ_Y will remain unconstrained. Since the group parameter of $\mathrm{U}(1)_{em}$ is a function of θ_Y, this leaves the unbroken group parameter unconstrained

and the *leader* generators[14]

$$\sigma^{\pm} := \frac{\sigma_1}{2} \pm i\frac{\sigma_2}{2} = \begin{bmatrix} 0 & \delta_{-,\pm} \\ \delta_{+,\pm} & 0 \end{bmatrix}, \tag{2.90}$$

From these fields A^{γ}_{μ} corresponds to the gauge field of the unbroken symmetry, and we therefore expect this component to remain massless. Indeed, the Higgs mechanism generates masses for the gauge field components through the scalar kinetic term

$$\left(D_\mu\phi\right)^\dagger \left(D^\mu\phi\right) = \left(D_\mu\tilde{\phi}\right)^\dagger \left(D^\mu\tilde{\phi}\right). \tag{2.91}$$

But $\langle\phi\rangle \in \mathrm{Ker}(Q^+)$ implies

$$D_\mu\tilde{\phi} \ni i\frac{g}{2}A^{\gamma}_{\mu}Q^+\tilde{\phi} = 0, \tag{2.92}$$

meaning no quadratic (mass) term for A^{γ}_{μ} will appear. Let us write the covariant derivative in terms of the components (2.88).

$$D_\mu\tilde{\phi} = \partial\tilde{\phi} - i\frac{g_L}{\sqrt{2}}\left(W^+_\mu\sigma^+ + W^-_\mu\sigma^-\right)\tilde{\phi} - i\frac{g_L}{2\cos(\theta_W)}Z_\mu Q^-\tilde{\phi} = \begin{bmatrix} i\frac{g_L}{2}W^+(h+v) \\ \frac{1}{\sqrt{2}}\partial_\mu h + i\frac{g_L}{2\sqrt{2}\cos(\theta_W)}Z_\mu(h+v) \end{bmatrix}$$

In this basis, computing (2.91) is trivial

$$\left(D_\mu\phi\right)^\dagger \left(D^\mu\phi\right) = \frac{1}{2}(\partial_\mu h)(\partial^\mu h) + \frac{g_L^2(h+v)^2}{4}W^-_\mu W^{+\mu} + \frac{g_L(h+v)^2}{8\cos(\theta_W)}Z_\mu Z^\mu. \tag{2.93}$$

It is clear from this expression that the W and Z components have acquired mass

$$M_W^2 W^-_\mu W^{+\mu} + \frac{1}{2}M_Z^2 Z_\mu Z^\mu \in \left(D_\mu\phi\right)^\dagger \left(D^\mu\phi\right) \tag{2.94}$$

with

$$M_W := \frac{g_L v}{2} \quad \wedge \quad M_Z := \frac{g_L v}{2\cos(\theta_W)}. \tag{2.95}$$

The appearance of these terms is possible purely because we expressed the scalar ϕ in terms of deviations from the vacuum. Distributivity of the gauge field components in factors like $W^a_\mu(h+v)$ yield "couplings" between the gauge field components and v, eventually resulting in quadratic terms $\sim v^2 W^a_\mu W^{\mu a}$. We can write this symbolically as

$$\phi = \phi_h + \langle\phi\rangle \implies \left(D_\mu\phi\right)^\dagger \left(D^\mu\phi\right) \ni M^2 \mathrm{Tr}\left\{V_\mu V^\mu\right\} \propto \left(V_\mu\langle\phi\rangle\right)^\dagger \left(V^\mu\langle\phi\rangle\right). \tag{2.96}$$

For ϕ_h the deviation from the vacuum and V_μ a generic gauge field component that acquires mass M. In contrast, those components that remain as stabilizer gauge fields will annihilate the vacuum yielding no mass terms.

[14]Additional information in why this basis (i.e. defining $\sigma^{\pm}$) is necessary to obtain the physical fields can be found in Appendix C.

2.3.5 Unitarity of spontaneously broken Yang-Mills theories

Unitarity is a fundamental property of any quantum theory, as it ensures that the inner product of the state space can be interpreted probabilistically. In high energy physics, most systems involve scattering, decay, or annihilation processes, making unitarity essential for maintaining the physical interpretation of transition probabilities. In particular, unitarity leads to the *optical theorem*, which is briefly discussed in Appendix D.

Despite its fundamental role in quantum field theory, not all classical field theories present unitarity after quantization. In fact, many quantum field theories exhibit unitarity only within a specific energy range, beyond which the optical theorem is no longer satisfied, leading to non-physical results. We call this type of models *effective theories*. Given this, the question of unitarity in quantum gauge theories, especially those incorporating spontaneous symmetry breaking, becomes particularly relevant. Although this section will not provide a review of the mathematical famework that leads to the conclusion that gauge theories with spontaneously broken symmetries *respect unitarity and renormalizability*[15], we will outline in a historical manner the elements from which it was constructed.

The study of unitarity in gauge theories gained prominence with the development of Yang-Mills theories. Early concerns arose regarding whether such theories could be consistently quantized while maintaining unitarity. The introduction of gauge fixing techniques by Faddeev and Popov [45] provided a systematic approach to dealing with redundant degrees of freedom, leading to the identification of *ghost fields* necessary to preserve unitarity at the quantum level. These are unphysical, anticommuting scalar fields introduced in the quantization of gauge theories to cancel unphysical degrees of freedom associated with gauge redundancies.

A significant breakthrough came with the work of Becchi, Rouet, Stora, and Tyutin (BRST) [46, 47], who developed a symmetry that ensures the consistency of gauge fixing and ghost field contributions. The BRST formalism provides a framework for handling gauge symmetries in quantum field theory, introducing a conserved quantity known as the BRST charge. This charge generates transformations that mix physical and ghost fields in a way that maintains gauge invariance at the quantum level. The presence of this symmetry ensures that unphysical states, including ghosts, do not contribute to physical observables, *preserving both unitarity and renormalizability*. Further work by Lee and Zinn-Justin [48] extended these results, showing that Yang-Mills theories remain renormalizable under perturbative quantum corrections.

With a firm understanding of unitarity in massless gauge theories, attention turned to spontaneous symmetry breaking. As discuses earlier and in Appendix B, the Higgs mechanism provides

[15] Doing so in a proper way would would take a great amount of space in this thesis, making it way longer than it is.

a means to give gauge bosons mass without actually breaking the group invariance. However, at the time it was not clear whether theories with massive gauge bosons retained unitarity. The proof of renormalizability in the presence of spontaneous symmetry breaking was a achieved by 't Hooft [49], who showed that despite the introduction of mass terms, gauge theories remain renormalizable when treated within a defined framework. The idea involved working in a renormalizable gauge and proving that all divergences could be consistently absorbed into redefinitions of the theory's parameters. 't Hooft's work demonstrated that despite the introduction of mass terms, gauge theories remain renormalizable when treated within this approach. This result was later extended in collaboration with Veltman [50], establishing that gauge theories with spontaneous symmetry breaking, such as the Standard Model, are consistent and preserve unitarity.

The fact that gauge theories do have such properties makes them ideal for building fundamental theories as well as ultraviolet extensions of effective theories when possible[16]. This validates the use of gauge theories as an ideal formalism from which the end of this thesis can be achieved.

[16] A historical example could very well be the electroweak sector of the standard model as a completion of the 4−Fermi theory.

Chapter 3

Minimal Spin-one isotriplet model

In this chapter, we focus on presenting the effective model that serves as the foundation for the dark−electroweak model, by leaning heavily in the original article that introduced it [1][1]. Known as the *Minimal Spin-one isotriplet dark matter model* or simply as the *vector isotriplet model*, this framework extends the Standard Model in a simple way. It introduces a triplet vector field V_μ, transforming under the adjoint representation of $SU(2)_L$, along with two new parameters: the mass of V_μ and its coupling to the Higgs boson. These additions establish a Higgs portal interaction, creating a connection between the dark sector and the Standard Model.

The dark matter candidate in this model is the neutral component of the vector triplet, V_μ^0, which classifies as a WIMP. Radiative corrections ensure that V_μ^0 is lighter than the charged components of the triplet, stabilizing it against decay. Importantly, as this chapter will ilustrate, V_μ^0 can reproduce the correct relic density and remains consistent with current experimental constraints, provided its mass lies within the range of 2.8 to 3.8 TeV.

Nevertheless, we will see that the model exhibits sources of unitarity violation, rendering it nonphysical beyond a certain energy cutoff. This limitation underlines the necessity of the ultraviolet completion that this thesis aims to develop. Chapter 4 will be concerned with showing that the model we are about to present is contained within a larger theory and can be extended. But first, we need to understand the features of this isotriplet model.

[1] Most non−Feymann diagram figures (3.5, 3.6, 3.7, 3.9) of this chapter have also been extracted from this article.

3.1 Model structure

The Lagrangian density for the model can be understood as

$$\mathscr{L} = \mathscr{L}_{sm} + \mathscr{L}_V, \tag{3.1}$$

with $\mathscr{L}_{sm}$ the standard model Lagrangian density and

$$\begin{aligned}\mathscr{L}_V :=& -\mathrm{Tr}\left\{D_\mu V_\nu D^\mu V^\nu\right\} + \mathrm{Tr}\left\{D_\mu V_\nu D^\nu V^\mu\right\} - \frac{g^2}{2}\mathrm{Tr}\left\{\left[V_\mu, V_\nu\right]\left[V^\mu, V^\nu\right]\right\} \\ & - ig\,\mathrm{Tr}\left\{W_{\mu\nu}\left[V^\mu, V^\nu\right]\right\} + \tilde{M}^2\,\mathrm{Tr}\left\{V_\mu V^\mu\right\} + a\left(\phi^\dagger\phi\right)\mathrm{Tr}\left\{V_\mu V^\mu\right\}.\end{aligned} \tag{3.2}$$

As stated before, the new fields correspond to the triplet components V_μ^a and we can see that only the parameters $\tilde{M}$ and a have been introduced to the standard model. It is also worth noticing that, since V_μ transforms under the adjoint representation (group conjugation), it is an algebra-valued field, which means that its components V_μ^a $(a = 1, 2, 3)$ are defined by

$$V_\mu \equiv V_\mu^a T_a, \tag{3.3}$$

with T_a the SU(2) generators in their fundamental representation.

3.2 Triplet mass

Although an explicit mass term for this object can be found in (3.2), an extra contribution will appear as the $\mathrm{SU}(2)_L$ symmetry spontaneously breaks. We can see that this is the case by allowing the Higgs doublet $\phi(x)$ to acquire its vacuum expectation value $v \approx 246$ GeV. As with the standard model, we do this by witting it as an object of the $\mathrm{SU}(2)_L$ orbit of $[0\ \ \alpha(x)]^T$ with

$$\alpha(x) := \frac{v + h(x)}{\sqrt{2}}, \tag{3.4}$$

such that

$$\phi^\dagger\phi = \frac{1}{2}(v + h)^2. \tag{3.5}$$

This implies that, once the symmetry has been broken, we obtain the following mass term for the triplet components

$$\frac{1}{2}\left(\tilde{M}^2 + \frac{1}{2}av^2\right)V_\mu^a V^{\mu a} \in \tilde{M}^2\,\mathrm{Tr}\left\{V_\nu V^\nu\right\} + a\left(\phi^\dagger\phi\right)\mathrm{Tr}\left\{V_\mu V^\mu\right\}, \tag{3.6}$$

which implies a square mass of

$$M_V^2 := \tilde{M}^2 + \frac{1}{2}av^2. \tag{3.7}$$

We also can see that from (3.6) new physical fields can be defined. In analogy to the $W^{\pm}$ bosons, the eigenfields of the electric charge operator[2] are

$$V_\mu^{\pm} := \frac{1}{\sqrt{2}}\left(V_\mu^1 \pm i V_\mu^2\right) \quad \wedge \quad V_\mu^0 := V_\mu^3. \tag{3.8}$$

This means that the mass term can be rewritten as

$$M_V^2 V_\mu^a V^{\mu a} = M_V^2 V_\mu^+ V^{\mu -} + \frac{1}{2} M_V^2 V_\mu^0 V^{\mu 0}. \tag{3.9}$$

From this expression, we can see that the masses for the charged and neutral fields result to be equal at tree level. However, radiative corrections due to electroweak interactions at the one− and higher−loop orders induce mass splitting, making V^0 lighter than $V^{\pm}$. This can be evidenced by considering the pole masses up to one−loop to be

$$\left(M_p^{\pm}\right)^2 = M_V^2 - \Sigma^{\pm}\left(M_V^2\right) \quad \wedge \quad \left(M_p^0\right)^2 = M_V^2 - \Sigma^0\left(M_V^2\right), \tag{3.10}$$

with $\Sigma^{\pm}$ and Σ^0 the correspondent self−energies. The mass splitting can be obtained by expanding the pole masses

$$M_p^i = M_V \sqrt{1 - \frac{\Sigma^i\left(M_V^2\right)}{M_V^2}} = M_V \sum_{n=0}^{\infty} (-1)^n \begin{pmatrix} 1/2 \\ n \end{pmatrix} \left(\frac{\Sigma^i\left(M_V^2\right)}{M_V^2}\right)^n \quad \Big/ i = 0, \pm \tag{3.11}$$

and taking the difference

$$\Delta M_V = M_p^{\pm} - M_p^0 = M_V \sum_{n=0}^{\infty} (-1)^n \begin{pmatrix} 1/2 \\ n \end{pmatrix} \left[\left(\frac{\Sigma^{\pm}\left(M_V^2\right)}{M_V^2}\right)^n - \left(\frac{\Sigma^0\left(M_V^2\right)}{M_V^2}\right)^n\right]. \tag{3.12}$$

Contributions to $\Sigma^{\pm}$ will come from diagrams that link $V^{\pm}$ to the photon, Z^0 and $W^{\pm}$ bosons (see Figure 3.1); while the only contribution to Σ^0 will come diagrams that link V^0 to the $W^{\pm}$ bosons (see Figure 3.2).

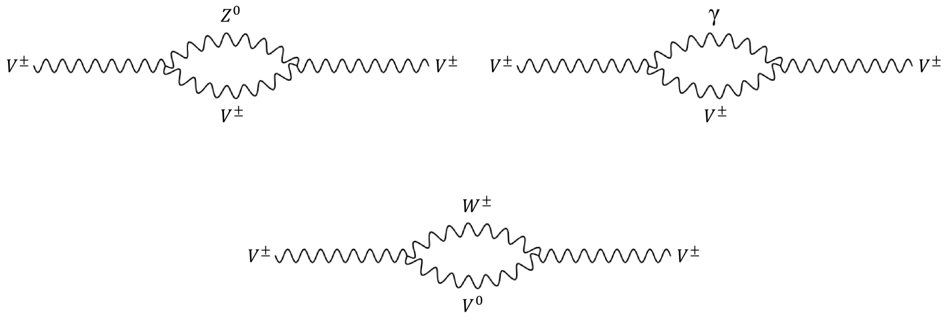

Figure 3.1: Diagrams corresponding to the self−energies $\Sigma^{\pm}$ at one loop.

[2]See Appendix C

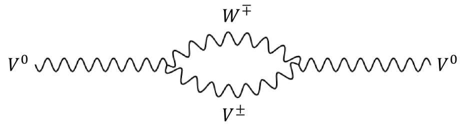

Figure 3.2: Diagram corresponding to the self–energy $\Sigma^{\pm}$ at one loop.

This difference in diagrams evidences the difference in self–energies and in masses. The explicit calculations of these corrections lie beyond the scope of this thesis, but an important result to highlight comes from the limit in which the triplet mass is much greater that those of the $W^{\pm}$ and Z^0 bosons (i.e. $M_V \gg M_W, M_Z$). In this case, the splitting limit converges towards a fixed value

$$\Delta M \to \frac{5g_L^2\left(M_W - \cos^2\theta_W M_Z\right)}{32\pi} \approx 217.3\text{ MeV}. \tag{3.13}$$

These results are important since they show how V^0 is lighter than its charged counterparts, which means that no kinematically allowed decay channels exist. In contrast, the heavier components can decay into V^0 plus electroweak bosons.

3.3 Perturbative unitarity

In a previous work it has also been shown that models with a Lagrangian similar to $\mathscr{L}_V$ do not present violation of perturbative unitarity, at least at tree level [51]. Nevertheless, the incorporation of a coupling

$$a\left(\phi^\dagger\phi\right)\text{Tr}\left\{V_\mu V^\mu\right\} \tag{3.14}$$

will prove to reintroduce unitarity violation, differentiating this model from those discussed in said work. Violation of unitarity can be evidenced by studying the forward process $V^+V^- \to V^+V^-$ in the high energy limit. At tree level, seven diagrams contribute to this process (see Figure 3.3). The matrix element $\mathscr{M}$ for large center–of–mass energies is given by

$$\mathscr{M}(V^+V^- \to V^+V^-) \to -i\frac{\left(16a^2\sin^4\theta_W + 3e^4\right)(\cos\theta+1)M_W^2 s}{8e^2\sin^2\theta_W M_V^4}, \tag{3.15}$$

with θ_W, θ and e the Weinberg angle, scattering angle and electron charge respectively. notice that this final expression is linear in the squared center of mass energy s. Therefore, as the momentum increases, so does the matrix element. Since the expression is unbounded, unitarity violation follows. To determine an energy cutoff for unitarity, we expand $\mathscr{M}$ in Legendre polynomials [3]. We find that the most stringent unitarity constraint comes from the partial wave unitarity bound

$$|\text{Re}(a_0)| \le 1 \tag{3.16}$$

[3] For more detail on this expansion (D.18) and the approach of partial wave unitary bounds, see Appendix D

This coefficient is calculated as

$$a_0 = \frac{1}{32\pi}\int_{-1}^{1} \mathcal{M} d(\cos\theta) \tag{3.17}$$

Computing this integral and replacing in (3.16) yields the energy cutoff

$$\sqrt{s} < \Lambda = \frac{2^{13/4} e M_V^2 \sqrt{\pi} \sin\theta_W}{\sqrt{16a^2 M_W^2 \sin^4\theta_W - 3e^4 M_Z^4 \sin^2\theta_W + 3e^4 M_Z^2 - 4e^4 M_V^2}}. \tag{3.18}$$

Figure 3.4 shows the order of magnitude of this scale as a function of M_V for different values of a. Notice that, for all of these values, the curves converge to scales $\Lambda \sim 12M_V$ from above. This is important because $\sqrt{s} \propto M_V$, which means that a cutoff too close to M_V render the theory inconsistent. Fortunately, this model grantees that will always be valid for at least $\sqrt{s} < 12M_V$.

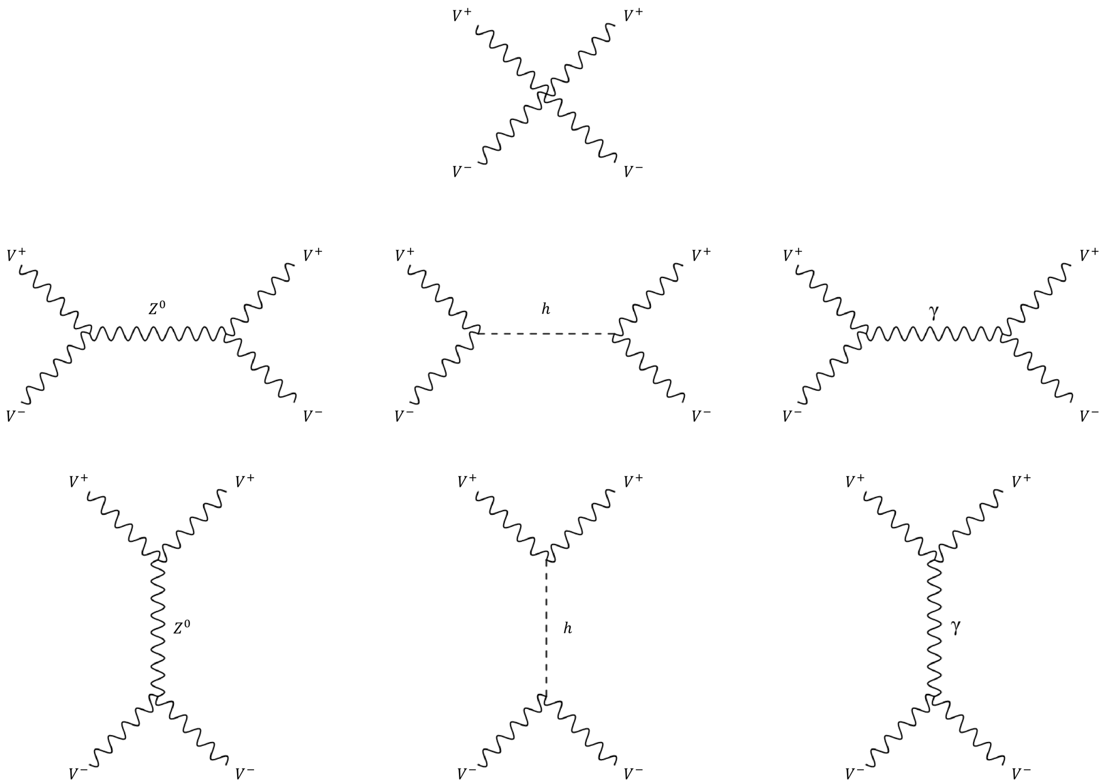

Figure 3.3: The forward matrix element is determined by the exchange of a Higgs boson, a $Z-$boson or a photon in the s and the t channels and a contact $V^+V^-V^+V^-$ diagram.

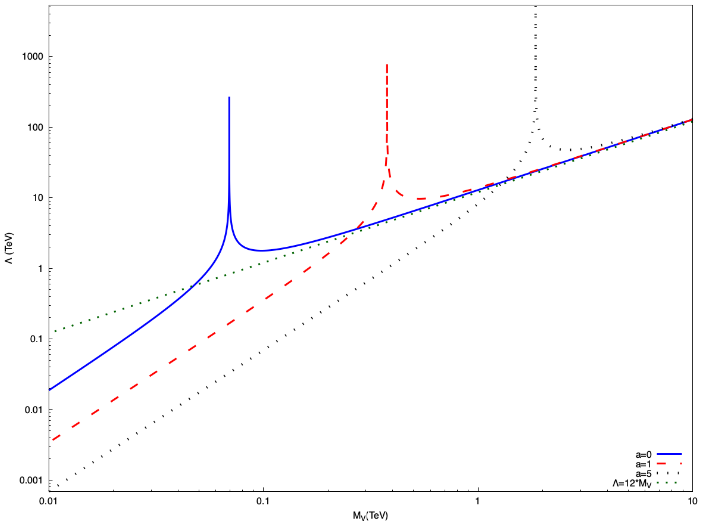

Figure 3.4: Order of magnitude plot for the energy cutoff Λ as function of M_V for $a = 0, 1, 5$. A dotted straight line for $\Lambda = 12M_V$ has been plotted, which all curves asymptotically approach from above.

3.4 Phenomenological generalities

We now shift our attention to the phenomenological aspects of this model. The authors of the original article present a few testable consequences that we will cover. Comparison between the experimental data at the time of publication (2019) and the possible predictions given by the model allow the parameter space to be restricted. We will show the results obtained originally and comment how they compare with most recent experimental data and projections.

Computing the relic density for the dark matter candidate V^0 yields the results shown in Figure 3.5. Various curves have been obtained for fixed values of a. We find that the relic abundance can always match the Planck experiment result $\Omega_{PLANCK}h^2 = 0.1186 \pm 0.0020$ [52], for sufficiently large values of M_V. The lowest of these can be found at $M_V \approx 2.85$ TeV, corresponding to $a = 0$. For increasing values of a, the mass M_V reaches a maximal value of several tens of TeV obtained at the perturbative bound of $a = 4\pi$. More recent results [22] do not alter these observations.

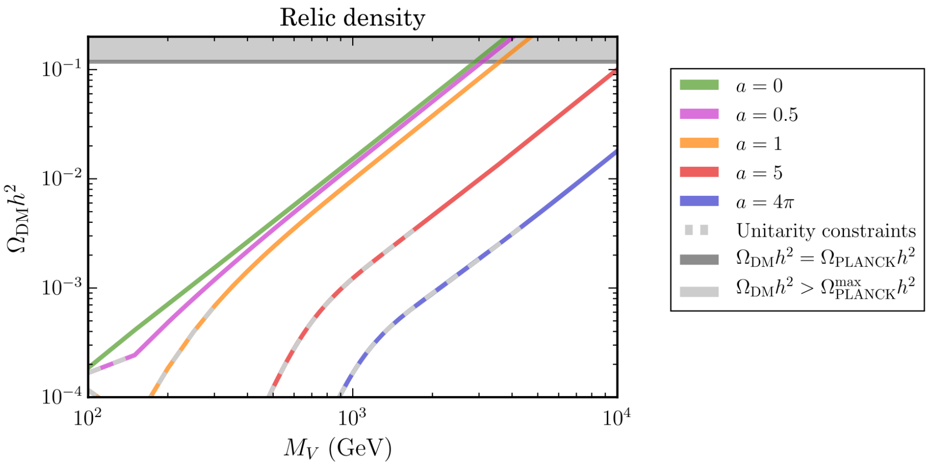

Figure 3.5: Thermal relic density $\Omega_{DM}h^2$ for V^0 as function of M_V for various values of a. The dark gray horizontal band corresponds to the region where dark matter relic density is within 1σ of the value measured by the Planck experiment. Dashed segments of each curve correspond to segments where unitarity is violated.

Additional constrains to the parameter space come from interactions between the vector dark matter and Standard Model nucleons through the Higgs coupling a (i.e. direct detection). Computing the spin−independent cross section on protons $\sigma_{\rm SI}$ allows to place bounds on a. To achieve this, the original work made use of the 2018 results form the XENON1T experiment [53, 54]. The results for these bounds are shown in Figure 3.6. Notice that the XENON1T results exclude values of $|a| > 1$, even for masses that present a deficit in relic density. This leaves a region of $M_V \lesssim 4$ TeV. More recent results like those of LUX-ZEPLIN (2024) [55] leave a much tighter space, since it has been shown to exclude values of $a \gtrsim 0.1$. Projections for XENONnT [56] and

DARWIN [57] significantly lower this cotes, making the experiment, becoming sensitive to the remaining region.

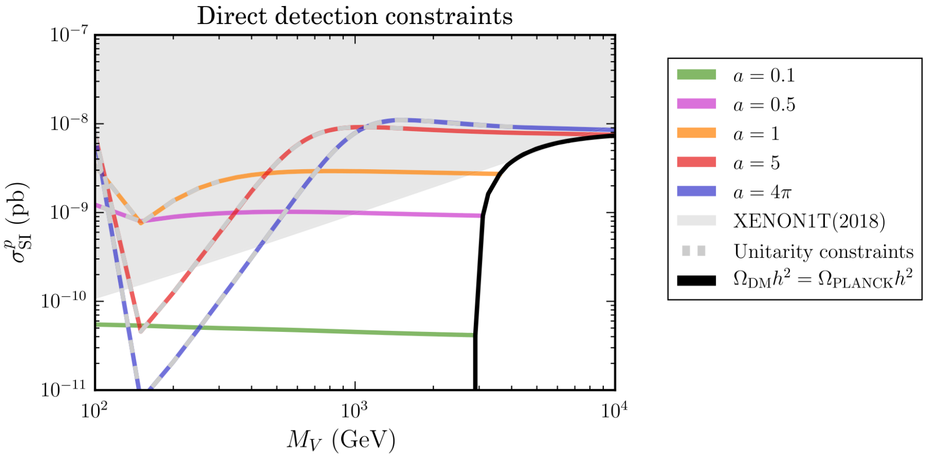

Figure 3.6: Spin−Independent for V^0−nucleon elastic scattering as function of M_V. The black curve represents the set of point (M_V, σ_{SI}) that reproduce the Relic density measured by the Planck experiment. The gray area Correspond to the region ruled out by the XENON1T experiment. The dashed lines highlight segments where unitarity is violated.

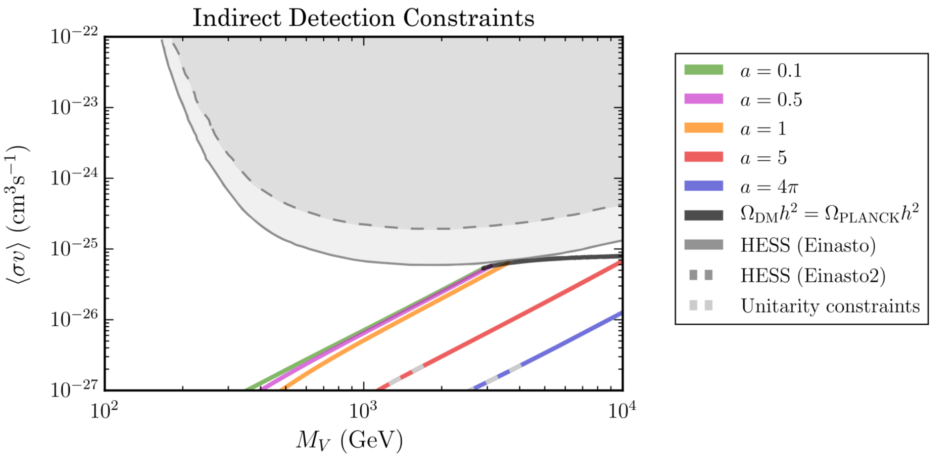

Figure 3.7: velocity averaged anihilation cross section of V^0 compared to the HESS bounds imposed for the W^+W^- channel. Each gray band corresponds to a different choice of dark matter profile model (Einasto profile). Once more, the black and dashed regions correspond to relic density measured by the Planck experiment and regions of unitarity violation respectively.

Results for indirect detection can also be obtained from the dominant annihilation channel of V^0 into a W boson pair to later obtain a $ZZhh$ final state. This process implies that the model can be studied by the search of cosmic rays produced by these resulting particles. Photons emitted by decay of the W, Z and Higgs bosons proves to be the most sensitive channel, so the velocity averaged cross section of the dark matter is compared with the results of HESS for the W^+W^-

decay channel [58]. Figure 3.7 shows this comparison. When it comes to future experiments, projections of the CTA [59] also lower the cotes to the region of allowed values.

A final observable consequence exposed by the authors was the contribution of this model to the decay process $h \to \gamma\gamma$ through loop effects. The diagram for this term is shown in Figure 3.8, and it consist on a loop composed by the charged components of the vector isotriplet. The explicit expression for the decay width $\Gamma(h \to \gamma\gamma)$ can be found in the original article. For our discussion, we emphasize the results for the decay ratio

$$R_{\gamma\gamma} := \frac{\Gamma(h \to \gamma\gamma)}{\Gamma(h \to \gamma\gamma)_{sm}}. \tag{3.19}$$

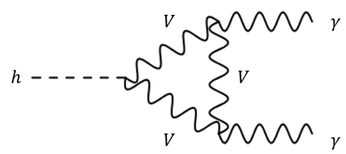

Figure 3.8: Diagram of the new decay contribution. this diagram has the same structure of the W boson's contribution.

In this last expression, $\Gamma(h \to \gamma\gamma)_{sm}$ is considered to be the predictions of the standard model for the same process. The experimental results used to compare the model's results are those of ATLAS [60] and CMS [61], obtained from an integrated luminosity of about 36 fb^{-1}. These results give intervals of

$$R_{\gamma\gamma}^{\text{ATLAS}} = 0.99 \pm 0.14 \;\wedge\; R_{\gamma\gamma}^{\text{CMS}} = 1.18^{+0.17}_{-0.12}. \tag{3.20}$$

Figure 3.9 shows the ratio $R_{\gamma\gamma}$ as a function of M_V for different different values of a ($a = \pm 1, \pm 5$). The colored bands are experimentally allowed regions at confidence of 2σ from ATLAS (pink) and CMS (yellow). In this regard, we can see that all values $|a| \in [1,5]$ present a region of $M_V \gtrsim 1.5$ TeV allowed by experiments. Since 2019, the intervals allowed by ATLAS and CMS have become tighter [62], with

$$R_{\gamma\gamma}^{\text{ATLAS}} = 1.04 \pm 0.10 \;\wedge\; R_{\gamma\gamma}^{\text{CMS}} = 1.08^{+0.12}_{-0.11}. \tag{3.21}$$

Nevertheless, small values for a are still well within these experimental constrains since the associated $R_{\gamma\gamma}(M_V)$ curves converge faster to the unity.

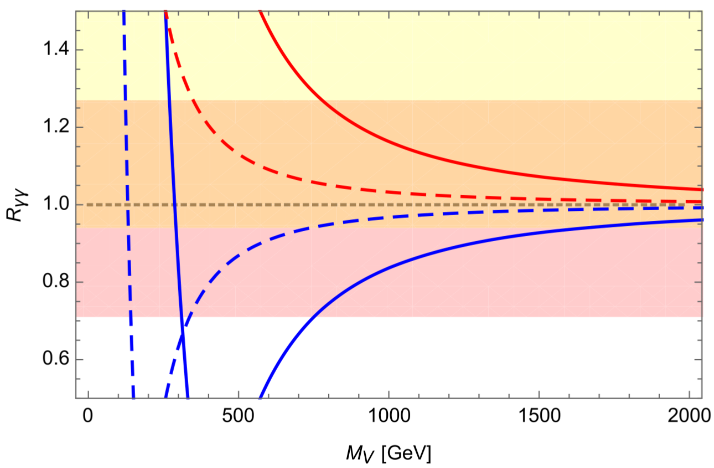

Figure 3.9: $R_{\gamma\gamma}$ as function of M_V for $a = \pm 1$ (dashed) and $a = \pm 5$ (solid). Different colors have been used to distinguish positive (red) values of a from negative ones (blue). Colored regions refer to zones allowed by the atlas (PINK) and CMS (yellow) experiments. The orange band corresponds to the intersection between the two regions defined by the experiments.

3.5 Final remarks

During this chapter, the most important features of the vector isotriplet model have been covered, but some others have been left out. Still, the aspects reviewed here should be enough to familiarize the reader with the simplicity of the model and the main constrains that its parameter space has. As it stands, all projections for future experiments render it falsifiable and within reach. Nevertheless, knowing that $\mathscr{L}_V$ represents a yet phenomenologically safe addition to the standard model motivates the search for extensions that do not suffer loss of unitarity. In next chapter we will do this by embedding the structure of $\mathscr{L}_V$ it into the larger structure of a gauge theory. Doing so will certainly include new features in the phenomenology that will have to be studied in the future. But for now, we trust in the solid base provided by the effective model and take this task of model building into our hands.

As a motivation and a glimpse of what will follow in the next chapter, the reader can verify that, by adding the $SU(2)_L$ gauge dynamic term to the isotriplet Lagrangian density (3.2), algebraic manipulation yields

$$-\frac{1}{2}\mathrm{Tr}\left\{W_{\mu\nu}W^{\mu\nu}\right\} + \mathscr{L}_V = -\frac{1}{2}\mathrm{Tr}\left\{F^{(4)}_{\mu\nu}F^{(4)\mu\nu}\right\} + \tilde{M}^2\,\mathrm{Tr}\left\{V_\mu V^\mu\right\} + a\left(\phi^\dagger\phi\right)\mathrm{Tr}\left\{V_\mu V^\mu\right\} \tag{3.22}$$

Under the definition of

$$F^{(4)}_{\mu\nu} := \partial_\mu(W_\nu + V_\nu) - \partial_\nu\left(W_\mu + V_\mu\right) - ig\left[W_\mu + V_\mu; W_\nu + V_\nu\right]. \tag{3.23}$$

Which is clearly the structure of a gauge field strength tensor. This hints the possibility of the original model being a an effective description of a gauge theory at low scales. If this was the case, the fact that V_μ is a massive triplet that does not transform like a gauge field suggests that this hypothetical gauge theory was constructed from a group that has been spontaneously broken down to $SU(2)_L$ somehow.

CHAPTER 4

ULTRAVIOLET COMPLETION: THE DARK–ELECTROWEAK MODEL

4.1 Model structure

The core idea behind the model is that both dark (non-gauge) and electroweak (gauge) sectors can be combined into one unified $\mathfrak{so}(4)$ gauge sector. Despite this, to construct a scalar sector that spontaneously breaks the symmetry in a proper way, the fields will be required to transform under the SO(4) universal cover SU(2) × SU(2). By doing it, we identify the complete dark-electroweak group to be $G_{DEW} := \mathrm{SU}(2)_{L1} \times \mathrm{SU}(2)_{L2} \times \mathrm{U}(1)_Y$ [1]. This group presents an exchange symmetry $1 \leftrightarrow 2$ since both SU(2) copies come from one unified $\mathfrak{so}(4)_L$ algebra. To preserve this symmetry, we require two fermion singlets (e_1, e_2) two fermion doublets (ψ_1, ψ_2), two scalar doublets (ϕ_1,ϕ_2) and a scalar bidoublet (Φ). Fields that should transform in the following way under the $\mathrm{SU}(2)_{L1} \times \mathrm{SU}(2)_{2L}$ subgroup:

Field	$\mathrm{SU}(2)_{L1}$	$\mathrm{SU}(2)_{L2}$
e_1	1	1
ϕ_1	2	1
ψ_1	2	1
e_2	1	1
ϕ_2	1	2
ψ_2	1	2
Φ	2^*	2

On the other hand, all fields should transform under different 1−dimensional representation of $\mathrm{U}(1)_Y$ given by their hypercharge. By construction Φ will have zero hypercharge ($Y = 0$) so that it

[1] We adopt the subscript notation L and Y to denote the groups and algebras associated with left-handed interactions and hypercharge, respectively.

transforms under the trivial representation. The list of each correspondent hypercharge can be found in the following table:

Field	Y
e_1	-2
ϕ_1	1
ψ_1	-1
e_2	-2
ϕ_2	1
ψ_2	-1
Φ	0

As stated previously, we also consider the existance of an exchange symmetry between the SU(2) subgroup indices. This means the following exchanges between fields leave the Lagrangian invariant:

$$e_1 \leftrightarrow e_2 \;\wedge\; \phi_1 \leftrightarrow \phi_2 \;\wedge\; \psi_1 \leftrightarrow \psi_2 \;\wedge\; \Phi \leftrightarrow \Phi^\dagger \tag{4.1}$$

Having a general invariant Lagrangian under the group, spontaneous symmetry breaking follows. This process can be divided in two steps. First, we allow the bidoublet Φ to acquire a non–zero vacuum expectation value. By doing this, the dark vector will appear as a massive non–gauge field and we will obtain the electroweak vectors as the gauges fields for our new symmetry. Secondly, we allow both scalar doublets to have a non–trivial vacuum expectation value, such that the electroweak sector acquires mass. The complete process can be visualized by the following diagram:

$$\begin{array}{c} \mathrm{SU}(2)_{L1} \times \mathrm{SU}(2)_{L2} \times \mathrm{U}(1)_Y \\ \big\downarrow \; \langle \Phi \rangle \neq 0 \\ \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y \\ \big\downarrow \; \langle \phi_1 \rangle = \langle \phi_2 \rangle \neq 0 \\ \mathrm{U}(1)_{em} \end{array}$$

Although the construction of this model extends the standard model, it follows the same formalism: A Yang-Mlls theory over a semisimple group with spontaneous symmetry breaking of the gauge symmetry via the Higgs mechanism. Since the difference lies only in the group from which the gauge symmetry is constructed, *no problems of unitarity or renormalization will arise*. Thus, if the model contains the original Lagrangian density (3.2) for the isotriplet model, then our gauge theory will be a valid ultraviolet completion of it.

(4.2)

4.2 The Lagrangian density

The Lagrangian density for the theory of dark-electroweak interactions can be divided in four terms:

$$\mathscr{L}_{DEW} = \mathscr{L}_F + \mathscr{L}_{\varphi,w} + \mathscr{L}_y + \mathscr{L}_s. \tag{4.3}$$

The first term corresponds to the gauge dynamic term:

$$\mathscr{L}_F = -\frac{1}{2}\mathrm{Tr}\left\{F_{\mu\nu}F^{\mu\nu}\right\}. \tag{4.4}$$

The second represents the fermion (doublet and singlet) kinetic terms:

$$\mathscr{L}_{\psi,e} = i\overline{\psi}_k \not{D}\psi_k + i\overline{e}_k \not{D} e_k. \tag{4.5}$$

In third place, we have the Yukawa terms:

$$\mathscr{L}_y = -y_a\left(\overline{\psi}_1\phi_1 e_1 + \overline{\psi}_2\phi_2 e_2\right) - y_b\left(\overline{\psi}_1\phi_1 e_2 + \overline{\psi}_2\phi_2 e_1\right) + \text{h.c.} \tag{4.6}$$

And lastly, the scalar sector:

$$\mathscr{L}_s = \mathrm{Tr}\left\{\left(D_\mu\Phi\right)^\dagger\left(D^\mu\Phi\right)\right\} + \left(D_\mu\phi_k\right)^\dagger\left(D^\mu\phi_k\right) + V\left(\Phi,\phi_1,\phi_2\right), \tag{4.7}$$

with the most general potential built by adding all renormalizable group invariant terms

$$\begin{aligned} V\left(\Phi,\phi_1,\phi_2\right) = & -\mu_0^2\,\mathrm{tr}\left\{\Phi^\dagger\Phi\right\} - \lambda_0\,\mathrm{tr}\left\{\Phi^\dagger\Phi\right\}^2 - \mu^2\phi_k^\dagger\phi_k + \lambda\left(\phi_k^\dagger\phi_k\right)^2 \\ & - \lambda_a\left(\phi_1^\dagger\phi_1\right)\left(\phi_2^\dagger\phi_2\right) - \lambda_b\,\mathrm{tr}\left\{\Phi^\dagger\Phi\right\}\phi_k^\dagger\phi_k - \lambda_c\left(\phi_2^\dagger\Phi\phi_1 + \phi_1^\dagger\Phi^\dagger\phi_2\right). \end{aligned} \tag{4.8}$$

By construction, this Lagrangian density will be invariant under the G_{DEW} transformations

$$U = \exp\left[i\left(\theta_a(x)J_a + \eta_a(x)K_a\right) \oplus i\theta_Y(x)T_Y\right], \tag{4.9}$$

with J_a, K_a $(a = 1,2,3)$ generators of $\mathfrak{so}(4)$ and T_Y the $\mathfrak{u}(1)$ generator. We will soon enough see that the isotriplet model (3.2) is recovered as

$$\mathscr{L}_V \in \mathscr{L}_F + \mathscr{L}_s \tag{4.10}$$

once the symmetry has been spontaneously broken.

4.3 Gauge sector

The dark electroweak sector is constructed under the algebra $\mathfrak{g}_{dew} := \mathfrak{so}(4)_L \oplus \mathfrak{u}(1)_Y$. Meaning that the $\mathfrak{g}_{dew}$ gauge field A_μ can be written as

$$A_\mu = \cos(\theta_W)A_\mu^{\mathfrak{so}(4)} \oplus \sin(\theta_W)A_\mu^{\mathfrak{u}(1)}, \tag{4.11}$$

where $A_\mu^{\mathfrak{so}(4)}$ and $A_\mu^{\mathfrak{u}(1)}$ are the subalgebra associated fields and θ_W the Weinberg angle. Recall that the gauge fields from the subalgebras correspond to linear combinations of the correspondent generators:

$$\begin{aligned} \mathfrak{so}(4):\ & A_\mu^{\mathfrak{so}(4)} = W_\mu + V_\mu \equiv W_\mu^a J_a + V_\mu^b K_b \\ \mathfrak{u}(1):\ & A_\mu^{\mathfrak{u}(1)} = B_\mu T_Y, \end{aligned} \tag{4.12}$$

Which leaves the covariant derivative

$$\begin{aligned} D_\mu &= \partial_\mu - igA_\mu \\ &= \partial_\mu - i\left(g_L\left(W_\mu^a J_a + V_\mu^a K_a\right) \oplus g_Y B_\mu T_Y\right), \end{aligned} \tag{4.13}$$

with the coupling constants $g_L := g\cos(\theta_W)$ and $g_Y := g\sin(\theta_W)$. For most fields, we consider the direct sum to be given by the Kronecker sum of representations as well as $T_Y = Y/2$ the one–dimensional representation indexed by the hypercharge. In these cases we have

$$D_\mu = \partial_\mu - ig_L\left(W_\mu^a J_a + V_\mu^a K_a\right) - ig_Y B_\mu \frac{Y}{2}\mathbb{1}. \tag{4.14}$$

For constructing the gauge dynamic term we consider instead the direct sum of matrix representations such that

$$g_L\left(W_\mu^a J_a + V_\mu^a K_a\right) \oplus g_Y B_\mu T_Y = g_L\left(W_\mu^a J_a + V_\mu^a K_a\right) + g_Y B_\mu T_Y, \tag{4.15}$$

with the generators represented by

$$J_a = \begin{bmatrix} \frac{\sigma_a}{2} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{\sigma_a}{2} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & 0 \end{bmatrix} \wedge K_a = \begin{bmatrix} \frac{\sigma_a}{2} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\frac{\sigma_a}{2} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & 0 \end{bmatrix} \wedge T_Y = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \frac{1}{\sqrt{2}} \end{bmatrix}. \tag{4.16}$$

This representation ensures the correct normalization of the dynamic terms. Let us write the field strength tensor in this representation as

$$F_{\mu\nu} = F_{\mu\nu}^{(4)} + B_{\mu\nu} T_Y, \tag{4.17}$$

with the subalgebra strength tensors being

$$F_{\mu\nu}^{(4)} := \partial_\mu A_\nu^{\mathfrak{so}(4)} - \partial_\nu A_\mu^{\mathfrak{so}(4)} - ig_L\left[A_\mu^{\mathfrak{so}(4)}, A_\nu^{\mathfrak{so}(4)}\right] \quad \wedge \quad B_{\mu\nu} := \partial_\mu B_\nu - \partial_\nu B_\mu. \tag{4.18}$$

This correctly normalized representation yields

$$\mathscr{L}_F = -\frac{1}{2}\mathrm{Tr}\left\{F_{\mu\nu}F^{\mu\nu}\right\} = -\frac{1}{2}\mathrm{Tr}\left\{F_{\mu\nu}^{(4)}F^{(4)\mu\nu}\right\} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} \tag{4.19}$$

As claimed before, from this dynamic sector $\mathscr{L}_V$ can be partially recovered. In fact, all but two terms (i.e. the mass term and its coupling with the Higgs doublet) are contained in $\mathscr{L}_F$. Appendix E provides proof that the following decomposition can be carried out

$$-\frac{1}{2}\mathrm{Tr}\left\{F^{(4)}_{\mu\nu}F^{(4)\mu\nu}\right\} = -\frac{1}{2}\mathrm{Tr}\left\{W_{\mu\nu}W^{\mu\nu}\right\} - \mathrm{Tr}\left\{D_\mu V_\nu D^\mu V^\nu\right\} + \mathrm{Tr}\left\{D_\mu V_\nu D^\nu V^\mu\right\} - \frac{g_L^2}{2}\mathrm{Tr}\left\{\left[V_\mu, V_\nu\right]\left[V^\mu, V^\nu\right]\right\} - i g_L \mathrm{Tr}\left\{W_{\mu\nu}\left[V^\mu, V^\nu\right]\right\}, \tag{4.20}$$

with the new field strength tensor

$$W_{\mu\nu} := \partial_\mu W_\nu - \partial_\nu W_\mu - i g_L \left[W_\mu, W_\nu\right] \tag{4.21}$$

and the adjoint covariant derivative

$$D_\mu V_\nu = \partial_\mu V_\nu - i g_L \left[W_\mu, V_\nu\right]. \tag{4.22}$$

Meaning we have recovered almost all terms from the original model:

$$-\mathrm{Tr}\left\{D_\mu V_\nu D^\mu V^\nu\right\} + \mathrm{Tr}\left\{D_\mu V_\nu D^\nu V^\mu\right\} - \frac{g_L^2}{2}\mathrm{Tr}\left\{\left[V_\mu, V_\nu\right]\left[V^\mu, V^\nu\right]\right\} - i g_L \mathrm{Tr}\left\{W_{\mu\nu}\left[V^\mu, V^\nu\right]\right\} \in \mathscr{L}_V.$$

This arises from writing $A_\mu^{\mathfrak{so}(4)} = W_\mu^a J_a + V_\mu^b K_b$. The fact that the J_a generators form a simple subalgebra while the K_a do not, forbids them from decoupling completely. This gives rise to many interaction terms, a $\mathfrak{su}(2)_L$ gauge dynamic term for W_μ and no gauge dynamic term for V_μ. From this perspective, it is straightforward that by spontaneously breaking the symmetry to the $\mathfrak{su}(2)_L$ subalgebra spanned by $\{J_1, J_2, J_3\}$, V_μ will stop behaving as gauge field. To see that this is the case, consider the $\mathrm{SU}(2)_{L1} \times \mathrm{SU}(2)_{L2}$ transformations writen as (2.8):

$$U = \exp\left[i\left(\theta_a J_a + \eta_a K_a\right)\right] = \exp\left[i\left(\theta_a^1 J_a^1 \oplus \theta_a^2 J_b^2\right)\right] \tag{4.23}$$

From the fact that $J_a = J_a^1 \oplus J_a^2$ and $K_a = J_a^1 \ominus J_a^2$ one can relate the group parameters as

$$\theta_a = \theta_a^1 + \theta_a^2 \qquad \wedge \qquad \eta_a = \theta_a^1 - \theta_a^2. \tag{4.24}$$

The transformation rules for the unbroken subgroup are obtained by setting $\eta_a = 0$, $\forall\, a$. This implies that the transformation:

$$\left(A_\mu^{\mathfrak{so}(4)}\right)' = \exp\left[i\left(\theta_a J_a + \eta_a K_a\right)\right]\left(W_\mu + V_\mu\right)\exp\left[-i\left(\theta_a J_a + \eta_a K_a\right)\right] - \frac{i}{g_L}\left(\partial_\mu U\right)U^{-1} \tag{4.25}$$

can be replaced by the pair

$$W_\mu' = \exp[i\theta_a J_a] W_\mu \exp[-i\theta_a J_a] - \frac{i}{g_L}\left(\partial_\mu \exp[i\theta_a J_a]\right)\exp[-i\theta_a J_a] \tag{4.26}$$

$$V_\mu' = \exp[i\theta_a J_a] V_\mu \exp[-i\theta_a J_a]. \tag{4.27}$$

This shows that V_μ no longer transforms as a gauge field. By setting $\eta_a = 0$ we also enforce $\theta_a^1 = \theta_a^2$. Therefore, the unbroken group will be the diagonal subgroup of $SU(2)_{L1} \times SU(2)_{L2}$ defined as

$$SU(2)_L := \left\{ (g_1, g_2) \in SU(2)_{L1} \times SU(2)_{L2} \;/ g_1 = g_2 \right\}. \tag{4.28}$$

Which means that, if U_1 and U_2 are representations of g_1 and g_2 respectively, then the associate $SU(2)_L$ transformations are those that satisfies

$$\left(U_1 = \exp\left[i\theta_a^1 T_a \right] \quad \wedge \quad U_2 = \exp\left[i\theta_a^2 T_a \right] \right) \implies \theta_a^1 = \theta_a^2 \tag{4.29}$$

A particular consequence of this is that $(2,1)$ and $(1,2)$ representations of the original group correspond to the same 2–dimensional representation of $SU(2)_L$. Thus, the standard model electroweak group $SU(2)_L \times U(1)_Y$ can be obtained through spontaneously breaking G_{DEW}.

4.3.1 The covariant derivative

While the gauge sector is constructed from the fragmentation of the $\mathfrak{so}(4)_L$ algebra into the diagonal subalgebra $\mathfrak{su}(2)_L$ and the remaining complement set, non-gauge fields are concerned with the way this algebra is divided as $\mathfrak{su}(2)_{L1} \oplus \mathfrak{su}(2)_{L2}$. Thus, most fields will be sensitive to only one of these $\mathfrak{su}(2)$ and the corresponding SU(2) group. This is important not only because invariant terms on the Lagrangian are built with this idea in mind, but also because the specific form for the covariant derivative relies on this fact. For instance, an object which transforms under the (2,1) representation will have a covariant derivative of the form

$$D_\mu = \partial_\mu - i g_L \left(W_\mu^a + V_\mu^a \right) \frac{\sigma_a}{2} - i g_Y \frac{Y}{2} B_\mu \mathbb{1}_2, \tag{4.30}$$

since

$$\begin{aligned} J_a &= J_a^1 \oplus_K J_a^2 = \frac{\sigma_a}{2} \oplus_K 0 = \frac{\sigma_a}{2} \\ K_a &= J_a^1 \ominus_K J_a^2 = \frac{\sigma_a}{2} \ominus_K 0 = \frac{\sigma_a}{2}. \end{aligned} \tag{4.31}$$

Similarly, for objects under a $(1,2)$ representation we have

$$D_\mu = \partial_\mu - i g_L \left(W_\mu^a - V_\mu^a \right) \frac{\sigma_a}{2} - i g_Y \frac{Y}{2} B_\mu \mathbb{1}_2, \tag{4.32}$$

Clearly, the presence of a sum over Pauli matrices imply their role as generators of each SU(2) copy. The other relevant representation corresponds to $(2^*, 2)$, for which

$$\begin{aligned} J_a &= J_a^1 \oplus_K J_a^2 = -\frac{\sigma_a^*}{2} \oplus_K \frac{\sigma_a}{2} \\ K_a &= J_a^1 \ominus_K J_a^2 = -\frac{\sigma_a^*}{2} \ominus_K \frac{\sigma_a}{2}. \end{aligned} \tag{4.33}$$

To this representation acting over $\mathbb{C}^4$ corresponds a covariant derivative

$$D_\mu = \partial_\mu + i \frac{g_L}{2} \left[\left(W_\mu^a + V_\mu^a \right) \sigma_a^* \ominus_K \left(W_\mu^a - V_\mu^a \right) \sigma_a \right] - i g_Y \frac{Y}{2} B_\mu \mathbb{1}_4. \tag{4.34}$$

Equivalently, a bidoublet representation can be rewritten as

$$D_\mu\Phi = \partial_\mu\Phi + i\frac{g_L}{2}\left[\left(W_\mu^a + V_\mu^a\right)\Phi\sigma_a - \sigma_a\Phi\left(W_\mu^a - V_\mu^a\right)\right] - ig_Y\frac{Y}{2}B_\mu\Phi, \tag{4.35}$$

with $\Phi \in \mathscr{M}_{2\times2}(\mathbb{C})$. In our particular case, the only object under this representation is the scalar bidoublet, so this form for the covariant derivative only applies to it. Moreover, by construction we demand it's hypercharge to be zero, so the last term vanishes:

$$D_\mu\Phi = \partial_\mu\Phi + i\frac{g_L}{2}\left[\left(W_\mu^a + V_\mu^a\right)\Phi\sigma_a - \sigma_a\Phi\left(W_\mu^a - V_\mu^a\right)\right]. \tag{4.36}$$

4.4 Scalar sector

Motivated by the fact that spontaneously breaking the original symmetry returns most terms of $\mathscr{L}_V$, one seeks a scalar field which presents a vacuum that has $SU(2)_L$ as its stabilizer. To do this, we seek an object $\rho \in \mathbb{C}^4$ that transforms under the $(2^*, 2)$ representation of $SU(2)_{L1}\times SU(2)_{L2}$[2], such that $\rho \in \mathrm{Ker}(J_a),\ \forall a = 1,2,3$. One finds that the subspace shared by the kernel of all J_a corresponds to the one−dimensional space spanned by $[1\,0\,0\,1]^T = \mathrm{Vec}(\mathbb{1})$. In other words

$$J_a\rho = \left(-\frac{\sigma_a^*}{2}\oplus_K\frac{\sigma_a}{2}\right)\mathrm{Vec}\left(\begin{bmatrix}\alpha & 0\\ 0 & \alpha\end{bmatrix}\right) = 0,\quad \forall a \ \wedge\ \forall\alpha\in\mathbb{C}. \tag{4.37}$$

Thus, the objet

$$\begin{aligned}\mathrm{Vec}(\Phi(x)) &= \exp\left[-i\chi^0\pi_a^0(x)K_a\right]\begin{bmatrix}\alpha_0(x)\\ 0\\ 0\\ \alpha_0(x)\end{bmatrix}\\ &= \mathrm{Vec}\left(\exp\left[i\chi^0\pi_a^0(x)\frac{\sigma_a}{2}\right]\begin{bmatrix}\alpha_0(x) & 0\\ 0 & \alpha_0(x)\end{bmatrix}\exp\left[i\chi^0\pi_a^0(x)\frac{\sigma_a}{2}\right]\right),\end{aligned} \tag{4.38}$$

or equivalently the $\mathscr{M}_{2\times2}(\mathbb{C})$ bidoublet

$$\Phi(x) = \alpha_0(x)\exp\left[i\chi^0\pi_a^0(x)\sigma_a\right], \tag{4.39}$$

with the normalization factor $\chi^0 \in \mathbb{R}$ and the real function

$$\alpha_0(x) := \frac{v_0 + h_0(x)}{\sqrt{2}}, \tag{4.40}$$

spontaneously breaks the symmetry to the diagonal subgroup. It is evident from this last equation that whatever the potential for the Lagrangian is, we must impose a minimum to be in the vacuum $\langle\Phi\rangle = \frac{v_0}{\sqrt{2}}\mathbb{1}$. We also can see that $\pi_a^0(x)$ correspond to the Goldstone bosons, since they can be

[2]It is only necessary to consider transformations under this $SU(2)\times SU(2)$ since this scalar will transform trivially under $U(1)_Y$ (Y=0).

eliminated from the Lagrangian by working on the unitary gauge. This fixing implies writing he general transformation (4.23) as

(4.41) $$U = \exp[i\tilde{\eta}_a K_a]\exp[i\tilde{\theta}_a J_a],$$

and imposing $\tilde{\eta}_a(x) = \chi^0 \pi^0_a(x)$. We now choose to replace $A^{\mathfrak{so}(4)}_\mu$ by W_μ and V_μ with subgroup transformations (4.26) and (4.27). We also replace all instates of Φ with $\Phi \to \tilde{\Phi} := \alpha_0(x)\mathbb{1}$, since

(4.42) $$\text{Vec}(\tilde{\Phi}) = \left(U\big|_{\tilde{\eta}_a=\chi^0\pi^0_a}\right)^{-1}\text{Vec}(\Phi).$$

As a last note, the reader can later verify that this action will act over objects under $(2,1)$ and $(1,2)$ representations as a $\text{SU}(2)_L$ gauge transformation by considering a new orbit parametrization in (4.46) (i.e redefining π_a, ξ_a to account for the non–gauge local transformation).

Once the original symmetry has been spontaneously broken, the second instance of spontaneous symmetry breaking follows. This is realized in a similar way as that of the standard model, with the difference lying on the need of two $\mathbb{C}^2$ scalar doublets in order to maintain the $1 \leftrightarrow 2$ exchange symmetry. As stated before, this implies that one transforms under the $(2,1)$ representation, while the other transforms under the $(1,2)$ representation. Nevertheless, since we require both doublets ϕ_1 and ϕ_2 to spontaneously break $\text{SU}(2)_L \times U(1)_Y$ in the same manner as the standard model, we neeed both to be able to define $\text{U}(1)_{em}$ as common a stabilizer. As it is the case with the standard model, this is equivalent to demanding the vacua $\langle\phi_1\rangle, \langle\phi_2\rangle \in \text{Ker}(Q^+)$, where the $\text{U}(1)_{em}$ generator is

(4.43) $$Q^+ := J_3 + \frac{Y}{2}\mathbb{1},$$

with $Y = 1$ for the scalar doublets. In both the $(2,1)$ and the $(1,2)$ representations of $\text{SU}(2)_{L1} \times \text{SU}(2)_{L2}$, we have

(4.44) $$J_a = \frac{\sigma_a}{2},$$

meaning that we demand vacua of the form

(4.45) $$\langle\phi_k\rangle = \begin{bmatrix} 0 \\ \frac{u_k}{\sqrt{2}} \end{bmatrix}$$

for $k = 1,2$. In the same manner as the standard model, we can write both doublets

(4.46) $$\phi_1 = \exp\left[i\chi^1\pi_a(x)\frac{\sigma_a}{2}\right]\begin{bmatrix} 0 \\ \alpha_1(x) \end{bmatrix} \wedge \phi_2 = \exp\left[i\chi^2\xi_a(x)\frac{\sigma_a}{2}\right]\begin{bmatrix} 0 \\ \alpha_2(x) \end{bmatrix},$$

with χ^k normalization parameters and the real functions

(4.47) $$\alpha_k(x) = \frac{u_k + h_k(x)}{\sqrt{2}}.$$

Despite their similarities, the model presents a notorious difference to the standard model. There is clearly a total of 6 different orbit−parameter functions (π_a, ξ_a, $a = 1,2,3$). In contrast, a general $SU(2)_L \times U(1)_Y$ transformation only allows 3 group parameters to be constrained through gauge fixing[3]. Thus, only 3 of Goldstone bosons can exist, leaving 3 new fields besides h_k that cannot be eliminated by the unitary gauge. This situation is similar to those found in *2 Higgs doublet models* [63–66], so a similar treatment for gauge fixing will be carried out. To keep the analysis simple, we will work in the *Higgs basis* [65, 66]. In simple terms, this means assuming π_a to be the goldstone bososns, such that the same unitary gauge used for the standard model can eliminate them. This forbids us from eliminating the fields ξ_a fields, instead adding a contribution coming from the fixed group parameters. Concretely, let us choose the gauge for (4.41) to be $\theta_a(x) = \chi^1 \pi_a(x)$. This implies that all instances of ϕ_k are replaced by

$$\phi_1 \to \tilde{\phi}_1 = \begin{bmatrix} 0 \\ \alpha_1(x) \end{bmatrix} \tag{4.48}$$

$$\phi_2 \to \tilde{\phi}_2 = \exp\left[i\left(\chi^2 \xi_a(x) - \chi^1 \pi_a(x)\right)\frac{\sigma_a}{2}\right]\begin{bmatrix} 0 \\ \alpha_2(x) \end{bmatrix} \equiv \begin{bmatrix} \phi^+(x) \\ \alpha_2(x) + i\dfrac{w(x)}{\sqrt{2}} \end{bmatrix}.$$

In this last expression, the linear combination $\left(\chi^2 \xi_a(x) - \chi^1 \pi_a(x)\right)$ has been replaced by the complex field ϕ^+ and the real field w. These fields will also prove to be eigenfields of the mass matrix (i.e. physical fields) that will not mix with h_0, h_1 or h_2.

4.4.1 The potential

Since all the terms in the Lagrangian are group invariant, let us evaluate the potential (4.8) considering the transformed fields given by the unitary gauge

$$V\left(\Phi, \phi_1, \phi_2\right) = V\left(\tilde{\Phi}, \tilde{\phi}_1, \tilde{\phi}_2\right) \tag{4.49}$$

This yields a polynomial over the fields[4]

$$V = P(h_0, h_1, h_2, \phi^+, \phi^-, w), \tag{4.50}$$

with $\phi^- := (\phi^+)^*$. We aim to constrain the values of v_0, u_1 and u_2 by demanding these to be extrema of the function. To that end, we impose

$$\nabla V|_{h_0=0,\, h_k=0} = 0. \tag{4.51}$$

Demanding a extremum to exist at this point forces u_1 and u_2 to be related and puts constrains on μ_0 and μ. The study of this extremum is detailed in appendix F. For now it suffices to say that

[3] Remember that one of the 4 group parameters should remain unconstrained in order to keep the gauge freedom of $U(1)_{em}$.

[4] The expression is long and the inclusion of its explicit form does not provide further insight. For this reason we will not add it here.

the condition

$$u_1 = u_2 \equiv u, \tag{4.52}$$

together with the expressions

$$\begin{aligned} \mu_0^2 &= -\frac{8v_0^3\lambda_0 + 4v_0u^2\lambda_b + \sqrt{2}u^2\lambda_c}{4v_0} \\ \mu^2 &= -\left(v_0^2\lambda_b - \frac{v_0\lambda_c}{\sqrt{2}} - \frac{1}{2}u^2(2\lambda + \lambda_a)\right) \end{aligned} \tag{4.53}$$

ensures (4.51) is satisfied. One can find constrains for the rest of parameters in the potential by imposing $\left(v_0/\sqrt{2}\,,\, u\,,\, u\right)$ to be not only an extremum, but a local minimum of V. Evaluating the Hessian with respect to h_0, h_k and demanding all the principal minors of the matrix to be positive yields an extensive expression for the allowed region. Figure 4.1 shows a random sample of $\sim 10^4$ possible values for $\lambda_0, \lambda, \lambda_a, \lambda_b, \lambda_c \in [-10, 10]$ which lie within it[5]. Possible values of μ_0^2 and μ^2 as functions of v_0 are also presented.

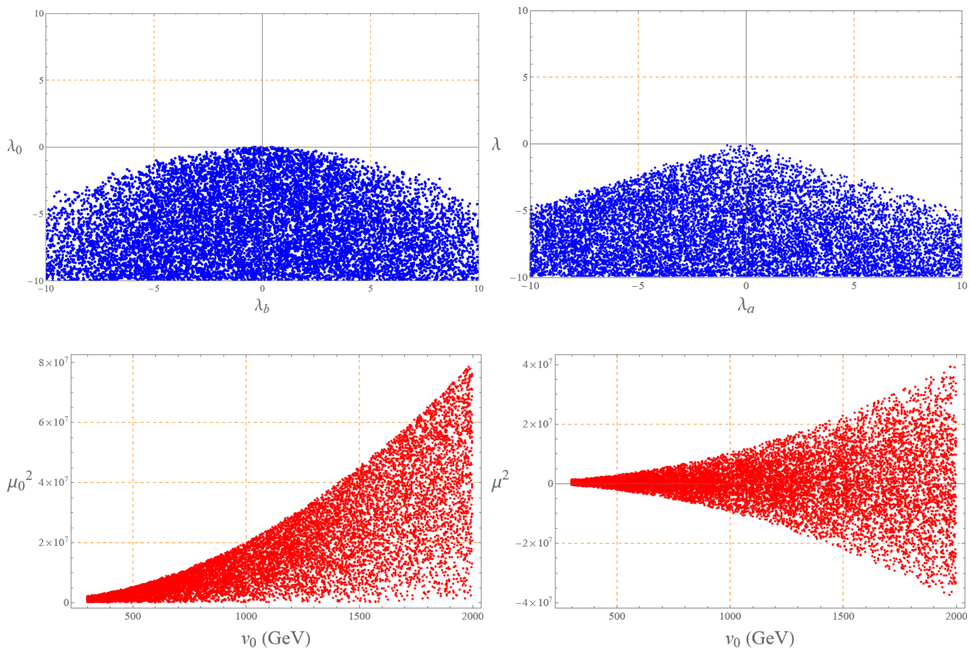

Figure 4.1: Region of potential parameters allowed for a positively defined Hessian. All panels are the result of the same remaining set of $\sim 10^4$ points from an initial random sample of 10^5 tuples. For the potential to present the desired local minima it is required that $\lambda_0, \lambda < 0$ while λ_a, λ_b remain mostly unconstrained (top left and right). μ_0^2 and μ^2 have been computed using (4.53) (bottom left and right). Notice that it required for $\mu_0^2 > 0$ in all cases.

[5] λ_c does not appear since it remains mostly unconstrained, providing no additional information to Figure 4.1

4.4.2 Massive scalars

As it can be evidenced from the existence of mixed quadratic terms in (4.8). The potential will not yield a polynomial with a diagonal quadratic form. That is, if the quadratic form of $P(h_0,h_1,h_2,\phi^+,\phi^-,w)$ is given by

$$\begin{bmatrix} h_0 \\ h_1 \\ h_2 \\ \phi^- \\ w \end{bmatrix}^T \mathbb{M} \begin{bmatrix} h_0 \\ h_1 \\ h_2 \\ \phi^+ \\ w \end{bmatrix} \in P(h_0,h_1,h_2,\phi^+,\phi^-,w), \tag{4.54}$$

then $\mathbb{M}$ is not diagonal. 5×5 matrices like these do not always have analytic solutions for their eigenvalues and vectors; fortunately, this is not our case. We can write (4.54) as

$$\begin{bmatrix} h_0 \\ h_1 \\ h_2 \\ \phi^- \\ w \end{bmatrix}^T \mathbb{M} \begin{bmatrix} h_0 \\ h_1 \\ h_2 \\ \phi^+ \\ w \end{bmatrix} = \frac{1}{2} \begin{bmatrix} H_0 \\ \chi_1 H_1 \\ \chi_2 H_2 \\ \phi^- \\ w \end{bmatrix}^T \operatorname{diag}\{\tilde{M}_0^2,\tilde{M}_1^2,\tilde{M}_2^2,2M_w^2,M_w^2\} \begin{bmatrix} H_0 \\ \chi_1 H_1 \\ \chi_2 H_2 \\ \phi^+ \\ w \end{bmatrix}, \tag{4.55}$$

with the corresponding square masses are given by

$$\begin{aligned} M_0^2 &:= \frac{1}{2}\left(\sqrt{2}v_0\lambda_c + u^2(\lambda_a - 2\lambda)\right) \\ \tilde{M}_1^2 &:= -\frac{\zeta_1+\zeta_2+\zeta_3}{8v_0} \\ \tilde{M}_2^2 &:= -\frac{\zeta_1+\zeta_2-\zeta_3}{8v_0} \\ M_w &:= \frac{v_0\lambda_c}{\sqrt{2}}, \end{aligned} \tag{4.56}$$

and the new massive fields

$$\begin{aligned} H_0 &:= \frac{1}{\sqrt{2}}(h_2-h_1) \\ \chi_1 H_1 &:= \frac{\zeta_4}{\sqrt{2\zeta_4^2+(\zeta_1-\zeta_2+\zeta_3)^2}}(h_2+h_1) + \frac{(\zeta_1-\zeta_2+\zeta_3)}{\sqrt{2\zeta_4^2+(\zeta_1-\zeta_2+\zeta_3)^2}}h_0 \\ \chi_2 H_2 &:= \frac{\zeta_4}{\sqrt{2\zeta_4^2+(\zeta_1-\zeta_2-\zeta_3)^2}}(h_2+h_1) + \frac{(\zeta_1-\zeta_2-\zeta_3)}{\sqrt{2\zeta_4^2+(\zeta_1-\zeta_2-\zeta_3)^2}}h_0. \end{aligned} \tag{4.57}$$

The constants ζ_j $(j=1,2,3,4)$ are functions of the parameters in the potential given by

$$\begin{aligned} \zeta_4 &:= 2uv_0\left(4v_0\lambda_b + \sqrt{2}\lambda_c\right) \\ \zeta_3 &:= \sqrt{256v_0^6\lambda_0^2 - 32\sqrt{2}u^2v_0^3(\lambda_0-2\lambda_b)\lambda_c + 2u^4\lambda_c^2 + 4\sqrt{2}u^4v_0\lambda_c(2\lambda+\lambda_a) + 64u^2v_0^4(2\lambda_b^2 - \lambda_0(2\lambda+\lambda_a)) + 4v_0^2(4u^2\lambda_c^2 + u^4(2\lambda+\lambda_b)^2)} \\ \zeta_2 &:= 2u^2v_0(2\lambda+\lambda_a) \\ \zeta_1 &:= 16v_0^3\lambda_0 - \sqrt{2}u^2\lambda_c. \end{aligned} \tag{4.58}$$

We have also added normalization factors χ_1 and χ_2 to the massive fields H_1,H_2, in order to obtain the correctly normalized scalar dynamic term $\frac{1}{2}\partial_\mu H_k \partial^\mu H_k$ $(k=1,2)$ once these fields are substituted in the Lagrangian. It is clear then, that the squared masses of these fields are not given by $\tilde{M}_k^2$ $(k=1,2)$, but rather by

$$M_k^2 := \chi_k^2 \tilde{M}_k^2. \tag{4.59}$$

One can find these factors rather quickly without computing the full covariant derivative by noticing that

$$\frac{1}{2}\partial_\mu h_k \partial^\mu h_k \in \left(D_\mu \phi_k\right)^\dagger \left(D^\mu \phi_k\right). \tag{4.60}$$

It is then a matter of substituting the (4.57) inverted relations

$$\begin{aligned}
h_0 &= \frac{\chi_2}{2\zeta_3}\sqrt{2\zeta_4^2+(\zeta_1-\zeta_2-\zeta_3)^2}\,H_2 + \frac{\chi_1}{2\zeta_3}\sqrt{2\zeta_4^2+(\zeta_1-\zeta_2+\zeta_3)^2}\,H_1\\
h_1 &= \frac{\chi_2(\zeta_1-\zeta_2+\zeta_3)}{4\zeta_3\zeta_4^2}\sqrt{2\zeta_4^2+(\zeta_1-\zeta_2-\zeta_3)^2}\,H_2 - \frac{\chi_1(\zeta_1-\zeta_2-\zeta_3)}{4\zeta_3\zeta_4^2}\sqrt{2\zeta_4^2+(\zeta_1-\zeta_2+\zeta_3)^2}\,H_1 - \frac{1}{\sqrt{2}}H_0\\
h_2 &= \frac{\chi_2(\zeta_1-\zeta_2+\zeta_3)}{4\zeta_3\zeta_4^2}\sqrt{2\zeta_4^2+(\zeta_1-\zeta_2-\zeta_3)^2}\,H_2 - \frac{\chi_1(\zeta_1-\zeta_2-\zeta_3)}{4\zeta_3\zeta_4^2}\sqrt{2\zeta_4^2+(\zeta_1-\zeta_2+\zeta_3)^2}\,H_1 + \frac{1}{\sqrt{2}}H_0
\end{aligned}$$

into these dynamic terms and imposing the resultant coefficient for the terms in $\partial_\mu H_k \partial^\mu H_k$ to be $\frac{1}{2}$. Doing this yields the result

$$\chi_1 = \frac{8\zeta_3\zeta_4}{\sqrt{\left(2\zeta_4^2+(\zeta_1-\zeta_2+\zeta_3)^2\right)\left(4\zeta_4^2+(\zeta_1-\zeta_2-\zeta_3)^2\right)}}. \tag{4.61}$$

$$\chi_2 = \frac{8\zeta_3\zeta_4}{\sqrt{\left(4\zeta_4^2+(\zeta_1-\zeta_2+\zeta_3)^2\right)\left(2\zeta_4^2+(\zeta_1-\zeta_2-\zeta_3)^2\right)}}. \tag{4.62}$$

4.4.3 Scalar dynamic terms and gauge field masses

Now we study how the process of spontaneous symmetry breaking affects the scalar dynamic terms

$$\mathscr{L}_s \ni \mathrm{Tr}\left\{\left(D_\mu \Phi\right)^\dagger \left(D^\mu \Phi\right)\right\} + \left(D_\mu \phi_k\right)^\dagger \left(D^\mu \phi_k\right). \tag{4.63}$$

We will show that the mass terms from the original isotriplet model (both before and after $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ is spontaneously broken) are contained within these dynamic terms. Let us start with the bidoublet dynamic term and carry on the first instance of symmetry breaking

$$\mathrm{SU}(2)_{L1} \times \mathrm{SU}(2)_{L2} \times \mathrm{U}(1)_Y \longrightarrow \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y. \tag{4.64}$$

Consider the covariant derivative (4.36)

$$D_\mu \Phi = \partial_\mu \Phi + i\frac{g_L}{2}\left[\left(W_\mu^a + V_\mu^a\right)\Phi\sigma_a - \sigma_a \Phi\left(W_\mu^a - V_\mu^a\right)\right]. \tag{4.65}$$

By working in the unitary gauge, we replace $D_\mu\Phi \to D_\mu\tilde{\Phi} = \alpha_0(x)\mathbb{1}_2$ so that

$$D_\mu\tilde{\Phi} = (\partial_\mu\alpha_0)\mathbb{1}_2 + i\alpha_0 g_L V_\mu^a\sigma_a. \tag{4.66}$$

The resulting expression is then

$$\mathrm{Tr}\left\{\left(D_\mu\Phi\right)^\dagger\left(D^\mu\Phi\right)\right\} = \mathrm{Tr}\left\{\left(D_\mu\tilde{\Phi}\right)^\dagger\left(D^\mu\tilde{\Phi}\right)\right\} = 2(\partial_\mu\alpha_0)^2 + 2\alpha_0^2 g_L^2\left(V_\mu^a\right)^2, \tag{4.67}$$

where we adopt the notation $(Q_\mu^a)^2 = Q_\mu^a Q^{\mu a}$ for Q_μ^a an arbitrary four-vector component. We now write α_0 explicitly as (4.40) such that

$$\mathrm{Tr}\left\{\left(D_\mu\Phi\right)^\dagger\left(D^\mu\Phi\right)\right\} = (\partial_\mu h_0)^2 + h_0^2 g_L^2\left(V_\mu^a\right)^2 + v_0^2 g_L^2\left(V_\mu^a\right)^2 + 2v_0 g_L^2 h_0\left(V_\mu^a\right). \tag{4.68}$$

We have recovered a mass term for V_μ^a as expected. This term is equivalent to

$$v_0^2 g_L^2\left(V_\mu^a\right)^2 \equiv \tilde{M}^2\mathrm{Tr}\left\{V_\mu V^\mu\right\} \in \mathscr{L}_V \tag{4.69}$$

Here, we identify $\tilde{M}^2 \equiv 2v_0^2 g_L^2$. Thus, in our model $\tilde{M}$ stops being a free parameter, being replaced by the vacuum expectation value of Φ. Recall that this contribution does not contain the full expression for the mass of V_μ. In the same manner as the original model, we expect to get an additional contribution from spontaneous symmetry breaking of $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$. Therefore, we shift our attention towards the remaininng terms

$$\left(D_\mu\phi_k\right)^\dagger\left(D^\mu\phi_k\right). \tag{4.70}$$

To study this expression, we use $\phi_1 \to \tilde{\phi}_1$ and $\phi_2 \to \tilde{\phi}_2$ given by (4.48). We will also consider the decomposition for $\tilde{\phi}_2$

$$\tilde{\phi}_2 = \mathscr{A} + \mathscr{B}, \tag{4.71}$$

under the definitions

$$\mathscr{A} := \begin{bmatrix} 0 \\ \alpha_2 \end{bmatrix} \wedge \mathscr{B} := \begin{bmatrix} \phi^+ \\ i\dfrac{w}{\sqrt{2}} \end{bmatrix}. \tag{4.72}$$

Therefore, we can write

$$\begin{aligned} \left(D_\mu\phi_1\right)^\dagger\left(D^\mu\phi_1\right) &= \left(D_\mu\tilde{\phi}_1\right)^\dagger\left(D^\mu\tilde{\phi}_1\right) \\ \left(D_\mu\phi_2\right)^\dagger\left(D^\mu\phi_2\right) &= \left(D_\mu\tilde{\phi}_2\right)^\dagger\left(D^\mu\tilde{\phi}_2\right) = \left(D_\mu\mathscr{A}\right)^\dagger\left(D^\mu\mathscr{A}\right) + \left(D_\mu\mathscr{B}\right)^\dagger\left(D^\mu\mathscr{B}\right) + 2\left(D_\mu\mathscr{A}\right)^\dagger\left(D^\mu\mathscr{B}\right) \end{aligned} \tag{4.73}$$

From the last expression, we will not compute terms involving $\mathscr{B}$. The reason being that our current analysis seeks to recover the remaining terms of $\mathscr{L}_V$, and $D_\mu\mathscr{B}$ will not provide contributions to these. Thus, we will only calculate explicitly the terms

$$\left(D_\mu\tilde{\phi}_1\right)^\dagger\left(D^\mu\tilde{\phi}_1\right) + \left(D_\mu\mathscr{A}\right)^\dagger\left(D^\mu\mathscr{A}\right) \in \left(D_\mu\phi_k\right)^\dagger\left(D^\mu\phi_k\right). \tag{4.74}$$

Continuing with our computations, we obtain

$$D_\mu\tilde{\phi}_1 = \begin{bmatrix} -\frac{i\alpha_1 g_L}{2}\left(W^1_\mu + V^1_\mu\right) - \frac{\alpha_1 g_L}{2}\left(W^2_\mu + V^2_\mu\right) \\ \\ \partial_\mu\alpha_1 - \frac{i\alpha_1 g_Y}{2}B_\mu + \frac{i\alpha_1 g_L}{2}\left(W^3_\mu + V^3_\mu\right) \end{bmatrix} \tag{4.75}$$
$$D_\mu\mathscr{A} = \begin{bmatrix} -\frac{i\alpha_2 g_L}{2}\left(W^1_\mu - V^1_\mu\right) - \frac{\alpha_2 g_L}{2}\left(W^2_\mu - V^2_\mu\right) \\ \\ \partial_\mu\alpha_2 - \frac{i\alpha_2 g_Y}{2}B_\mu + \frac{i\alpha_1 g_L}{2}\left(W^3_\mu - V^3_\mu\right) \end{bmatrix}.$$

Expressions that yield the products

$$\left(D_\mu\tilde{\phi}_1\right)^\dagger\left(D^\mu\tilde{\phi}_1\right) = \left(\partial_\mu\alpha_1\right)^2 + \frac{\alpha_1^2 g_L^2}{4}\left(W^a_\mu + V^a_\mu\right)^2 + \frac{\alpha_1^2 g_Y^2}{4}B^2_\mu - \frac{\alpha_1^2 g_L g_Y}{2}\left(W^3_\mu + V^3_\mu\right)B^\mu \tag{4.76}$$
$$\left(D_\mu\mathscr{A}\right)^\dagger\left(D^\mu\mathscr{A}\right) = \left(\partial_\mu\alpha_2\right)^2 + \frac{\alpha_2^2 g_L^2}{4}\left(W^a_\mu - V^a_\mu\right)^2 + \frac{\alpha_2^2 g_Y^2}{4}B^2_\mu - \frac{\alpha_2^2 g_L g_Y}{2}\left(W^3_\mu - V^3_\mu\right)B^\mu$$

Addition of these two equations yields an explicit but extensive form for (4.74). For now, let us pay attention only to the emergent massive terms. These correspond to

$$\left(D_\mu\tilde{\phi}_1\right)^\dagger\left(D^\mu\tilde{\phi}_1\right)\big|_{h_1=0} + \left(D_\mu\mathscr{A}\right)^\dagger\left(D^\mu\mathscr{A}\right)\big|_{h_2=0} \in \left(D_\mu\phi_k\right)^\dagger\left(D^\mu\phi_k\right), \tag{4.77}$$

or explicitly

$$\frac{u^2 g_L^2}{4}\left(W^a_\mu\right)^2 + \frac{u^2 g_L^2}{4}\left(V^a_\mu\right)^2 + \frac{u^2 g_Y^2}{4}\left(B_\mu\right)^2 - \frac{u^2 g_L g_Y}{2}W^3_\mu B^\mu \in \left(D_\mu\phi_k\right)^\dagger\left(D^\mu\phi_k\right). \tag{4.78}$$

Within this expression, the quadratic form that gives mass to the standard model bosons can be found. To see that this is true, we simply set

$$u \equiv \frac{v}{\sqrt{2}}, \tag{4.79}$$

where v corresponds to the vacuum expectation value of the standard model Higgs boson. We also identify the physical $W^\pm_\mu$ fields as

$$W^\pm_\mu := \frac{1}{\sqrt{2}}\left(W^1_\mu \mp iW^2_\mu\right). \tag{4.80}$$

Which leads to

$$\frac{v^2 g_L^2}{4}W^+_\mu W^{\mu-} + \frac{v^2 g_L^2}{8}\left(V^a_\mu\right)^2 + \frac{v^2 g_L^2}{8}\left(W^3_\mu\right)^2 + \frac{v^2 g_Y^2}{8}\left(B_\mu\right)^2 - \frac{v^2 g_L g_Y}{4}W^3_\mu B^\mu \in \left(D_\mu\phi_k\right)^\dagger\left(D^\mu\phi_k\right). \tag{4.81}$$

The three last terms correspond to the mass quadratic form diagonalized by

$$Z_\mu := W^3_\mu\cos(\theta_W) - B_\mu\sin(\theta_W) \tag{4.82}$$
$$A^\gamma_\mu := W^3_\mu\sin(\theta_W) + B_\mu\cos(\theta_W).$$

with their masses being

(4.83) $$M_\gamma = 0 \quad \wedge \quad M_Z = \frac{g_L v}{2\cos(\theta_W)}.$$

Yielding a new expression for (4.81) in terms of physical fields:

(4.84) $$\frac{v^2 g_L^2}{4} W_\mu^+ W^{\mu-} + \frac{v^2 g_L^2}{4\cos^2(\theta_W)} (Z_\mu)^2 + \frac{v^2 g_L^2}{8} \left(V_\mu^a\right)^2 \in \left(D_\mu \phi_k\right)^\dagger \left(D^\mu \phi_k\right)$$

Notice that spontaneously breaking the $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ symmetry to $\mathrm{U}(1)_{em}$ added a new contribution to the mass of V_μ. This term corresponds to the mass contribution obtained in $\mathscr{L}_V$ once the standard model Higgs spontaneously breaks the electroweak symmetry:

(4.85) $$\frac{v^2 g_L^2}{8} \left(V_\mu^a\right)^2 \equiv \frac{1}{4} a v^2 \left(V_\mu^a\right)^2 \in a \left(\phi^\dagger \phi\right) \mathrm{Tr}\left\{V_\mu V^\mu\right\} \in \mathscr{L}_V,$$

where here ϕ corresponds to the standard model Higgs doublet in (3.2). Although this expression leaves the constant $a = g_L^2/2$ fixed, this does not constrain the final V_μ mass given by addition of (4.69) and (4.85):

(4.86) $$\frac{1}{2} M_V^2 \left(V_\mu^a\right)^2 \equiv \frac{1}{2} g_L^2 \left(2v_0^2 + \frac{v^2}{4}\right) \left(V_\mu^a\right)^2 \in \mathrm{Tr}\left\{\left(D_\mu \Phi\right)^\dagger \left(D^\mu \Phi\right)\right\} + \left(D_\mu \phi_k\right)^\dagger \left(D^\mu \phi_k\right)$$

Fixing the constant a does not constrain the interactions between the vector triplet and the scalar sector either, since their interactions will be given by the parameters in the potential[6]. We have yet to study these interactions from which analogs to the remaining terms

(4.87) $$\frac{1}{4} a \left(h^2 + 2vh\right) \left(V_\mu^a\right)^2 \in a \left(\phi^\dagger \phi\right) \mathrm{Tr}\left\{V_\mu V^\mu\right\},$$

should appear. It is important to emphasize that the interaction terms of our model and the effective model *will not be exactly the same*, since the will not depend on a. This is natural, since our scalar sector is fundamentally different to that of the standard model. It is not trivial to determine which physical field (H_0, H_1, H_2) plays the role of the standard model Higgs boson. In fact, this aspect of the model is left as future work, since it would require more analysis on the parameter space and scalar sector. Still, the potential presents degrees of freedom that allow to set at least one of them to be a Higgs−like boson. In any case, replacing $h_l = h_l(H_0, H_1, H_2)$ $(l = 0, 1, 2)$ for all of them will yield terms with the desired structure. This means that there exist coefficients $a_l = a_l(\lambda_0, \lambda, \lambda_a, \lambda_b, \lambda_c)$ and $b_l = b_l(\lambda_0, \lambda, \lambda_a, \lambda_b, \lambda_c)$ such that

(4.88) $$\left(a_l H_l^2 + b_l H_l\right) \left(V_\mu^a\right)^2 \in \mathrm{Tr}\left\{\left(D_\mu \Phi\right)^\dagger \left(D^\mu \Phi\right)\right\} + \left(D_\mu \phi_k\right)^\dagger \left(D^\mu \phi_k\right),$$

which are equivalent interaction terms.

[6] Remember that h_0, h_1, h_2 doo not represent physical (massive aigenfields) of the theory. The physical assoiated scalars correspond to H_0, H_1, H_2 given by (4.57).

4.5 Yukawa terms and fermion masses

We have verified that all terms of $\mathscr{L}_V$ can be contained in this gauge theory. This implies that, with the exception of additional degrees of freedom, this ultraviolet complete model should be able to reproduce similar phenomenological results to those of the isotriplet model. Thus, we now shift our attention to a feature of this model that cannot go unnoticed: The fermionic sector, which is composed by the pair of fields ψ_k ($k = 1,2$). This implies that we have *doubled* the number of fermions, and demands at least a superficial analysis of the new Yukawa terms to ensure that this can be consistent with the standard model. From (4.6) we have

$$\mathscr{L}_y = -y_a\left(\overline{\psi}_1\phi_1 e_1 + \overline{\psi}_2\phi_2 e_2\right) - y_b\left(\overline{\psi}_1\phi_1 e_2 + \overline{\psi}_2\phi_2 e_1\right) + \text{h.c.} \tag{4.89}$$

Although these terms are invariant under G_{DEW}, we can start our analysis by assuming that the $\text{SU}(2)_{L1} \times \text{SU}(2)_{L2}$ symmetry has been spontaneously broken. Therefore, our doublets will all transform under the 2−dimensional representation of the subgroup $\text{SU}(2)_L$ and so will any linear combination of the these. We write the fermions generically as

$$\psi_1 = \begin{bmatrix} \nu_1 \\ \epsilon_1 \end{bmatrix} \quad \wedge \quad \psi_2 = \begin{bmatrix} \nu_2 \\ \epsilon_2 \end{bmatrix} \tag{4.90}$$

with ν_k and ϵ_k left-handed Dirac spinors. To find the physical (massive) fermions, the second instance of symmetry breaking must be carried out. In the unitary gauge this implies $\phi_k \to \tilde{\phi}_k$, which yields

$$-y_a\overline{\psi}_1\phi_1 e_1 = -y_a\left[\overline{\nu}_1\ \overline{\epsilon}_1\right]\begin{bmatrix} 0 \\ \dfrac{u+h_1}{\sqrt{2}} \end{bmatrix} e_1 = -\frac{y_a(u+h_1)}{\sqrt{2}}\overline{\epsilon}_1 e_1 \tag{4.91}$$

$$-y_b\overline{\psi}_1\phi_1 e_2 = -y_b\left[\overline{\nu}_1\ \overline{\epsilon}_1\right]\begin{bmatrix} 0 \\ \dfrac{u+h_1}{\sqrt{2}} \end{bmatrix} e_2 = -\frac{y_b(u+h_1)}{\sqrt{2}}\overline{\epsilon}_k e_2$$

$$-y_a\overline{\psi}_2\phi_2 e_2 = -y_a\left[\overline{\nu}_2\ \overline{\epsilon}_2\right]\begin{bmatrix} \phi^+ \\ \dfrac{u+h_2+iw}{\sqrt{2}} \end{bmatrix} e_2 = -\frac{y_a(u+h_2+iw)}{\sqrt{2}}\overline{\epsilon}_2 e_2 - y_a\overline{\nu}_2\phi^+ e_2 \tag{4.92}$$

$$-y_b\overline{\psi}_2\phi_2 e_1 = -y_b\left[\overline{\nu}_2\ \overline{\epsilon}_2\right]\begin{bmatrix} \phi^+ \\ \dfrac{u+h_2+iw}{\sqrt{2}} \end{bmatrix} e_1 = -\frac{y_b(u+h_2+iw)}{\sqrt{2}}\overline{\epsilon}_2 e_1 - y_b\overline{\nu}_2\phi^+ e_1$$

leading to an explicit form for the Yukawa terms

$$\begin{aligned}\mathscr{L}_y = &-\frac{y_a(u+h_1)}{\sqrt{2}}\overline{\epsilon}_1 e_1 - \frac{y_a(u+h_2+iw)}{\sqrt{2}}\overline{\epsilon}_2 e_2 - \frac{y_b(u+h_1)}{\sqrt{2}}\overline{\epsilon}_1 e_2 - \frac{y_b(u+h_2+iw)}{\sqrt{2}}\overline{\epsilon}_2 e_1 \\ &- y_b\overline{\nu}_2\phi^+(e_1+e_2) + \text{h.c.}\end{aligned} \tag{4.93}$$

It is straight forward to obtain the mass quadratic form

$$\mathscr{L}_y \ni -\frac{y_a u}{\sqrt{2}}(\bar{\epsilon}_1 e_1 + \bar{\epsilon}_2 e_2) - \frac{y_b u}{\sqrt{2}}(\bar{\epsilon}_1 e_2 + \bar{\epsilon}_2 e_1) + \text{h.c.} = -\frac{u}{\sqrt{2}}\begin{bmatrix}\bar{\epsilon}_1\\ \bar{\epsilon}_2\end{bmatrix}^T \begin{bmatrix} y_a & y_b \\ y_b & y_a\end{bmatrix}\begin{bmatrix} e_1 \\ e_2\end{bmatrix} + \text{h.c.} \tag{4.94}$$

Diagonalization of this mass matrix yields

$$-\frac{u}{\sqrt{2}}\begin{bmatrix}\bar{\epsilon}_1\\ \bar{\epsilon}_2\end{bmatrix}^T \begin{bmatrix} y_a & y_b \\ y_b & y_a\end{bmatrix}\begin{bmatrix} e_1 \\ e_2\end{bmatrix} \equiv -\begin{bmatrix}\bar{e}_L\\ \bar{\xi}_L\end{bmatrix}^T \begin{bmatrix} m_e & 0 \\ 0 & M_\xi\end{bmatrix}\begin{bmatrix} e_R \\ \xi_R\end{bmatrix}, \tag{4.95}$$

where the eigenfields and their eigenvalues (masses) are

$$e_L := \frac{1}{\sqrt{2}}(\epsilon_2 - \epsilon_1) \quad \wedge \quad e_R := \frac{1}{\sqrt{2}}(e_2 - e_1) \quad \wedge \quad m_e := \frac{u}{\sqrt{2}}(y_a - y_b) \tag{4.96}$$

$$\xi_L := \frac{1}{\sqrt{2}}(\epsilon_2 + \epsilon_1) \quad \wedge \quad \xi_R := \frac{1}{\sqrt{2}}(e_2 + e_1) \quad \wedge \quad M_\xi := \frac{u}{\sqrt{2}}(y_a + y_b). \tag{4.97}$$

Thus, we can express (4.94) as

$$\mathscr{L}_y \ni -m_e(\bar{e}_L e_R + \bar{e}_R e_L) - M_\xi\left(\bar{\xi}_L\xi_R + \bar{\xi}_R\xi_L\right) \equiv -m_e\bar{e}e - M_\xi\bar{\xi}\xi \tag{4.98}$$

where we have also defined the physical dirac fermions as the sum of the left and right-handed fermions

$$e := e_L + e_R \quad \wedge \quad \xi := \xi_L + \xi_R. \tag{4.99}$$

4.5.1 Mass analysis

Clearly our Yukawa terms imply that there will be two types of physical fermions on this model (e and ξ). We know that one of these has to be a generic standard model fermion while the other is beyond standard model. To ensure consistency, we required these beyond standard model fields to have a large mass in order to justify the lack of observational evidence. Following this constrain, we claim that the e field will correspond to a standard model fermion and ξ to a heavy fermion. To see that this is the case, let us recall form (4.96) and (4.97) the correspondent masses

$$m_e = \frac{v}{2}(y_a - y_b) \quad \wedge \quad M_\xi = \frac{v}{2}(y_a + y_b) \tag{4.100}$$

and define new Yukawa couplings y_{sm} and y_Ω such that

$$y_a \equiv \sqrt{2}\left(y_{sm} + \frac{y_\Omega}{2}\right) \quad \wedge \quad y_b \equiv \sqrt{2}\frac{y_\Omega}{2}. \tag{4.101}$$

We can express the masses in terms of this parameters as

$$m_e = \frac{v}{\sqrt{2}}y_{sm} \quad \wedge \quad M_\xi = \frac{v}{\sqrt{2}}(y_{sm} + y_\Omega). \tag{4.102}$$

Setting y_{sm} to be the standard model Yukawa coupling implies that e correspond to a standard model fermion field with mass m_e. On the other hand, to obtain a heavy fermion able to decouple from the standard model, it is enough to impose $y_{sm} \ll y_\Omega$. This coupling, however, has an upper bound determined by the perturbative conditions, given by the constraint on the Yukawa couplings y_a and y_b. For the terms on $\mathscr{L}_y$ to be perturbative, we require y_a, $y_b < 4\pi$, which implies

$$y_\Omega < \frac{8\pi}{\sqrt{2}} - 2y_{sm} = \frac{1}{\sqrt{2}}\left(8\pi - 4\frac{m_e}{v}\right) \tag{4.103}$$

We can do an order of magnitude estimation for this upper in the case of the electron. Let us consider the approximate values of $m_e \approx 0.51\,\text{MeV}$ and $v \approx 246\,\text{GeV}$. This yields an estimate of

$$y_\Omega \lesssim 17.77 \implies M_\xi \lesssim 3.09\,\text{TeV}. \tag{4.104}$$

Which leaves the perturbative upper bound on the heavy fermion masses around the TeV scale.

Chapter 5

Conclusions and projections

In this thesis, we have explored the formulation of an ultraviolet completion for the Minimal Isotriplet Model, an effective theory describing a vector dark matter candidate. The effective model's simplicity makes it an appealing framework, and its phenomenology remains consistent with current experimental constraints, though within a tightly restricted parameter space. Importantly, future experimental searches will either falsify or confirm this model, reinforcing its status as a testable dark matter scenario; a characteristic that many alternative models lack.

To address the loss of unitarity at high energies, we have demonstrated that the effective model can be embedded within a gauge theory based on the algebra $\mathfrak{so}(4)$. This construction naturally resolves issues related to unitarity, as expected from a Yang-Mills theory with spontaneous symmetry breaking. We have seen that this extension requires the addition of new fields and parameters, making the model considerably more complex that its effective counterpart. Nevertheless, if phenomenologically viable, this formulation could offer an elegant and testable alternative to the original model, potentially exhibiting a more robust compatibility with experimental constraints.

The work presented here has focused on the theoretical construction of the model, ensuring its mathematical consistency within the framework of gauge theories. However, a detailed study of its phenomenology and constrains on the parameter space remains an open problem. Given its structural similarities to the effective model, we anticipate comparable phenomenological features, yet its extended framework may allow for new results or improved compatibility with future experimental bounds. Thus, a comprehensive study of these aspects is a crucial next step in assessing the viability of this ultraviolet completion as a viable dark matter model.

APPENDIX

A

LIE GROUPS, ALGEBRAS AND THEIR REPRESENTATIONS

Before we begin, it is important to clarify that this appendix does not aim to provide a comprehensive review of the theory of Lie groups, nor does it aspire to meet the highest standards of mathematical rigor. Furthermore, no proofs will be provided, as these lie beyond the intended scope. Readers seeking a deeper understanding of these topics, including detailed proofs and definitions, are encouraged to consult references [41, 67–72], which should prove insightful. The goal of this appendix is instead to offer a discussion that provides readers unfamiliar with the subject a broad overview of the mathematical formalism, with an emphasis on aspects that are fundamental to the main text. With these clarifications in mind, let us begin our discussion.

A good place to start is by recalling what a group is. A Group can be understood as a set of abstract elements that under a binary operation (often referred as a *product*) satisfy what we call the *group axioms*. Because it requires only abstract objects and an operation, a group is essentially *defined* by how these objects operate through it. In finite groups, this is expressed writing what we call a *multiplication table* (Cayley table) that portraits how different elements are obtained by combining other elements. For infinite groups (e.g. non-numerable) such as Lie groups one may describe this in a different way since a table cannot always be made. It is by understanding that groups are abstract entities defined this way that the idea of representations becomes clearer. Group objects are not by themselves matrices, numbers or differential operators but take the form of these objects depending on context. Algebras in the other hand, although not only provided with a product binary operation but also a vector space structure are not different: Abstract objects defined by how the combine and that can take different forms called representations. Despite this, it is customary to treat a group element and its representation as equivalent, much like how column vectors are often regarded as abstract vectors. While the main text is not exempt from this common abuse of language, in this appendix, we will strive to maintain this distinction.

A.1 Lie Groups

A Lie group G is a non-numerable set of objects that can be labeled by a list of continuous parameter (e.g. intervals of $\mathbb{R}$) α_k, $k = 1,\dots,N$ such that we write one of its elements as

$$g(\alpha_1, \alpha_2, \dots, \alpha_N) \in G. \tag{A.1}$$

This elements are usually parameterized in a specific way such that, if α_k and β_k $\forall k$ are two sets of parameters, then there exists δ_k set of parameters such that under the group operation

$$g(\alpha_1, \alpha_2, \dots, \alpha_N)g(\beta_1, \beta_2, \dots, \beta_N) = g(\delta_1, \delta_2, \dots, \delta_N). \tag{A.2}$$

$$g(\alpha, 0, \dots, 0)g(\beta, 0, \dots, 0) = g(\alpha + \beta, 0, \dots, 0). \tag{A.3}$$

This parametrization also satisfies

$$g(0, 0, \dots, 0) = e \tag{A.4}$$

with e an identity element. One can verify that the set of these parameterized elements satisfy all group axioms making it a group. These properties also implies that any group action can be decomposed in one parameter group actions as

$$g(\alpha_1, \alpha_2, \dots, \alpha_N) = g(\alpha'_1, 0, \dots, 0)g(0, \alpha'_2, \dots, 0)\cdots g(0, 0, \dots, \alpha'_N), \tag{A.5}$$

for some collection of α'_k. The parametrization also allows to understand it as differentiable manifold. Since this is the case, for parameter values arbitrarily close to cero ($\Delta\alpha_k \to 0$) one can write:

$$g(\Delta\alpha_1, \dots, \Delta\alpha_N) = e + \Delta\alpha_k \left.\frac{\partial g}{\partial \alpha_k}\right|_{\alpha_k = 0}. \tag{A.6}$$

From this, we define the *group generators* as objects proportional to the derivative

$$T_k := -i \left.\frac{\partial g}{\partial \alpha_k}\right|_{\alpha_k = 0} \tag{A.7}$$

with the $-i$ factor added by convention so that if a group representation is unitary, the representation of T_k will be hermitian. This means that (A.6) takes the form

$$g(\Delta\alpha_1, \dots, \Delta\alpha_N) = e + i\Delta\alpha_k T_k. \tag{A.8}$$

These generators do not need to be group elements but abstract objects which operate under the same binary operation of the group. Soon enough we will see that these objects conform an algebra, a *Lie algebra*. Note that, to first order on $\Delta\alpha_k, \Delta\beta_k \to 0$ the following limit is satisfied:

$$g(\Delta\alpha_1,\ldots,\Delta\alpha_N)g(\Delta\beta_1,\ldots,\Delta\beta_N) \to g(\Delta\alpha_1+\Delta\beta_1,\ldots,\Delta\alpha_N+\Delta\beta_N) \tag{A.9}$$

This impLies that by operating group objects close to the identity one can approach any object of the group. Thus, one can take the limit where elements label by a partition of the α_k parameters are operated continuously. We claim

$$g(\alpha_1,\ldots,\alpha_N) = \lim_{n\to\infty} g(\alpha_1/n,\ldots,\alpha_N/n)^n = \lim_{n\to\infty}\left(e + i\frac{\alpha_k}{n}T_k\right)^n. \tag{A.10}$$

This limit structure is the same as the one for the exponential function and series. For this reason we write

$$g(\alpha_1,\ldots,\alpha_N) = \exp[i\alpha_k T_k] \tag{A.11}$$

A.1.1 The Lie algebra

Nothing has been said about the structure of the set of generators other than these elements are abstract entities that are allowed to be combined by using the group operation. Nevertheless, since at least some generators (usually all of them) can be linearly independent objects, linear combinations of these and their products form a vector space $\mathfrak{g}$. One can also find that these linear combinations of generators have a very particular structure by imposing that (A.2) must hold. This last equation can be written by talking (A.11) into account:

$$\exp[i\alpha_a T_a]\exp\left[i\beta_b T_b\right] = \exp[i\delta_c T_c]. \tag{A.12}$$

We claim that, for this statement to be true, the generators must satisfy that

$$[T_a, T_b] = i f_{abc} T_c \tag{A.13}$$

Where the *Lie bracket* $[\cdot,\cdot] : \mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ is defined as

$$[x,y] = xy - yx \tag{A.14}$$

for $x, y \in \mathfrak{g}$, and f_{abc} some algebra dependent constant called the *structure constant* which must be antisymmetric

$$f_{abc} = -f_{bac}. \tag{A.15}$$

More concretely, if $x, y \in \mathfrak{g}$, the argument of the product of the exponential form is given by the Baker–Campbell–Hausdorff formula:

$$\exp[x]\exp[y] = \exp[z] \;\Big/ z = x + y + \frac{1}{2}[x,y] + \frac{1}{12}([x,[x,y]] - [y,[x,y]]) + \ldots \tag{A.16}$$

Equation (A.13) evidences that the vector space $\mathfrak{g}$ is closed under the Lie bracket. One can verify that this operation does have all the properties of a bilinear product, which allows to identify $\mathfrak{g}$ as an algebra with the Lie bracket as its product. Algebras that are defined like this are called *Lie algebras* and are crucial for understanding why physical transformations in the form of group actions behave the way they do. In principle more than one group can have the same generator algebra. This is an important observation because many times in physics one might think that what describes the way different entities transform is one group, when in fact different groups might govern the way different objects transform with the underlying link between them being a common algebra.

A.2 Representations

A representation of a group or algebra corresponds to the way this entities take the form of linear transformations (automorphisms) over vector spaces. This means that for each element of the group we identify a linear operator over a vector space V such that the group axioms are still satisfied by the linear operators. In other words, a representation of a group G is a mapping D of the elements of G onto a set of linear operators of the form $D(g)$, with $g \in G$. This mapping should satisfy the following properties:

1. $D(e) = \mathbb{1}$, where $\mathbb{1}$ is the identity operator in V.
2. $D(g_1)D(g_2) = D(g_1 g_2)$, which means the group multiplication is mapped to the multiplication or composition of these operators.

Once a valid representation of a group has been constructed, all properties of the group can be translated onto the language of products of operators. A new name is also given to the representation of the Lie bracket: the *commutator*. Since linear operators acting on finite-dimensional vector spaces can always be represented by matrices, the most common way to build group representations is by constructing the proper matrices for the vector space. In these cases, we say that if a representation is being carried out by $n \times n$ matrices, then the representation is $n-$dimensional. It is worth noting that not all groups will have an arbitrary dimensional representation.

In the case of Lie groups, one builds n-dimensional representations of the group by building n-dimensional representations of the algebra (i.e. the generators). These representations are build by identifying a set of operators that satisfy the algebra commutation relations (A.13). However, different representations of the algebra may yield representations of different groups since, as stated before, more than one group can have the same algebra. Groups with a same Lie algebra are closely related, usually by a *many to one* identification of its elements. We call groups which are in this sense "bigger" than a given group, but share its algebra *covering groups*. The typical example corresponds to SU(2) being covering group of the rotation group SO(3), both of which result from exponentiation of the same $\mathfrak{so}(3)$ algebra (also called $\mathfrak{su}(2)$ for the same reason).

A particular and very important type of representation is that which is build from others. We encounter this type of structure elements of vector spaces are composed by parts that each transform individually under different representations of the same group. When this happens we say that the representation is *completely reducible*. Concretely, let $V = \bigoplus_k V_k$ be a finite−dimensional vector space built from the direct sum of subspaces. If a generic element g of a group G acting V has a set of matrix representations $D_k(g)$ such that each of them act on one of the different subspaces V_k, then the representation correspondent to V takes the block diagonal form

$$D(g) = D_1(g) \boxplus D_2(g) \boxplus \cdots = \begin{bmatrix} D_1(g) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & D_2(g) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \ddots \end{bmatrix}, \tag{A.17}$$

called the *direct sum of representations*[1].

A.2.1 Product of groups and sum of algebras

Often in physics one finds that a system is sensitive to the transformations of more than one group. To describe this, the *direct product of groups* is defined. Let G_1 be a group with a group operation $\cdot : G_1 \times G_1 \to G_1$ and G_2 another group with a group operation $* : G_2 \times G_2 \to G_2$. We call direct product of the groups to the usual Cartesian product

$$G_1 \times G_2 := \{(g_1, g_2) \ / \ g_1 \in G_1 \ \wedge \ g_2 \in G_2\} \tag{A.18}$$

together with a new binary operation defined component-wise as

$$\begin{aligned} \star : \ & (G_1 \times G_2) \times (G_1 \times G_2) \to G_1 \times G_2 \\ & (g_1, g_2) \star (\tilde{g}_1, \tilde{g}_2) \to (g_1 \cdot \tilde{g}_1, g_2 * \tilde{g}_2). \end{aligned} \tag{A.19}$$

It is straightforward to see that $G_1 \times G_2$ is also a group. Since the action of both subgroups is carried out independently, representations of $G_1 \times G_2$ are mostly given in two ways: by *tensor product* (denoted $\otimes$) or *direct sum* of the individual representations. If the representations are given by matrices, then this is equivalent to talking either the *Kronecker product* or *matrix direct sum* between them:

$$\text{Direct product of matrix representations}: \ D_{\otimes}(g_1, g_2) = D_1(g_1) \otimes_K D_2(g_2) \tag{A.20}$$

$$\text{Direct sum of matrix representations}: \ D_{\boxplus}(g_1, g_2) = \begin{bmatrix} D_1(g_1) & \mathbf{0} \\ \mathbf{0} & D_2(g_2) \end{bmatrix} \tag{A.21}$$

[1] The operation of direct sum of matrices is not often denoted as $\boxplus$, but rather as $\oplus$. Nevertheless, we adopt this unconventional notation to avoid confusion between the direct sum of matrices, abstract algebras and the Kronecker sum.

Note that these representations are fundamentally different. Representations given by tensor product will always result in operators that act over the tensor space between of the individual correspondent vector spaces. Therefore, these representations are always $nm-$dimensional[2], built from a n-dimensional representation of G_1 and a $m-$dimensional representation of G_2. We will denote them as a pair (n,m). In the other hand, direct sums of representations will always yield operators that act over the direct sum of vector spaces. This means that, if we consider a $n-$dimensional representation of G_1 and a m-dimensional representation of G_2, the result will be an $(n+m)-$dimensional representation. We can denote these representations as $(n,0)\oplus(0,m)$ since they represent direct sums of vector spaces.

Since Lie groups can be thought as the exponentiation of a Lie algebra, it makes sense that a Cartesian product of algebras must also be defined. Although this is the case, we demand that the resulting space must have the structure of an algebra, which means not a product operation, but also an additive operation must be defined. Thus, we call the resulting structure a *direct sum* of the algebras. Concretely, let $(\mathfrak{g}_1,[\cdot,\cdot]_1,+_1)$ and $(\mathfrak{g}_2,[\cdot,\cdot]_2,+_2)$ be algebras with their correspondent Lie brackets and an additive operation. The direct sum of these algebras is defined as

$$\mathfrak{g}_1\oplus\mathfrak{g}_2 := \left\{(x_1,x_2)\ /\ x_1\in\mathfrak{g}_1\ \wedge\ x_2\in\mathfrak{g}_2\right\} \tag{A.22}$$

together with a new Lie bracket

$$\begin{aligned}[\cdot,\cdot]: (\mathfrak{g}_1\oplus\mathfrak{g}_2)\times(\mathfrak{g}_1\oplus\mathfrak{g}_2) &\to \mathfrak{g}_1\oplus\mathfrak{g}_2\\ [(x_1,x_2),(y_1,y_2)] &\to ([x_1,y_1]_1,[x_2,y_2]_2)\end{aligned} \tag{A.23}$$

as well as an additive operation

$$\begin{aligned}+: (\mathfrak{g}_1\oplus\mathfrak{g}_2)\times(\mathfrak{g}_1\oplus\mathfrak{g}_2) &\to \mathfrak{g}_1\oplus\mathfrak{g}_2\\ (x_1,x_2)+(y_1,y_2) &\to (x_1+_1 y_1, x_2+_2 y_2).\end{aligned} \tag{A.24}$$

For representations of these sums, we seek to express them in some way that allows for a group representation to be obtained via exponentiation. We can do this in two ways and we will see that these relate directly to the group representations presented earlier. The first way is by *Kronecker sum* of individual representations. This operation relates to the tensor product directly, since it is defined as

$$L_1\oplus_K L_2 := L_1\otimes\mathbb{1}_2 + \mathbb{1}_1\otimes L_2, \tag{A.25}$$

where L_1 and L_2 are linear operators, and $\mathbb{1}_1$ and $\mathbb{1}_2$ are the correspondent identity operators[3]. From this definition, a fundamental property that links the representation of Lie algebras with

[2] In the case of being finite.

[3] Clearly, in the case of L_1, L_2 being matrix representations, then the tensor product is replaced with the Kronecker product $\otimes_K$.

Lie groups can be derived. For simplicity, let $\alpha_a J_a^{(1)}$ and $\beta_b J_b^{(2)}$ be matrix representations of linear combinations of generators of the Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$. This means that a representation $\theta_k J_k$ of a linear combination of the $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ generators is written as

$$\theta_k J_k = \left(\alpha_a J_a^{(1)}\right) \oplus_K \left(\beta_b J_b^{(2)}\right) = \left(\alpha_a J_a^{(1)}\right) \otimes_K \mathbb{1} + \mathbb{1} \otimes_K \left(\beta_b J_b^{(2)}\right). \tag{A.26}$$

through exponentiation we find that

$$\exp[i\theta_k J_k] = \exp\left[i\left(\alpha_a J_a^{(1)}\right) \oplus_K \left(\beta_b J_b^{(2)}\right)\right] = \exp\left[i\alpha_a J_a^{(1)}\right] \otimes_K \exp\left[i\beta_b J_b^{(2)}\right] \tag{A.27}$$

which corresponds to the Kronecker (tensor) product of two matrix representations of groups generated by $\mathfrak{g}_1$ and $\mathfrak{g}_2$. This allows to conclude that, loosely speaking, "the addition of Lie algebras implies the the product of Lie groups". The alternative way of representing direct sums of algebras is also by using the direct sum of operators in the same way it was done in the context of groups. In this case, we can represent linear combinations of generators as

$$\theta_k J_k = \left(\alpha_a J_a^{(1)}\right) \boxplus \left(\beta_b J_b^{(2)}\right) = \begin{bmatrix} \alpha_a J_a^{(1)} & \mathbf{0} \\ \mathbf{0} & \beta_b J_b^{(2)} \end{bmatrix}. \tag{A.28}$$

Since the product of block diagonal matrices preserves the block diagonal structure, we have that

$$\exp[i\theta_k J_k] = \begin{bmatrix} \exp\left[i\alpha_a J_a^{(1)}\right] & \mathbf{0} \\ \mathbf{0} & \exp\left[i\beta_b J_b^{(2)}\right] \end{bmatrix}, \tag{A.29}$$

which corresponds to the direct sum of group representations. As a last note on the subject, one may be interested in a representation for the generators of the whole group $G_1 \times G_2$. In the case of a Kronecker sum representation, the answer is obtained from (A.26):

$$\begin{aligned} \theta_k J_k &= \alpha_a J_a^{(1)} \oplus_K \beta_b J_b^{(2)} \\ \theta_k J_k &= \alpha_a \left(J_a^{(1)} \otimes_K \mathbb{1}_2\right) + \beta_b \left(\mathbb{1}_1 \otimes_K J_b^{(2)}\right) \end{aligned} \tag{A.30}$$

Which mean that the different J_k are just the generators $J_a^{(1)}$ and $J_b^{(2)}$ extended to much the dimension of the tensorized space. Symbolically one may say that

$$J_k = \begin{cases} J_k^{(1)} \otimes_K \mathbb{1}_2\,, & k = 1, \ldots, n \\ \mathbb{1}_1 \otimes_K J_{k-n}^{(2)}\,, & k = n+1, \ldots, n+m \end{cases}. \tag{A.31}$$

Similarly, one can rewrite representations of the form (A.28) to find the correspondent representations:

$$\theta_k J_k = \begin{bmatrix} \alpha_a J_a^{(1)} & \mathbf{0} \\ \mathbf{0} & \beta_b J_b^{(2)} \end{bmatrix} \tag{A.32}$$

$$= \alpha_a \begin{bmatrix} J_a^{(1)} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} + \beta_b \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & J_b^{(2)} \end{bmatrix}. \tag{A.33}$$

This allows to write the generators as the properly sized block diagonal matrices

$$J_k = \begin{cases} \text{diag}\left[J_k^{(1)}, \mathbf{0}\right], & k = 1, \ldots, n \\ \\ \text{diag}\left[\mathbf{0}, J_{k-n}^{(1)}\right], & k = n+1, \ldots, n+m \end{cases} . \tag{A.34}$$

A.3 Orbits and stabilizers

A direct consequence of group representations acting as automorphisms on a vector space is that the group action naturally partitions the space into disjoint subsets called *orbits*. Let V be a vector space and G a Lie group such that representations $D(g)$ with $g \in G$ are automorphisms of V. Let us also consider a generic object $v \in V$. We call the *orbit of v under the action of G* to the set

$$G \cdot v := \left\{ x = D(g)v \ / g \in G \right\}. \tag{A.35}$$

An intuitive example of an orbit is a circle of radius r, which corresponds to the orbit of any vector $v \in \mathbb{R}^2$ with $|v| = r$ under the action of SO(2). Similarly, a spherical shell of radius r is the orbit of any vector $v \in \mathbb{R}^3$ with $|v| = r$ under the action of SO(3). While these orbits do not individually partition the space, the collection of all such orbits (i.e., varying r) does provide a natural way to describe the space. This structure allows us to describe the entire space using group parameters for motion within an orbit, along with an additional parameter to distinguish different orbits (such as the radius in this case). Understanding the number of independent group parameters required to span the space is particularly important in the context of *spontaneous symmetry breaking*, as it determines the number of *Goldstone bosons* that will emerge in a theory with global continuous symmetry. To understand how this is done, let us define the concept of orbit tangent space. We call the *tangent space to the orbit at $x \in G \cdot v$* to the space

$$T_x(G \cdot v) := \left\{ \alpha_k J_k x \ / \alpha_k \in \mathbb{R},\ k = 1, \ldots, N \right\}, \tag{A.36}$$

with J_a the correspondent representation of the group generators and $N = \dim(\mathfrak{g})$. Since all of these spaces are isomorphic it is useful to consider the particular case of $x = v$:

$$T_v(G \cdot v) := \left\{ \alpha_k J_k v \ / \alpha_k \in \mathbb{R},\ k = 1, \ldots, N \right\}. \tag{A.37}$$

In other words, the tangent space is the action of the algebra over the vector that defines the orbit. The dimension of all tangent spaces is given by the number of linearly independent vectors $\alpha_k J_k v$ and represents the number of degrees of freedom needed to parameterize the orbit. Note that $\dim(T_v(G \cdot v)) \leq N$ which is evident since that is the number of group parameters that define the Lie group. The strict inequality is obtained when there exists one or more generators such that

$$J_k v = 0. \tag{A.38}$$

When this happens, group parameters associated with these generators can be treated as redundant when acted upon the orbit−defining vector, allowing to identify a subgroup called the *stabilizer subgroup* or *little group*. A stabilizer subgroup H_v consists of all group elements that leave v unchanged:

$$H_v := \{\tilde{g} \in G \,/ D(\tilde{g})v = v\}. \tag{A.39}$$

In the context of spontaneous symmetry breaking, stabilizer groups are commonly refereed as *unbroken subgroups*, since they define the remanent explicit symmetry that fields in the la Lagrangian will present once the original group has been spontaneously broken. In this same way, generators that satisfy (A.38) are called *unbroken generators* with their counterparts being referred as *broken*. As implied before, existence of a stabilizer subgroup allows one to reduce the number of independent group parameters needed to describe the orbit. To see this, let $g \in G$ such that

$$D(g) = \exp\left[i\theta_a J_a + i\eta_b K_b\right], \tag{A.40}$$

with J_a $(a = 1,\ldots,n-1)$ generators that satisfy (A.38) and K_a $(b = n,\ldots,N)$ generators that do not. Let $G \cdot v$ the orbit of $v \in V$ such that any object within the orbit is written as

$$x = \exp\left[i\theta_a J_a + i\eta_b K_b\right] v \tag{A.41}$$

Equation (A.16) allows to define auxiliary group parameters $\tilde{\theta}_a$ and $\tilde{\eta}_a$ such that

$$\exp\left[i\tilde{\eta}_b K_b\right]\exp\left[i\tilde{\theta}_a J_a\right] = \exp\left[i\theta_a J_a + i\eta_b K_b\right] \tag{A.42}$$

Meaning that

$$x = \exp\left[i\theta_a J_a + i\eta_b K_b\right] v = \exp\left[i\tilde{\eta}_b K_b\right] v, \tag{A.43}$$

This expression corresponds to a parametrization of the orbit, with the number of parameters matching the dimension of the tangent space. It can be proven that orbits built in this way correspond to the coset−space G/H, which is the way they are often referred as in the literature. To fully describe the space additional degrees of freedom, preserved under group actions, must also be included as parameters. For instance, in SO(N) and SU(N) groups, the vector norm remains invariant and must be considered to allow movement between orbits. As stated earlier, this has profound implications in the context of spontaneous symmetry breaking of global continuous symmetries, as the number of Goldstone bosons is determined by the number of independent group parameters required to describe the orbit. This allows to conclude that

$$\#\text{Goldstone bosons} = \dim(\mathfrak{g}) - \dim(\mathfrak{h}_v), \tag{A.44}$$

with $\mathfrak{g}, \mathfrak{h}_v$ algebras of G and H_v correspondingly. In contrast, for gauge theories this equality is slightly modified since Goldstone bosons are by definition massless, while some of the fields associated with the orbit's degrees of freedom may acquire mass and not behave as bosons. Thus, one can state that, for gauge theories

$$\#\text{Goldstone bosons} \leq \dim(\mathfrak{g}) - \dim(\mathfrak{h}_v). \tag{A.45}$$

Appendix B

Spontaneous symmetry breaking: examples

This appendix aims to provide simple examples of how the process of spontaneous symmetry breaking takes place in different contexts. Three examples will be presented, each for a different type of symmetry: discrete, continuous global, and continuous local. For the sake of consistency, the same potential structure $V(x) = -ax^2 + bx^4$ will be used for each example.

Example 1: discrete symmetry

Let us consider the following Lagrangian density for a real scalar field ϕ:

$$\mathscr{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi) \tag{B.1}$$

with the introduced potential being

$$V(\phi) = -\frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4, \tag{B.2}$$

with $m, \lambda > 0$. Clearly this Lagrangian is invariant under the finite group Z_2, since the transformation $\phi' = -\phi$ leaves every term invariant. As a consequence, the symmetric function presents two local minima (also called *vacuum*) at

$$\langle\phi\rangle_\pm := \pm\sqrt{\frac{6m^2}{\lambda}} \tag{B.3}$$

Nevertheless, translation of the function such that the origin lies within one of the minima reproduces a new asymmetric potential. Thus, to spontaneously break the Z_2 symmetry, we may define the new field $h(x) = \phi(x) - \langle\phi\rangle_+$ such that the Lagrangian takes the new form

$$\mathscr{L} = \frac{1}{2}\partial_\mu h\partial^\mu h - m^2h^2 - \sqrt{\frac{\lambda}{6}}mh^3 - \frac{\lambda}{4!}h^4 + \frac{3m^4}{2\lambda}. \tag{B.4}$$

Note that writing the Lagrangian as a function of h instead of ϕ hides the original symmetry since the homogeneous transformation $h' = -h$ does not leave it unchanged as it did under the action $\phi' = -\phi$. This comes from the fact that the actual transformation rule for h under the actions of Z_2 is not the one just written, but

$$h' = -h - 2\langle\phi\rangle_+ . \tag{B.5}$$

This follows directly from the original Z_2 action, $\phi' = -\phi$. Although the symmetry is no longer manifest in its original form, it remains implicitly present through an inhomogeneous transformation rather than a group representation, which are linear. In spite of this, the Lagrangian remains invariant under the fundamental representation of the trivial group $Z_1 \subsetneq Z_2$ which yields the trivial transformation $h' = h$. When this happens, we say that the Z_2 symmetry has been spontaneously broken, in this case, to the trivial subgroup.

This case has helped to illustrate how a finite is spontaneously broken. Obviously, not all systems exhibiting a finite symmetry are left with only a trivial symmetry once the original has been spontaneously broken. Nevertheless, this illustrates that even when the symmetry has been completely hidden, there will always be a remanent subgroup.

Example 2: global continuous symmetry

Let us consider a similar Lagrangian density than the last one, but over a complex scalar field ϕ:

$$\mathscr{L} = (\partial_\mu\phi^*)(\partial^\mu\phi) - V(\phi) \tag{B.6}$$

with the potential being

$$V(\phi) = -\frac{1}{2}m^2\phi^*\phi + \frac{\lambda}{4}(\phi^*\phi)^2, \tag{B.7}$$

and $m, \lambda > 0$. In this case, the Lagrangian presents a global U(1) since the terms in the lagrangian are now invariant under the group action $\phi' = e^{i\theta}\phi$, for θ a real constant. As a consequence, this time we will find that the continuous set of points

$$\langle\phi\rangle_\theta := \sqrt{\frac{2m^2}{\lambda}}\exp[i\theta], \quad \forall\theta \in \mathbb{R}, \tag{B.8}$$

all are vacua of the potential. This is the same as saying that *the potential is minimized over the entire orbit of* $\langle\phi\rangle_0 = \sqrt{2m^2/\lambda}$ *under the action of* U(1). Once again, shifting of the function towards one specific vacuum of this set (say $\langle\phi\rangle_0$) implies that the new fields will not transform under a representation of U(1)), hiding the symmetry. We write

$$\phi(x) = \exp\left[i\beta\pi(x)\right]\left(\alpha h(x) + \phi_0\right), \tag{B.9}$$

where two real fields $h(x)$ and $\pi(x)$ had to be defined. The first one corresponds to the deviation field, but the second one corresponds to an additional degree of freedom correspondent to the extra internal degree of freedom coming from angular translations around the set of minima. This last field $\pi(x)$ will prove to the Goldstone boson. On another note, we have also added additional normalization factors $\alpha, \beta \in \mathbb{R}$ to ensure the new form for the Lagrangian has the correct dynamic structure. Replacing this new fields in the Lagrangian density yields

$$\begin{aligned}\mathscr{L} =&\alpha^2 \partial_\mu h \partial^\mu h + \beta^2 \langle \phi \rangle_0^2 \partial_\mu \pi \partial^\mu \pi + \left(\alpha^2 \beta^2 h^2 + 2\alpha\beta^2 \langle \phi \rangle_0 h\right) \partial_\mu \pi \partial^\mu \pi \\ &+ \frac{m^4}{\lambda} - 2\alpha^2 m^2 h^2 - \sqrt{2\lambda}\,\alpha^3 m h^3 - \frac{1}{4}\alpha^4 \lambda h^4\end{aligned} \tag{B.10}$$

From here, we set $\alpha = 1/\sqrt{2}$ and $\beta = 1/\left(\langle \phi \rangle_0 \sqrt{2}\right)$ such that the kinetic terms are properly normalized. More simplification gives

$$\mathscr{L} = \frac{1}{2}\partial_\mu h \partial^\mu h + \frac{1}{2}\partial_\mu \pi \partial^\mu \pi + \left(\frac{1}{2\langle \phi \rangle_0^2} h^2 + \frac{1}{\sqrt{2}\langle \phi \rangle_0} h\right) \partial_\mu \pi \partial^\mu \pi + \frac{m^4}{\lambda} - m^2 h^2 - \frac{1}{2}\sqrt{\lambda}\, m h^3 - \frac{\lambda}{16} h^4. \tag{B.11}$$

Note that as mentioned before, there is no mass term for the Goldstone boson despite the previous existence of a mass term for the original complex field ϕ from which this boson is extracted. Let us now obtain the transformation rule for our fields. We know that

$$\phi = \exp\left[i \frac{\pi}{\sqrt{2}\langle \phi \rangle_0}\right] \left(\frac{h}{\sqrt{2}} + \langle \phi \rangle_0\right) \tag{B.12}$$

then, we can write the original group action

$$\phi' = \exp[i\theta] \exp\left[i \frac{\pi}{\sqrt{2}\langle \phi \rangle_0}\right] \left(\frac{h}{\sqrt{2}} + \langle \phi \rangle_0\right) = \exp\left[i \frac{1}{\sqrt{2}\langle \phi \rangle_0}\left(\pi + \sqrt{2}\langle \phi \rangle_0 \theta\right)\right] \left(\frac{h}{\sqrt{2}} + \langle \phi \rangle_0\right). \tag{B.13}$$

This means that the h field transforms trivially while the Goldstone boson does not transform under a U(1) representation, but rather as

$$\pi' = \pi + \sqrt{2}\langle \phi \rangle_0 \theta. \tag{B.14}$$

So we have arrived at a Lagrangian where the fields do not transform under a group representation, which means that the U(1) symmetry has been spontaneously broken.

Example 3: local continuous symmetry

We now explore the case of gauge theories. In particular, we study how a Yang-Mills theory behaves when the symmetry of the gauge group is spontaneously broken. As an example, we consider a Lagrangian density that is invariant under the actions of SO(3). As before, we introduce a single non-gauge field ϕ, which in this case is an $\mathbb{R}^3$-valued scalar transforming under the fundamental representation of the group:

$$\phi' = \exp[i\theta_a(x) T_a] \phi, \tag{B.15}$$

where $\theta_a(x)$ ($a = 1,2,3$) corresponds to a spacetime−dependent group parameter and

$$T_a = i \begin{bmatrix} 0 & -\delta_{a3} & \delta_{a2} \\ \delta_{a3} & 0 & -\delta_{a1} \\ -\delta_{a2} & \delta_{a1} & 0 \end{bmatrix} \tag{B.16}$$

the fundamental representation of the generators. Since it is a Yang-Mills theory, this construction also includes a gauge field

$$A_\mu = A_\mu^a T_a, \tag{B.17}$$

such that the correspondent covariant derivative for ϕ can be written as

$$D_\mu\phi = \partial_\mu\phi - igA_\mu\phi. \tag{B.18}$$

Knowing this, let us consider the following Lagrangian density:

$$\mathscr{L} = \frac{1}{2}\left(D_\mu\phi\right)^T\left(D^\mu\phi\right) - \frac{1}{2}\mathrm{Tr}\left\{F_{\mu\nu}F^{\mu\nu}\right\} - V(\phi), \tag{B.19}$$

with the potential

$$V(\phi) = -m^2\phi^T\phi + \frac{\lambda}{4}\left(\phi^T\phi\right)^2, \tag{B.20}$$

and $m, \lambda > 0$. This potential is minimized over the entire orbit of

$$\langle\phi\rangle_0 := \sqrt{2m^2/\lambda}\,[0\,0\,1]^T \tag{B.21}$$

Note $T_3\langle\phi\rangle_0 = 0$, this is important because it means that this vacuum $\langle\phi\rangle_0$ defines a stabilizer subgroup formed by elements with a representation of the form $\exp[i\theta_3(x)T_3]$: SO(2). Therefore We say *the* SO(3) *symmetry will not be spontaneously broken to the trivial subgroup, but rather to* SO(2). The existence of this subgroup implies that only two group parameters are enough to characterize the orbit, meaning that only two independent Goldstone bosons are required:

$$\phi(x) = \exp\left[i\beta\pi_b(x)T_b\right]\left(\frac{h(x)}{\sqrt{2}} + \sqrt{2m^2/\lambda}\right)[0\,0\,1]^T, \tag{B.22}$$

with $b = 1,2$ and $\beta \in \mathbb{R}$ the appropriate normalization parameter. This implies that the potential can be written as

$$V = -\frac{m^4}{\lambda} + 2m^2h^2 + \sqrt{2\lambda}\,mh^3 + \frac{\lambda}{4}h^4. \tag{B.23}$$

Clearly, no explicit residual symmetry is apparent in this expression, as the deviation field $h(x)$ transforms trivially under the remaining unbroken subgroup. We therefore proceed to study the scalar kinetic term, as it will provide more insight. Replacing (B.22) in (B.18) yields

$$D_\mu\phi = \exp\left[i\beta\pi_b(x)T_b\right]\begin{bmatrix} 0 \\ 0 \\ \partial_\mu h \end{bmatrix} - ig\left(\frac{i}{g}\left(\partial_\mu\exp\left[i\beta\pi_bT_b\right]\right)\exp\left[-i\beta\pi_bT_b\right] + A_\mu\right)\phi, \tag{B.24}$$

We can always assume that the gauge field A_μ comes from a group action over a previous gauge field $\tilde{A}_\mu$ such that

(B.25) $$A_\mu = \exp[i\theta_a T_a]\tilde{A}_\mu \exp[-i\theta_a T_a] - \frac{i}{g}\left(\partial_\mu \exp[i\theta_a T_a]\right)\exp[-i\theta_a T_a]$$

Let us define $\tilde{\theta}_a$ such that

(B.26) $$\exp\left[i\tilde{\theta}_b T_b\right]\exp\left[i\tilde{\theta}_3 T_3\right] = \exp[i\theta_a T_a].$$

This means that

(B.27) $$\left(\partial_\mu \exp[i\theta_a T_a]\right) = \left(\partial_\mu \exp\left[i\tilde{\theta}_b T_b\right]\right)\exp\left[i\tilde{\theta}_3 T_3\right] + \exp\left[i\tilde{\theta}_b T_b\right]\left(\partial_\mu \exp[i\theta_3 T_3]\right).$$

Therefore, we write the gauge field as

(B.28) $$\begin{aligned} A_\mu =& \exp\left[i\tilde{\theta}_b T_b\right]\exp\left[i\tilde{\theta}_3 T_3\right]\tilde{A}_\mu \exp\left[-i\tilde{\theta}_3 T_3\right]\exp\left[-i\tilde{\theta}_b T_b\right] \\ &- \frac{i}{g}\left(\partial_\mu \exp\left[i\tilde{\theta}_b T_b\right]\right)\exp\left[-i\tilde{\theta}_b T_b\right] \\ &- \frac{i}{g}\exp\left[i\tilde{\theta}_b T_b\right]\left(\partial_\mu \exp[i\theta_3 T_3]\right)\exp\left[-i\tilde{\theta}_3 T_3\right]\exp\left[-i\tilde{\theta}_b T_b\right] \end{aligned}$$

Note that there is a particular gauge fixing for the group parameters θ_b that will simplify substantially the covariant derivative and will prove to eliminate all Goldstone bosons from the Lagrangian. Let us choose

(B.29) $$\tilde{\theta}_b(x) = \beta\pi_b(x),$$

for both $b = 1,2$. This is equivalent as stating that the scalar ϕ comes from a previously transformed $\tilde{\phi}$ such that

(B.30) $$\phi = \exp\left[i\tilde{\theta}_b T_b\right]\exp\left[i\tilde{\theta}_3 T_3\right]\tilde{\phi} = \exp\left[i\beta\pi_b(x)T_b\right]\left(\frac{h(x)}{\sqrt{2}} + \sqrt{2m^2/\lambda}\right)[0\,0\,1]^T$$

and fixing the gauge such that

(B.31) $$\tilde{\phi} = \left(\frac{h(x)}{\sqrt{2}} + \sqrt{2m^2/\lambda}\right)[0\,0\,1]^T.$$

The reader can verify that under this gauge, simple algebraic manipulation of (B.24) after replacing the gauge field yields

(B.32) $$D_\mu\phi = \exp\left[i\beta\pi_b T_b\right]\left(\begin{bmatrix} 0 \\ 0 \\ \partial_\mu h \end{bmatrix} - ig\left(V_\mu + A_\mu^{\mathfrak{so}(2)}\right)\begin{bmatrix} 0 \\ 0 \\ h + \sqrt{2m^2/\lambda} \end{bmatrix}\right).$$

Where we have defined a new non−gauge algebra−valued vector field

(B.33) $$V_\mu := \exp\left[i\tilde{\theta}_3 T_3\right]\tilde{V}_\mu \exp\left[-i\tilde{\theta}_3 T_3\right] \quad \Big/ \tilde{V}_\mu := \tilde{A}_\mu^b T_b,$$

together with the stabilizer SO(2) gauge field

$$A_\mu^{\mathfrak{so}(2)} := \exp\left[i\tilde{\theta}_3 T_3\right]\tilde{A}_\mu^{\mathfrak{so}(2)}\exp\left[-i\tilde{\theta}_3 T_3\right] - \frac{i}{g}\left(\partial_\mu \exp[i\theta_3 T_3]\right)\exp[-i\theta_3 T_3] \tag{B.34}$$
$$= \tilde{A}_\mu^{\mathfrak{so}(2)} + \frac{1}{g}\partial_\mu\theta_3 T_3.$$

for $\tilde{A}_\mu^{\mathfrak{so}(2)} := \tilde{A}_\mu^3 T_3$. Finally, we compute explicitly

$$D_\mu\phi = \exp\left[i\beta\pi_a T_a\right]\begin{bmatrix} gV_\mu^2\left(h+\sqrt{2m^2/\lambda}\right) \\ -gV_\mu^1\left(h+\sqrt{2m^2/\lambda}\right) \\ \partial_\mu h \end{bmatrix}, \tag{B.35}$$

in order to obtain the dynamic term

$$\frac{1}{2}\left(D_\mu\phi\right)^T\left(D^\mu\phi\right) = \frac{1}{2}\partial_\mu h\partial^\mu h + \frac{1}{2}g^2\left(h+\sqrt{2m^2/\lambda}\right)^2 V_\mu^1 V^{1\mu} + \frac{1}{2}g^2\left(h+\sqrt{2m^2/\lambda}\right)^2 V_\mu^2 V^{2\mu}. \tag{B.36}$$

We see that the Goldstone bosons have been entirely removed from the Lagrangian, as they no longer appear in either the kinetic terms or the potential[1]. We call the choice of gauge fixing that eliminates the goldstone bosons from the Lagrangian, the *unitary gauge*. Notice that this gauge fixing has an equivalent effect as replacing

$$A_\mu \to \tilde{V}_\mu + \tilde{A}_\mu^{\mathfrak{so}(2)} \quad \wedge \quad \phi \to \tilde{\phi} = \left(h(x)+\sqrt{2m^2/\lambda}\right)[0\;0\;1]^T \tag{B.37}$$

in the original Lagrangian density (B.19). It is also the same as considering all fields in the Lagrangian to transform only under the unbroken group ($\tilde{\theta}_b = \beta\pi_b = 0$). In the case of the gauge field, this can be seen from the transformation rule (B.25):

$$A_\mu = \exp[i\theta_3 T_3]\left(\tilde{V}_\mu + \tilde{A}_\mu^{\mathfrak{so}(2)}\right)\exp[-i\theta_3 T_3] - \frac{i}{g}\left(\partial_\mu \exp[i\theta_3 T_3]\right)\exp[-i\theta_3 T_3] = V_\mu + A_\mu^{\mathfrak{so}(2)}. \tag{B.38}$$

We must also point out that we have obtained a mass term for V_μ:

$$\frac{1}{2}\left(D_\mu\phi\right)^T\left(D^\mu\phi\right) \ni \frac{2g^2m^2}{\lambda}\mathrm{Tr}\{V_\mu V^\mu\}, \tag{B.39}$$

while none have been obtained for $\tilde{A}_\mu^{\mathfrak{so}(2)}$. This is important because terms of this type cannot be added to a Yang-Mills theory naturally from the start, since $\mathrm{Tr}\{A_\mu A^\mu\}$ is not group invariant for a given gauge field A_μ. The reason behind this result comes from the fact that, when a symmetry is spontaneously broken by a vacuum that defines a stabilizer subgroup, the gauge field components associated with said group generators will retain their character as gauge fields under the actions of this unbroken group. In contrast, all other components will not behave as gauge fields independently, instead acquiring a non−gauge group transformation rule under

[1]Although not calculated here, writing the gauge kinetic term by considering $F_{\mu\nu} = U\tilde{F}_{\mu\nu}U^{-1}$, with $\tilde{F}_{\mu\nu} := \partial_\mu\tilde{A}_\nu - \partial_\nu\tilde{A}_\mu - ig\left[\tilde{A}_\mu,\tilde{A}_\nu\right]$, also yields a decomposition where no Goldstone bosons will be found.

the stabilizer. The process of giving mass to gauge field components as well as other fields in the Lagrangian once through the process of spontaneous symmetry breaking is what we call the *Higgs mechanism*. Thus, our Lagrangian after the spontaneously breaking SO(3) will be composed by the fields $h(x)$ that transforms trivially under SO(2), $V_\mu = V_\mu^1(x)T_1 + V_\mu^2(x)T_2$ which transforms under the adjoint representation of SO(2) (group conjugation) and the gauge boson $A_\mu^{\mathfrak{so}(2)}(x)$ of SO(2), which transforms as such.

APPENDIX C

$W^{\pm}$ AS PHYSICAL BOSONS OF THE STANDARD MODEL

This appendix aims to clarify why the fields $W^{\pm}_{\mu}$ are introduced instead of directly working with W^1_{μ} and W^2_{μ} in the Lagrangian. While both sets of fields, (W^+, W^-) and (W^1, W^2), diagonalize the mass matrix, there is a deeper reason to prefer one over the other. This preference originates from how these fields transform under the unbroken subgroup $\mathrm{U}(1)_{em}$.

More generally, consider a gauge field A_μ associated with a gauge group G, which undergoes spontaneous symmetry breaking to a nontrivial subgroup H. The proper basis for the physical vector fields must satisfy two key conditions:

1. The gauge field components in this basis must diagonalize the mass quadratic form.
2. The gauge field components in this basis must transform independently (without mixing) under transformations U_H of the unbroken subgroup.

More precisely, if we write the gauge field as

$$A_\mu = A^a_\mu T_a = \mathscr{A}^b_\mu \mathscr{T}_b + \mathscr{B}^c_\mu \mathscr{K}_c, \tag{C.1}$$

where T_a ($a = 1, \ldots, \dim(\mathfrak{g})$) are the generators of the original group G, and $\{\mathscr{T}_b, \mathscr{K}_c\}$ ($b = 1, \ldots, n$, $c = n+1, \ldots, \dim(\mathfrak{g})$) form a new basis with $\mathscr{K}_c$ being the generators of the unbroken subgroup H, then:

1. The scalar kinetic terms in the Lagrangian contain only quadratic terms over these components of the form

   $$\left(D_\mu \phi\right)^\dagger \left(D^\mu \phi\right) \ni \frac{1}{2} M_b^2 \mathscr{A}^b_\mu \mathscr{A}^{\mu b}, \tag{C.2}$$

   ensuring that the mass terms remain diagonal.

2. Each gauge field component in the new basis transforms independently[1] under H:

$$U_H(\mathscr{A}_\mu^1\mathscr{T}_1)U_H^{-1} = (\mathscr{A}_\mu^1)'\mathscr{T}_1 \;\wedge\; U_H(\mathscr{A}_\mu^2\mathscr{T}_2)U_H^{-1} = (\mathscr{A}_\mu^2)'\mathscr{T}_2 \;\wedge\; \dots \tag{C.3}$$

which implies that the basis elements form an eigenbasis of U_H:

$$U_H\mathscr{T}_1U_H^{-1} = \lambda_1\mathscr{T}_1 \;\wedge\; U_H\mathscr{T}_2U_H^{-1} = \lambda_2\mathscr{T}_2 \;\wedge\; \dots \tag{C.4}$$

noindent This last condition can be simplified by the use of the *Campbell identity*[2] for group conjugation:

$$\begin{aligned}\exp\left[i\theta_H^c\mathscr{K}_c\right]\mathscr{T}_b\exp\left[-i\theta_H^c\mathscr{K}_c\right] &= \mathscr{T}_b + \left[i\theta_H^c\mathscr{K}_c\,,\,\mathscr{T}_b\right] + \frac{1}{2}\left[i\theta_H^c\mathscr{K}_c\,,\,\left[i\theta_H^c\mathscr{K}_c\,,\,\mathscr{T}_b\right]\right] + ...\\ &= \lambda_b\mathscr{T}_b.\;\text{(no sum over } b).\end{aligned} \tag{C.5}$$

This leads to the condition

$$\left[i\theta_H^c\mathscr{K}_c\,,\,\mathscr{T}_a\right] \propto \mathscr{T}_a \quad \forall a \tag{C.6}$$

The physical fields are those that respect both (C.2) and (C.6). In the case of the Standard Model, the fields $W_\mu^{\pm}$ are the appropriate physical fields since they transform as eigenfields of $U(1)_{em}$, unlike W_μ^1 and W_μ^2, which mix under its transformations. Let us dive deeper into this particular case. We start from (2.88) written in the form

$$A_\mu = \cos(\theta_W)W_\mu^1\frac{\sigma_1}{2} + \cos(\theta_W)W_\mu^2\frac{\sigma_2}{2} + \frac{1}{2}Z_\mu Q^- + \frac{1}{2}A_\mu^\gamma Q^+. \tag{C.7}$$

Here $\mathscr{B}_\mu^4\mathscr{K}_4 = \frac{1}{2}A_\mu^\gamma Q^+$, while

$$\mathscr{T}_1 = \frac{\sigma_1}{2} \;\wedge\; \mathscr{T}_2 = \frac{\sigma_2}{2} \;\wedge\; \mathscr{T}_3 = Q^- \tag{C.8}$$

Computation of the covariant derivative and the dynamic term proves that this basis ensures that the mass quadratic form is indeed diagonalized without introducing the fields $W^{\pm}$. Nevertheless

$$\left[\frac{\sigma_1}{2}, Q^+\right] = -i\frac{\sigma_2}{2} \;\wedge\; \left[\frac{\sigma_2}{2}, Q^+\right] = i\frac{\sigma_1}{2} \tag{C.9}$$

Meaning that condition (C.6) is not satisfied and the the associated fields W_μ^1 and W_μ^2 will not transform independently under U_{em} given by (2.82). This means that two linear combinations between $\sigma_1/2$ and $\sigma_2/2$ should be defined such that (C.6) is satisfied. Here we construct the generators $\sigma^{\pm}$ as (2.90). These satisfy

$$\left[\sigma^{\pm}, Q^+\right] = \mp\sigma^{\pm} \tag{C.10}$$

Meaning that the associated vector fields ($W^{\pm}$) will transform independently from each other.

[1] This preserves the structure of (2.50), with the difference being that these are not gauge fields, so they only transform under group conjugation.

[2] This formula can be obtained as an special case of the Baker–Campbell–Hausdorff formula (A.16).

APPENDIX D

NOTES ON UNITARITY

The construction of quantum mechanics and the theory of quantum fields is rooted[1] in the idea that systems can be described by *state vectors* $|\psi\rangle$ of a Hilbert space $\mathscr{E}$. Each of these objects corresponds to a different configuration ψ the system might have, with the inner product of the space representing conditional probabilities for these configurations. In particular, this means that all observable states of the system need to be properly normalized such that product $\langle\psi|\psi\rangle = P(\psi \mid \psi) = 1$ at all times. Since the evolution of the states is determined by the *time evolution operator*

$$\mathscr{U} = \exp\left[-i(t-t_0)\frac{H}{\hbar}\right], \tag{D.1}$$

such that the states transform as

$$|\psi(t)\rangle = \mathscr{U}\,|\psi(t_0)\rangle. \tag{D.2}$$

We demand this operator to be unitary in order to preserve the space inner product

$$\mathscr{U}^\dagger\mathscr{U} = \mathbb{1}. \tag{D.3}$$

This essential property is what we call *unitarity*, and it is often said that it is the property that "ensures all probabilities add up to one". In Quantum field theories, this imposes conditions on the transfer matrix $\mathscr{T}$, which is defined by the relation

$$\mathscr{U} \equiv \mathbb{1} + i\mathscr{T}. \tag{D.4}$$

Concretely, (D.3) implies that

$$i(\mathscr{T}^\dagger - \mathscr{T}) = \mathscr{T}^\dagger\mathscr{T}. \tag{D.5}$$

[1]At least when formulated through canonical quantization.

Let $|f\rangle, |i\rangle \in \mathscr{E}$ be the final and initial asymptotic states of a system with total momenta p_f and p_i respectively. This implies that

$$i\langle f|(\mathscr{T}^\dagger - \mathscr{T})|i\rangle = i\langle i|\mathscr{T}|f\rangle^* - i\langle f|\mathscr{T}|i\rangle = i(2\pi)^2\delta^4\left(p_i - p_f\right)\left(\mathscr{M}^*(f\to i) - \mathscr{M}(i\to f)\right), \tag{D.6}$$

with the matrix elements $\mathscr{M}$ defined by

$$\langle A|\mathscr{T}|B\rangle \equiv (2\pi)\delta^4(p_A - p_B)\mathscr{M}(A\to B). \tag{D.7}$$

Let also A_j be an arbitrary particle with 4−momentum $p_j = [p_j^0 \;\; \vec{p}_j]^T$. We can write any generic N−particle state as the union of duplets (A_j, p_j):

$$X := \bigcup_{j=1}^{N} \{(A_j, p_j)\}. \tag{D.8}$$

The set of all possible states $|X\rangle$ form an orthogonal basis of $\mathscr{E}$. As a result, the completeness relation of the many−particle Hilbert space[2] is

$$\sum_X \int_{\mathbb{R}^{3N}} |X\rangle\langle X| d\Pi_X = \mathbb{1} \tag{D.9}$$

where

$$d\Pi_X := \prod_{j=1}^{N} \frac{d^3\vec{p}_j}{(2\pi)^3}\frac{1}{2E_j} \tag{D.10}$$

is the Lorentz invariant measure of the space. We can make use of this relation we see that

$$\langle f|\mathscr{T}^\dagger\mathscr{T}|i\rangle = \sum_X \int \langle f|\mathscr{T}^\dagger|X\rangle\langle X|\mathscr{T}|i\rangle d\Pi_X = \sum_X \int (2\pi)^4\delta^4(p_f - p_X)(2\pi)^4\delta^4(p_i - p_X)\mathscr{M}(i\to X)\mathscr{M}^*(f\to X) d\Pi_X. \tag{D.11}$$

Combining this result with (D.6) yields the *Generalized optical theorem*

$$\mathscr{M}(i\to f) - \mathscr{M}^*(f\to i) = i\sum_X \int (2\pi)^4\delta^4(p_i - p_X)\mathscr{M}(i\to X)\mathscr{M}^*(f\to X) d\Pi_X. \tag{D.12}$$

This result imposes constraints on the matrix elements, providing a means to verify whether unitarity is preserved in the theory. These constraints must hold regardless of whether the theory permits a perturbative expansion of $\mathscr{M}$. A particular and useful case is when $|i\rangle = |f\rangle \equiv |Y\rangle$:

$$\mathrm{Im}(\mathscr{M}(Y\to Y)) = \frac{1}{2}\sum_X \int (2\pi)^4\delta^4(p_Y - p_X)|\mathscr{M}(Y\to X)|^2. \tag{D.13}$$

[2]Also called Fock space.

If $Y = \{(A,p);(B,-p)\} \equiv AB$ is a 2−particle state on the center of mass fame of reference, then the total cross section is given by

$$\sigma(AB \to X) = \frac{1}{4\sqrt{s}|\vec{p}|} \int (2\pi)^4 \delta^4(p - p_X) |\mathcal{M}(AB \to X)|^2 d\Pi_X, \tag{D.14}$$

with s the square center of mass energy (Mandelstam variable). Meaning that (D.13) can be expressed as

$$\mathrm{Im}(\mathcal{M}(AB \to AB)) = 2\sqrt{s}|\vec{p}| \sum_X \sigma(AB \to X) \tag{D.15}$$

This last result is known as *the* optical theorem. A direct consequence is that it imposes a lower bound on the imaginary part of $\mathcal{M}$ given by the elastic scattering cross section:

$$\begin{gathered} \sum_X \sigma(AB \to X) \geq \sigma(AB \to AB') \\ \big\downarrow \\ \mathrm{Im}(\mathcal{M}(AB \to AB)) \geq 2\sqrt{s}|\vec{p}|\sigma(AB \to AB'), \end{gathered} \tag{D.16}$$

with $AB' = \{(A,p');(B,-p')\}$. The total cross section is given by

$$\sigma(AB \to AB') = \frac{1}{32\pi s} \int_{-1}^{1} \left|\mathcal{M}(AB \to AB')\right|^2 d(\cos\theta) \quad \Big/ \cos\theta = \frac{\vec{p}\cdot\vec{p}'}{|\vec{p}||\vec{p}'|}. \tag{D.17}$$

Now, since the matrix element is a function of the scattering angle θ, we can expand in Legendre Polynomials[3] as

$$\mathcal{M}(AB \to AB') = 16\pi \sum_{n=0}^{\infty} a_n (2n+1) P_n(\cos\theta). \tag{D.18}$$

In particular, for the forward process we have

$$\mathcal{M}(AB \to AB) = 16\pi \sum_{n=0}^{\infty} a_n (2n+1). \tag{D.19}$$

To obtain the cross section, we must compute $|\mathcal{M}|^2 = \mathcal{M}^*\mathcal{M}$. To do it, we can use the orthogonality of the Legendre polynomials

$$\int_{-1}^{1} P_n(x) P_l(x) dx = \frac{2}{2n+1}\delta_{nl}. \tag{D.20}$$

Once calculated, we arrive at the following result

$$\sigma(AB \to AB') = \frac{16\pi}{s} \sum_{n=0}^{\infty} |a_n|^2 (2n+1). \tag{D.21}$$

We can now substitute (D.21) and (D.19) into (D.16) to obtain

$$\sum_{n=0}^{\infty} \mathrm{Im}(a_n)(2n+1) \geq \frac{2|\vec{p}|}{\sqrt{s}} \sum_{n=0}^{\infty} |a_n|^2 (2n+1) \tag{D.22}$$

[3] Using the standard normalization where $P_n(1) = 1 \; \forall n$.

It can be proven using angular momentum eigenstates that the sum over n can be dropped, allowing to obtain the following inequality of each coefficient of the series[4]

$$\mathrm{Im}(a_n) \geq \frac{2|\vec{p}|}{\sqrt{s}} |a_n|^2. \tag{D.23}$$

This inequality describes the region inside a circumference of diameter $\mathrm{d} = \frac{\sqrt{s}}{2|\vec{p}|}$ in the complex plane, meaning that the coefficients are bounded by the energy−momenta ratio of the process. Thus, for the high center of mass energy limit we (i.e. $\sqrt{s} \gg m_A, m_B$) in which $|\vec{p}| \to \frac{1}{2}\sqrt{s}$, the diameter of said sphere converges to the unity:

$$\mathrm{Im}(a_n) \geq |a_n|^2. \tag{D.24}$$

This limit is of much interest, since many theories that present violation of unitarity do so above an energy threshold or *cutoff*. Equation (D.24) implicitly provides the following inequalities that can prove to be useful when finding such cutoff:

$$|a_n| \leq 1 \qquad \wedge \qquad 0 \leq \mathrm{Im}(a_n) \leq 1 \qquad \wedge \qquad |\mathrm{Re}(a_n)| \leq \frac{1}{2}. \tag{D.25}$$

Bounds like these (i.e. defined by restricting the expansion coefficients of $\mathcal{M}$) are called *partial wave unitary bounds*.

[4]The complete derivation is more involved and lies beyond the scope of this appendix. The complete derivation of (D.23) can be found on [73], Section 5.3.

APPENDIX E

VECTOR ISOTRIPLET MODEL AS GAUGE SECTOR OF A $\mathfrak{so}(4)$ THEORY

This appendix provides the derivation of the decomposition that yields most of the vector isotriplet dark matter model. We start by considering the gauge dynamic term for a Yang-Mills theory over a group with an $\mathfrak{so}(4)$

$$-\frac{1}{2}\mathrm{Tr}\left\{F^{(4)}_{\mu\nu}F^{(4)\mu\nu}\right\}. \tag{E.1}$$

The field strength tensor is given by

$$F^{(4)}_{\mu\nu} := \partial_\mu A^{\mathfrak{so}(4)}_\nu - \partial_\nu A^{\mathfrak{so}(4)}_\mu - ig_L\left[A^{\mathfrak{so}(4)}_\mu, A^{\mathfrak{so}(4)}_\nu\right], \tag{E.2}$$

with the associated gauge field

$$A^{\mathfrak{so}(4)}_\mu := W^a_\mu J_a + V^a_\mu K_a \equiv W_\mu + V_\mu. \tag{E.3}$$

Where J_a, K_a $(a = 1,2,3)$ are the algebra's generators that satisfy (2.2). We now replace this form for the gauge field onto (E.2) to obtain

$$F_{\mu\nu} = W_{\mu\nu} + V_{\mu\nu} + \tilde{V}_{\mu\nu}, \tag{E.4}$$

where we have defined

$$\begin{aligned} W_{\mu\nu} &:= \partial_\mu W_\nu - \partial_\nu W_\mu - ig_L\left[W_\mu, W_\nu\right] \\ V_{\mu\nu} &:= D_\mu V_\nu - D_\nu V_\mu \\ \tilde{V}_{\mu\nu} &:= -ig_L\left[V_\mu, V_\nu\right], \end{aligned} \tag{E.5}$$

together with the adjoint covariant derivative

$$D_\mu V_\nu = \partial_\mu V_\nu - ig_L\left[W_\mu, V_\nu\right]. \tag{E.6}$$

From the $\mathfrak{so}(4)$ commutation relations (2.2), we can conclude that the objects just defined are linear combinations of either the J_a or K_a, but not both. The reader can verify using these relations that

$$W_{\mu\nu} \propto J_a \;\wedge\; V_{\mu\nu} \propto K_a \;\wedge\; \tilde{V}_{\mu\nu} \propto J_a. \tag{E.7}$$

This is important because the generators are orthogonal under the trace (see (2.59)). As a consequence, we get the following relations

$$\mathrm{Tr}\left\{W_{\mu\nu}V^{\mu\nu}\right\} = \mathrm{Tr}\left\{\tilde{V}_{\mu\nu}V^{\mu\nu}\right\} = 0 \tag{E.8}$$

Therefore, if the product $F^{(4)}_{\mu\nu}F^{(4)\mu\nu}$ yields

$$\begin{aligned} F^{(4)}_{\mu\nu}F^{(4)\mu\nu} =& W_{\mu\nu}W^{\mu\nu} + V_{\mu\nu}V^{\mu\nu} + \tilde{V}_{\mu\nu}\tilde{V}^{\mu\nu} + W_{\mu\nu}\tilde{V}^{\mu\nu} + \tilde{V}_{\mu\nu}W^{\mu\nu} \\ &+ W_{\mu\nu}V^{\mu\nu} + V_{\mu\nu}W^{\mu\nu} + \tilde{V}_{\mu\nu}V^{\mu\nu} + V_{\mu\nu}\tilde{V}^{\mu\nu}. \end{aligned} \tag{E.9}$$

Its trace will be reduced to

$$\mathrm{Tr}\left\{F^{(4)}_{\mu\nu}F^{(4)\mu\nu}\right\} = \mathrm{Tr}\left\{W_{\mu\nu}W^{\mu\nu}\right\} + \mathrm{Tr}\left\{V_{\mu\nu}V^{\mu\nu}\right\} + \mathrm{Tr}\left\{\tilde{V}_{\mu\nu}\tilde{V}^{\mu\nu}\right\} + 2\mathrm{Tr}\left\{W_{\mu\nu}\tilde{V}^{\mu\nu}\right\} \tag{E.10}$$

Substituting the definitions (E.5) into this last equation yields the explicit form

$$\mathrm{Tr}\left\{F^{(4)}_{\mu\nu}F^{(4)\mu\nu}\right\} = \mathrm{Tr}\left\{W_{\mu\nu}W^{\mu\nu}\right\} + \mathrm{Tr}\left\{\left(D_\mu V_\nu - D_\nu V_\mu\right)\left(D^\mu V^\nu - D^\nu V^\mu\right)\right\} \tag{E.11}$$

$$- g_L^2 \mathrm{Tr}\left\{\left[V_\mu, V_\nu\right]\left[V^\mu, V^\nu\right]\right\} - 2ig_L \mathrm{Tr}\left\{W_{\mu\nu}\left[V^\mu, V^\nu\right]\right\} \tag{E.12}$$

We can further simplify this expression noticing that

$$\left(D_\mu V_\nu - D_\nu V_\mu\right)\left(D^\mu V^\nu - D^\nu V^\mu\right) = 2D_\mu V_\nu D^\mu V_\nu - 2D_\mu V_\nu D^\nu V_\mu. \tag{E.13}$$

Simply replacing this last equation on (E.11) and adding the factor of $\frac{1}{2}$ yields (E)

$$\begin{aligned} -\frac{1}{2}\mathrm{Tr}\left\{F^{(4)}_{\mu\nu}F^{(4)\mu\nu}\right\} =& -\frac{1}{2}\mathrm{Tr}\left\{W_{\mu\nu}W^{\mu\nu}\right\} - \mathrm{Tr}\left\{D_\mu V_\nu D^\mu V^\nu\right\} + \mathrm{Tr}\left\{D_\mu V_\nu D^\nu V^\mu\right\} \\ & - \frac{g_L^2}{2}\mathrm{Tr}\left\{\left[V_\mu, V_\nu\right]\left[V^\mu, V^\nu\right]\right\} - ig_L \mathrm{Tr}\left\{W_{\mu\nu}\left[V^\mu, V^\nu\right]\right\}. \end{aligned} \tag{E.14}$$

This decomposition was only possible because the generators K_a do not form a simple subalgebra. *If they did*, they would be closed under Lie bracket

$$\left[K_a, K_b\right] \propto K_c. \tag{E.15}$$

They would also be orthogonal under the Lie bracket to the generators J_a

$$\left[K_a, J_b\right] = 0. \tag{E.16}$$

The reader can verify that, if these two conditions were satisfied, then the dynamic term (E.14) would be completely reduced to

$$-\frac{1}{2}\mathrm{Tr}\left\{F^{(4)}_{\mu\nu}F^{(4)\mu\nu}\right\} = -\frac{1}{2}\mathrm{Tr}\left\{W_{\mu\nu}W^{\mu\nu}\right\} - \frac{1}{2}\mathrm{Tr}\left\{\mathcal{V}_{\mu\nu}\mathcal{V}^{\mu\nu}\right\} \tag{E.17}$$

with this *hypothetical* $K-$subalgebra field strength tensor

$$\mathcal{V}_{\mu\nu} := \partial_\mu V_\nu - \partial_\nu V_\mu - ig_L\left[V_\mu, V_\nu\right]. \tag{E.18}$$

APPENDIX

F

CONDITIONS OVER THE EXTREMA OF THE SCALAR POTENTIAL

This appendix aims to explain how demanding the point $\left(v_0/\sqrt{2}\,,\, u_1\,,\, u_2\right)$ (i.e. $h_0 = h_1 = h_2 = 0$) to be an extremum of the potential has equations (4.52) and (4.53) as consequences. First, let us write the potential (4.8) as

$$V_{12}\left(\Phi,\phi_1,\phi_2\right) = -\mu_0^2 \operatorname{tr}\left\{\Phi^\dagger\Phi\right\} - \lambda_0 \operatorname{tr}\left\{\Phi^\dagger\Phi\right\}^2 - \mu_1^2\phi_1^\dagger\phi_1 - \mu_2^2\phi_2^\dagger\phi_2 + \lambda\left(\phi_k^\dagger\phi_k\right)^2 - \lambda_a\left(\phi_1^\dagger\phi_1\right)\left(\phi_2^\dagger\phi_2\right) - \lambda_b \operatorname{tr}\left\{\Phi^\dagger\Phi\right\}\phi_k^\dagger\phi_k - \lambda_c\left(\phi_2^\dagger\Phi\phi_1 + \phi_1^\dagger\Phi^\dagger\phi_2\right). \tag{F.1}$$

Here, we have considered the quadratic terms $\phi_1^\dagger\phi_1$ and $\phi_2^\dagger\phi_2$ to have differently named coefficients μ_1^2 and μ_2^2. We know the $1 \leftrightarrow 2$ exchange symmetry demands these two constants to be equal, but let us ignore that for now. Once the fields are replace, we will obtained a polynomial over the different fields. Let us name this polynomial differently from (4.50) just to emphasize that we have not considered μ_1^2 and μ_2^2 to be equal.

$$P_{12}(h_0,h_1,h_2,\phi^+,\phi^-,w) \tag{F.2}$$

We ask an extremum to be situated at $h_0 = h_k = 0$ ($k = 1,2$). Therefore, we demand the derivatives (i.e. the gradient) to vanish at this point

$$\left.\frac{\partial P_{12}}{\partial h_0}\right|_{h_0=h_k=0} = 0 \quad\wedge\quad \left.\frac{\partial P_{12}}{\partial h_1}\right|_{h_0=h_k=0} = 0 \quad\wedge\quad \left.\frac{\partial P_{12}}{\partial h_2}\right|_{h_0=h_k=0} = 0 \tag{F.3}$$

This three expressions form a system of equations over μ^2 and μ_k^2, since the evaluated derivatives are themselves polynomials over these variables. The reader can verify that solving for μ_0^2, μ_1^2

and μ_2^2 yields

$$\begin{aligned}
\mu_0^2 &= -\frac{8v_0^3\lambda_0 + 2v_0(u_1^2+u_2^2)\lambda_b + \sqrt{2}u_1u_2\lambda_c}{4v_0} \\
\mu_1^2 &= -\frac{2v_0^2u_1\lambda_b + \sqrt{2}v_0u_2\lambda_c + 2u_1^3\lambda + u_1u_2^2\lambda_a}{2u_1} \\
\mu_2^2 &= -\frac{2v_0^2u_2\lambda_b + \sqrt{2}v_0u_1\lambda_c + 2u_2^3\lambda + u_2u_1^2\lambda_a}{2u_2}
\end{aligned} \tag{F.4}$$

Now, we set $\mu_1^2 = \mu_2^2$ and solve for u_1. Doing so will yield three possible relations between u_1 and u_2:

$$\begin{aligned}
u_1 &= u_2 \\
u_1 &= -u_2 \\
u_1 &= \frac{\sqrt{2}\lambda_c v_0}{2\lambda - \lambda_a}\frac{1}{u_2}.
\end{aligned} \tag{F.5}$$

Choosing the first relation ((4.52)) is required by construction. Applying this condition to (F.4) gives (4.53).

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