



Numerical Methods for Stochastic Functional Differential Equations Driven by a Fractional Brownian Motion

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A thesis submitted for the degree of Doctor of Philosophy in Mathematics

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Resumen

En esta tesis se estudia la aproximación numérica de una clase de ecuaciones diferenciales funcionales estocásticas dirigidas por un movimiento Browniano fraccionario con parámetro de Hurst $H > 1/2$, en las cuales el efecto de memoria es descrito mediante un término integral con núcleo. Debido a la naturaleza del ruido fraccionario, las integrales estocásticas involucradas son interpretadas en el sentido de la integral de Riemann–Stieltjes generalizada, definida mediante herramientas de cálculo fraccionario.

En primer lugar, se presentan los resultados preliminares necesarios para el desarrollo del trabajo, incluyendo algunas propiedades fundamentales del movimiento Browniano fraccionario, elementos de cálculo fraccionario y la teoría básica de las ecuaciones diferenciales funcionales estocásticas consideradas. Asimismo, se realiza una revisión del estado del arte sobre métodos numéricos para ecuaciones diferenciales estocásticas y ecuaciones diferenciales funcionales estocásticas dirigidas tanto por movimiento Browniano estándar como por movimiento Browniano fraccionario, con especial énfasis en los esquemas de Euler, backward Euler, θ -Maruyama y Milstein.

La primera contribución original de esta tesis consiste en la construcción y análisis de un esquema de Euler explícito para la aproximación de la solución de las ecuaciones estudiadas. Bajo hipótesis adecuadas sobre los coeficientes y el término de memoria, se demuestra la consistencia del método y se establece su convergencia hacia la solución exacta. Estos resultados constituyen la base de un artículo de investigación actualmente sometido para publicación.

La segunda contribución corresponde al desarrollo y análisis de un esquema implícito de backward Euler para la misma clase de ecuaciones. Se demuestra que las aproximaciones obtenidas mediante este método son consistentes y convergen hacia la solución exacta con el mismo orden de convergencia que el esquema explícito. Estos resultados amplían la teoría existente sobre métodos implícitos para ecuaciones con retardo impulsadas por movimiento Browniano fraccionario y forman parte de un trabajo actualmente en preparación.

Finalmente, se presentan diversas simulaciones numéricas que permiten ilustrar el comportamiento de ambos esquemas y visualizar la aproximación de las trayectorias numéricas a la solución exacta. Los resultados obtenidos muestran que tanto el método de Euler como el método backward Euler constituyen herramientas adecuadas para la aproximación de ecuaciones diferenciales funcionales estocásticas dirigidas por movimiento Browniano fraccionario, y abren la puerta al estudio de métodos de orden superior y de problemas más generales.

Abstract

This thesis deals with the numerical approximation of a class of stochastic functional differential equations driven by a fractional Brownian motion with Hurst parameter $H > 1/2$, where the memory effect is described through an integral kernel. Owing to the fractional nature of the driving noise, the stochastic integrals involved are interpreted pathwise by means of the generalized Riemann–Stieltjes integral, which is defined using tools from fractional calculus.

First, the mathematical background required throughout the thesis is presented, including some fundamental properties of fractional Brownian motion, elements of fractional calculus, and the basic theory of the stochastic functional differential equations under consideration. A review of the existing literature on numerical methods for stochastic differential equations and stochastic functional differential equations driven by both standard and fractional Brownian motions is also provided, with particular emphasis on Euler, backward Euler, θ -Maruyama, and Milstein-type schemes.

The first original contribution of this thesis consists in the construction and analysis of an explicit Euler scheme for approximating the solution of the equations under study. Under suitable assumptions on the coefficients and on the memory term, consistency and convergence of the method are established. These results constitute the basis of a research article currently submitted for publication.

The second contribution concerns the development and analysis of an implicit backward Euler scheme for the same class of equations. It is shown that the approximations produced by this method are consistent and converge to the exact solution with the same order of convergence as the explicit Euler scheme. These results extend the existing theory of implicit methods for delay equations driven by fractional Brownian motion and constitute part of a work currently in preparation.

Finally, several numerical simulations are presented to illustrate the behavior of both schemes and to visualize the approximation of the numerical trajectories to the exact solution. The results obtained show that both the Euler and backward Euler methods provide effective tools for approximating stochastic functional differential equations driven by fractional Brownian motion and open the way to the study of higher-order methods and more general problems.

Keywords: Fractional Brownian motion; stochastic functional differential equations; delay equations; numerical methods; Euler method; backward Euler method; convergence analysis.

INTRODUCTION

Stochastic differential equations with delay or memory have been widely used to model dynamical systems whose future evolution depends not only on the present state but also on their past history. Such equations arise naturally in several areas, including mathematical biology, economics, engineering, and physics, where memory effects play a fundamental role (see, for example, [17, 15]). In particular, stochastic functional differential equations with memory allow us to describe systems with long-range historical dependence, providing a more realistic framework for modeling persistent phenomena.

From a mathematical perspective, the presence of memory introduces a non-Markovian structure that significantly complicates both the analysis of the solutions and the design of numerical approximation schemes, since the evolution depends on the entire past trajectory.

In the classical setting, randomness is typically modeled by standard Brownian motion, leading to Markovian stochastic differential equations and well-developed theory from both analytical and numerical perspectives [12]. However, empirical evidence in many applications reveals the presence of long-range dependence and temporal correlations that cannot be adequately captured by classical Brownian noise. In this context, fractional Brownian motion (fBm), introduced by Mandelbrot and Van Ness [14], has emerged as a fundamental tool for modeling processes with long-range and memory dependence.

The fractional Brownian motion is a centered Gaussian process characterized by a Hurst parameter $H \in (0, 1)$, whose sample paths are Hölder continuous of any order strictly less than H . In the regime $H > 1/2$, the process exhibits positively correlated increments, making it particularly suitable for modeling persistent behavior [14, 2]. Nevertheless, when $H \neq 1/2$, the fractional Brownian motion is not a semimartingale, which prevents the direct application of the calculus Itô and introduces substantial analytical and numerical challenges.

For stochastic differential equations driven by fractional Brownian motion with $H > 1/2$, a widely used approach consists in defining the stochastic integral in the sense of generalized Lebesgue-Stieltjes or Young integrals, exploiting the Hölder regularity of the trajectories [23, 24]. This framework has allowed the establishment of uniqueness and existence results for various classes of stochastic differential equations, including functional equations with memory and delay, under suitable regularity assumptions for the coefficients [8, 3].

In this thesis, we study a class of stochastic functional differential equations with distributed memory driven by fractional Brownian motion. The evolution of the system depends not only on its present state but also on a functional of its past trajectory over a finite delay interval. This hereditary structure is modeled through a distributed delay term, allowing us to capture

finite-memory effects within a non-Markovian stochastic framework. The precise mathematical formulation of the models are presented in Chapters 3 and 4.

The combination of fractional noise and distributed delay creates a challenging setting for numerical approximation, since temporal roughness and non-local memory effects must be controlled simultaneously.

Equations of the form (4.1)–(4.2), involving distributed memory terms, have been previously studied in the context of stochastic functional differential equations driven by standard Brownian motion. In particular, Buckwar [5], [6] investigated stochastic functional differential equations with distributed delay and proposed the θ -Maruyama scheme for their numerical approximation. Strong convergence results were established under suitable Lipschitz-type conditions on the coefficients, highlighting the impact of the memory term on both the stability and convergence properties of the numerical method.

From a numerical point of view, the approximation of solutions to stochastic differential equations driven by fractional Brownian motion has attracted increasing attention over the past decades. For equations without delay, several Euler- and Milstein-type schemes have been proposed and analyzed, producing strong convergence results and explicit convergence rates under different assumptions on coefficients and the Hurst parameter [18, 11]. In particular, in the case $H > 1/2$, the convergence of the Euler scheme was systematically studied in [16], who proved strong convergence under appropriate regularity conditions using Riemann-Stieltjes integration and Hölder-type estimates. Later, Neuenkirch and Nourdin obtained further refinements and optimal convergence rates [18].

Despite these advances, most of the available numerical results concern stochastic differential equations without memory or, at most, with finite delay. The numerical analysis of stochastic functional differential equations with hereditary structures driven by fractional Brownian motion remains comparatively underdeveloped. The presence of memory significantly complicates both the analytical and numerical treatment of such systems, since the evolution of the solution depends on its past trajectory rather than on its current state alone. Existing work in this area has focused mainly on qualitative properties of solutions, such as existence, uniqueness, and stability, whereas rigorous convergence analyses of numerical schemes and the derivation of explicit convergence rates have received less attention [7].

The main objective of this thesis is to develop and analyze numerical methods for approximating the solutions of stochastic functional differential equations with distributed delay driven by a fractional Brownian motion with Hurst parameter $H > 1/2$. The presence of both hereditary effects and long-range dependence substantially increases the mathematical complexity of the problem, making the extension of classical numerical techniques far from straightforward. Consequently, establishing rigorous approximation results for this class of equations requires combining tools from fractional stochastic calculus, functional differential equations, and numerical analysis.

The original contributions of this thesis are twofold. The first contribution consists in the construction and convergence analysis of an explicit Euler scheme for stochastic functional differential equations driven by a fractional Brownian motion. Under suitable regularity assumptions on the drift, diffusion, and memory kernel, the proposed approximation is shown to converge to the exact solution by means of estimates based on fractional calculus and the generalized Riemann–Stieltjes integral. This result extends existing convergence analyses for stochastic differential equations without memory to a broader class of stochastic functional differential equations with distributed delay. The mathematical developments corresponding to this contribution are presented in Chapter 3 and constitute the basis of a research article that has been submitted for publication.

The second contribution of this thesis is the formulation and analysis of a backward Euler discretization for the same class of stochastic functional differential equations. Although implicit numerical methods have been extensively investigated for stochastic differential equations driven by the standard Brownian motion, analogous results for equations with memory driven by a fractional Brownian motion are still very limited. In this work, a backward Euler approximation is proposed and its local consistency is established under appropriate assumptions on the coefficients and the memory functional. This analysis, developed in Chapter 4, provides a first theoretical step toward a complete convergence and stability theory for implicit methods in this setting and forms the basis of ongoing research currently under preparation.

The remainder of the thesis is organized as follows. Chapter 1 provides the mathematical background required throughout the manuscript. In particular, it recalls some fundamental properties of fractional Brownian motion, introduces the tools from fractional calculus needed to define the generalized Riemann–Stieltjes integral, and presents the class of stochastic functional differential equations considered in this work. Chapter 2 is devoted to a review of the existing literature on numerical methods for stochastic differential equations and stochastic functional differential equations driven by both standard and fractional Brownian motions, with particular emphasis on Euler-type methods, backward Euler schemes, the θ -Maruyama method, and Milstein-type approximations.

Chapter 3 develops the explicit Euler approximation for the stochastic functional differential equation under consideration and establishes its consistency and convergence towards the exact solution. Chapter 4 is devoted to the corresponding backward Euler scheme, where analogous convergence results are obtained in the implicit setting. Chapter 5 presents several numerical simulations illustrating the performance of both methods and providing a visual comparison between the numerical approximations and the exact solution whenever it is available. Finally, the appendices collect a number of auxiliary results, technical lemmas, propositions, corollaries, and detailed proofs that support the theoretical developments presented throughout the thesis.

CAPÍTULO 1

PRELIMINARIES

This chapter introduces the mathematical framework required for the analysis of stochastic functional differential equations driven by fractional Brownian motion. We recall the definition and main properties of fractional Brownian motion, present the fractional integration theory introduced by Zähle, discuss Hölder regularity, and review stochastic differential equations and stochastic functional differential equations. These concepts form the theoretical foundation for the results developed in the subsequent chapters.

1.1. Fractional Brownian motion

Fractional Brownian motion was first introduced in the Hilbert space setting by Kolmogorov in 1940 [13], where it was referred to as the **Wiener helix**. It was later studied in detail by Yaglom [22]. The name *fractional Brownian motion* was introduced by Mandelbrot and Van Ness, who in 1968 provided a stochastic integral representation of this process in terms of standard Brownian motion [14].

1.1.1. Definition and basic properties

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. A fractional Brownian motion (fBm) with Hurst parameter $H \in (0, 1)$ is a centered continuous Gaussian process $(B^H(t))_{t \geq 0}$ with covariance function

$$\mathbb{E}[B^H(t)B^H(s)] = \frac{1}{2} (t^{2H} + s^{2H} - |t - s|^{2H}). \quad (1.1)$$

The Hurst parameter H determines both the regularity of the sample paths and the dependence structure of the process.

An equivalent construction of fractional Brownian motion is given by the Mandelbrot–Van Ness representation:

$$B^H(t) = \frac{1}{C_H} \left(\int_{-\infty}^0 [(t-s)^{H-\frac{1}{2}} - (-s)^{H-\frac{1}{2}}] dB(s) + \int_0^t (t-s)^{H-\frac{1}{2}} dB(s) \right), \quad (1.2)$$

where B denotes a two-sided standard Brownian motion and C_H is a normalization constant depending on H .

The following properties are fundamental:

- $B^H(0) = 0$ and $\mathbb{E}[B^H(t)] = 0$ for all $t \geq 0$.
- B^H has stationary increments, i.e.,

$$B^H(t+s) - B^H(s) \stackrel{d}{=} B^H(t), \quad \forall s, t \geq 0.$$

- The variance satisfies

$$\mathbb{E}[B^H(t)^2] = t^{2H}, \quad \forall t \geq 0.$$

When $H = 1/2$, fractional Brownian motion coincides with standard Brownian motion and therefore has independent increments. However, when $H \neq 1/2$, the increments are no longer independent. Indeed, consider the increments $B^H(t+h) - B^H(t)$ and $B^H(s+h) - B^H(s)$ with $s+h \leq t$ and $t-s = nh$. Their covariance is given by

$$\rho_H(n) = \frac{1}{2} h^{2H} [(n+1)^{2H} + (n-1)^{2H} - 2n^{2H}]. \quad (1.3)$$

Consequently,

- when $H > 1/2$, the increments are positively correlated (persistent behavior),
- when $H < 1/2$, the increments are negatively correlated (anti-persistent behavior).

Processes exhibiting the first property model systems with memory and persistence, whereas those exhibiting the second property describe systems with anti-persistence.

We now introduce the notion of long-range dependence.

Definition 1.1.1. A stationary sequence $(X_n)_{n \in \mathbb{N}}$ exhibits long-range dependence if its autocovariance function $\rho(n) = \text{Cov}(X_k, X_{k+n})$ satisfies

$$\lim_{n \rightarrow \infty} \frac{\rho(n)}{cn^{-\alpha}} = 1,$$

for some constant $c > 0$ and $\alpha \in (0, 1)$. In this case,

$$\sum_{n=1}^{\infty} \rho(n) = \infty.$$

If $H > 1/2$, fractional Brownian motion exhibits long-range dependence. Indeed, defining the increments

$$X_k = B^H(k) - B^H(k-1),$$

one obtains

$$\rho_H(n) = \frac{1}{2} [(n+1)^{2H} + (n-1)^{2H} - 2n^{2H}].$$

A Taylor expansion shows that

$$\rho_H(n) \sim H(2H-1)n^{2H-2}, \quad n \rightarrow \infty.$$

Thus,

$$\sum_{n=1}^{\infty} |\rho_H(n)| = \begin{cases} \infty, & H > 1/2, \\ < \infty, & H < 1/2. \end{cases}$$

Therefore, fractional Brownian motion exhibits long-range dependence when $H > 1/2$ and short-range dependence when $H < 1/2$.

Another fundamental property of fractional Brownian motion is self-similarity.

Definition 1.1.2. A stochastic process $(X_t)_{t \geq 0}$ is said to be self-similar with index H if for every $a > 0$,

$$(X_{at})_{t \geq 0} \stackrel{d}{=} (a^H X_t)_{t \geq 0}.$$

Theorem 1.1.1. Fractional Brownian motion is self-similar with index H .

Proof. Using the covariance function,

$$\mathbb{E}[B^H(at)B^H(as)] = a^{2H} \mathbb{E}[B^H(t)B^H(s)].$$

Since both processes are centered Gaussian, the result follows.

Fractional Brownian motion has almost surely nowhere differentiable sample paths.

Moreover, fractional Brownian motion is not a semimartingale unless $H = 1/2$. This follows from its p -variation, whose index equals $1/H$. Since semimartingales have p -variation index equal to 2, fractional Brownian motion is a semimartingale only when $H = 1/2$. For more details see [2].

1.1.2. Hölder regularity

Definition 1.1.3. A function $f : [0, T] \rightarrow \mathbb{R}$ is said to be Hölder continuous of order $\gamma \in (0, 1)$ if there exists $C > 0$ such that

$$|f(t) - f(s)| \leq C|t - s|^\gamma.$$

The space of Hölder continuous functions is denoted by $C^\gamma([0, T])$.

A fundamental regularity property of fractional Brownian motion is given by the following theorem.

Theorem 1.1.2. Let $H \in (0, 1)$. Then fractional Brownian motion admits a version whose sample paths are almost surely Hölder continuous of order γ for every $\gamma < H$.

Proof. Using self-similarity,

$$\mathbb{E}|B^H(t) - B^H(s)|^\alpha = C|t - s|^{\alpha H}.$$

The Kolmogorov continuity theorem yields the result.

This Hölder regularity plays a fundamental role in defining stochastic integrals with respect to fractional Brownian motion for $H > 1/2$.

1.2. Fractional calculus and the Zähle integral

Since fractional Brownian motion is not a semimartingale when $H \neq 1/2$, Itô calculus cannot be used. Instead, stochastic integrals can be defined pathwise using fractional calculus techniques introduced by Feyel and de la Pradelle [9] and further developed by Zähle [24].

1.2.1. Fractional Calculus

Let $a, b \in \mathbb{R}$ with $a < b$. For $f \in L^1([a, b])$ and $\alpha > 0$, the left-sided and right-sided fractional Riemann-Liouville integrals of f of order α are defined for almost all $t \in (a, b)$ by

$$I_{a+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds, \quad I_{b-}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} f(s) ds.$$

respectively, where $(-1)^{-\alpha} = e^{-\pi\alpha i}$ and Γ is the gamma function. When $\alpha = k \in \mathbb{N}$, $I_{a+}^{\alpha} f$ is the k -th iterated integral.

Fractional derivatives are defined as inverse operators of fractional integrals. Let $0 < \alpha < 1$ and $p \geq 1$. For $f \in I_{a+}^{\alpha}(L^p([a, b]))$, the Weyl derivative is defined as

$$D_{a+}^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(t)}{(t-a)^{\alpha}} + \alpha \int_a^t \frac{f(t) - f(s)}{(t-s)^{\alpha+1}} ds \right),$$

and the corresponding Weyl derivative for $f \in I_{b-}^{\alpha}(L^p)$ is defined as

$$D_{b-}^{\alpha} f(t) = \frac{(-1)^{\alpha}}{\Gamma(1-\alpha)} \left(\frac{f(t)}{(b-t)^{\alpha}} + \alpha \int_t^b \frac{f(t) - f(s)}{(s-t)^{\alpha+1}} ds \right).$$

Both $D_{a+}^{\alpha} f(t)$ and $D_{b-}^{\alpha} f(t)$ are defined for almost all $t \in (a, b)$. The convergence of the integrals at the singularity $s = t$ holds pointwise for almost all $t \in (a, b)$ if $p = 1$ and L^p -sense if $p > 1$.

1.2.2. Zähle integral

Let's define

$$f_{a+}(t) = (f(t) - f(a+))1_{(a,b)}(t), \quad g_{b-}(t) = (g(t) - g(b-))1_{(a,b)}(t),$$

where $f(a+) = \lim_{\delta \rightarrow 0+} f(a + \delta)$ and $g(b-) = \lim_{\delta \rightarrow 0+} g(b - \delta)$.

Definition 1.2.1. Suppose that f and g are functions such that $f(a+), g(a+)$ and $g(b-)$ exist, $f_{a+} \in I_{a+}^{\alpha}(L^p)$ and $g_{b-} \in I_{b-}^{1-\alpha}(L^q)$ for some $p, q \geq 1$ with $1/p + 1/q \leq 1$, $\alpha \in (0, 1)$. The generalized Riemann-Stieltjes integral of f with respect to g is defined by

$$\int_a^b f dg = (-1)^{\alpha} \int_a^b D_{a+}^{\alpha} f_{a+}(t) D_{b-}^{1-\alpha} g_{b-}(t) dt + f(a+)(g(b-) - g(a+)).$$

Proposition 1.2.1. If $\alpha p < 1$, from the above definition we get

$$\int_a^b f dg = (-1)^{\alpha} \int_a^b D_{a+}^{\alpha} f(t) D_{b-}^{1-\alpha} g(t) dt.$$

for general functions $f \in I_{a+}^{\alpha}(L^p)$ and $g_{b-} \in I_{b-}^{1-\alpha}(L^q)$.

If $f \in C^{\lambda}([a, b])$ and $g \in C^{\mu}([a, b])$ with $\lambda + \mu > 1$, it is proved in Zähle's paper [24] that the conditions of the definition of generalized Riemann-Stieltjes integral and the proposition 1.2.1 are satisfied and it is possible to choose $p = q = \infty$ and $\alpha < \lambda, 1 - \alpha < \mu$. Furthermore, the integral $\int_a^b f dg$ exists and coincides with the Riemann-Stieltjes integral. Also, Zähle's paper gives the explicit expression in terms of the fractional Weyl derivatives.

Proposition 1.2.2. *Let $f \in C^\lambda([a, b])$, $g \in C^\mu([a, b])$ with $\lambda + \mu > 1$ and $\alpha < \lambda, 1 - \alpha < \mu$.*

$$\int_a^b f dg = (-1)^\alpha \int_a^b D_{a+}^\alpha f(t) D_{b-}^{1-\alpha} g_{b-}(t) dt.$$

where $g_{b-} = g(t) - g(b)$. Furthermore, we have

$$\left| \int_a^b f dg \right| \leq C(\omega) \left(\int_a^b \frac{|f(s)|}{(s-a)^\alpha} ds + \int_a^b \int_a^s \frac{|f(s) - f(u)|}{(s-u)^{\alpha+1}} du ds \right)$$

where

$$C(\omega) := \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \sup_{a < s < t < sb} \left(\frac{|g(t) - g(s)|}{(t-s)^{1-\alpha}} + \int_a^b \frac{|g(u) - g(s)|}{(u-s)^{2-\alpha}} du \right) < \infty.$$

The proof of this proposition can be found in [19].

Thus, when $B^H(t)$ is a fractional Brownian motion with parameter Hurst $H \in (1/2, 1)$, the pathwise integral is defined as

$$\int_a^b f dB^H = \int_a^b D_{a+}^\alpha f(t) D_{b-}^{1-\alpha} B_{b-}^H(t) dt.$$

where

$$D_{b-}^{1-\alpha} B_{b-}^H(t) = \frac{(-1)^\alpha}{\Gamma(\alpha)} \left(\frac{B_{b-}^H(t)}{(b-t)^{1-\alpha}} + (1-\alpha) \int_t^b \frac{B_{b-}^H(t) - B_{b-}^H(s)}{(s-t)^{2-\alpha}} ds \right).$$

and $B_{b-}^H = B^H(t) - B^H(b)$. The integral exists for any $\alpha \in (1-H, \nu)$ if, for example, $f \in C^\nu(a, b)$ with $\nu + H > 1$. Moreover, in this case the pathwise integral admits an estimate

$$\left| \int_a^b f dB^H \right| \leq C(\omega) \left(\int_a^b \frac{|f(s)|}{(s-a)^\alpha} ds + \int_a^b \int_a^s \frac{|f(s) - f(u)|}{(s-u)^{\alpha+1}} du ds \right) \quad (1.4)$$

where

$$C(\omega) = C \sup_{a < t < b} |D_{b-}^{1-\alpha} B_{b-}^H(t)| < \infty \text{ a.s.} \quad (1.5)$$

1.3. Stochastic differential equations (SDE) and stochastic functional differential equations (SFDE)

1.3.1. SDE driven by fractional Brownian motion

Fix a time interval $[0, T]$ and a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Suppose that $B^H = \{B^H(t), t \in [0, T]\}$ is a fractional Brownian motion with Hurst parameter $H \in (1/2, 1)$. Consider the stochastic differential equation

$$X(t) = X(0) + \int_0^t b(s, X(s)) ds + \int_0^t \sigma(s, X(s)) dB^H(s), \quad t \geq 0. \quad (1.6)$$

Let's consider the following assumptions on the coefficients.

(H.b) There exist constants L_1 and L_2 such that for all $x, y \in \mathbb{R}$ and $s, t \in [0, T]$

$$\begin{aligned} |b(t, x) - b(t, y)| &\leq L_1 |x - y| \\ |b(t, x)| &\leq L_2 (1 + |x|). \end{aligned}$$

(H. σ) The function $\sigma(t, \xi)$ is differentiable in the variable x . Moreover, there exist constants L_3, L_4, L_5 and L_6 such that for all $\xi, \eta \in C([-r, 0], \mathbb{R})$ and $s, t \in [0, T]$

$$\begin{aligned} |\sigma(t, x) - \sigma(t, y)| &\leq L_3|x - y| \\ |\sigma_x(t, x) - \sigma_y(t, y)| &\leq L_4|x - y| \\ |\sigma(t, x) - \sigma(s, x)| + |\sigma_x(t, x) - \sigma_x(s, x)| &\leq L_5|t - s| \\ |\sigma(t, x)| &\leq L_6(1 + |x|). \end{aligned}$$

The following theorem provide the existence and uniqueness of the solution of equation (1.6).

Theorem 1.3.1. *Suppose that $X(0)$ is a random variable, the coefficients b, σ satisfy assumptions (H.b) and (H. σ). If α satisfies $1 - H < \alpha < H$, then there exists a unique stochastic process X solution of the equation (1.6) with paths in $C^{1-\alpha}([0, T])$, $\mathbb{P}$ -a.s.*

The proof of this theorem can be found in [19].

1.3.2. Stochastic functional differential equations

Fix a time interval $[0, T]$ and a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Suppose that $B^H = \{B^H(t), t \in [0, T]\}$ is a fractional Brownian motion with Hurst parameter $H \in (1/2, 1)$. Consider the stochastic functional differential equation

$$\begin{aligned} X(t) &= \phi(0) + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dB^H(s), \quad t \geq 0 \\ X_0 &= \phi \in C([-r, 0], \mathbb{R}). \end{aligned} \tag{1.7}$$

Here $X_s \in C([-r, 0], \mathbb{R})$ denote the function defined by $X_s(u) = X(s + u)$, $\forall u \in [-r, 0]$. Let's consider the following assumptions on the coefficients.

(H.b) The function $b(t, \xi)$ is continuous. Moreover, there exist constants L_1 and L_2 such that for all $\xi, \eta \in C([-r, 0], \mathbb{R})$ and $t \in [0, T]$

$$\begin{aligned} |b(t, \xi) - b(t, \eta)| &\leq L_1\|\xi - \eta\| \\ |b(t, \xi)| &\leq L_2(1 + \|\xi\|). \end{aligned}$$

(H. σ) The function $\sigma(t, \xi)$ is continuous and Fréchet differentiable in the variable ξ . Moreover, there exist constants L_3, L_4 and L_5 such that for all $\xi, \eta \in C([-r, 0], \mathbb{R})$ and $s, t \in [0, T]$

$$\begin{aligned} |D_\xi \sigma(t, \xi)|_{L(C([-r, 0], \mathbb{R}), \mathbb{R})} &\leq L_3 \\ |D_\xi \sigma(t, \xi) - D_\xi \sigma(t, \eta)|_{L(C([-r, 0], \mathbb{R}), \mathbb{R})} &\leq L_4\|\xi - \eta\| \\ |\sigma(t, \xi) - \sigma(s, \xi)| + |D_\xi \sigma(t, \xi) - D_\xi \sigma(s, \xi)|_{L(C([-r, 0], \mathbb{R}), \mathbb{R})} &\leq L_5|t - s|. \end{aligned}$$

The following theorem provide the uniqueness, existence and dependence of the solution of equation (1.7) on the initial condition.

Theorem 1.3.2. *Assume that the coefficients b, σ satisfy the assumptions (H.b) and (H. σ). If α satisfies $1 - H < \alpha < H$ and if ϕ is a stochastic process whose trajectories belong to the space $C^{1-\alpha}([-r, 0])$, almost surely. Then there exists a unique solution X of equation (1.7) with paths in $C^{1-\alpha}([-r, T])$, $\mathbb{P}$ -a.s.*

The proof of this theorem can be found in [3].

CAPÍTULO 2

NUMERICAL METHODS FOR STOCHASTIC DIFFERENTIAL EQUATIONS DRIVEN BY FRACTIONAL BROWNIAN MOTION

The numerical approximation of stochastic differential equations (SDEs) and stochastic functional differential equations (SFDEs) has received considerable attention over the last decades due to its relevance in physics, biology, finance, engineering, and applied sciences. In many situations, explicit solutions are not available, making numerical schemes essential tools for the qualitative and quantitative analysis of stochastic systems.

When the stochastic perturbation is generated by a standard Brownian motion, classical Itô calculus provides a natural framework for the development of numerical methods such as Euler–Maruyama, Milstein, and θ -Maruyama schemes. However, when the driving noise is a fractional Brownian motion (fBm), the problem becomes substantially more delicate because the process is not a semimartingale whenever $H \neq \frac{1}{2}$. Consequently, the classical Itô theory is no longer applicable and alternative integration frameworks, such as fractional calculus, rough paths theory, and Malliavin calculus, must be employed.

This chapter presents a review of the main numerical methods developed for stochastic differential equations and stochastic functional differential equations driven by fractional Brownian motion. Special attention is devoted to Euler-type methods, backward Euler methods, θ -Maruyama schemes, and Milstein-type approximations, emphasizing their convergence properties and stability behavior. We also discuss several important contributions in the literature that motivated the development of the numerical analysis carried out in the subsequent chapters of this thesis.

2.1. Euler-Type Methods for SDEs Driven by Fractional Brownian Motion

One of the first systematic studies concerning Euler approximations for stochastic differential equations driven by fractional Brownian motion was carried out by Mishura and Shevchenko. In [16], the authors analyzed the convergence rate of Euler approximations for stochastic differential equations driven by a fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$. They

considered equations of the form

$$X(t) = X_0 + \int_0^t b(s, X(s)) ds + \int_0^t \sigma(s, X(s)) dB^H(s),$$

where the stochastic integral is interpreted pathwise in the sense of fractional calculus.

The corresponding Euler approximation is given by

$$X_{n+1} = X_n + b(t_n, X_n)h + \sigma(t_n, X_n)\Delta B_n^H,$$

where

$$\Delta B_n^H = B^H(t_{n+1}) - B^H(t_n).$$

Using Hölder regularity properties of the trajectories of the fractional Brownian motion and estimates derived from fractional calculus, the authors proved that the Euler approximation converges with rate

$$O(h^{2H-1}),$$

for pathwise solutions. This result established one of the foundational convergence analyses for numerical methods associated with fractional stochastic differential equations.

Later, Mishura and Shevchenko extended these ideas to mixed stochastic differential equations involving both standard Brownian motion and fractional Brownian motion. In [21], they studied the equation

$$X_t = X_0 + \int_0^t a(s, X_s) ds + \int_0^t b(s, X_s) dW_s + \int_0^t c(s, X_s) dB_s^H,$$

and established mean-square convergence rates for the Euler approximation scheme. Their work showed that Euler-type methods remain effective even in the presence of mixed Gaussian noises.

2.2. Backward Euler Methods

Implicit numerical methods play an important role in the approximation of stochastic differential equations because of their enhanced stability properties, especially in the presence of stiff coefficients or nonlinear drift terms. Among these methods, the backward Euler scheme constitutes one of the most classical implicit discretizations.

For stochastic differential equations driven by fractional Brownian motion, implicit Euler methods have been investigated in several contexts. In particular, Zhang and Yuan studied in [25] stochastic differential equations with locally Lipschitz drift driven by fractional Brownian motion and analyzed the corresponding implicit Euler approximation. They considered equations of the form

$$X(t) = X_0 + \int_0^t b(s, X(s))ds + \int_0^t \sigma dB^H(s), \quad X(0) \geq 0,$$

where the stochastic integral is interpreted pathwise in the sense of fractional calculus.

The corresponding backward-Euler approximation is given by

$$X_{n+1} = X_n + b(t_{n+1}, X_{n+1})h + \sigma\Delta B_n^H,$$

where

$$\Delta B_n^H = B^H(t_{n+1}) - B^H(t_n).$$

They established existence, uniqueness, positivity, and convergence properties of the numerical scheme under suitable assumptions on the coefficients.

The implicit nature of the backward Euler method makes it particularly suitable for equations with memory effects and delayed interactions, since these systems may exhibit strong stability constraints. For this reason, backward Euler-type methods have become natural candidates for the approximation of stochastic functional differential equations driven by fractional Brownian motion.

2.3. The θ -Maruyama Method

The θ -Maruyama method generalizes the Euler-Maruyama and backward Euler schemes through the introduction of a parameter $\theta \in [0, 1]$. The numerical analysis of θ -Maruyama schemes for stochastic delay differential equations was extensively developed by Evelyn Buckwar. In [6], Buckwar investigated θ -Maruyama approximations for stochastic functional differential equations with distributed memory terms and driven by a standard Brownian motion. She considered equations of the form

$$\begin{aligned} X(t) &= X(0) + \int_0^t b(s, X(s), Y(s))ds + \int_0^t \sigma(s, X(s), Y(s))dB(s), \quad t \in [0, T], \\ X(t) &= \phi(t), \quad t \in [-r, 0]. \end{aligned}$$

Here, $Y(t)$ represents a memory term of the type

$$Y(t) = \int_{-r}^0 K(t, s, X(t+s))ds = \int_{t-r}^t K(t, s-t, X(s))ds.$$

where the stochastic integral is the Wiener integral.

The corresponding θ -Maruyama approximation is given by

$$\begin{aligned} \tilde{X}^h(t_{n+1}) &= \tilde{X}^h(t_n) + h(\theta b(t_{n+1}, \tilde{X}^h(t_{n+1}), \tilde{Y}(t_{n+1})) + (1-\theta)b(t_n, \tilde{X}^h(t_n), \tilde{Y}(t_n))) \\ &\quad + \sigma(t_n, \tilde{X}^h(t_n), \tilde{Y}(t_n))\Delta B_n, \end{aligned}$$

where

$$\Delta B_n = B(t_{n+1}) - B(t_n).$$

The expression $\tilde{Y}(t_m)$ provides an approximation to the integral $Y(t_m)$.

For $\theta = 0$, the scheme reduces to the Euler-Maruyama method, whereas $\theta = 1$ yields the backward-Euler method. The work established consistency and convergence properties of the method while emphasizing the role of quadrature formulas in the approximation of delay integrals. In addition, in [1], Halanay-type inequalities were employed to investigate exponential mean-square stability for stochastic delay differential equations discretized by the θ -Maruyama method. Their results demonstrated the strong connection between the choice of the parameter θ and the asymptotic behavior of the numerical approximation.

2.4. Milstein-Type Methods

Milstein-type methods provide higher-order approximations for stochastic differential equations by incorporating additional information related to iterated stochastic integrals. In the

classical Brownian framework, these schemes improve the strong convergence order of Euler-type methods.

For stochastic functional differential equations with memory terms, Buckwar analyzed Milstein approximations in [4]. In that work, Euler–Maruyama and Milstein schemes were studied for stochastic functional differential equations involving distributed delays. Strong convergence results and numerical simulations were established under suitable regularity assumptions.

The extension of Milstein-type methods to equations driven by fractional Brownian motion is substantially more involved due to the non-semimartingale nature of the noise. In this setting, the construction of higher-order schemes requires additional analytical tools, including rough path techniques and Malliavin calculus. Nevertheless, these methods continue to be an active area of research because of their improved convergence properties.

2.5. Numerical Methods for SFDEs Driven by Fractional Brownian Motion

Compared with the theory of stochastic differential equations driven by fractional Brownian motion, the numerical analysis of stochastic functional differential equations in the fractional framework is still relatively recent. The presence of memory terms introduces additional difficulties, since the approximation must simultaneously capture the Hölder regularity of the driving process and the delayed dependence structure of the equation.

Recent contributions have begun to address this problem. For example, Garzón, León, Lozada, and Torres proposed in [10] an Euler approximation scheme for stochastic functional differential equations driven by fractional Brownian motion using rough paths techniques and Young integration. They considered equations of the form

$$\begin{aligned} dX(t) &= f(\overline{X}_t)dB^H(t), \quad t \in [0, T], \\ \overline{X}(0) &= \psi. \end{aligned}$$

Here, $f : C^\gamma([-\tau, 0]; \mathbb{R}) \rightarrow \mathbb{R}$ and $\overline{X}_t : [-r, 0] \rightarrow \mathbb{R}$ is given by $\overline{X}_t(\theta) = X(t + \theta)$. They established convergence rates depending on the Hölder regularity of the solution and provided numerical simulations illustrating the theoretical results.

These works constitute the main theoretical background upon which the numerical analysis developed in this thesis is built since it lies at the intersection of the two aforementioned directions: the fractional-noise framework studied by Mishura for SDEs driven by fractional Brownian motion and the functional (delay) framework studied by Buckwar for equations driven by standard Brownian motion. The results presented in the following chapters contribute to this line of research by establishing convergence results for Euler and backward-Euler numerical schemes applied to stochastic functional differential equations driven by fractional Brownian motion with Hurst parameter $H > 1/2$.

CAPÍTULO 3

EULER-MARUYAMA SCHEME FOR SFDES DRIVEN BY A FRACTIONAL BROWNIAN MOTION

3.1. The stochastic functional differential equation

We study a class of stochastic functional differential equation (SFDE) with memory driven by fractional Brownian motion. More precisely, we consider the equation

$$X(t) = \begin{cases} X(0) + \int_0^t b(s, X(s), Y(s)) ds + \int_0^t \sigma(s, X(s), Y(s)) dB^H(s), & t \in [0, T] \\ \phi(t), & t \in [-r, 0] \end{cases} \quad (3.1)$$

where $B^H = (B^H(t))_{t \geq 0}$ is a fractional Brownian motion with Hurst parameter $H > 1/2$. The stochastic integral with respect to B^H is understood in the sense of the generalized Lebesgue-Stieltjes integral as introduced by Zähle [24]. The drift and diffusion coefficients $b, \sigma : [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ are assumed to be continuous functions, and the initial condition $\phi : [-r, 0] \rightarrow \mathbb{R}$ is assumed to be Hölder continuous. The memory effect is modeled through the functional $Y(t)$ defined by

$$Y(t) = \int_{-r}^0 K(t, s, X(t+s)) ds = \int_{t-r}^t K(t, s-t, X(s)) ds, \quad t \geq 0, \quad (3.2)$$

where the kernel $K : [0, T] \times [-r, 0] \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function. This formulation allows us to capture hereditary effects over a finite window of the past while maintaining a functional dependence on the entire trajectory segment.

We consider the following assumptions on the coefficients b, σ , the initial condition ϕ and the Kernel K in (3.1) and (3.2).

The assumptions imposed throughout this work are standard in the analysis of stochastic functional differential equations, but we briefly comment on their role and necessity. The Lipschitz-type conditions guarantee the well-posedness of the equation and allow us to obtain uniform bounds, while the growth conditions ensure that the solution does not explode in finite time.

The regularity assumptions on the coefficients are also required in order to apply the stochastic calculus tools used in the proofs, in particular when dealing with the memory term and the fractional noise.

Hypothesis ϕ There exists a positive constant L such that

$$|\phi(t) - \phi(s)| \leq L|t - s|^{H-\rho}, \quad 0 < \rho < H. \quad (3.3)$$

Hypothesis 1 There exist positive constants L_1, L_2 and L_3 such that the following properties hold.

$$|b(t, x, y) - b(s, x, y)| + |\sigma(t, x, y) - \sigma(s, x, y)| \leq L_1|t - s|. \quad (3.4)$$

$$|b(t, x_1, y_1) - b(t, x_2, y_2)| + |\sigma(t, x_1, y_1) - \sigma(t, x_2, y_2)| \leq L_2(|x_1 - x_2| + |y_1 - y_2|). \quad (3.5)$$

$$|b(t, x, y)| + |\sigma(t, x, y)| \leq L_3(1 + |x| + |y|). \quad (3.6)$$

Hypothesis 2 The function $\sigma(t, \xi)$ is Fréchet differentiable in the variable ξ . Also, there exist positive constants M_1, M_2 and M_3 such that for all $\xi, \eta \in C_r$ and $t \in [0, T]$,

$$\begin{aligned} |D_\xi \sigma(t, \xi)|_{\mathcal{L}(C_r, \mathbb{R})} &\leq M_1. \\ |D_\xi \sigma(t, \xi) - D_\xi \sigma(t, \eta)|_{\mathcal{L}(C_r, \mathbb{R})} &\leq M_2 \|\xi - \eta\|. \\ |D_\xi \sigma(t, \xi) - D_\xi \sigma(s, \xi)|_{\mathcal{L}(C_r, \mathbb{R})} &\leq M_3 |t - s|. \end{aligned} \quad (3.7)$$

Hypothesis 3 For each $t \in [0, T]$, the map

$$(s, x) \longmapsto K(t, s, x), \quad (s, x) \in [-r, 0] \times \mathbb{R},$$

is continuously differentiable. Moreover, there exist positive constants K_1, K_2, K_3, K_4 such that

$$\begin{aligned} |K(t, s, x) - K(u, v, x)| &\leq K_1(|t - u| + |s - v|), \\ |K(t, s, x) - K(t, s, y)| &\leq K_2|x - y|, \\ |K(t, s, x)| &\leq K_3(1 + |x|), \\ |\nabla_{(s, x)} K(t, s, x)| &\leq K_4, \\ |\nabla_{(s, x)} K(t, s, x) - \nabla_{(s, x)} K(u, v, y)| &\leq K_4(|t - u| + |s - v| + |x - y|). \end{aligned} \quad (3.8)$$

The following theorem establishes the existence and uniqueness of the solution to (3.1) (the proof is analogous to those given in [3] and [20]), and is therefore left to the reader.

Theorem 3.1.1. *Under hypotheses 3.3 to 3.8. If ϕ is a stochastic process whose trajectories belong to the space $C^{H-\rho}[0, T]$ a.s. for any $0 < \rho < H$, then Equation (3.1) has a unique solution $X \in C^{H-\rho}[-r, T]$ a.s.,*

$$|X(u) - X(v)| \leq C(\omega)|u - v|^{H-\rho}. \quad (3.9)$$

We obtain the next a priori estimate of the process X , the proof is given in Appendix 7.2. Such bounds will be crucial in controlling both the exact solution and the error terms arising in the discretization scheme.

Proposition 3.1.1. *Under Hypotheses 1 and 3, for any $\varepsilon > 0$ and $0 < \rho < H$ there exists $h_0 > 0$ and $\Omega_{\varepsilon, h_0, \rho} \subset \Omega$ such that $\mathbb{P}(\Omega_{\varepsilon, h_0, \rho}) > 1 - \varepsilon$ and for each $\omega \in \Omega_{\varepsilon, h_0, \rho}$, $h < h_0$*

$$|X(t)| < C(\omega)$$

and for $0 \leq s < t \leq T$,

$$|X(t) - X(s)| \leq C(\omega)(t - s)^{H-\rho}.$$

3.1.1. Structural Estimates for the Delay Functional and Fractional Integral

In this subsection, we establish several structural estimates that will be used repeatedly in the sequel. The first result concerns the regularity properties of the delay functional $Y(t)$ defined in (3.2), while the second recalls the fundamental bound associated with the fractional integral with respect to B^H . Isolating these estimates allows us to streamline subsequent arguments and to avoid repeating technical computations. The proofs are deferred to the Appendix 7.1.

The following lemma shows how the delay functional $Y(t)$ inherits regularity from the process X , while also capturing the contribution of the past trajectory over the memory window.

Lemma 3.1.1 (Structural estimate for the delay functional). *Let Y be defined by (3.2) and assume Hypothesis 3 holds. Then for all $0 \leq s < t \leq T$,*

$$|Y(t) - Y(s)| \leq C_1|t - s| + K_3 \int_{s-r}^{t-r} |X(u)| du.$$

3.2. Euler scheme for SFDE

In this section, we introduce the Euler approximation of the SFDE and describe how the fractional noise and the memory term are discretized. Particular attention is paid to the delay functional, whose approximation involves the past trajectory of the numerical solution.

We assume, without loss of generality, that there exists $N_T \in \mathbb{N}$ such that $T = N_T r$. Let $h := T/N$ for some $N \in \mathbb{N}$. For simplicity, we're going to take $N_T = 1$, that is, $T = r$. Now, we consider the uniform partition of $[-r, T]$ given by

$$t_n = nh, \quad n = -N, \dots, -1, 0, 1, \dots, N.$$

The Euler scheme associated with (3.1) is defined recursively by

$$\begin{cases} \tilde{X}^h(t_n) = \phi(t_n), & n = -N, \dots, 0, \\ \tilde{X}^h(t_{n+1}) = \tilde{X}^h(t_n) + h b(t_n, \tilde{X}^h(t_n), \tilde{Y}(t_n)) + \sigma(t_n, \tilde{X}^h(t_n), \tilde{Y}(t_n)) \Delta B^H(t_n), & n = 0, \dots, N-1, \end{cases}$$

where

$$\Delta B^H(t_n) = B^H(t_{n+1}) - B^H(t_n).$$

In contrast to the classical Markovian setting, the scheme depends not only on the current value $\tilde{X}^h(t_n)$ but also on the discrete approximation of the delay functional $\tilde{Y}(t_n)$, which encodes the past trajectory of the process.

The continuous interpolation is defined for $t \in [t_n, t_{n+1})$ by

$$\tilde{X}^h(t) = \tilde{X}^h(t_n) + b(t_n, \tilde{X}^h(t_n), \tilde{Y}(t_n))(t - t_n) + \sigma(t_n, \tilde{X}^h(t_n), \tilde{Y}(t_n))(B^H(t) - B^H(t_n)).$$

This continuous-time interpolation allows us to rewrite the discrete scheme in an integral form, which is more suitable for comparison with the exact solution and for the derivation of error estimates.

Then for $t \geq 0$,

$$\tilde{X}^h(t) = X(0) + \int_0^t b(C_h(u), \tilde{X}^h(C_h(u)), \tilde{Y}(C_h(u))) du + \int_0^t \sigma(C_h(u), \tilde{X}^h(C_h(u)), \tilde{Y}(C_h(u))) dB^H(u), \quad (3.10)$$

where

$$C_h(u) = \left\lfloor \frac{u}{h} \right\rfloor h. \quad (3.11)$$

Clearly, if $u \in [t_i, t_{i+1})$ then $C_h(u) = t_i$.

The operator $C_h(u)$ selects the last discretization point before u , ensuring that the coefficients are evaluated along the piecewise constant approximation of the trajectory.

The discrete delay functional is given by

$$\tilde{Y}(t) = \sum_{i=-N}^{-1} h K(t, t_i, \tilde{X}^h(t + t_i)). \quad (3.12)$$

Equivalently,

$$\begin{aligned} \tilde{Y}(t) &= \sum_{i=-N}^{-1} \int_{t_i}^{t_{i+1}} K(t, C_h(s), \tilde{X}^h(t + C_h(s))) ds \\ &= \int_{-r}^0 K(t, C_h(s), \tilde{X}^h(t + C_h(s))) ds. \end{aligned}$$

This representation makes explicit the link between the exact delay term and its discrete counterpart, and will be instrumental in the convergence analysis, where the discrepancy induced by the discretization of both the kernel and the trajectory must be carefully controlled.

3.3. Estimates for the numerical scheme

In this section, we derive a priori estimates for the numerical scheme that will be used in the convergence analysis.

First, we introduce some technical lemmas needed to estimate $|\tilde{X}^h(t)|$ for $0 \leq t \leq T$.

We define the function $\tilde{F} : [-r, T] \rightarrow \mathbb{R}$ by

$$\tilde{F}(t) = \sup_{u \in [-r, t]} |\tilde{X}^h(u)|. \quad (3.13)$$

The following result is the discrete structural counterpart of Lemma 3.1.1. It shows that the Euler approximation preserves the incremental structure of the delay functional uniformly with respect to the mesh size. For clarity, we defer the proofs of the lemmas to the Appendix 7.1.

Lemma 3.3.1. *Let $Y(t)$ given by (3.2). If $\tilde{Y}(t)$ is the approximation of the integral $Y(t)$, by Riemann sums (3.12),*

$$\tilde{Y}(t) = \sum_{i=-N}^{-1} h \cdot K(t, t_i, \tilde{X}^h(t + t_i)),$$

then

$$|\tilde{Y}(C_h(u))| \leq r K_3 (1 + \tilde{F}(u - h)).$$

Observe that the constants in Lemma 3.3.1 are independent of h . Hence the discrete delay functional satisfies the same structural increment bound uniformly in the discretization parameter.

The following lemma gives a relation between $\Delta \tilde{Y}$ and $\Delta \tilde{X}$. This relation will be used to transfer regularity estimates from $\tilde{X}^h$ to the delay term.

Lemma 3.3.2. *Let $\tilde{Y}$ be the approximation defined by (3.12). Then*

$$|\tilde{Y}(u) - \tilde{Y}(v)| \leq rK_1|u - v| + K_2 \int_{-r}^0 |\tilde{X}^h(u + C_h(s)) - \tilde{X}^h(v + C_h(s))| ds,$$

for $0 \leq v \leq u \leq T$.

We next control the local discretization error between grid points.

Lemma 3.3.3. *Let $\tilde{X}^h(t)$ be the Euler scheme given by (3.10). Then, for $v \in [0, T]$*

$$|\tilde{X}^h(v) - \tilde{X}^h(C_h(v))| \leq C(\omega) \left(1 + |\tilde{X}^h(C_h(v))| + rK_3(1 + \tilde{F}(v - h)) \right) (v - C_h(v))^{H-\rho},$$

where $C(\omega)$ is a constant independent of h .

Remark 3.3.1. *For all $0 \leq i \leq N - 1$ and C_h defined in (3.11), we have*

$$\int_{t_i}^{t_{i+1}} (u - C_h(u))^{-\beta} du = \int_{t_i}^{t_{i+1}} (u - t_i)^{-\beta} du = \frac{1}{1 - \beta} h^{1-\beta}.$$

Consequently,

$$\int_0^t (u - C_h(u))^{-\beta} du \leq \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} (u - C_h(u))^{-\beta} du = \frac{N}{1 - \beta} h^{1-\beta} = \frac{T}{1 - \beta} h^{-\beta}. \quad (3.14)$$

Lemma 3.3.4. *Let $\tilde{X}^h(t)$ be the Euler scheme given by (3.10). Then, for $0 \leq v \leq C_h(u)$,*

$$\begin{aligned} & \int_0^{C_h(u)} |\tilde{X}^h(C_h(u)) - \tilde{X}^h(v)| (u - v)^{-\beta-1} dv \\ & \leq C(\omega) \left(1 + \int_0^{C_h(u)} |\tilde{X}^h(w)| (u - w)^{-2\beta} dw + \int_0^{C_h(u)} \tilde{F}(w) (u - w)^{-2\beta} dw \right. \\ & \quad + \int_0^{C_h(u)} (u - w)^{-\beta} \int_0^{C_h(w)} \frac{|\tilde{X}^h(C_h(w)) - \tilde{X}^h(z)|}{(w - z)^{\beta+1}} dz dw + \tilde{F}(C_h(u)) h^{H-\rho-\beta} \\ & \quad \left. + \int_0^{C_h(u)} (u - w)^{-\beta} \int_0^{C_h(w)} \int_{-r}^0 \frac{|\tilde{X}^h(C_h(w) + C_h(s)) - \tilde{X}^h(C_h(z) + C_h(s))| ds}{(w - z)^{\beta+1}} dz dw \right). \end{aligned}$$

Lemma 3.3.5. *Let $\tilde{X}^h(t)$ be the Euler scheme given by (3.10) and for $w \in [0, T]$ define*

$$I(w) := \int_0^{C_h(w)} (w - z)^{-\beta-1} \int_{-r}^0 |\tilde{X}^h(C_h(w) + C_h(s)) - \tilde{X}^h(C_h(z) + C_h(s))| ds dz.$$

Then, for $w \in [0, T]$

$$\begin{aligned} I(w) & \leq C_1(\omega) \left(1 + \tilde{F}(C_h(w)) + h^{H-\rho} (w - C_h(w))^{-\beta} + h (w - C_h(w))^{-\beta} \tilde{F}(C_h(w)) \right. \\ & \quad \left. + \int_0^{C_h(w)} (w - z)^{-\beta-1} \int_{-C_h(z)}^0 |\tilde{X}^h(C_h(w) + C_h(s)) - \tilde{X}^h(C_h(z) + C_h(s))| ds dz \right). \end{aligned}$$

Lemma 3.3.6. *Let $\tilde{X}^h(t)$ be the Euler scheme given by (3.10), z, w such that $0 < z < C_h(w)$, and $s \in [-r, 0]$, then*

$$\begin{aligned} & \int_0^{C_h(w)} \int_{-C_h(z)}^0 \frac{|\tilde{X}^h(C_h(w) + C_h(s)) - \tilde{X}^h(C_h(z) + C_h(s))|}{(w - z)^{\beta+1}} ds dz \\ & \leq C(\omega) \left(1 + h^{1-\beta} + h^{H-\rho-\beta} + h^{H-\rho-\beta} \tilde{F}(w) + h^{1-\beta} \tilde{F}(C_h(w)) + \tilde{F}(C_h(w)) w^{1-\beta} \right. \\ & \quad + \int_0^{C_h(w)} \frac{|\tilde{F}^h(C_h(z))|}{(w - z)^{2\beta}} dz + \int_0^{C_h(w)} \int_0^{C_h(u)} (w - u)^{-\beta} \frac{|\tilde{X}^h(C_h(u)) - \tilde{X}^h(v)|}{(u - v)^{\beta+1}} dv du \\ & \quad + \int_0^{C_h(w)} (w - u)^{-\beta} \int_0^{C_h(u)} (u - v)^{-\beta-1} \\ & \quad \left. \int_{-C_h(v)}^0 |\tilde{X}^h(C_h(u) + C_h(s)) - \tilde{X}^h(C_h(v) + C_h(s))| ds dv du \right). \end{aligned}$$

We now establish uniform bounds for the Euler scheme. These estimates are the key ingredient for the convergence analysis and rely on the previous lemmas.

Theorem 3.3.1. *According to Hypotheses 1 and 3, for any $\varepsilon > 0$ and $0 < \rho < H$ there exist $h_0 > 0$ and $\Omega_{\varepsilon, h_0, \rho} \subset \Omega$ such that $\mathbb{P}(\Omega_{\varepsilon, h_0, \rho}) > 1 - \varepsilon$, and for each $\omega \in \Omega_{\varepsilon, h_0, \rho}$, $h < h_0$ and $C(\omega) > 0$, such that*

$$|\tilde{X}^h(t)| < C(\omega)$$

and for $0 \leq s < t \leq T$,

$$|\tilde{X}^h(t) - \tilde{X}^h(s)| \leq C(\omega)(t - s)^{H-\rho}.$$

Proof. From the Euler scheme (3.10) we have the following,

$$\tilde{X}^h(t) = X(0) + \int_0^t b(C_h(u), \tilde{X}^h(C_h(u)), \tilde{Y}(C_h(u))) du + \int_0^t \sigma(C_h(u), \tilde{X}^h(C_h(u)), \tilde{Y}(C_h(u))) dB_u^H.$$

Applying inequality 1.4 to the function $f(u) = \sigma(C_h(u), \tilde{X}^h(C_h(u)), \tilde{Y}(C_h(u)))$, we obtain

$$\begin{aligned} |\tilde{X}^h(t)| & \leq |X(0)| + \int_0^t |b(C_h(u), \tilde{X}^h(C_h(u)), \tilde{Y}(C_h(u)))| du \\ & \quad + C(\omega) \int_0^t \frac{|\sigma(C_h(u), \tilde{X}^h(C_h(u)), \tilde{Y}(C_h(u)))|}{u^\beta} du \\ & \quad + C(\omega) \int_0^t \int_0^u \frac{|\sigma(C_h(u), \tilde{X}^h(C_h(u)), \tilde{Y}(C_h(u))) - \sigma(C_h(v), \tilde{X}^h(C_h(v)), \tilde{Y}(C_h(v)))|}{(u - v)^{\beta+1}} dv du. \end{aligned}$$

Due to the linear growth property of b and σ , the Lipschitz property of σ (see Hypothesis 1 (3.5) and (3.6)) we derive

$$\begin{aligned} |\tilde{X}^h(t)| & \leq |X(0)| + \int_0^t L_3(1 + |\tilde{X}^h(C_h(u))| + |\tilde{Y}(C_h(u))|) du \\ & \quad + C(\omega) \int_0^t \frac{L_3(1 + |\tilde{X}^h(C_h(u))| + |\tilde{Y}(C_h(u))|)}{u^\beta} du \\ & \quad + C(\omega) \int_0^t \int_0^u \frac{L_1|C_h(u) - C_h(v)|}{(u - v)^{\beta+1}} dv du \\ & \quad + C(\omega) \int_0^t \int_0^u \frac{L_2|\tilde{X}^h(C_h(u)) - \tilde{X}^h(C_h(v))| + L_2|\tilde{Y}(C_h(u)) - \tilde{Y}(C_h(v))|}{(u - v)^{\beta+1}} dv du. \end{aligned}$$

Using Lemma 3.3.1 to control the delay term, and splitting the inner integral over $[0, u]$ as $[0, u] = [0, C_h(u)] \cup [C_h(u), u]$, we obtain

$$\begin{aligned} |\tilde{X}^h(t)| \leq & |X(0)| + L_3 t + L_3 \int_0^t |\tilde{X}^h(C_h(u))| du + L_3 r K_3 t + L_3 r K_3 \int_0^t \tilde{F}(u-h) du \\ & + C(\omega) L_3 \int_0^t \frac{du}{u^\beta} + C(\omega) L_3 \int_0^t \frac{|\tilde{X}^h(C_h(u))|}{u^\beta} du + C(\omega) L_3 r K_3 \int_0^t \frac{1 + \tilde{F}(u-h)}{u^\beta} du \\ & + C(\omega) \int_0^t \int_0^u \frac{L_1 |C_h(u) - C_h(v)|}{(u-v)^{\beta+1}} dv du \\ & + C(\omega) \int_0^t \int_0^u \frac{L_2 |\tilde{X}^h(C_h(u)) - \tilde{X}^h(C_h(v))| + L_2 |\tilde{Y}(C_h(u)) - \tilde{Y}(C_h(v))|}{(u-v)^{\beta+1}} dv du. \end{aligned}$$

For $v \in [C_h(u), u]$ we have $C_h(v) = C_h(u)$, so the integral over $[C_h(u), u]$ vanishes. Collecting the previous bounds, and using

$$\int_0^t u^{-\beta} du = \frac{t^{1-\beta}}{1-\beta},$$

we have that

$$\begin{aligned} |\tilde{X}^h(t)| \leq & |X(0)| + L_3(1+rK_3) \left(T + C(\omega) \frac{T^{1-\beta}}{1-\beta} \right) + L_3 \int_0^t |\tilde{X}^h(C_h(u))| du + L_3 r K_3 \int_0^t \tilde{F}(u-h) du \\ & + C(\omega) L_3 \int_0^t \frac{|\tilde{X}^h(C_h(u))|}{u^\beta} du + C_\beta C(\omega) L_3 r K_3 \int_0^t \frac{\tilde{F}(u-h)}{u^\beta} du \\ & + C(\omega) \int_0^t \int_0^{C_h(u)} \frac{L_1 |C_h(u) - C_h(v)| + L_2 |\tilde{X}^h(C_h(u)) - \tilde{X}^h(C_h(v))| + L_2 |\tilde{Y}(C_h(u)) - \tilde{Y}(C_h(v))|}{(u-v)^{\beta+1}} dv du. \end{aligned}$$

Therefore,

$$\begin{aligned} |\tilde{X}^h(t)| \leq & |X(0)| + L_3(1+rK_3) \left(T + C(\omega) \frac{T^{1-\beta}}{1-\beta} \right) + L_3(T^\beta + C(\omega)) \int_0^t \frac{|\tilde{X}^h(C_h(u))|}{u^\beta} du \\ & + L_3 r K_3(T^\beta + C(\omega)) \int_0^t \frac{\tilde{F}(u-h)}{u^\beta} du + C(\omega) \int_0^t \int_0^{C_h(u)} \frac{L_1 |C_h(u) - C_h(v)|}{(u-v)^{\beta+1}} dv du \\ & + C(\omega) \int_0^t \int_0^{C_h(u)} \frac{L_2 |\tilde{X}^h(C_h(u)) - \tilde{X}^h(v)|}{(u-v)^{\beta+1}} dv du + C(\omega) \int_0^t \int_0^{C_h(u)} \frac{L_2 |\tilde{X}^h(v) - \tilde{X}^h(C_h(v))|}{(u-v)^{\beta+1}} dv du \\ & + C(\omega) \int_0^t \int_0^{C_h(u)} \frac{L_2 |\tilde{Y}(C_h(u)) - \tilde{Y}(C_h(v))|}{(u-v)^{\beta+1}} dv du \\ & = C(\beta, \omega)(1 + I_1 + I_2 + I_3 + I_4 + I_5 + I_6), \end{aligned} \tag{3.15}$$

where $C(\beta, \omega) = (L_1 + L_2 + L_3(1+rK_3))(1+T^\beta)(1+C_\beta C(\omega))$, and

$$\begin{aligned} I_1 &= \int_0^t \frac{|\tilde{X}^h(C_h(u))|}{u^\beta} du, \quad I_2 = \int_0^t \frac{\tilde{F}(u-h)}{u^\beta} du, \quad I_3 = \int_0^t \int_0^{C_h(u)} \frac{|C_h(u) - C_h(v)|}{(u-v)^{\beta+1}} dv du, \\ I_4 &= \int_0^t \int_0^{C_h(u)} \frac{|\tilde{X}^h(C_h(u)) - \tilde{X}^h(v)|}{(u-v)^{\beta+1}} dv du, \quad I_5 = \int_0^t \int_0^{C_h(u)} \frac{|\tilde{X}^h(v) - \tilde{X}^h(C_h(v))|}{(u-v)^{\beta+1}} dv du, \\ I_6 &= \int_0^t \int_0^{C_h(u)} \frac{|\tilde{Y}(C_h(u)) - \tilde{Y}(C_h(v))|}{(u-v)^{\beta+1}} dv du. \end{aligned}$$

To estimate I_3 , we use $C_h(u) - C_h(v) \leq u - v + h$ if $u > v$, from (7.22) and Remark (3.3.1)

eq-(3.14)], then is easy to see that there exists a constant $C_3 > 0$ depending only on T and β such that

$$I_3 \leq \frac{1}{(1-\beta)(2-\beta)} t^{2-\beta} + \frac{h}{\beta} \int_0^t (u - C_h(u))^{-\beta} du \leq C_3 \quad (3.16)$$

From Lemma 3.3.4

$$\begin{aligned} I_4 &= \int_0^t \int_0^{C_h(u)} |\tilde{X}^h(C_h(u)) - \tilde{X}^h(v)|(u-v)^{-\beta-1} dv du \\ &\leq C_4(\omega) \left(1 + \int_0^t \tilde{F}(u) u^{1-2\beta} du + \tilde{F}(C_h(t)) h^{H-\rho-\beta} \right. \\ &\quad + \int_0^t \int_0^{C_h(u)} (u-w)^{-\beta} \int_0^{C_h(w)} \frac{|\tilde{X}^h(C_h(w)) - \tilde{X}^h(z)|}{(w-z)^{\beta+1}} dz dw du \\ &\quad \left. + \int_0^t \int_0^{C_h(u)} (u-w)^{-\beta} \int_0^{C_h(w)} \int_{-r}^0 \frac{|\tilde{X}^h(C_h(w) + C_h(s)) - \tilde{X}^h(C_h(z) + C_h(s))| ds}{(w-z)^{\beta+1}} dz dw du \right). \end{aligned}$$

From Lemma 3.3.5, (7.22) and Remark [3.3.1 (3.14)]

$$\begin{aligned} I_4 &\leq C_4(\omega) \left(1 + \int_0^t \tilde{F}(u) u^{1-2\beta} du + \tilde{F}(C_h(t)) h^{H-\rho-\beta} \right. \\ &\quad + \int_0^t \int_0^{C_h(u)} (u-w)^{-\beta} \int_0^{C_h(w)} \frac{|\tilde{X}^h(C_h(w)) - \tilde{X}^h(z)|}{(w-z)^{\beta+1}} dz dw du \\ &\quad + \int_0^t \int_0^{C_h(u)} (u-w)^{-\beta} \left[1 + \tilde{F}(C_h(w)) + h^{H-\rho} (w - C_h(w))^{-\beta} + h(w - C_h(w))^{-\beta} \tilde{F}(C_h(w)) \right] dw du \\ &\quad + \int_0^t \int_0^{C_h(u)} (u-w)^{-\beta} \int_0^{C_h(w)} (w-z)^{-\beta-1} \int_{-C_h(z)}^0 |\tilde{X}^h(C_h(w) + C_h(s)) - \tilde{X}^h(C_h(z) + C_h(s))| ds dz dw du \Big) \\ &\leq C_4(\omega) \left(1 + \int_0^t \tilde{F}(u) u^{1-2\beta} du + \tilde{F}(C_h(t)) h^{H-\rho-\beta} + \int_0^t u^{1-\beta} \tilde{F}(u) du + h^{H-\rho-\beta} + h^{1-\beta} \tilde{F}(t) \right. \\ &\quad + \int_0^t \int_0^{C_h(u)} (u-w)^{-\beta} \int_0^{C_h(w)} \frac{|\tilde{X}^h(C_h(w)) - \tilde{X}^h(z)|}{(w-z)^{\beta+1}} dz dw du \\ &\quad \left. + \int_0^t \int_0^{C_h(u)} (u-w)^{-\beta} \int_0^{C_h(w)} (w-z)^{-\beta-1} \int_{-C_h(z)}^0 |\tilde{X}^h(C_h(w) + C_h(s)) - \tilde{X}^h(C_h(z) + C_h(s))| ds dz dw du \right) \end{aligned} \quad (3.17)$$

For I_5 , we use Lemma 3.3.3 and (3.13) to deduce

$$\begin{aligned} I_5 &\leq C_5(\omega) \int_0^t \int_0^{C_h(u)} \frac{\left(1 + |\tilde{X}^h(C_h(v))| + rK_3(1 + \tilde{F}(v-h)) \right) (v - C_h(v))^{H-\rho}}{(u-v)^{\beta+1}} dv du \\ &\leq C_5(\omega) h^{H-\rho} \int_0^t \int_0^{C_h(u)} \frac{1 + \tilde{F}(C_h(v)) + rK_3(1 + \tilde{F}(v-h))}{(u-v)^{\beta+1}} dv du \\ &\leq C_5(\omega) h^{H-\rho-\beta} + C(\omega) h^{H-\rho} \int_0^t \int_0^{C_h(u)} \frac{\tilde{F}(C_h(v))}{(u-v)^{\beta+1}} dv du \\ &\leq C_5(\omega) h^{H-\rho-\beta} + C(\omega) h^{H-\rho-\beta} \tilde{F}(t). \end{aligned} \quad (3.18)$$

We now analyze I_6 , using Lemma 3.3.2 and the upper bound of I_3 given in (3.16), we can get

$$\begin{aligned}
 I_6 &\leq \int_0^t \int_0^{C_h(u)} \frac{rK_1|C_h(u) - C_h(v)| + K_2 \int_{-r}^0 |\tilde{X}^h(C_h(u) + C_h(s)) - \tilde{X}^h(C_h(v) + C_h(s))| ds}{(u-v)^{\beta+1}} dv du \\
 &\leq rK_1 I_3 + K_2 \int_0^t \int_0^{C_h(u)} \int_{-r}^0 \frac{|\tilde{X}^h(C_h(u) + C_h(s)) - \tilde{X}^h(C_h(v) + C_h(s))|}{(u-v)^{\beta+1}} ds dv du \\
 &\leq C + K_2 \int_0^t \int_0^{C_h(u)} \int_{-r}^0 \frac{|\tilde{X}^h(C_h(u) + C_h(s)) - \tilde{X}^h(C_h(v) + C_h(s))|}{(u-v)^{\beta+1}} ds dv du.
 \end{aligned}$$

Now, from Lemma 3.3.5

$$\begin{aligned}
 I_6 &\leq C_6(\omega) \left(1 + \int_0^t \tilde{F}(C_h(u)) du + h^{H-\rho} \int_0^t (u - C_h(u))^{-\beta} du \right. \\
 &\quad + h \int_0^t (u - C_h(u))^{-\beta} \tilde{F}(C_h(u)) du \\
 &\quad \left. + \int_0^t \int_0^{C_h(u)} (u-z)^{-\beta-1} \int_{-C_h(z)}^0 |\tilde{X}^h(C_h(u) + C_h(s)) - \tilde{X}^h(C_h(z) + C_h(s))| ds dz du \right).
 \end{aligned}$$

From Remark [3.3.1 (3.14)]

$$\begin{aligned}
 I_6 &\leq C_6(\omega) \left(1 + \int_0^t \tilde{F}(C_h(u)) du + h^{H-\rho-\beta} + h^{1-\beta} \tilde{F}(C_h(t)) \right. \\
 &\quad \left. + \int_0^t \int_0^{C_h(u)} (u-z)^{-\beta-1} \int_{-C_h(z)}^0 |\tilde{X}^h(C_h(u) + C_h(s)) - \tilde{X}^h(C_h(z) + C_h(s))| ds dz du \right).
 \end{aligned}$$

From Lemma 3.3.6

$$\begin{aligned}
 I_6 &\leq C_6(\omega) \left(1 + \int_0^t \tilde{F}(C_h(u)) du + h^{H-\rho-\beta} \tilde{F}(C_h(t)) + h^{1-\beta} \tilde{F}(C_h(t)) + \int_0^t \tilde{F}(C_h(w)) w^{1-\beta} dw \right. \\
 &\quad + \int_0^t \int_0^{C_h(u)} \frac{|\tilde{F}^h(C_h(z))|}{(u-z)^{2\beta}} dz du \\
 &\quad + \int_0^t \int_0^{C_h(u)} \int_0^{C_h(w)} (u-w)^{-\beta} \frac{|\tilde{X}^h(C_h(w) - \tilde{X}^h(v))|}{(w-v)^{\beta+1}} dv dw du \\
 &\quad \left. + \int_0^t \int_0^{C_h(u)} (u-w)^{-\beta} \int_0^{C_h(w)} (w-v)^{-\beta-1} \int_{-C_h(v)}^0 |\tilde{X}^h(C_h(w) + C_h(s)) - \tilde{X}^h(C_h(v) + C_h(s))| ds dv dw du \right) \quad (3.19)
 \end{aligned}$$

Using (3.15), (3.16), (3.17), (3.18), (3.19), (3.14) and from (3.13) and Equation (8) in [16], considering $H - \rho - \beta > 0$, we can obtain

$$\begin{aligned}
 \tilde{F}(t) &\leq C_0(\omega) \left(1 + h^{H-\rho-\beta} \tilde{F}(t) + \int_0^t \frac{|\tilde{X}^h(C_h(u))|}{u^\beta} du + \int_0^t \frac{\tilde{F}(u)}{u^\beta} du \right. \quad (3.20) \\
 &\quad + \int_0^t \int_0^{C_h(u)} \int_0^{C_h(w)} (u-w)^{-\beta} \frac{|\tilde{X}^h(C_h(w) - \tilde{X}^h(v))|}{(w-v)^{\beta+1}} dv dw du \\
 &\quad \left. + \int_0^t \int_0^{C_h(u)} (u-w)^{-\beta} \int_0^{C_h(w)} (w-v)^{-\beta-1} \int_{-C_h(v)}^0 |\tilde{X}^h(C_h(w) + C_h(s)) - \tilde{X}^h(C_h(v) + C_h(s))| ds dv dw du \right)
 \end{aligned}$$

By Lemma 3.3.4 ,

$$\begin{aligned}
 & \int_0^{C_h(t)} \frac{|\tilde{X}^h(C_h(t)) - \tilde{X}^h(v)|}{(t-v)^{\beta+1}} dv \\
 & \leq C_0(\omega) \left(1 + h^{H-\rho-\beta} \tilde{F}(t) + \int_0^{C_h(t)} (t-w)^{-\beta} \tilde{F}(w) dw \right. \\
 & + \int_0^{C_h(t)} (t-w)^{-\beta} \int_0^{C_h(w)} \frac{|\tilde{X}^h(C_h(w)) - \tilde{X}^h(z)|}{(w-z)^{\beta+1}} dz dw \\
 & \left. + \int_0^{C_h(t)} (t-w)^{-\beta} \int_0^{C_h(w)} \int_{-C_h(z)}^0 \frac{|\tilde{X}^h(C_h(w) + C_h(s)) - \tilde{X}^h(C_h(z) + C_h(s))|}{(w-z)^{\beta+1}} ds dz dw \right). \tag{3.21}
 \end{aligned}$$

From Lemma 3.3.6,

$$\begin{aligned}
 & \int_0^{C_h(t)} \int_{-C_h(v)}^0 \frac{|\tilde{X}^h(C_h(t) + C_h(s)) - \tilde{X}^h(C_h(v) + C_h(s))|}{(t-v)^{\beta+1}} ds dv \\
 & \leq C_0(\omega) \left(1 + h^{H-\rho-\beta} \tilde{F}(t) + \int_0^{C_h(t)} (t-u)^{-\beta} \tilde{F}(u) du \right. \\
 & + \int_0^{C_h(t)} (t-u)^{-\beta} \int_0^{C_h(u)} \frac{|\tilde{X}^h(C_h(u)) - \tilde{X}^h(v)|}{(u-v)^{\beta+1}} dv du \\
 & \left. + \int_0^{C_h(t)} (t-u)^{-\beta} \int_0^{C_h(u)} \int_{-C_h(v)}^0 \frac{|\tilde{X}^h(C_h(u) + C_h(s)) - \tilde{X}^h(C_h(v) + C_h(s))|}{(u-v)^{\beta+1}} ds dv du \right). \tag{3.22}
 \end{aligned}$$

We define the function $\varphi(t)$ by

$$\begin{aligned}
 \varphi(t) := & \tilde{F}(t) + \int_0^{C_h(t)} \frac{|\tilde{X}^h(C_h(t)) - \tilde{X}^h(v)|}{(t-v)^{\beta+1}} dv \\
 & + \int_0^{C_h(t)} \int_{-C_h(v)}^0 \frac{|\tilde{X}^h(C_h(t) + C_h(s)) - \tilde{X}^h(C_h(v) + C_h(s))|}{(t-v)^{\beta+1}} ds dv. \tag{3.23}
 \end{aligned}$$

Summing (3.20), (3.21) and (3.22), we obtain

$$\begin{aligned}
 \varphi(t) \leq & C_0(\omega) \left(1 + h^{H-\rho-\beta} \tilde{F}(t) + \int_0^t u^{-\beta} \tilde{F}(u) du + \int_0^t (t-u)^{-\beta} \tilde{F}(u) du \right. \\
 & + \int_0^t u^{-\beta} \int_0^{C_h(u)} \frac{|\tilde{X}^h(C_h(u)) - \tilde{X}^h(v)|}{(u-v)^{\beta+1}} dv du \\
 & + \int_0^t (t-u)^{-\beta} \int_0^{C_h(u)} \frac{|\tilde{X}^h(C_h(u)) - \tilde{X}^h(v)|}{(u-v)^{\beta+1}} dv du \\
 & + \int_0^t u^{-\beta} \int_0^{C_h(u)} \int_{-C_h(v)}^0 \frac{|\tilde{X}^h(C_h(u) + C_h(s)) - \tilde{X}^h(C_h(v) + C_h(s))|}{(u-v)^{\beta+1}} ds dv du \\
 & \left. + \int_0^t (t-u)^{-\beta} \int_0^{C_h(u)} \int_{-C_h(v)}^0 \frac{|\tilde{X}^h(C_h(u) + C_h(s)) - \tilde{X}^h(C_h(v) + C_h(s))|}{(u-v)^{\beta+1}} ds dv du \right).
 \end{aligned}$$

Hence

$$\varphi(t) \leq C_0(\omega) \left(1 + h^{H-\rho-\beta} \tilde{F}(t) + \int_0^t (u^{-\beta} + (t-u)^{-\beta}) \varphi(u) du \right).$$

By definition of $\varphi(t)$, we have $\tilde{F}(t) \leq \varphi(t)$. Then

$$\varphi(t) \leq C_0(\omega) \left(1 + h^{H-\rho-\beta} \varphi(t) + \int_0^t (u^{-\beta} + (t-u)^{-\beta}) \varphi(u) du \right).$$

Since $H - \rho - \beta > 0$, we may choose $h_0 > 0$ and a set $\Omega_{\varepsilon, h_0, \rho}$ with $\mathbb{P}(\Omega_{\varepsilon, h_0, \rho}) > 1 - \varepsilon$ such that for $h < h_0$ and $\omega \in \Omega_{\varepsilon, h_0, \rho}$,

$$C_0(\omega) h^{H-\rho-\beta} \leq \frac{1}{2}.$$

Therefore,

$$\varphi(t) \leq C_0(\omega) \left(1 + \int_0^t (u^{-\beta} + (t-u)^{-\beta}) \varphi(u) du \right).$$

The kernel $u^{-\beta} + (t-u)^{-\beta}$ is integrable on $[0, t]$ for $\beta < 1$. By the fractional Gronwall inequality (see Equation (8) in [16]), there exists $C(\omega) > 0$ such that

$$\sup_{t \in [0, T]} \varphi(t) \leq C(\omega).$$

In particular,

$$|\tilde{X}^h(t)| \leq C(\omega), \quad \forall t \in [0, T].$$

Let $0 \leq s < t \leq T$. Subtracting (3.10) at times t and s , we obtain

$$\begin{aligned} \tilde{X}^h(t) - \tilde{X}^h(s) &= \int_s^t b(C_h(u), \tilde{X}^h(C_h(u)), \tilde{Y}(C_h(u))) du \\ &\quad + \int_s^t \sigma(C_h(u), \tilde{X}^h(C_h(u)), \tilde{Y}(C_h(u))) dB^H(u). \end{aligned}$$

We have established the uniform bound of $\tilde{X}^h$, and of $\tilde{Y}$ follows from Lemma 3.3.1.

Because the coefficients b and σ have linear growth, there exists $C'(\omega) > 0$ such that for all $u \in [0, T]$,

$$\begin{aligned} |b(C_h(u), \tilde{X}^h(C_h(u)), \tilde{Y}(C_h(u)))| &\leq L_3(1 + |\tilde{X}^h(t)| + |\tilde{Y}(t)|) \leq C'(\omega) \\ |\sigma(t, \tilde{X}^h(t), \tilde{Y}(t))| &\leq L_3(1 + |\tilde{X}^h(t)| + |\tilde{Y}(t)|) \leq C'(\omega). \end{aligned}$$

Since the integrand is bounded, we have

$$\left| \int_s^t b(C_h(u), \tilde{X}^h(C_h(u)), \tilde{Y}(C_h(u))) du \right| \leq C'(\omega)(t-s).$$

The integrand

$$u \mapsto \sigma(C_h(u), \tilde{X}^h(C_h(u)), \tilde{Y}(C_h(u)))$$

is uniformly bounded. Moreover, almost surely, B^H is $(H - \rho)$ -Hölder continuous. Using the fractional integral estimate, inequality 1.4, we obtain

$$\left| \int_s^t \sigma(C_h(u), \tilde{X}^h(C_h(u)), \tilde{Y}(C_h(u))) dB^H(u) \right| \leq C(\omega)(t-s)^{H-\rho}.$$

Combining both estimates yields

$$|\tilde{X}^h(t) - \tilde{X}^h(s)| \leq C(\omega) [(t-s) + (t-s)^{H-\rho}].$$

Since $H \in (1/2, 1)$ and $\rho > 0$ is chosen such that $H - \rho > 0$, we have $H - \rho < 1$, hence $1 - (H - \rho) > 0$. Therefore, for $0 < t - s \leq T$,

$$(t-s) = (t-s)^{H-\rho} (t-s)^{1-(H-\rho)} \leq T^{1-(H-\rho)} (t-s)^{H-\rho}.$$

Consequently, for $C(\omega) = T^{1-(H-\rho)} C'(\omega)$, we obtain

$$|\tilde{X}^h(t) - \tilde{X}^h(s)| \leq C(\omega) (t-s)^{H-\rho}, \quad 0 \leq s < t \leq T.$$

The following corollary is an immediate consequence of Lemma 3.3.2 and Theorem 3.3.1. It will be used in the following section.

Corollary 3.3.1. *Let Y and $\tilde{Y}$ defined by (4.2) and (3.12), respectively. Under hypotheses of Theorem 3.3.1, for $0 \leq v \leq u \leq T$,*

$$|Y(u) - Y(v)| + |\tilde{Y}(u) - \tilde{Y}(v)| \leq rK_1|u-v| + rK_2C(\omega)|u-v|^{H-\rho}.$$

3.4. Convergence of the Euler scheme

In this section, we prove that the Euler scheme introduced in Section 3.2 converges to the solution of (3.1).

To quantify the error between the exact solution and its approximation, we introduce increment-type quantities measuring their local discrepancies.

Denoting

$$\Delta_{u,s}(X, \tilde{X}^h) = |X(s) - \tilde{X}^h(s) - X(u) + \tilde{X}^h(u)| \quad (3.24)$$

and

$$\Delta_{u,s}(Y, \tilde{Y}) = |Y(s) - \tilde{Y}(s) - Y(u) + \tilde{Y}(u)|. \quad (3.25)$$

Theorem 3.4.1. *Under Hypotheses 1, 2 and 3, for any $\varepsilon > 0$, there exists $h_0 > 0$ and a set $\Omega_{\varepsilon, h_0, \rho} \subset \Omega$ such that $\mathbb{P}(\Omega_{\varepsilon, h_0, \rho}) > 1 - \varepsilon$ and for each $\omega \in \Omega_{\varepsilon, h_0, \rho}$ and all $h < h_0$,*

$$\begin{aligned} U_h &:= \sup_{y \in [0, T]} \left(|X(y) - \tilde{X}^h(y)| + \int_0^y \frac{\Delta_{v,y}(X, \tilde{X}^h)}{(y-v)^{\beta+1}} dv + \int_0^{C_h(y)} \int_{-C_h(v)}^0 \frac{\Delta_{v+s,y+s}(X, \tilde{X}^h)}{(y-v)^{\beta+1}} ds dv \right) \\ &\leq C(\omega) h^{2H-1}, \end{aligned} \quad (3.26)$$

for $\beta \in (1-H, 1/2)$ and small enough $\rho > 0$ such that $1/2 < H - \rho < H$, where $C(\omega)$ does not depend on h and ε , but may depend on ρ .

The proof is based on controlling the difference between the exact solution and the numerical scheme through increment-type quantities. In particular, we introduce suitable measures of local error that allow us to handle both the fractional integral terms and the delay contribution in a unified way. The control of the delay terms follows the structural estimate of the functional Y given in Lemma 3.1.1, together with its discrete counterpart.

Before proving Theorem 3.4.1, we establish a sequence of technical lemmas controlling separately each component of the error.

Lemma 3.4.1. *Under the conditions of Theorem 3.4.1, taking*

$$Z^h(t) := \sup_{y \in [0, t]} |X(y) - \tilde{X}^h(y)|. \quad (3.27)$$

Then

$$\begin{aligned} Z^h(t) &\leq C(\omega)(h^{H-\rho-\beta} + \int_0^t Z^h(u)u^{-\beta}du + \int_0^t \int_0^u \Delta_{v,u}(X, \tilde{X}^h)(u-v)^{-\beta-1}dvdu \\ &\quad + \int_0^t \int_0^u \Delta_{v,u}(Y, \tilde{Y})(u-v)^{-\beta-1}dvdu). \end{aligned}$$

Proof.

$$\begin{aligned} Z^h(t) &\leq \sup_{y \in [0, t]} \int_0^y |b(u, X(u), Y(u)) - b(C_h(u), \tilde{X}^h(C_h(u)), \tilde{Y}(C_h(u)))| du \\ &\quad + \sup_{y \in [0, t]} \left| \int_0^y (\sigma(u, X(u), Y(u)) - \sigma(C_h(u), \tilde{X}^h(C_h(u)), \tilde{Y}(C_h(u)))) dB^H(u) \right| \\ &\leq I_1 + I_2 + I_3 + I_4 + I_5 + I_6, \end{aligned} \quad (3.28)$$

where

$$\begin{aligned} I_1 &= \int_0^t |b(u, X(u), Y(u)) - b(u, \tilde{X}^h(u), \tilde{Y}(u))| du, \\ I_2 &= \int_0^t |b(u, \tilde{X}^h(u), \tilde{Y}(u)) - b(C_h(u), \tilde{X}^h(u), \tilde{Y}(u))| du, \\ I_3 &= \int_0^t |b(C_h(u), \tilde{X}^h(u), \tilde{Y}(u)) - b(C_h(u), \tilde{X}^h(C_h(u)), \tilde{Y}(C_h(u)))| du, \\ I_4 &= \sup_{y \in [0, t]} \left| \int_0^y (\sigma(u, X(u), Y(u)) - \sigma(u, \tilde{X}^h(u), \tilde{Y}(u))) dB^H(u) \right|, \\ I_5 &= \sup_{y \in [0, t]} \left| \int_0^y (\sigma(u, \tilde{X}^h(u), \tilde{Y}(u)) - \sigma(C_h(u), \tilde{X}^h(u), \tilde{Y}(u))) dB^H(u) \right|, \\ I_6 &= \sup_{y \in [0, t]} \left| \int_0^y (\sigma(C_h(u), \tilde{X}^h(u), \tilde{Y}(u)) - \sigma(C_h(u), \tilde{X}^h(C_h(u)), \tilde{Y}(C_h(u)))) dB^H(u) \right|. \end{aligned}$$

From (3.5),

$$I_1 \leq L_2 \int_0^t (|X(u) - \tilde{X}^h(u)| + |Y(u) - \tilde{Y}(u)|) du \leq L_2 \int_0^t (Z^h(u) + |Y(u) - \tilde{Y}(u)|) du. \quad (3.29)$$

The control of the term involving Y follows the structural estimate provided by Lemma 3.1.1, combined with the discretization of the delay functional.

According to (3.2), (3.12) and Hypothesis 3 (3.8),

$$\begin{aligned} |Y(u) - \tilde{Y}(u)| &= \left| \int_{-r}^0 K(u, s, X(u+s)) ds - \sum_{i=-N}^{-1} h \cdot K(u, t_i, \tilde{X}^h(u+t_i)) \right| \\ &\leq \int_{-r}^0 (K_1 |s - C_h(s)| + K_2 |X(u+s) - \tilde{X}^h(u+C_h(s))|) ds \\ &\leq \int_{-r}^0 (K_1 |s - C_h(s)| + K_2 (|X(u+s) - \tilde{X}^h(u+s)| + |\tilde{X}^h(u+s) - \tilde{X}^h(u+C_h(s))|)) ds \end{aligned}$$

by Theorem 3.3.1, together with inequality

$$|X(u+s) - \tilde{X}^h(u+s)| \leq \sup_{w \in [-r, u]} |X(w) - \tilde{X}^h(w)| = \sup_{w \in [0, u]} |X(w) - \tilde{X}^h(w)| = Z^h(u)$$

for all $s \in [-r, 0]$, we obtain

$$\begin{aligned} |Y(u) - \tilde{Y}(u)| &\leq \int_{-r}^0 \left(K_1 |s - C_h(s)| + K_2 |X(u+s) - \tilde{X}^h(u+s)| + K_2 C(\omega) |s - C_h(s)|^{H-\rho} \right) ds \\ &= Ch + C(\omega) h^{H-\rho} + CZ^h(u). \end{aligned} \quad (3.30)$$

Thus, from (3.29)

$$I_1 \leq Ch + C(\omega) h^{H-\rho} + C \int_0^t Z^h(u) du. \quad (3.31)$$

Now, using (3.4), we obtain the following

$$I_2 \leq \int_0^t L_1 |u - C_h(u)| du \leq Ch. \quad (3.32)$$

By Hypothesis 1 (3.5), and using Theorem 3.3.1 and Corollary 3.3.1,

$$\begin{aligned} I_3 &\leq L_2 \int_0^t \left(|\tilde{X}^h(u) - \tilde{X}^h(C_h(u))| + |\tilde{Y}(u) - \tilde{Y}(C_h(u))| \right) du \\ &\leq L_2 \int_0^t \left(C(\omega) |u - C_h(u)|^{H-\rho} + rK_1 |u - C_h(u)| + rK_2 C(\omega) |u - C_h(u)|^{H-\rho} \right) du \\ &\leq L_2 \int_0^t \left(C(\omega) h^{H-\rho} + rK_1 h + rK_2 C(\omega) h^{H-\rho} \right) du \\ &\leq Ch + C(\omega) h^{H-\rho}. \end{aligned} \quad (3.33)$$

Using the estimate (1.4), we get

$$\begin{aligned} I_4 &\leq \sup_{y \in [0, t]} C(\omega) \left(\int_0^y \frac{|\sigma(u, X(u), Y(u)) - \sigma(u, \tilde{X}^h(u), \tilde{Y}(u))|}{u^\beta} du \right. \\ &\quad \left. + \int_0^y \int_0^u \frac{|\sigma(u, X(u), Y(u)) - \sigma(u, \tilde{X}^h(u), \tilde{Y}(u)) - \sigma(v, X(v), Y(v)) + \sigma(v, \tilde{X}^h(v), \tilde{Y}(v))|}{(u-v)^{\beta+1}} dv du \right) \\ &\leq I_{4-1} + I_{4-2}. \end{aligned} \quad (3.34)$$

Using (3.5) and (3.30), we obtain,

$$\begin{aligned} I_{4-1} &= \sup_{y \in [0, t]} C(\omega) \int_0^y \frac{|\sigma(u, X(u), Y(u)) - \sigma(u, \tilde{X}^h(u), \tilde{Y}(u))|}{u^\beta} du \\ &\leq C(\omega) \int_0^t \frac{|\sigma(u, X(u), Y(u)) - \sigma(u, \tilde{X}^h(u), \tilde{Y}(u))|}{u^\beta} du \\ &\leq C(\omega) \int_0^t \frac{L_2 r (K_1 h + K_2 C(\omega) h^{H-\rho}) + L_2 (1 + rK_2) Z^h(u)}{u^\beta} du \\ &\leq C(\omega) h + C(\omega) h^{H-\rho} + C(\omega) \int_0^t Z^h(u) u^{-\beta} du. \end{aligned} \quad (3.35)$$

To estimate I_{4-2} , we use the following corollary, whose proof is given in Appendix 7.1.

Corollary 3.4.1. *Under Hypothesis 1 ((4.4) to (4.6)), if $|x_1|, |x_2|, |x_3|, |x_4| \leq C$, for some positive constant C , then*

$$\begin{aligned} & |\sigma(t_1, x_1, y_1) - \sigma(t_1, x_2, y_2) - \sigma(t_2, x_3, y_3) + \sigma(t_2, x_4, y_4)| \\ & \leq L_2|x_1 - x_2 - x_3 + x_4| + L_1|x_1 - x_2||t_1 - t_2| + L_2|x_1 - x_2||x_2 - x_4| + L_2|x_1 - x_2||x_1 - x_3| \\ & \quad + L_2|x_1 - x_2||y_2 - y_4| + L_2|x_1 - x_2||y_1 - y_3| + L_2|y_1 - y_2 - y_3 + y_4| + L_1|y_1 - y_2||t_1 - t_2| \\ & \quad + L_2|y_1 - y_2||x_2 - x_4| + L_2|y_1 - y_2||x_1 - x_3| + L_2|y_1 - y_2||y_2 - y_4| + L_2|y_1 - y_2||y_1 - y_3|. \end{aligned}$$

Thus,

$$\begin{aligned} I_{4-2} & \leq C(\omega) \int_0^t \int_0^u \frac{|\sigma(u, X(u), Y(u)) - \sigma(u, \tilde{X}^h(u), \tilde{Y}(u)) - \sigma(v, X(v), Y(v)) + \sigma(v, \tilde{X}^h(v), \tilde{Y}(v))|}{(u-v)^{\beta+1}} dv du \\ & \leq C(\omega) \sum_{k=7}^{18} I_k, \end{aligned} \tag{3.36}$$

where

$$\begin{aligned} I_7 &= \int_0^t \int_0^u |X(u) - \tilde{X}^h(u) - X(v) + \tilde{X}^h(v)|(u-v)^{-\beta-1} dv du \\ I_8 &= \int_0^t \int_0^u |X(u) - \tilde{X}^h(u)||u-v|(u-v)^{-\beta-1} dv du \\ I_9 &= \int_0^t \int_0^u |X(u) - \tilde{X}^h(u)||\tilde{X}^h(u) - \tilde{X}^h(v)|(u-v)^{-\beta-1} dv du \\ I_{10} &= \int_0^t \int_0^u |X(u) - \tilde{X}^h(u)||X(u) - X(v)|(u-v)^{-\beta-1} dv du \\ I_{11} &= \int_0^t \int_0^u |X(u) - \tilde{X}^h(u)||\tilde{Y}(u) - \tilde{Y}(v)|(u-v)^{-\beta-1} dv du \\ I_{12} &= \int_0^t \int_0^u |X(u) - \tilde{X}^h(u)||Y(u) - Y(v)|(u-v)^{-\beta-1} dv du \\ I_{13} &= \int_0^t \int_0^u |Y(u) - \tilde{Y}(u) - Y(v) + \tilde{Y}(v)|(u-v)^{-\beta-1} dv du \\ I_{14} &= \int_0^t \int_0^u |Y(u) - \tilde{Y}(u)||u-v|(u-v)^{-\beta-1} dv du \\ I_{15} &= \int_0^t \int_0^u |Y(u) - \tilde{Y}(u)||\tilde{X}^h(u) - \tilde{X}^h(v)|(u-v)^{-\beta-1} dv du \\ I_{16} &= \int_0^t \int_0^u |Y(u) - \tilde{Y}(u)||X(u) - X(v)|(u-v)^{-\beta-1} dv du \\ I_{17} &= \int_0^t \int_0^u |Y(u) - \tilde{Y}(u)||\tilde{Y}(u) - \tilde{Y}(v)|(u-v)^{-\beta-1} dv du \\ I_{18} &= \int_0^t \int_0^u |Y(u) - \tilde{Y}(u)||Y(u) - Y(v)|(u-v)^{-\beta-1} dv du. \end{aligned}$$

From (3.24) and (3.25)

$$I_7 + I_{13} = \int_0^t \int_0^u \Delta_{v,u}(X, \tilde{X}^h)(u-v)^{-\beta-1} dv du + \int_0^t \int_0^u \Delta_{v,u}(Y, \tilde{Y})(u-v)^{-\beta-1} dv du. \tag{3.37}$$

By (3.27),

$$I_8 \leq \int_0^t \int_0^u Z^h(u)(u-v)^{-\beta} dv du \leq C \int_0^t Z^h(u) du. \quad (3.38)$$

Moreover, by Theorems 3.1.1 and 3.3.1,

$$I_9 + I_{10} \leq C(\omega) \int_0^t \int_0^u Z^h(u)(u-v)^{H-\rho-\beta-1} dv du \leq C(\omega) \int_0^t Z^h(u) u^{H-\rho-\beta} du \leq C(\omega) \int_0^t Z^h(u) du. \quad (3.39)$$

From Corollary 3.3.1 and (3.27),

$$\begin{aligned} I_{11} + I_{12} &= \int_0^t \int_0^u |X(u) - \tilde{X}^h(u)| |\tilde{Y}(u) - \tilde{Y}(v)| (u-v)^{-\beta-1} dv du \\ &\leq \int_0^t \int_0^u Z^h(u) (rK_1|u-v| + rK_2C(\omega)|u-v|^{H-\rho}) (u-v)^{-\beta-1} dv du \\ &\leq C \int_0^t \int_0^u Z^h(u)(u-v)^{-\beta} dv du + C(\omega) \int_0^t \int_0^u Z^h(u)(u-v)^{H-\rho-\beta-1} dv du \\ &\leq C(\omega) \int_0^t Z^h(u) du. \end{aligned} \quad (3.40)$$

Using (3.30), Theorem 3.1.1 and Theorem 3.3.1

$$\begin{aligned} I_{14} + I_{15} + I_{16} &= \int_0^t \int_0^u |Y(u) - \tilde{Y}(u)| \left[|u-v| + |\tilde{X}^h(u) - \tilde{X}^h(v)| + |X(u) - X(v)| \right] (u-v)^{-\beta-1} dv du \\ &\leq C(\omega) \int_0^t \int_0^u (h + h^{H-\rho} + Z^h(u)) \left[|u-v| + 2(u-v)^{H-\rho} \right] (u-v)^{-\beta-1} dv du \\ &\leq C(\omega) (h + h^{H-\rho} + \int_0^t Z^h(u) du). \end{aligned} \quad (3.41)$$

In the same manner, from (3.30) and Corollary 3.3.1

$$\begin{aligned} I_{17} + I_{18} &= \int_0^t \int_0^u |Y(u) - \tilde{Y}(u)| \left[|\tilde{Y}(u) - \tilde{Y}(v)| + |Y(u) - Y(v)| \right] (u-v)^{-\beta-1} dv du \\ &\leq C(\omega) \int_0^t \int_0^u (h + h^{H-\rho} + Z^h(u)) |u-v|^{H-\rho-\beta-1} dv du \\ &\leq C(\omega) (h + h^{H-\rho} + \int_0^t Z^h(u) du). \end{aligned} \quad (3.42)$$

Finally, from (3.36) and the bounds (3.37) to (3.42)

$$I_{4-2} \leq C(\omega) \left(\int_0^t \int_0^u (\Delta_{v,u}(X, \tilde{X}^h) + \Delta_{v,u}(Y, \tilde{Y}))(u-v)^{-\beta-1} dv du + \int_0^t Z^h(u) du + h + h^{H-\rho} \right). \quad (3.43)$$

Therefore, from (3.34), (3.35) and (3.43) we obtain

$$\begin{aligned} I_4 &\leq C(\omega) \left(h + h^{H-\rho} + \int_0^t Z^h(u) du + \int_0^t Z^h(u) u^{-\beta} du \right. \\ &\quad \left. + \int_0^t \int_0^u (\Delta_{v,u}(X, \tilde{X}^h) + \Delta_{v,u}(Y, \tilde{Y}))(u-v)^{-\beta-1} dv du \right). \end{aligned} \quad (3.44)$$

Let us now return to I_5 in (3.28), from (1.4),

$$\begin{aligned} I_5 &\leq C(\omega) \left(\int_0^t \left| \sigma(u, \tilde{X}^h(u), \tilde{Y}(u)) - \sigma(C_h(u), \tilde{X}^h(u), \tilde{Y}(u)) \right| u^{-\beta} du \right. \\ &\quad + \int_0^t \int_0^u \left| \sigma(u, \tilde{X}^h(u), \tilde{Y}(u)) - \sigma(C_h(u), \tilde{X}^h(u), \tilde{Y}(u)) \right. \\ &\quad \left. \left. - \left(\sigma(v, \tilde{X}^h(v), \tilde{Y}(v)) - \sigma(C_h(v), \tilde{X}^h(v), \tilde{Y}(v)) \right) \right| (u-v)^{-\beta-1} dv du \right) \\ &= I_{5-1} + I_{5-2}, \end{aligned}$$

where,

$$\begin{aligned} I_{5-1} &= \int_0^t |\sigma(u, \tilde{X}^h(u), \tilde{Y}(u)) - \sigma(C_h(u), \tilde{X}^h(u), \tilde{Y}(u))| u^{-\beta} du, \\ I_{5-2} &= \int_0^t \int_0^u |\sigma(u, \tilde{X}^h(u), \tilde{Y}(u)) - \sigma(C_h(u), \tilde{X}^h(u), \tilde{Y}(u)) - (\sigma(v, \tilde{X}^h(v), \tilde{Y}(v)) \\ &\quad + \sigma(C_h(v), \tilde{X}^h(v), \tilde{Y}(v)))| (u-v)^{-\beta-1} dv du. \end{aligned}$$

From Hypothesis 1 and (3.4),

$$I_{5-1} \leq L_1 \int_0^t |u - C_h(u)| u^{-\beta} du \leq L_1 \int_0^t h u^{-\beta} du \leq L_1 h. \quad (3.45)$$

To study I_{5-2} , we split the integral over $[0, u]$ into $[0, C_h(u)]$ and $[C_h(u), u]$. Using Hypothesis 1, (3.4), Teorema (3.3.1), Corollary (3.3.1), (3.14) and Remark (3.3.1), we obtain

$$\begin{aligned} I_{5-2} &\leq L_1 \int_0^t \int_0^{C_h(u)} (|u - C_h(u)| + |v - C_h(v)|) (u-v)^{-\beta-1} dv du \\ &\quad + \int_0^t \int_{C_h(u)}^u (|\sigma(u, \tilde{X}^h(u), \tilde{Y}(u)) - \sigma(v, \tilde{X}^h(v), \tilde{Y}(v))| \\ &\quad + |\sigma(C_h(u), \tilde{X}^h(u), \tilde{Y}(u)) - \sigma(C_h(u), \tilde{X}^h(v), \tilde{Y}(v))|) (u-v)^{-\beta-1} dv du \\ &\leq C \int_0^t \int_0^{C_h(u)} h (u-v)^{-\beta-1} dv du \\ &\quad + C \int_0^t \int_{C_h(u)}^u (|u-v| + |\tilde{X}^h(u) - \tilde{X}^h(v)| + |\tilde{Y}(u) - \tilde{Y}(v)|) (u-v)^{-\beta-1} dv du \\ &\leq C h^{1-\beta} + C(\omega) \int_0^t \int_{C_h(u)}^u (2|u-v| + 2|u-v|^{H-\rho}) (u-v)^{-\beta-1} dv du \\ &\leq C(\omega) h^{1-\beta} + C(\omega) h^{H-\rho-\beta}. \end{aligned} \quad (3.46)$$

Thus, from (3.45) and (3.46) we obtain

$$I_5 \leq C(\omega) h^{H-\rho-\beta}. \quad (3.47)$$

Similarly to I_5 , we have for I_6 that

$$I_6 \leq C(\omega) (I_{6-1} + I_{6-2}),$$

where

$$\begin{aligned} I_{6-1} &= \int_0^t |\sigma(C_h(u), \tilde{X}^h(u), \tilde{Y}(u)) - \sigma(C_h(u), \tilde{X}^h(C_h(u)), \tilde{Y}(C_h(u)))| u^{-\beta} du, \\ I_{6-2} &= \int_0^t \int_0^u \left| \sigma(C_h(u), \tilde{X}^h(u), \tilde{Y}(u)) - \sigma(C_h(u), \tilde{X}^h(C_h(u)), \tilde{Y}(C_h(u))) \right. \\ &\quad \left. - \left(\sigma(C_h(v), \tilde{X}^h(v), \tilde{Y}(v)) - \sigma(C_h(v), \tilde{X}^h(C_h(v)), \tilde{Y}(C_h(v))) \right) \right| (u-v)^{-\beta-1} dv du. \end{aligned}$$

Using Hypothesis 1 (3.5), Theorem (3.3.1) and Corollary 3.3.1 we have,

$$I_{6-1} \leq L_2 \int_0^t (|\tilde{X}^h(u) - \tilde{X}^h(C_h(u))| + |\tilde{Y}(u) - \tilde{Y}(C_h(u))|) u^{-\beta} du \leq C(\omega) h^{H-\rho}. \quad (3.48)$$

Again, splitting the integral over $[0, u]$ into $[0, C_h(u)]$ and $[C_h(u), u]$, from Hypothesis 1 (3.5), Theorem 3.3.1, Corollary 3.3.1 and (3.14), we have

$$\begin{aligned} I_{6-2} &\leq \int_0^t \int_0^{C_h(u)} \left[|\sigma(C_h(u), \tilde{X}^h(u), \tilde{Y}(u)) - \sigma(C_h(u), \tilde{X}^h(C_h(u)), \tilde{Y}(C_h(u)))| \right. \\ &\quad \left. + |\sigma(C_h(v), \tilde{X}^h(v), \tilde{Y}(v)) - \sigma(C_h(v), \tilde{X}^h(C_h(v)), \tilde{Y}(C_h(v)))| \right] (u-v)^{-\beta-1} dv du \\ &\quad + \int_0^t \int_{C_h(u)}^u |\sigma(C_h(u), \tilde{X}^h(u), \tilde{Y}(u)) - \sigma(C_h(u), \tilde{X}^h(v), \tilde{Y}(v))| (u-v)^{-\beta-1} dv du \\ &\leq L_2 \int_0^t \int_0^{C_h(u)} \left[|\tilde{X}^h(u) - \tilde{X}^h(C_h(u))| + |\tilde{Y}(u) - \tilde{Y}(C_h(u))| \right. \\ &\quad \left. + |\tilde{X}^h(v) - \tilde{X}^h(C_h(v))| + |\tilde{Y}(v) - \tilde{Y}(C_h(v))| \right] (u-v)^{-\beta-1} dv du \\ &\quad + L_2 \int_0^t \int_{C_h(u)}^u \left[|\tilde{X}^h(u) - \tilde{X}^h(v)| + |\tilde{Y}(u) - \tilde{Y}(v)| \right] (u-v)^{-\beta-1} dv du \\ &\leq C(\omega) \int_0^t \int_0^{C_h(u)} (h + h^{H-\rho}) (u-v)^{-\beta-1} dv du \\ &\quad + C(\omega) \int_0^t \int_{C_h(u)}^u [|u-v| + |u-v|^{H-\rho}] (u-v)^{-\beta-1} dv du \\ &\leq C(\omega) h^{H-\rho-\beta} + C(\omega) h^{1-\beta}. \end{aligned} \quad (3.49)$$

Therefore, from (3.48) and (3.49)

$$I_6 \leq C(\omega) h^{H-\rho-\beta}. \quad (3.50)$$

Summarizing, from (3.28), (3.31), (3.32), (3.33), (3.44), (3.47), (3.50), for $\omega \in \Omega_{\varepsilon, h_0, \rho}$ (see Theorem 3.3.1), and recalling that $1 - H < \beta < \frac{1}{2}$ and that $\rho > 0$ is sufficiently small so that $H - \rho - \beta > 0$,

$$\begin{aligned} Z^h(t) &\leq C(\omega) \left(h + h^{H-\rho} + h^{1-\beta} + h^{H-\rho-\beta} + \int_0^t Z^h(u) du + \int_0^t Z^h(u) u^{-\beta} du \right. \\ &\quad \left. + \int_0^t \int_0^u \Delta_{v,u}(X, \tilde{X}^h)(u-v)^{-\beta-1} dv du + \int_0^t \int_0^u \Delta_{v,u}(Y, \tilde{Y})(u-v)^{-\beta-1} dv du \right) \\ &\leq C(\omega) \left(h^{H-\rho-\beta} + \int_0^t Z^h(u) u^{-\beta} du + \int_0^t \int_0^u \Delta_{v,u}(X, \tilde{X}^h)(u-v)^{-\beta-1} dv du \right. \\ &\quad \left. + \int_0^t \int_0^u \Delta_{v,u}(Y, \tilde{Y})(u-v)^{-\beta-1} dv du \right). \end{aligned} \quad (3.51)$$

We continue our analysis by studying the term in (3.51), $\int_0^t \int_0^u \Delta_{v,u}(Y, \tilde{Y})(u-v)^{-\beta-1} dv$ in the following Lemma.

Lemma 3.4.2. *Under the hypotheses of Theorem 3.4.1. Let $\Delta_{v,u}(X, \tilde{X}^h)$ and $\Delta_{v,u}(Y, \tilde{Y})$ given in (3.24) and (3.25), respectively, then*

$$\int_0^t \int_0^u \Delta_{v,u}(Y, \tilde{Y})(u-v)^{-\beta-1} dv du \leq C(\omega) h^{H-\rho-\beta} + C(\omega) \int_0^t Z^h(u) du + C(\omega) \int_0^t \kappa(u) du, \quad (3.52)$$

where

$$\kappa(u) = \int_0^{C_h(u)} \int_{-C_h(v)}^0 (u-v)^{-\beta-1} \Delta_{v+s, u+s}(X, \tilde{X}^h) ds dv. \quad (3.53)$$

Proof. We combine the structural properties of the functional Y from Lemma 3.1.1 with the discretization errors introduced by the Euler scheme.

From (3.25)

$$\begin{aligned} \Delta_{v,u}(Y, \tilde{Y}) \leq & \int_{-r}^0 |K(u, s, X(u+s)) - K(u, C_h(s), \tilde{X}^h(u+C_h(s))) \\ & - K(v, s, X(v+s)) + K(v, C_h(s), \tilde{X}^h(v+C_h(s)))| ds, \end{aligned}$$

by properly applying Corollary (3.4.1) to the function K , we obtain

$$\int_0^u \Delta_{v,u}(Y, \tilde{Y})(u-v)^{-\beta-1} dv \leq C \sum_{i=1}^7 M_i(u), \quad (3.54)$$

where

$$\begin{aligned} M_1(u) &= \int_0^u \int_{-r}^0 |u-v| |s - C_h(s)| (u-v)^{-\beta-1} ds dv \\ M_2(u) &= \int_0^u \int_{-r}^0 |u-v| |X(v+s) - \tilde{X}^h(v+C_h(s))| (u-v)^{-\beta-1} ds dv \\ M_3(u) &= \int_0^u \int_{-r}^0 |u-v| |X(u+s) - \tilde{X}^h(u+C_h(s))| (u-v)^{-\beta-1} ds dv \\ M_4(u) &= \int_0^u \int_{-r}^0 |X(u+s) - X(v+s) - \tilde{X}^h(u+C_h(s)) + \tilde{X}^h(v+C_h(s))| (u-v)^{-\beta-1} ds dv \\ M_5(u) &= \int_0^u \int_{-r}^0 |X(u+s) - X(v+s)| |s - C_h(s)| (u-v)^{-\beta-1} ds dv \\ M_6(u) &= \int_0^u \int_{-r}^0 |X(u+s) - X(v+s)| |X(v+s) - \tilde{X}^h(v+C_h(s))| (u-v)^{-\beta-1} ds dv \\ M_7(u) &= \int_0^u \int_{-r}^0 |X(u+s) - X(v+s)| |X(u+s) - \tilde{X}^h(u+C_h(s))| (u-v)^{-\beta-1} ds dv. \end{aligned} \quad (3.55)$$

For M_1 we have,

$$M_1(u) \leq hr \int_0^u (u-v)^{-\beta} dv du \leq Ch. \quad (3.56)$$

From Theorem 3.3.1 and recalling that $X(s) = \tilde{X}^h(s)$ for $-r \leq s \leq 0$, as both correspond to the initial condition, and following the analysis on the different subintervals as in Lemma (3.3.5), we obtain

$$\begin{aligned}
M_2(u) &\leq \int_0^u (u-v)^{-\beta} \int_{-r}^0 \left[|X(v+s) - \tilde{X}^h(v+s)| + |\tilde{X}^h(v+s) - \tilde{X}^h(v+C_h(s))| \right] ds dv \\
&\leq \int_0^u (u-v)^{-\beta} \int_{-v}^0 Z^h(v+s) ds dv + \int_0^u (u-v)^{-\beta} \left\{ \int_{-r}^{-v} |\phi(v+s) - \phi(v+C_h(s))| ds \right. \\
&\quad + \int_{-v}^{-C_h(v)} \left[|\tilde{X}^h(v+s) - \tilde{X}^h(0)| + |\tilde{X}^h(0) - \phi(v+C_h(s))| \right] ds \\
&\quad \left. + \int_{-C_h(v)}^0 |\tilde{X}^h(v+s) - \tilde{X}^h(v+C_h(s))| ds \right\} dv \\
&\leq C \int_0^u (u-v)^{-\beta} Z^h(v) dv + C(\omega) \int_0^u (u-v)^{-\beta} \left\{ \int_{-r}^{-v} (s-C_h(s))^{H-\rho} ds \right. \\
&\quad \left. + \int_{-v}^{-C_h(v)} [(v-C_h(v))^{H-\rho} + (s-C_h(s))^{H-\rho}] ds + \int_{-C_h(v)}^0 (s-C_h(s))^{H-\rho} ds \right\} dv \\
&\leq C Z^h(u) \int_0^u (u-v)^{-\beta} dv + C(\omega) h^{H-\rho} \int_0^u (u-v)^{-\beta} dv \\
&\leq C Z^h(u) + C(\omega) h^{H-\rho}.
\end{aligned} \tag{3.57}$$

Analogously,

$$\begin{aligned}
M_3(u) &\leq \int_0^u (u-v)^{-\beta} \int_{-r}^0 \left[|X(u+s) - \tilde{X}^h(u+s)| + |\tilde{X}^h(u+s) - \tilde{X}^h(u+C_h(s))| \right] ds dv \\
&\leq C Z^h(u) + C(\omega) h^{H-\rho}.
\end{aligned} \tag{3.58}$$

From Theorem (3.1.1) and since $H - \rho - \beta > 0$,

$$\begin{aligned}
M_5(u) &\leq h \int_0^u (u-v)^{-\beta-1} \left[\int_{-r}^{-u} |\phi(u+s) - \phi(v+s)| ds \right. \\
&\quad \left. + \int_{-u}^{-v} |X(u+s) - X(0)| + |X(0) - \phi(v+s)| ds + \int_{-v}^0 |X(u+s) - X(v+s)| ds \right] dv \\
&\leq C(\omega) h \int_0^u (u-v)^{-\beta-1} 4(u-v)^{H-\rho} dv \leq C(\omega) h,
\end{aligned} \tag{3.59}$$

We now study M_6 . With an analysis similar to that used for M_2 ,

$$\begin{aligned}
 M_6(u) &\leq \int_0^u (u-v)^{-\beta-1} \left[\int_{-r}^{-u} |\phi(u+s) - \phi(v+s)| |\phi(v+s) - \phi(v+C_h(s))| ds \right. \\
 &\quad + \int_{-u}^{-v} (|X(u+s) - X(0)| + |X(0) - \phi(v+s)|) |\phi(v+s) - \phi(v+C_h(s))| ds \\
 &\quad + \int_{-v}^{-C_h(v)} |X(u+s) - X(v+s)| [|X(v+s) - X(0)| + |X(0) - \phi(v+C_h(s))|] ds \\
 &\quad \left. + \int_{-C_h(v)}^0 |X(u+s) - X(v+s)| \left[|X(v+s) - \tilde{X}^h(v+s)| + |\tilde{X}^h(v+s) - \tilde{X}^h(v+C_h(s))| \right] dv \right] \\
 &\leq C(\omega) \int_0^u (u-v)^{-\beta-1} \left[\int_{-r}^{-u} (u-v)^{H-\rho} (s-C_h(s))^{H-\rho} ds + \int_{-u}^{-v} 2(u-v)^{H-\rho} (s-C_h(s))^{H-\rho} ds \right. \\
 &\quad + \int_{-v}^{-C_h(v)} (u-v)^{H-\rho} [(v-C_h(v))^{H-\rho} + (s-C_h(s))^{H-\rho}] ds \\
 &\quad \left. + \int_{-C_h(v)}^0 (u-v)^{H-\rho} [Z^h(v+s) + (s-C_h(s))^{H-\rho}] dv \right] dv.
 \end{aligned} \tag{3.60}$$

Hence,

$$M_6(u)du \leq C(\omega)h^{H-\rho} + C(\omega)Z^h(u). \tag{3.61}$$

Analogously to $M_6(u)$,

$$M_7(u) \leq C(\omega)h^{H-\rho} + C(\omega)Z^h(u) \tag{3.62}$$

To study $M_4(u)$, we introduce the following notation

$$\Upsilon(u, v, s) = |X(u+s) - X(v+s) - \tilde{X}^h(u+C_h(s)) + \tilde{X}^h(v+C_h(s))|. \tag{3.63}$$

Then, by splitting the interval $[-r, 0]$ into subintervals we get

$$\begin{aligned}
 M_4(u) &= \int_0^u \int_{-r}^0 \Upsilon(u, v, s)(u-v)^{-\beta-1} ds dv \\
 &= \int_0^{C_h(u)} \int_{-r}^{-u} \Upsilon(u, v, s)(u-v)^{-\beta-1} ds dv + \int_{C_h(u)}^u \int_{-r}^{-u} \Upsilon(u, v, s)(u-v)^{-\beta-1} ds dv \\
 &\quad + \int_0^{C_h(u)} \int_{-u}^{-C_h(u)} \Upsilon(u, v, s)(u-v)^{-\beta-1} ds dv + \int_{C_h(u)}^u \int_{-u}^{-C_h(u)} \Upsilon(u, v, s)(u-v)^{-\beta-1} ds dv \\
 &\quad + \int_0^{C_h(u)} \int_{-C_h(u)}^0 \Upsilon(u, v, s)(u-v)^{-\beta-1} ds dv + \int_{C_h(u)}^u \int_{-C_h(u)}^0 \Upsilon(u, v, s)(u-v)^{-\beta-1} ds dv \\
 &= \sum_{i=1}^6 A_i(u).
 \end{aligned} \tag{3.64}$$

We consider each of these cases separately.

Case 1: (A_1 and A_2) We have $0 \leq v \leq u$ and $-r \leq s \leq -u$. Then $u+s, v+s, u+C_h(s)$ and $v+C_h(s)$ are non positive; thus

$$\Upsilon(u, v, s) = |\phi(u+s) - \phi(v+s) - \phi(u+C_h(s)) + \phi(v+C_h(s))|.$$

Using the Hölder continuity of ϕ we obtain

$$\begin{aligned}
A_1(u) &\leq \int_0^{C_h(u)} \int_{-r}^{-u} (u-v)^{-\beta-1} [|\phi(u+s) - \phi(u+C_h(s))| + |\phi(v+s) - \phi(v+C_h(s))|] ds dv \\
&\leq C(\omega) \int_0^{C_h(u)} \int_{-r}^{-u} (u-v)^{-\beta-1} 2(s-C_h(s))^{H-\rho} ds dv \\
&\leq C(\omega) h^{H-\rho} \int_0^{C_h(u)} (u-v)^{-\beta-1} dv \\
&\leq C(\omega) h^{H-\rho} (u-C_h(u))^{-\beta}
\end{aligned} \tag{3.65}$$

and

$$\begin{aligned}
A_2(u) &\leq \int_{C_h(u)}^u \int_{-r}^{-u} (u-v)^{-\beta-1} [|\phi(u+s) - \phi(v+s)| + |\phi(u+C_h(s)) - \phi(v+C_h(s))|] ds dv \\
&\leq C(\omega) \int_{C_h(u)}^u \int_{-r}^{-u} (u-v)^{-\beta-1} 2(u-v)^{H-\rho} ds dv \\
&\leq C(\omega) \int_{C_h(u)}^u (u-v)^{H-\rho-\beta-1} dv \\
&\leq C(\omega) (u-C_h(u))^{H-\rho-\beta} \\
&\leq C(\omega) h^{H-\rho-\beta}.
\end{aligned} \tag{3.66}$$

Case 2: (A_3 and A_4) We have $-u \leq s \leq -C_h(u)$, hence $u+s \geq 0$ and $u+C_h(s) \leq 0$. Then $|u+C_h(s)| = -u-C_h(s) \leq s-C_h(s)$. For A_3 , we have $0 \leq v \leq C_h(u)$, then $v+s \leq 0$ and $v+C_h(s) \leq 0$. Consequently,

$$\Upsilon(u, v, s) = |X(u+s) - \phi(v+s) - \phi(u+C_h(s)) + \phi(v+C_h(s))|.$$

Theorem 3.1.1 and the Hölder continuity of ϕ implies

$$\begin{aligned}
A_3(u) &\leq \int_0^{C_h(u)} \int_{-u}^{-C_h(u)} (u-v)^{-\beta-1} [|X(u+s) - X(0)| + |\phi(v+s) - \phi(v+C_h(s))| \\
&\quad + |X(0) - \phi(u+C_h(s))|] ds dv \\
&\leq C(\omega) \int_0^{C_h(u)} \int_{-u}^{-C_h(u)} (u-v)^{-\beta-1} [(u+s)^{H-\rho} + (s-C_h(s))^{H-\rho} + |u+C_h(s)|^{H-\rho}] ds dv \\
&\leq C(\omega) \int_0^{C_h(u)} \int_{-u}^{-C_h(u)} (u-v)^{-\beta-1} [(u-C_h(u))^{H-\rho} + 2(s-C_h(s))^{H-\rho}] ds dv \\
&\leq C(\omega) h^{H-\rho+1} \int_0^{C_h(u)} (u-v)^{-\beta-1} dv \\
&\leq C(\omega) h^{H-\rho+1} (u-C_h(u))^{-\beta}.
\end{aligned} \tag{3.67}$$

Now, we divide the integral in $A_4(u)$ as follows

$$\begin{aligned}
A_4(u) &= \int_{C_h(u)}^u \int_{-u}^{-v} (u-v)^{-\beta-1} \Upsilon(u, v, s) ds dv + \int_{C_h(u)}^u \int_{-v}^{-C_h(u)} (u-v)^{-\beta-1} \Upsilon(u, v, s) ds dv \\
&= A_{4-1}(u) + A_{4-2}(u).
\end{aligned} \tag{3.68}$$

For $A_{4-1}(u)$, $C_h(u) < v < u$ and $-u \leq s \leq -v$, then $v + s \leq 0$ and $v + C_h(s) \leq 0$, since $C_h(s) < s$. Analogously to $A_3(u)$,

$$\begin{aligned}
 A_{4-1}(u) &\leq \int_{C_h(u)}^u \int_{-u}^{-v} (u-v)^{-\beta-1} [|X(u+s) - X(0)| + |\phi(v+s) - \phi(v+C_h(s))| \\
 &\quad + |X(0) - \phi(u+C_h(s))|] ds dv \\
 &\leq C(\omega) \int_{C_h(u)}^u \int_{-u}^{-v} (u-v)^{-\beta-1} [(u+s)^{H-\rho} + (s-C_h(s))^{H-\rho} + |u+C_h(s)|^{H-\rho}] ds dv \\
 &\leq C(\omega) \int_{C_h(u)}^u \int_{-u}^{-v} (u-v)^{-\beta-1} [(u-C_h(u))^{H-\rho} + (s-C_h(s))^{H-\rho} + (s-C_h(s))^{H-\rho}] ds dv \\
 &\leq C(\omega) h^{H-\rho} \int_{C_h(u)}^u (u-v)^{-\beta} dv \leq C(\omega) h^{H-\rho} (u-C_h(u))^{1-\beta} \leq C(\omega) h^{H-\rho+1-\beta}. \quad (3.69)
 \end{aligned}$$

For $A_{4-2}(u)$, $C_h(u) < v < u$ and $-v \leq s \leq -C_h(u)$, then $v + s \geq 0$ and $v + C_h(s) \leq 0$ then

$$\begin{aligned}
 \Upsilon(u, v, s) &= |X(u+s) - X(v+s) - \phi(u+C_h(s)) + \phi(v+C_h(s))| \\
 &\leq |X(u+s) - X(v+s)| + |\phi(u+C_h(s)) - \phi(v+C_h(s))| \leq C(\omega)(u-v)^{H-\rho}.
 \end{aligned}$$

Hence,

$$A_{4-2}(u) \leq C(\omega) \int_{C_h(u)}^u (u-v)^{H-\rho-\beta-1} \int_{-v}^{-C_h(u)} ds dv \leq C(\omega) h^{H-\rho-\beta}. \quad (3.70)$$

Thus, from (3.68), (3.69) and (3.70),

$$A_4(u) \leq C(\omega) h^{H-\rho-\beta}. \quad (3.71)$$

Case 3: (A_5 and A_6) We have $0 \leq v \leq u$ and $-C_h(u) \leq s \leq 0$, then $u + s \geq 0$ and $u + C_h(s) \geq 0$. We divide the integral in $A_5(u)$ as follows,

$$\begin{aligned}
 A_5(u) &= \int_0^{C_h(u)} \int_{-C_h(u)}^{-v} (u-v)^{-\beta-1} \Upsilon(u, v, s) ds dv + \int_0^{C_h(u)} \int_{-v}^{-C_h(v)} (u-v)^{-\beta-1} \Upsilon(u, v, s) ds dv \\
 &\quad + \int_0^{C_h(u)} \int_{-C_h(v)}^0 (u-v)^{-\beta-1} \Upsilon(u, v, s) ds dv \\
 &= A_{5-1}(u) + A_{5-2}(u) + A_{5-3}(u). \quad (3.72)
 \end{aligned}$$

For $A_{5-1}(u)$, $-C_h(u) \leq s \leq -v$, then $v + s \leq 0$ and $v + C_h(s) \leq 0$. From Theorem 3.3.1, (3.27) and the Hölder continuity of ϕ

$$\begin{aligned}
 \Upsilon(u, v, s) &= |X(u+s) - \phi(v+s) - \tilde{X}^h(u+C_h(s)) + \phi(v+C_h(s))| \\
 &\leq |X(u+s) - \tilde{X}^h(u+s)| + |\tilde{X}^h(u+s) - \tilde{X}^h(u+C_h(s))| + |\phi(v+s) - \phi(v+C_h(s))| \\
 &\leq Z^h(u+s) + C(\omega)(s-C_h(s))^{H-\rho}.
 \end{aligned}$$

Therefore, by $C_h(u) - v \leq u - v$,

$$\begin{aligned}
 A_{5-1}(u) &\leq \int_0^{C_h(u)} \int_{-C_h(u)}^{-v} (u-v)^{-\beta-1} [Z^h(u+s) + C(\omega)(s-C_h(s))^{H-\rho}] ds dv \\
 &\leq [Z^h(u) + C(\omega)h^{H-\rho}] \int_0^{C_h(u)} (u-v)^{-\beta} dv \\
 &\leq CZ^h(u) + C(\omega)h^{H-\rho}. \quad (3.73)
 \end{aligned}$$

For $A_{5-2}(u)$, $-v \leq s \leq -C_h(v)$, then $-C_h(u) \leq C_h(s) \leq -v \leq s \leq -C_h(v)$. Thus, $v + s \geq 0$, $v + C_h(s) \leq 0$ and $|v + C_h(s)| = -v - C_h(s) \leq u - v$. Hence,

$$\begin{aligned} \Upsilon(u, v, s) &= |X(u + s) - X(v + s) - \tilde{X}^h(u + C_h(s)) + \phi(v + C_h(s))| \\ &\leq |X(u + s) - X(v + s)| + |\tilde{X}^h(u + C_h(s)) - \tilde{X}^h(0)| + |\tilde{X}^h(0) - \phi(v + C_h(s))| \\ &\leq C(\omega)[(u - v)^{H-\rho} + (u + C_h(s))^{H-\rho} + |v + C_h(s)|^{H-\rho}] \\ &\leq C(\omega)(u - v)^{H-\rho}. \end{aligned}$$

Then

$$\begin{aligned} A_{5-2}(u) &\leq C(\omega) \int_0^{C_h(u)} \int_{-v}^{-C_h(v)} (u - v)^{H-\rho-\beta-1} ds dv \\ &\leq hC(\omega) \int_0^{C_h(u)} (u - v)^{H-\rho-\beta-1} dv \\ &\leq C(\omega)h. \end{aligned} \tag{3.74}$$

For $A_{5-3}(u)$, $-C_h(v) \leq s \leq 0$, then $v + s \geq 0$ and $v + C_h(s) \geq 0$, by the definition of Υ given in (3.63)

$$\begin{aligned} \Upsilon(u, v, s) &= |X(u + s) - X(v + s) - \tilde{X}^h(u + C_h(s)) + \tilde{X}^h(v + C_h(s))| \\ &\leq |X(u + s) - X(v + s) - \tilde{X}^h(u + s) + \tilde{X}^h(v + s)| \\ &\quad + |\tilde{X}^h(u + s) - \tilde{X}^h(u + C_h(s))| + |\tilde{X}^h(v + C_h(s)) - \tilde{X}^h(v + s)| \\ &= \Delta_{v+s, u+s}(X, \tilde{X}^h) + |\tilde{X}^h(u + s) - \tilde{X}^h(u + C_h(s))| + |\tilde{X}^h(v + C_h(s)) - \tilde{X}^h(v + s)|. \end{aligned}$$

By the definition of κ given in (3.53), Theorem 3.3.1 and (3.9)

$$\begin{aligned} A_{5-3}(u) &\leq \kappa(u) + \int_0^{C_h(u)} \int_{-C_h(v)}^0 (u - v)^{-\beta-1} [|\tilde{X}^h(u + s) - \tilde{X}^h(u + C_h(s))| \\ &\quad + |\tilde{X}^h(v + C_h(s)) - \tilde{X}^h(v + s)|] ds dv \\ &\leq \kappa(u) + C(\omega)h^{H-\rho} \int_0^{C_h(u)} \int_{-C_h(v)}^0 (u - v)^{-\beta-1} ds dv \\ &\leq \kappa(u) + C(\omega)h^{H-\rho}(u - C_h(u))^{-\beta} \end{aligned} \tag{3.75}$$

Hence, from (3.72), (3.73), (3.74), and (3.75), we obtain

$$A_5(u) \leq CZ^h(u) + C(\omega)h^{H-\rho} + \kappa(u) + C(\omega)h^{H-\rho}(u - C_h(u))^{-\beta}. \tag{3.76}$$

For A_6 , $-C_h(u) \leq s \leq 0$ and $C_h(u) \leq v \leq u$, then $v + s \geq 0$ and $v + C_h(s) \geq 0$. Then,

$$\Upsilon(u, v, s) \leq |X(u + s) - X(v + s)| + |\tilde{X}^h(u + C_h(s)) - \tilde{X}^h(v + C_h(s))| \leq C(\omega)(u - v)^{H-\rho}.$$

Hence,

$$\begin{aligned} A_6 &\leq C(\omega) \int_0^t \int_{C_h(u)}^u \int_{-C_h(u)}^0 (u - v)^{-\beta-1} (u - v)^{H-\rho} ds dv du \\ &\leq C(\omega) \int_0^t \int_{C_h(u)}^u (u - v)^{H-\rho-\beta-1} dv du \\ &\leq C(\omega)h^{H-\rho-\beta}. \end{aligned} \tag{3.77}$$

Using the equality (3.64) and the inequalities (3.65), (3.66), (3.67), (3.71), (3.76) and (3.77), we obtain

$$M_4(u) \leq CZ^h(u) + C(\omega)h^{H-\rho-\beta} + \kappa(u) + C(\omega)h^{H-\rho}(u - C_h(u))^{-\beta}. \quad (3.78)$$

In summary, from (3.54)

$$\int_0^t \int_0^u \Delta_{v,u}(Y, \tilde{Y})(u-v)^{-\beta-1} dv du \leq C \sum_{i=1}^7 \int_0^t M_i(u) du \quad (3.79)$$

and by bounds (3.56) to (3.62) and (3.78), together with (3.14), we obtain the result (3.52).

Let us now study $\int_0^u \Delta_{v,u}(X, \tilde{X}^h)(u-v)^{-\beta-1} dv$ in the following lemma.

Lemma 3.4.3. *Under the conditions of Theorem 3.4.1. Let $0 \leq v < u \leq T$ and $\Delta_{v,u}(X, \tilde{X}^h)$ be given in (3.24), defining*

$$\theta(u) := \int_0^u \Delta_{v,u}(X, \tilde{X}^h)(u-v)^{-\beta-1} dv,$$

then

$$\theta(u) \leq C(\omega) \left(h^{H-\rho-\beta} + \int_0^u (Z^h(s) + \theta(s) + \kappa(s))(u-s)^{-\beta} ds + \int_0^u Z^h(s)(u-s)^{-2\beta} ds \right).$$

Proof. Let $0 \leq v < u \leq T$, by (1.4),

$$\begin{aligned} \Delta_{v,u}(X, \tilde{X}^h) &\leq \int_v^u |b(s, X(s), Y(s)) - b(C_h(s), \tilde{X}^h(C_h(s)), \tilde{Y}(C_h(s)))| ds \\ &\quad + C(\omega) \left(\int_v^u |\sigma(s, X(s), Y(s)) - \sigma(C_h(s), \tilde{X}^h(C_h(s)), \tilde{Y}(C_h(s)))| (s-v)^{-\beta} ds \right. \\ &\quad + \int_v^u \int_v^s |\sigma(s, X(s), Y(s)) - \sigma(C_h(s), \tilde{X}^h(C_h(s)), \tilde{Y}(C_h(s))) \\ &\quad \left. - (\sigma(z, X(z), Y(z)) - \sigma(C_h(z), \tilde{X}^h(C_h(z)), \tilde{Y}(C_h(z))))| (s-z)^{-\beta-1} dz ds \right). \end{aligned} \quad (3.80)$$

Then

$$\theta_u = \int_0^u \Delta_{v,u}(X, \tilde{X}^h)(u-v)^{-\beta-1} dv \leq \sum_{i=1}^9 J_i, \quad (3.81)$$

where

$$\begin{aligned}
 J_1(u) &= \int_0^u \int_v^u |b(s, X(s), Y(s)) - b(s, \tilde{X}^h(s), \tilde{Y}(s))| ds (u-v)^{-\beta-1} dv, \\
 J_2(u) &= \int_0^u \int_v^u |b(s, \tilde{X}^h(s), \tilde{Y}(s)) - b(C_h(s), \tilde{X}^h(s), \tilde{Y}(s))| ds (u-v)^{-\beta-1} dv, \\
 J_3(u) &= \int_0^u \int_v^u |b(C_h(s), \tilde{X}^h(s), \tilde{Y}(s)) - b(C_h(s), \tilde{X}^h(C_h(s)), \tilde{Y}(C_h(s)))| ds (u-v)^{-\beta-1} dv, \\
 J_4(u) &= \int_0^u C(\omega) \int_v^u |\sigma(s, X(s), Y(s)) - \sigma(s, \tilde{X}^h(s), \tilde{Y}(s))| (s-v)^{-\beta} ds (u-v)^{-\beta-1} dv, \\
 J_5(u) &= \int_0^u C(\omega) \int_v^u |\sigma(s, \tilde{X}^h(s), \tilde{Y}(s)) - \sigma(C_h(s), \tilde{X}^h(s), \tilde{Y}(s))| (s-v)^{-\beta} ds (u-v)^{-\beta-1} dv, \\
 J_6(u) &= \int_0^u C(\omega) \int_v^u |\sigma(C_h(s), \tilde{X}^h(s), \tilde{Y}(s)) - \sigma(C_h(s), \tilde{X}^h(C_h(s)), \tilde{Y}(C_h(s)))| (s-v)^{-\beta} ds (u-v)^{-\beta-1} dv, \\
 J_7(u) &= \int_0^u C(\omega) \int_v^u \int_v^s |\sigma(s, X(s), Y(s)) - \sigma(s, \tilde{X}^h(s), \tilde{Y}(s)) \\
 &\quad - (\sigma(z, X(z), Y(z)) - \sigma(z, \tilde{X}^h(z), \tilde{Y}(z)))| (s-z)^{-\beta-1} dz ds (u-v)^{-\beta-1} dv, \\
 J_8(u) &= \int_0^u C(\omega) \int_v^u \int_v^s |\sigma(s, \tilde{X}^h(s), \tilde{Y}(s)) - \sigma(C_h(s), \tilde{X}^h(s), \tilde{Y}(s)) \\
 &\quad - (\sigma(z, \tilde{X}^h(z), \tilde{Y}(z)) - \sigma(C_h(z), \tilde{X}^h(z), \tilde{Y}(z)))| (s-z)^{-\beta-1} dz ds (u-v)^{-\beta-1} dv, \\
 J_9(u) &= \int_0^u C(\omega) \int_v^u \int_v^s |\sigma(C_h(s), \tilde{X}^h(s), \tilde{Y}(s)) - \sigma(C_h(s), \tilde{X}^h(C_h(s)), \tilde{Y}(C_h(s))) \\
 &\quad - (\sigma(C_h(z), \tilde{X}^h(z), \tilde{Y}(z)) - \sigma(C_h(z), \tilde{X}^h(C_h(z)), \tilde{Y}(C_h(z))))| (s-z)^{-\beta-1} dz ds (u-v)^{-\beta-1} dv.
 \end{aligned} \tag{3.82}$$

From Hypothesis 1 (3.5), changing the limits of integration and (3.30), we obtain for J_1 in (3.82)

$$\begin{aligned}
 J_1(u) &\leq L_2 \int_0^u \int_v^u (|X(s) - \tilde{X}^h(s)| + |Y(s) - \tilde{Y}(s)|) ds (u-v)^{-\beta-1} dv \\
 &\leq C(\omega) \int_0^u \int_v^u (Z^h(s) + h + h^{H-\rho}) ds (u-v)^{-\beta-1} dv \\
 &\leq C(\omega) \int_0^u (Z^h(s) + h + h^{H-\rho}) \int_0^s (u-v)^{-\beta-1} dv ds \\
 &\leq C(\omega) \int_0^u Z^h(s) (u-s)^{-\beta} ds + C(\omega) (h + h^{H-\rho}).
 \end{aligned} \tag{3.83}$$

and

$$J_2(u) \leq L_1 \int_0^u \int_v^u |s - C_h(s)| ds (u-v)^{-\beta-1} dv \leq L_1 h \int_0^u (u-v)^{-\beta} dv \leq Ch. \tag{3.84}$$

By Theorem 3.3.1 and Corollary 3.3.1,

$$\begin{aligned}
 J_3(u) &\leq L_2 \int_0^u \int_v^u (|\tilde{X}^h(s) - \tilde{X}^h(C_h(s))| + |\tilde{Y}(s) - \tilde{Y}(C_h(s))|) ds (u-v)^{-\beta-1} dv \\
 &\leq L_2 \int_0^u \int_v^u (C(\omega) h^{H-\rho} + r K_1 h + C(\omega) h^{H-\rho}) ds (u-v)^{-\beta-1} dv \\
 &\leq C(\omega) (h^{H-\rho} + h).
 \end{aligned} \tag{3.85}$$

For the next calculus, recall that

$$\int_0^s (s-v)^{-\beta}(u-v)^{-\beta-1}dv \leq C_0(u-s)^{-2\beta}, \quad (3.86)$$

where $C_0 = \int_0^\infty (1+y)^{-\beta-1}y^{-\beta}dy$.

Using Hypothesis 1 (3.5), (3.30), (3.86) and since $\beta < 1/2$, we obtain

$$\begin{aligned} J_4 &\leq L_2 C(\omega) \int_0^u \int_v^u (|X(s) - \tilde{X}^h(s)| + |Y(s) - \tilde{Y}(s)|)(s-v)^{-\beta}ds(u-v)^{-\beta-1}dv \\ &\leq C(\omega) \int_0^u \int_v^u (Z^h(s) + rK_1h + rK_2Z^h(s))(s-v)^{-\beta}ds(u-v)^{-\beta-1}dv \\ &\leq C(\omega) \int_0^u \int_v^u Z^h(s)(s-v)^{-\beta}ds(u-v)^{-\beta-1}dv + C(\omega)h \int_0^u \int_v^u (s-v)^{-\beta}ds(u-v)^{-\beta-1}dv \\ &\leq C(\omega) \int_0^u Z^h(s) \int_0^s (s-v)^{-\beta}(u-v)^{-\beta-1}dvds + C(\omega)h \int_0^u (u-v)^{-2\beta}dv \\ &\leq C(\omega) \int_0^u Z^h(s)(u-s)^{-2\beta}ds + C(\omega)h, \end{aligned} \quad (3.87)$$

We estimate J_5 similarly to how we obtained J_2 , and J_6 similarly to how we obtained J_3 , then

$$J_5 \leq C(\omega)h \quad (3.88)$$

and

$$J_6 \leq C(\omega)(h^{H-\rho} + h). \quad (3.89)$$

To study J_7 , we apply Corollary (3.4.1), then

$$J_7 \leq C(\omega) \sum_{k=1}^5 J_{7-k}, \quad (3.90)$$

where

$$\begin{aligned} J_{7-1} &= \int_0^u \int_v^u \int_v^s |X(s) - \tilde{X}^h(s) - X(z) + \tilde{X}^h(z)|(s-z)^{-\beta-1}dzds(u-v)^{-\beta-1}dv, \\ J_{7-2} &= \int_0^u \int_v^u \int_v^s |X(s) - \tilde{X}^h(s)||s-z|(s-z)^{-\beta-1}dzds(u-v)^{-\beta-1}dv, \\ J_{7-3} &= \int_0^u \int_v^u \int_v^s |X(s) - \tilde{X}^h(s)| \left[|\tilde{X}^h(s) - \tilde{X}^h(z)| + |X(s) - X(z)| \right. \\ &\quad \left. |\tilde{Y}(s) - \tilde{Y}(z)| + |Y(s) - Y(z)| \right] (s-z)^{-\beta-1}dzds(u-v)^{-\beta-1}dv, \\ J_{7-4} &= \int_0^u \int_v^u \int_v^s |Y(s) - \tilde{Y}(s) - Y(z) + \tilde{Y}(z)|(s-z)^{-\beta-1}dzds(u-v)^{-\beta-1}dv, \\ J_{7-5} &= \int_0^u \int_v^u \int_v^s |Y(s) - \tilde{Y}(s)| \left[|s-z| + |\tilde{X}^h(s) - \tilde{X}^h(z)| + |X(s) - X(z)| \right. \\ &\quad \left. |\tilde{Y}(s) - \tilde{Y}(z)| + |Y(s) - Y(z)| \right] (s-z)^{-\beta-1}dzds(u-v)^{-\beta-1}dv. \end{aligned}$$

By (3.24) and changing the limits of integration, we write

$$\begin{aligned}
J_{7-1} &= \int_0^u \int_0^s \Delta_{z,s}(X, \tilde{X})(s-z)^{-\beta-1} \int_0^z (u-v)^{-\beta-1} dv dz ds \\
&\leq \int_0^u \int_0^s (u-z)^{-\beta} \Delta_{z,s}(X, \tilde{X})(s-z)^{-\beta-1} dz ds \\
&\leq \int_0^u (u-s)^{-\beta} \int_0^s \Delta_{z,s}(X, \tilde{X})(s-z)^{-\beta-1} dz ds \\
&= \int_0^u (u-s)^{-\beta} \theta_s ds,
\end{aligned}$$

and

$$\begin{aligned}
J_{7-2} &\leq \int_0^u \int_v^u Z^h(s) \int_v^s (s-z)^{-\beta} dz ds (u-v)^{-\beta-1} dv \\
&\leq \int_0^u \int_v^u Z^h(s) (s-v)^{1-\beta} ds (u-v)^{-\beta-1} dv \\
&\leq C \int_0^u Z^h(s) \int_0^s (u-v)^{-\beta-1} dv ds \\
&\leq C \int_0^u Z^h(s) (u-s)^{-\beta} ds.
\end{aligned}$$

Due to Theorem 3.3.1, Corollary 3.3.1 and (3.9),

$$\begin{aligned}
J_{7-3} &\leq C(\omega) \int_0^u \int_v^u \int_v^s Z^h(s) [2(s-z)^{H-\rho} + 2rK_1(s-z) + 2rK_2(s-z)^{H-\rho}] \\
&\quad \times (s-z)^{-\beta-1} dz ds (u-v)^{-\beta-1} dv \\
&\leq C(\omega) \int_0^u \int_v^u Z^h(s) \int_v^s (s-z)^{H-\rho-\beta-1} dz ds (u-v)^{-\beta-1} dv \\
&\leq C(\omega) \int_0^u Z^h(s) (u-s)^{-\beta} ds.
\end{aligned}$$

Using Theorems 3.1.1 and 3.3.1, Corollary 3.3.1 and (3.30); and calculations analogous to those done for J_{7-3} , we get

$$\begin{aligned}
J_{7-5} &\leq C(\omega) \int_0^u \int_v^u \int_v^s [h + h^{H-\rho} + Z^h(s)] (|s-z| + |s-z|^{H-\rho}) (s-z)^{-\beta-1} dz ds (u-v)^{-\beta-1} dv \\
&\leq C(\omega) \int_0^u \int_v^u Z^h(s) \int_v^s (s-z)^{H-\rho-\beta-1} dz ds (u-v)^{-\beta-1} dv + C(\omega) h^{H-\rho} \\
&\leq C(\omega) \int_0^u Z^h(s) (u-s)^{-\beta} ds + C(\omega) h^{H-\rho}.
\end{aligned}$$

Now, we study the term J_{7-4} , similarly to J_{7-1} , by (3.25)

$$\begin{aligned}
J_{7-4} &= \int_0^u \int_0^s \Delta_{z,s}(Y, \tilde{Y})(s-z)^{-\beta-1} \int_0^z (u-v)^{-\beta-1} dv dz ds \\
&\leq \int_0^u (u-s)^{-\beta} \int_0^s \Delta_{z,s}(Y, \tilde{Y})(s-z)^{-\beta-1} dz ds.
\end{aligned} \tag{3.91}$$

Recalling the calculation made in Lemma 3.4.2, applying the inequality (3.79) to $\int_0^s \Delta_{z,s}(Y, \tilde{Y})(s-z)^{-\beta-1} dz$, we have

$$J_{7-4} \leq \int_0^u (u-s)^{-\beta} \sum_{i=1}^7 M_i(s) ds,$$

where $M_i(s)$ are the corresponding integrals given in (3.55).

Using the bound found in (3.56) to (3.62), we obtain

$$\int_0^u (u-s)^{-\beta} [M_1(s) + M_2(s) + M_3(s) + M_5(s) + M_6(s) + M_7(s)] ds \leq C \int_0^u Z^h(s) (u-s)^{-\beta} ds + C(\omega) h^{\rho} \quad (3.92)$$

We will study now $\int_0^u (u-s)^{-\beta} M_4(s) ds$, from (3.78),

$$\int_0^u (u-s)^{-\beta} M_4(s) ds \leq \int_0^u (u-s)^{-\beta} [CZ^h(s) + C(\omega) h^{H-\rho-\beta} + \kappa(s) + C(\omega) h^{H-\rho} (s-C_h(s))^{-\beta}] ds, \quad (3.93)$$

where $\kappa(s)$ is defined in (3.53). We will study the integral of the last term in (3.93),

$$\int_0^u (u-s)^{-\beta} (s-C_h(s))^{-\beta} ds = \int_0^{C_h(u)} (u-s)^{-\beta} (s-C_h(s))^{-\beta} ds + \int_{C_h(u)}^u (u-s)^{-\beta} (s-C_h(s))^{-\beta} ds$$

From the proof of Lemma 2.2. in [16], we have that

$$\int_0^{C_h(u)} (u-s)^{-\beta} (s-C_h(s))^{-\beta} ds \leq Ch^{-\beta}.$$

Now

$$\int_{C_h(u)}^u (u-s)^{-\beta} (s-C_h(s))^{-\beta} ds \leq h^{-\beta} \int_{C_h(u)}^u (u-s)^{-\beta} ds \leq Ch^{-\beta} (u-C_h(u))^{1-\beta} \leq Ch^{1-2\beta}.$$

Hence, we obtain that

$$\int_0^u (u-s)^{-\beta} (s-C_h(s))^{-\beta} ds \leq Ch^{-\beta}. \quad (3.94)$$

Therefore, from (3.93)

$$\int_0^u (u-s)^{-\beta} M_4(s) ds \leq C \int_0^u Z^h(s) (u-s)^{-\beta} ds + \int_0^u \kappa(s) (u-s)^{-\beta} ds + C(\omega) h^{H-\rho-\beta} \quad (3.95)$$

Hence, from (3.91), (3.92) and (3.95), we obtain

$$J_{7-4} \leq C(\omega) h^{H-\rho-\beta} + C \int_0^u Z^h(s) (u-s)^{-\beta} ds + \int_0^u \kappa(s) (u-s)^{-\beta} ds.$$

Finally, from (3.90)

$$J_7 \leq C(\omega) \left(h^{H-\rho-\beta} + \int_0^u \theta_s (u-s)^{-\beta} ds + \int_0^u Z^h(s) (u-s)^{-\beta} ds + \int_0^u \kappa(s) (u-s)^{-\beta} ds \right). \quad (3.96)$$

We now study J_8 ,

$$\begin{aligned}
J_8(u) &\leq \int_0^u C(\omega) \int_v^u \int_0^s |\sigma(s, \tilde{X}^h(s), \tilde{Y}(s)) - \sigma(C_h(s), \tilde{X}^h(s), \tilde{Y}(s)) \\
&\quad - (\sigma(z, \tilde{X}^h(z), \tilde{Y}(z)) - \sigma(C_h(z), \tilde{X}^h(z), \tilde{Y}(z)))| (s-z)^{-\beta-1} dz ds (u-v)^{-\beta-1} dv \\
&\leq C(\omega) \left[\int_0^u \int_v^u \int_0^{C_h(s)} \cdots dz ds dv + \int_0^u \int_v^u \int_{C_h(s)}^s \cdots dz ds dv \right] \\
&= C(\omega) [J_{8-1}(u) + J_{8-2}(u)].
\end{aligned}$$

By Hypothesis 1, (3.4) - (3.6),

$$\begin{aligned}
J_{8-1}(u) &= \int_0^u \int_v^u \int_0^{C_h(s)} |\sigma(s, \tilde{X}^h(s), \tilde{Y}(s)) - \sigma(C_h(s), \tilde{X}^h(s), \tilde{Y}(s)) \\
&\quad - (\sigma(z, \tilde{X}^h(z), \tilde{Y}(z)) - \sigma(C_h(z), \tilde{X}^h(z), \tilde{Y}(z)))| (s-z)^{-\beta-1} dz ds (u-v)^{-\beta-1} dv \\
&\leq C \int_0^u \int_v^u \int_0^{C_h(s)} [|s - C_h(s)| + |z - C_h(z)|] (s-z)^{-\beta-1} dz ds (u-v)^{-\beta-1} dv \\
&\leq Ch \int_0^u \int_v^u \int_0^{C_h(s)} (s-z)^{-\beta-1} dz ds (u-v)^{-\beta-1} dv \\
&\leq Ch \int_0^u \int_v^u (s - C_h(s))^{-\beta} ds (u-v)^{-\beta-1} dv,
\end{aligned}$$

making the change of variables and using the inequality (3.94),

$$\begin{aligned}
J_{8-1}(u) &\leq Ch \int_0^u \int_0^s (u-v)^{-\beta-1} dv (s - C_h(s))^{-\beta} ds \\
&\leq Ch \int_0^u (u-s)^{-\beta} (s - C_h(s))^{-\beta} ds \\
&\leq Ch^{1-\beta}.
\end{aligned}$$

To bound $J_{8-2}(u)$, by hypothesis 1 and recalling that if $z \in [C_h(s), s]$, then $C_h(z) = C_h(s)$

$$\begin{aligned}
J_{8-2}(u) &= \int_0^u \int_v^u \int_{C_h(s)}^s |\sigma(s, \tilde{X}^h(s), \tilde{Y}(s)) - \sigma(C_h(s), \tilde{X}^h(s), \tilde{Y}(s)) \\
&\quad - (\sigma(z, \tilde{X}^h(z), \tilde{Y}(z)) - \sigma(C_h(s), \tilde{X}^h(z), \tilde{Y}(z)))| (s-z)^{-\beta-1} dz ds (u-v)^{-\beta-1} dv \\
&\leq \int_0^u \int_v^u \int_{C_h(s)}^s (s-z)^{-\beta-1} (u-v)^{-\beta-1} [|\sigma(s, \tilde{X}^h(s), \tilde{Y}(s)) - \sigma(s, \tilde{X}^h(z), \tilde{Y}(z))| \\
&\quad + |\sigma(s, \tilde{X}^h(z), \tilde{Y}(z)) - \sigma(z, \tilde{X}^h(z), \tilde{Y}(z))| \\
&\quad + |\sigma(C_h(s), \tilde{X}^h(z), \tilde{Y}(z)) - (\sigma(z, \tilde{X}^h(z), \tilde{Y}(z)))|] dz ds dv \\
&\leq C \int_0^u \int_v^u \int_{C_h(s)}^s [|s-z| + |\tilde{X}^h(s) - \tilde{X}^h(z)| + |\tilde{Y}(s) - \tilde{Y}(z)|] (s-z)^{-\beta-1} (u-v)^{-\beta-1} dz ds dv
\end{aligned}$$

and from Theorem 3.3.1 and Corollary 3.3.1

$$\begin{aligned}
 J_{8-2}(u) &\leq C(\omega) \int_0^u \int_v^u \int_{C_h(s)}^s (s-z)^{H-\rho-\beta-1} (u-v)^{-\beta-1} dz ds dv \\
 &\leq C(\omega) \int_0^u \int_v^u (s-C_h(s))^{H-\rho-\beta} (u-v)^{-\beta-1} ds dv \\
 &\leq C(\omega) h^{H-\rho-\beta} \int_0^u (u-v)^{-\beta} dv \\
 &\leq C(\omega) h^{H-\rho-\beta}.
 \end{aligned}$$

Hence, we obtain that

$$J_8 \leq C(\omega) h^{H-\rho-\beta}. \quad (3.97)$$

Similarly to J_8 ,

$$\begin{aligned}
 J_9 &\leq C(\omega) \int_0^u \int_v^u \int_0^s |\sigma(C_h(s), \tilde{X}^h(s), \tilde{Y}(s)) - \sigma(C_h(s), \tilde{X}^h(C_h(s)), \tilde{Y}(C_h(s))) \\
 &\quad - (\sigma(C_h(z), \tilde{X}^h(z), \tilde{Y}(z)) - \sigma(C_h(z), \tilde{X}^h(C_h(z)), \tilde{Y}(C_h(z))))| (s-z)^{-\beta-1} dz ds (u-v)^{-\beta-1} dv \\
 &= C(\omega) \left[\int_0^u \int_v^u \int_0^{C_h(s)} \cdots dz ds dv + \int_0^u \int_v^u \int_{C_h(s)}^s \cdots dz ds dv \right] \\
 &= C(\omega) [J_{9-1}(u) + J_{9-2}(u)],
 \end{aligned}$$

where

$$\begin{aligned}
 J_{9-1}(u) &= \int_0^u \int_v^u \int_0^{C_h(s)} |\sigma(C_h(s), \tilde{X}^h(s), \tilde{Y}(s)) - \sigma(C_h(s), \tilde{X}^h(C_h(s)), \tilde{Y}(C_h(s))) \\
 &\quad - (\sigma(C_h(z), \tilde{X}^h(z), \tilde{Y}(z)) - \sigma(C_h(z), \tilde{X}^h(C_h(z)), \tilde{Y}(C_h(z))))| (s-z)^{-\beta-1} dz ds (u-v)^{-\beta-1} dv \\
 &\leq C \int_0^u \int_v^u \int_0^{C_h(s)} [|s-C_h(s)|^{H-\rho} + |z-C_h(z)|^{H-\rho}] (s-z)^{-\beta-1} dz ds (u-v)^{-\beta-1} dv \\
 &\leq Ch^{H-\rho} \int_0^u \int_v^u \int_0^{C_h(s)} (s-z)^{-\beta-1} dz ds (u-v)^{-\beta-1} dv \\
 &\leq Ch^{H-\rho-\beta}
 \end{aligned}$$

and

$$\begin{aligned}
 J_{9-2}(u) &= \int_0^u \int_v^u \int_{C_h(s)}^s |\sigma(C_h(s), \tilde{X}^h(s), \tilde{Y}(s)) - \sigma(C_h(s), \tilde{X}^h(C_h(s)), \tilde{Y}(C_h(s))) \\
 &\quad - (\sigma(C_h(z), \tilde{X}^h(z), \tilde{Y}(z)) - \sigma(C_h(z), \tilde{X}^h(C_h(z)), \tilde{Y}(C_h(z))))| (s-z)^{-\beta-1} dz ds (u-v)^{-\beta-1} dv \\
 &= \int_0^u \int_v^u \int_{C_h(s)}^s |\sigma(C_h(s), \tilde{X}^h(s), \tilde{Y}(s)) - \sigma(C_h(s), \tilde{X}^h(z), \tilde{Y}(z))| (s-z)^{-\beta-1} dz ds (u-v)^{-\beta-1} dv \\
 &\leq C(\omega) \int_0^u \int_v^u \int_{C_h(s)}^s (s-z)^{H-\rho-\beta-1} (u-v)^{-\beta-1} dz ds dv \\
 &\leq C(\omega) h^{H-\rho-\beta}.
 \end{aligned}$$

Thus,

$$J_9 \leq C(\omega) h^{H-\rho-\beta}. \quad (3.98)$$

In summary, by (3.81) and bounds from (3.83) to (3.85), (3.87) to (3.90), (3.96), (3.97) and (3.98), we obtain that

$$\begin{aligned} \theta(u) \leq & C(\omega) \left(h^{H-\rho-\beta} + \int_0^u Z^h(s)(u-s)^{-\beta} ds + \int_0^u Z^h(s)(u-s)^{-2\beta} ds + \int_0^u \theta(s)(u-s)^{-\beta} ds \right. \\ & \left. + \int_0^u \kappa(s)(u-s)^{-\beta} ds \right). \end{aligned}$$

Hence, we obtain the result.

We will study now $\kappa(u)$.

Lemma 3.4.4. *Under hypotheses of Theorem 3.4.1. Let $0 \leq u \leq T$ and $\kappa(u)$ defined by (3.53),*

$$\kappa(u) \leq C(\omega) \left(h^{H-\rho-\beta} + \int_0^u (Z^h(s) + \theta(s) + \kappa(s))(u-s)^{-\beta} ds + \int_0^u Z^h(s)(u-s)^{-2\beta} ds \right).$$

Proof. We follow the proof of Lemma 3.4.3, calculous are similar except some technical changes. From (3.80) and (3.81),

$$\begin{aligned} \Delta_{v+s, u+s}(X, \tilde{X}^h) \leq & \int_{v+s}^{u+s} |b(w, X(w), Y(w)) - b(C_h(w), \tilde{X}^h(C_h(w)), \tilde{Y}(C_h(w)))| dw \\ & + C(\omega) \left(\int_{v+s}^{u+s} |\sigma(s, X(w), Y(w)) - \sigma(C_h(w), \tilde{X}^h(C_h(w)), \tilde{Y}(C_h(w)))| (w-v)^{-\beta} dw \right. \\ & + \int_{v+s}^{u+s} \int_{v+s}^w |\sigma(w, X(w), Y(w)) - \sigma(C_h(w), \tilde{X}^h(C_h(w)), \tilde{Y}(C_h(w))) \\ & \left. - (\sigma(z, X(z), Y(z)) - \sigma(C_h(z), \tilde{X}^h(C_h(z)), \tilde{Y}(C_h(z))))| (w-z)^{-\beta-1} dz dw \right). \end{aligned}$$

Then

$$\kappa(u) = \int_0^{C_h(u)} \int_{-C_h(v)}^0 (u-v)^{-\beta-1} \Delta_{v+s, u+s}(X, \tilde{X}^h) (u-v)^{-\beta-1} dv \leq \sum_{i=1}^9 \tilde{J}_i, \quad (3.99)$$

where

$$\begin{aligned} \tilde{J}_1(u) &= \int_0^{C_h(u)} \int_{-C_h(v)}^0 (u-v)^{-\beta-1} \int_{v+s}^{u+s} |b(w, X(w), Y(w)) - b(w, \tilde{X}^h(w), \tilde{Y}(w))| dw ds dv, \\ \tilde{J}_2(u) &= \int_0^{C_h(u)} \int_{-C_h(v)}^0 (u-v)^{-\beta-1} \int_{v+s}^{u+s} |b(w, \tilde{X}^h(w), \tilde{Y}(w)) - b(C_h(w), \tilde{X}^h(w), \tilde{Y}(w))| dw ds dv, \\ \tilde{J}_3(u) &= \int_0^{C_h(u)} \int_{-C_h(v)}^0 (u-v)^{-\beta-1} \int_{v+s}^{u+s} |b(C_h(w), \tilde{X}^h(w), \tilde{Y}(w)) - \\ & \quad - b(C_h(w), \tilde{X}^h(C_h(w)), \tilde{Y}(C_h(w)))| dw ds dv, \\ \tilde{J}_4(u) &= \int_0^{C_h(u)} \int_{-C_h(v)}^0 (u-v)^{-\beta-1} \int_{v+s}^{u+s} |\sigma(w, X(w), Y(w)) \\ & \quad - \sigma(w, \tilde{X}^h(w), \tilde{Y}(w))| (w-v)^{-\beta} dw ds dv, \\ \tilde{J}_5(u) &= C(\omega) \int_0^{C_h(u)} \int_{-C_h(v)}^0 (u-v)^{-\beta-1} \int_{v+s}^{u+s} |\sigma(w, \tilde{X}^h(w), \tilde{Y}(w)) \\ & \quad - \sigma(C_h(w), \tilde{X}^h(w), \tilde{Y}(w))| (w-v)^{-\beta} dw ds dv, \end{aligned}$$

$$\begin{aligned}
 \tilde{J}_6(u) &= C(\omega) \int_0^{C_h(u)} \int_{-C_h(v)}^0 (u-v)^{-\beta-1} \int_{v+s}^{u+s} |\sigma(C_h(w), \tilde{X}^h(w), \tilde{Y}(w)) \\
 &\quad - \sigma(C_h(w), \tilde{X}^h(C_h(w)), \tilde{Y}(C_h(w)))| (w-v)^{-\beta} dw ds dv, \\
 \tilde{J}_7(u) &= C(\omega) \int_0^{C_h(u)} \int_{-C_h(v)}^0 (u-v)^{-\beta-1} \int_{v+s}^{u+s} \int_{v+s}^w |\sigma(w, X(w), Y(w)) - \sigma(w, \tilde{X}^h(w), \tilde{Y}(w)) \\
 &\quad - (\sigma(z, X(z), Y(z)) - \sigma(z, \tilde{X}^h(z), \tilde{Y}(z)))| (w-z)^{-\beta-1} dz dw ds dv, \\
 \tilde{J}_8(u) &= C(\omega) \int_0^{C_h(u)} \int_{-C_h(v)}^0 (u-v)^{-\beta-1} \int_{v+s}^{u+s} \int_{v+s}^w |\sigma(w, \tilde{X}^h(w), \tilde{Y}(w)) - \sigma(C_h(w), \tilde{X}^h(w), \tilde{Y}(w)) \\
 &\quad - (\sigma(z, \tilde{X}^h(z), \tilde{Y}(z)) - \sigma(C_h(z), \tilde{X}^h(z), \tilde{Y}(z)))| (w-z)^{-\beta-1} dz dw ds dv, \\
 \tilde{J}_9(u) &= C(\omega) \int_0^{C_h(u)} \int_{-C_h(v)}^0 (u-v)^{-\beta-1} \int_{v+s}^{u+s} \int_{v+s}^w |\sigma(C_h(w), \tilde{X}^h(w), \tilde{Y}(w)) \\
 &\quad - \sigma(C_h(w), \tilde{X}^h(C_h(w)), \tilde{Y}(C_h(w))) - (\sigma(C_h(z), \tilde{X}^h(z), \tilde{Y}(z)) \\
 &\quad - \sigma(C_h(z), \tilde{X}^h(C_h(z)), \tilde{Y}(C_h(z))))| (w-z)^{-\beta-1} dz dw ds dv.
 \end{aligned} \tag{3.100}$$

From Hypothesis 1 and (3.30), changing the limits of integration

$$\begin{aligned}
 \tilde{J}_1(u) &\leq C(\omega) \int_0^u \int_{-v}^0 (u-v)^{-\beta-1} \int_{v+s}^{u+s} |X(w) - \tilde{X}^h(w)| + |Y(w) - \tilde{Y}(w)| dw ds dv, \\
 &\leq C(\omega) \int_0^u [Z^h(w) + h^{H-\rho}] \int_{w-u}^0 \int_{-s}^{w-s} (u-v)^{-\beta-1} dv ds dw \\
 &\leq C(\omega) \int_0^u Z^h(w) dw + C(\omega) h^{H-\rho}.
 \end{aligned} \tag{3.101}$$

and

$$\tilde{J}_2(u) \leq \int_0^{C_h(u)} \int_{-C_h(v)}^0 (u-v)^{-\beta-1} \int_{v+s}^{u+s} |w - C_h(w)| dw ds dv \leq Ch. \tag{3.102}$$

Also,

$$\begin{aligned}
 \tilde{J}_3(u) &\leq \int_0^{C_h(u)} \int_{-C_h(v)}^0 (u-v)^{-\beta-1} \int_{v+s}^{u+s} |\tilde{X}^h(w) - \tilde{X}^h(C_h(w))| + |\tilde{Y}(w) - \tilde{Y}(C_h(w))| dw ds dv, \\
 &\leq C(\omega) h^{H-\rho}.
 \end{aligned} \tag{3.103}$$

Similarly to (3.87), using the change of variable, the inequality (3.86) and $\beta < 1/2$, we obtain

$$\begin{aligned}
 \tilde{J}_4(u) &\leq C(\omega) \int_0^u \int_{-v}^0 (u-v)^{-\beta-1} \int_{v+s}^{u+s} [|X(w) - \tilde{X}^h(w)| + |Y(w) - \tilde{Y}(w)|] (w-v)^{-\beta} dw ds dv \\
 &\leq C(\omega) \int_0^u [Z^h(w) + h^{H-\rho}] \int_0^w \int_{\max\{-v, w-u\}}^0 (u-v)^{-\beta-1} (w-v)^{-\beta} ds dv dw \\
 &\leq C(\omega) \int_0^u Z^h(w) (u-w)^{-2\beta} dw + C(\omega) h^{H-\rho}.
 \end{aligned} \tag{3.104}$$

Analogously to $\tilde{J}_2(u)$ and $\tilde{J}_3(u)$, we obtain

$$\tilde{J}_5(u) \leq Ch \quad \text{and} \quad \tilde{J}_6(u) \leq C(\omega) h^{H-\rho}. \tag{3.105}$$

To study $\tilde{J}_7$, we apply Corollary (3.4.1), then

$$\tilde{J}_7 \leq C(\omega) \sum_{k=1}^5 \tilde{J}_{7-k}, \quad (3.106)$$

where

$$\begin{aligned} \tilde{J}_{7-1} &= \int_0^{C_h(u)} \int_{-C_h(v)}^0 (u-v)^{-\beta-1} \int_{v+s}^{u+s} \int_{v+s}^w |X(w) - \tilde{X}^h(w) - X(z) + \tilde{X}^h(z)| (w-z)^{-\beta-1} dz dw ds dv, \\ \tilde{J}_{7-2} &= \int_0^{C_h(u)} \int_{-C_h(v)}^0 (u-v)^{-\beta-1} \int_{v+s}^{u+s} \int_{v+s}^w |X(w) - \tilde{X}^h(w)| |w-z| (w-z)^{-\beta-1} dz dw ds dv, \\ \tilde{J}_{7-3} &= \int_0^{C_h(u)} \int_{-C_h(v)}^0 (u-v)^{-\beta-1} \int_{v+s}^{u+s} \int_{v+s}^w |X(w) - \tilde{X}^h(w)| \left[|\tilde{X}^h(w) - \tilde{X}^h(z)| \right. \\ &\quad \left. + |X(w) - X(z)| + |\tilde{Y}(w) - \tilde{Y}(z)| + |Y(w) - Y(z)| \right] (w-z)^{-\beta-1} dz dw ds dv, \\ \tilde{J}_{7-4} &= \int_0^{C_h(u)} \int_{-C_h(v)}^0 (u-v)^{-\beta-1} \int_{v+s}^{u+s} \int_{v+s}^w |Y(w) - \tilde{Y}(w) - Y(z) + \tilde{Y}(z)| (w-z)^{-\beta-1} dz dw ds dv, \\ \tilde{J}_{7-5} &= \int_0^{C_h(u)} \int_{-C_h(v)}^0 (u-v)^{-\beta-1} \int_{v+s}^{u+s} \int_{v+s}^w |Y(w) - \tilde{Y}(w)| \left[|w-z| + |\tilde{X}^h(w) - \tilde{X}^h(z)| \right. \\ &\quad \left. + |X(w) - X(z)| + |\tilde{Y}(w) - \tilde{Y}(z)| + |Y(w) - Y(z)| \right] (w-z)^{-\beta-1} dz dw ds dv. \end{aligned}$$

By (3.24) and changing the limits of integration, we have

$$\begin{aligned} \tilde{J}_{7-1} &\leq \int_0^u \int_{-v}^0 (u-v)^{-\beta-1} \int_{v+s}^{u+s} \int_{v+s}^w \Delta_{z,w}(X, \tilde{X}) (w-z)^{-\beta-1} dz dw ds dv, \\ &= \int_0^u \int_0^w \Delta_{z,w}(X, \tilde{X}) (w-z)^{-\beta-1} \int_{w-u}^0 \int_{-s}^{z-s} (u-v)^{-\beta-1} dv ds dz dw \\ &\leq C \int_0^u \int_0^w (w-z)^{-\beta-1} \Delta_{z,w}(X, \tilde{X}) \int_{w-u}^0 (u-z+s)^{-\beta} ds dz dw \\ &\leq C \int_0^u \int_0^w \Delta_{z,w}(X, \tilde{X}) (w-z)^{-\beta-1} dz dw \\ &= C \int_0^u \theta(w) dw. \end{aligned}$$

Also,

$$\begin{aligned} \tilde{J}_{7-2} &\leq \int_0^u \int_{-v}^0 (u-v)^{-\beta} \int_{v+s}^{u+s} \int_{v+s}^w |X(w) - \tilde{X}^h(w)| |w-z| (w-z)^{-\beta-1} dz dw ds dv \\ &= \int_0^u Z^h(w) \int_0^w (w-z)^{-\beta} \int_{w-u}^0 \int_{-s}^{z-s} (u-v)^{-\beta-1} dv ds dz dw \\ &\leq C \int_0^u Z^h(w) dw. \end{aligned}$$

Due to Theorem 3.3.1, Corollary 3.3.1 and (3.9),

$$\begin{aligned}
 \tilde{J}_{7-3} &\leq C(\omega) \int_0^u \int_{-v}^0 (u-v)^{-\beta} \int_{v+s}^{u+s} \int_{v+s}^w Z^h(w)(w-z)^{H-\rho-\beta-1} dz dw ds dv \\
 &= C(\omega) \int_0^u Z^h(w) \int_0^w (w-z)^{H-\rho-\beta-1} \int_{w-u}^0 \int_{-s}^{z-s} (u-v)^{-\beta-1} dv ds dz dw \\
 &\leq C(\omega) \int_0^u Z^h(w) dw.
 \end{aligned}$$

And similarly, from Theorems 3.1.1 and 3.3.1, Corollary 3.3.1 and (3.30)

$$\begin{aligned}
 \tilde{J}_{7-5} &\leq C(\omega) \int_0^u \int_{-v}^0 (u-v)^{-\beta} \int_{v+s}^{u+s} \int_{v+s}^w [Z^h(w) + h^{H-\rho}](w-z)^{H-\rho-\beta-1} dz dw ds dv \\
 &= C(\omega) \int_0^u [Z^h(w) + h^{H-\rho}] \int_0^w (w-z)^{H-\rho-\beta-1} \int_{w-u}^0 \int_{-s}^{z-s} (u-v)^{-\beta-1} dv ds dz dw \\
 &\leq C(\omega) \int_0^u Z^h(w) dw + C(\omega) h^{H-\rho}.
 \end{aligned}$$

Now we study $\tilde{J}_{7-4}$, similarly to $\tilde{J}_{7-1}$, by (3.25)

$$\begin{aligned}
 \tilde{J}_{7-4} &\leq \int_0^u \int_{-v}^0 (u-v)^{-\beta-1} \int_{v+s}^{u+s} \int_{v+s}^w \Delta_{z,w}(Y, \tilde{Y})(w-z)^{-\beta-1} dz dw ds dv, \\
 &\leq C \int_0^u \int_0^w \Delta_{z,w}(Y, \tilde{Y})(w-z)^{-\beta-1} dz dw \\
 &\leq C \int_0^u (u-w)^{-\beta} \int_0^w \Delta_{z,w}(Y, \tilde{Y})(w-z)^{-\beta-1} dz dw.
 \end{aligned}$$

The last bound of $\tilde{J}_{7-4}$ is equal to the bound (3.91) obtained in Lemma 3.4.3 for J_{7-4} . Hence,

$$\tilde{J}_{7-4} \leq C(\omega) h^{H-\rho-\beta} + \int_0^u Z^h(w)(u-w)^{-\beta} dw + C(\omega) \int_0^u \kappa(w)(u-w)^{-\beta} dw. \quad (3.107)$$

Thus, from (3.106)

$$\tilde{J}_7 \leq C(\omega) \left(h^{H-\rho-\beta} + \int_0^u \theta(w) dw + \int_0^u Z^h(w)(u-w)^{-\beta} dw + \int_0^u \kappa(w)(u-w)^{-\beta} dw \right). \quad (3.108)$$

Now, we study $\tilde{J}_8(u)$, similar to the last calculus, making a change of variables and integrating, we obtain the following result.

$$\begin{aligned}
 \tilde{J}_8(u) &\leq \int_0^u \int_{-v}^0 (u-v)^{-\beta-1} \int_{v+s}^{u+s} \int_{v+s}^w |\sigma(w, \tilde{X}^h(w), \tilde{Y}(w)) - \sigma(C_h(w), \tilde{X}^h(w), \tilde{Y}(w)) \\
 &\quad - (\sigma(z, \tilde{X}^h(z), \tilde{Y}(z)) - \sigma(C_h(z), \tilde{X}^h(z), \tilde{Y}(z)))| (w-z)^{-\beta-1} dz dw ds dv \\
 &\leq C \int_0^u \int_0^w (w-z)^{-\beta-1} |\sigma(w, \tilde{X}^h(w), \tilde{Y}(w)) - \sigma(C_h(w), \tilde{X}^h(w), \tilde{Y}(w)) \\
 &\quad - (\sigma(z, \tilde{X}^h(z), \tilde{Y}(z)) - \sigma(C_h(z), \tilde{X}^h(z), \tilde{Y}(z)))| \int_{w-u}^0 \int_{-s}^{z-s} (u-v)^{-\beta-1} dv ds dz dw \\
 &\leq C \int_0^u \int_0^w (w-z)^{-\beta-1} |\sigma(w, \tilde{X}^h(w), \tilde{Y}(w)) - \sigma(C_h(w), \tilde{X}^h(w), \tilde{Y}(w)) \\
 &\quad - (\sigma(z, \tilde{X}^h(z), \tilde{Y}(z)) - \sigma(C_h(z), \tilde{X}^h(z), \tilde{Y}(z)))| dz dw \\
 &\leq C \left[\int_0^u \int_0^{C_h(w)} \cdots dz dw + \int_0^u \int_{C_h(w)}^w \cdots dz dw \right] \\
 &= C[\tilde{J}_{8-1}(u) + \tilde{J}_{8-2}(u)].
 \end{aligned} \tag{3.109}$$

From Hypothesis 1 and 3.14,

$$\begin{aligned}
 \tilde{J}_{8-1}(u) &= \int_0^u \int_0^{C_h(w)} (w-z)^{-\beta-1} |\sigma(w, \tilde{X}^h(w), \tilde{Y}(w)) - \sigma(C_h(w), \tilde{X}^h(w), \tilde{Y}(w)) \\
 &\quad - (\sigma(z, \tilde{X}^h(z), \tilde{Y}(z)) - \sigma(C_h(z), \tilde{X}^h(z), \tilde{Y}(z)))| dz dw \\
 &\leq C \int_0^u \int_0^{C_h(w)} (w-z)^{-\beta-1} [|w - C_h(w)| + |z - C_h(z)|] dz dw \\
 &\leq C(\omega)h \int_0^u \int_0^{C_h(w)} (w-z)^{-\beta-1} dz dw \\
 &\leq C(\omega)h \int_0^u (w - C_h(w))^{-\beta} dw \\
 &\leq C(\omega)h^{1-\beta}.
 \end{aligned}$$

From Hypothesis 1, Theorem 3.3.1 and Corollary 3.3.1,

$$\begin{aligned}
 \tilde{J}_{8-2}(u) &= \int_0^u \int_{C_h(w)}^w (w-z)^{-\beta-1} |\sigma(w, \tilde{X}^h(w), \tilde{Y}(w)) - \sigma(C_h(w), \tilde{X}^h(w), \tilde{Y}(w)) \\
 &\quad - (\sigma(z, \tilde{X}^h(z), \tilde{Y}(z)) - \sigma(C_h(z), \tilde{X}^h(z), \tilde{Y}(z)))| dz dw \\
 &\leq \int_0^u \int_{C_h(w)}^w [|\sigma(w, \tilde{X}^h(w), \tilde{Y}(w)) - \sigma(w, \tilde{X}^h(z), \tilde{Y}(z))| + |\sigma(w, \tilde{X}^h(z), \tilde{Y}(z)) - \sigma(z, \tilde{X}^h(z), \tilde{Y}(z))| \\
 &\quad + |\sigma(C_h(w), \tilde{X}^h(z), \tilde{Y}(z)) - \sigma(C_h(w), \tilde{X}^h(w), \tilde{Y}(w))|] (w-z)^{-\beta-1} dz dw \\
 &\leq C(\omega) \int_0^u \int_{C_h(w)}^w (w-z)^{H-\rho-\beta-1} dz dw \\
 &\leq C(\omega) \int_0^u (w - C_h(w))^{H-\rho-\beta} dw \\
 &\leq C(\omega)h^{H-\rho-\beta}.
 \end{aligned}$$

Hence,

$$\tilde{J}_8(u) \leq C(\omega)h^{H-\rho-\beta}. \tag{3.110}$$

Similarly, we study $\tilde{J}_9(u)$.

$$\begin{aligned}
 \tilde{J}_9(u) &\leq \int_0^u \int_{-v}^0 (u-v)^{-\beta-1} \int_{v+s}^{u+s} \int_{v+s}^w |\sigma(C_h(w), \tilde{X}^h(w), \tilde{Y}(w)) \\
 &\quad - \sigma(C_h(w), \tilde{X}^h(C_h(w)), \tilde{Y}(C_h(w))) - (\sigma(C_h(z), \tilde{X}^h(z), \tilde{Y}(z)) \\
 &\quad - \sigma(C_h(z), \tilde{X}^h(C_h(z)), \tilde{Y}(C_h(z))))|(w-z)^{-\beta-1} dz dw ds dv \\
 &\leq C \int_0^u \int_0^w (w-z)^{-\beta-1} |\sigma(C_h(w), \tilde{X}^h(w), \tilde{Y}(w)) - \sigma(C_h(w), \tilde{X}^h(C_h(w)), \tilde{Y}(C_h(w))) \\
 &\quad - (\sigma(C_h(z), \tilde{X}^h(z), \tilde{Y}(z)) - \sigma(C_h(z), \tilde{X}^h(C_h(z)), \tilde{Y}(C_h(z))))| dz dw \\
 &\leq C \left[\int_0^u \int_0^{C_h(w)} \cdots dz dw + \int_0^u \int_{C_h(w)}^w \cdots dz dw \right] \\
 &= C[\tilde{J}_{9-1}(u) + \tilde{J}_{9-2}(u)],
 \end{aligned}$$

where

$$\begin{aligned}
 \tilde{J}_{9-1}(u) &= \int_0^u \int_0^{C_h(w)} (w-z)^{-\beta-1} |\sigma(C_h(w), \tilde{X}^h(w), \tilde{Y}(w)) - \sigma(C_h(w), \tilde{X}^h(C_h(w)), \tilde{Y}(C_h(w))) \\
 &\quad - (\sigma(C_h(z), \tilde{X}^h(z), \tilde{Y}(z)) - \sigma(C_h(z), \tilde{X}^h(C_h(z)), \tilde{Y}(C_h(z))))| dz dw \\
 &\leq C \int_0^u \int_0^{C_h(w)} (w-z)^{-\beta-1} [|w - C_h(w)| + |z - C_h(z)|] dz dw \\
 &\leq Ch^{1-\beta}
 \end{aligned}$$

and

$$\begin{aligned}
 \tilde{J}_{9-2}(u) &= \int_0^u \int_{C_h(w)}^w (w-z)^{-\beta-1} |\sigma(C_h(w), \tilde{X}^h(w), \tilde{Y}(w)) - \sigma(C_h(w), \tilde{X}^h(C_h(w)), \tilde{Y}(C_h(w))) \\
 &\quad - (\sigma(C_h(z), \tilde{X}^h(z), \tilde{Y}(z)) - \sigma(C_h(z), \tilde{X}^h(C_h(z)), \tilde{Y}(C_h(z))))| dz dw \\
 &= \int_0^u \int_{C_h(w)}^w (w-z)^{-\beta-1} |\sigma(C_h(w), \tilde{X}^h(w), \tilde{Y}(w)) - \sigma(C_h(w), \tilde{X}^h(z), \tilde{Y}(z))| dz dw \\
 &\leq C(\omega) \int_0^u \int_0^{C_h(w)} (w-z)^{H-\rho-\beta-1} dz dw \\
 &\leq C(\omega) h^{H-\rho-\beta}.
 \end{aligned}$$

Thus,

$$\tilde{J}_9(u) \leq C(\omega) h^{H-\rho-\beta}. \quad (3.111)$$

Finally, from 3.99 to 3.111 we obtain the result.

Proof (Theorem 3.4.1). From Lemmas 3.4.1, 3.4.2, 3.4.3 and 3.4.4,

$$\begin{aligned}
 &Z^h(t) + \theta(t) + \kappa(t) \\
 &\leq C(\omega) \left(h^{H-\rho-\beta} + \int_0^t Z^h(u) u^{-\beta} du + \int_0^t (Z^h(u) + \theta(u) + \kappa(u))(t-u)^{-\beta} du + \int_0^t Z^h(u)(t-u)^{-2\beta} du \right) \\
 &\leq C(\omega) \left(h^{H-\rho-\beta} + \int_0^t (Z^h(u) + \theta(u) + \kappa(u)) u^{-\beta} du + \int_0^t (t-u)^{-2\beta} (Z^h(u) + \theta(u) + \kappa(u)) du \right) \\
 &\leq C(\omega) \left(h^{H-\rho-\beta} + \int_0^t (Z^h(u) + \theta(u) + \kappa(u))(u^{-\beta} + (t-u)^{-2\beta}) du \right) \\
 &\leq C(\omega) \left(h^{H-\rho-\beta} + t^{2\beta} \int_0^t (Z^h(u) + \theta(u) + \kappa(u)) u^{-2\beta} (t-u)^{-2\beta} du \right).
 \end{aligned}$$

For $\omega \in \Omega_{\varepsilon, h_0, \rho}$ (see Theorem 3.3.1) and recalling that $1 - H < \beta < 1/2$ and $\rho > 0$ is small enough to $1/2 < H - \rho < H$. Due to Gronwall's Lemma, we have that

$$Z^h(t) + \theta(t) + \kappa(t) \leq C(\omega)h^{2H-1}\exp\{Ct\}.$$

Recalling the definition of U_h in (3.26), the previous bounds on Z^h , θ and κ imply the desired estimate.

CAPÍTULO 4

CONSISTENCY OF THE BACKWARD-EULER SCHEME FOR SFDES DRIVEN BY FRACTIONAL BROWNIAN MOTION

In the previous chapter, we established the convergence of the explicit Euler scheme for a class of stochastic functional differential equations driven by a fractional Brownian motion with Hurst parameter $H > 1/2$. The analysis relied on the regularity properties of the exact solution, the approximation of the memory term, and the pathwise interpretation of the stochastic integral in the sense of Zähle. These results provide a theoretical justification for the explicit Euler approximation within the framework considered throughout this thesis.

The purpose of the present chapter is to investigate the corresponding implicit discretization, namely the backward-Euler scheme. Implicit numerical methods constitute an important class of time-stepping procedures due to their enhanced stability properties and their successful application to deterministic and stochastic differential equations. Extending these methods to stochastic functional differential equations driven by fractional Brownian motion is, however, considerably more delicate because of the simultaneous presence of memory effects, the non-semimartingale nature of the driving noise, and the implicit dependence of the drift term.

The main objective of this chapter is therefore to establish the consistency of the backward-Euler approximation. More precisely, we derive an estimate for the local truncation error by comparing one step of the numerical method with the exact solution evaluated at the mesh points. The proof combines the Hölder regularity of the exact solution, the Lipschitz assumptions imposed on the coefficients and the kernel defining the delay term, together with the fractional calculus estimates associated with the generalized Riemann–Stieltjes integral.

Unlike the explicit Euler scheme, obtaining a complete convergence analysis for the backward-Euler method requires overcoming additional technical difficulties arising from the implicit character of the discretization and the recursive dependence introduced by the delay term. For this reason, the present chapter is devoted exclusively to the consistency analysis, which constitutes the first theoretical step toward a complete convergence theory for implicit schemes applied to stochastic functional differential equations driven by fractional Brownian motion.

4.1. The stochastic functional differential equation

We study the stochastic functional differential equation again:

$$X(t) = \begin{cases} X(0) + \int_0^t b(s, X(s), Y(s))ds + \int_0^t \sigma(s, X(s), Y(s))dB^H(s), & t \in [0, T] \\ \phi(t), & t \in [-r, 0], \end{cases} \quad (4.1)$$

where $B^H = (B^H(t))_{t \geq 0}$ is a fractional Brownian motion with Hurst parameter $H > 1/2$. The stochastic integral with respect to B^H is understood in the sense of the generalized Lebesgue-Stieltjes integral as introduced by Zähle [24]. The drift and diffusion coefficients $b, \sigma : [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ are assumed to be continuous functions, and the initial condition $\phi : [-r, 0] \rightarrow \mathbb{R}$ is assumed to be Hölder continuous. The memory effect is modeled through the functional $Y(t)$ defined by

$$Y(t) = \int_{-r}^0 K(t, s, X(t+s)) ds = \int_{t-r}^t K(t, s-t, X(s)) ds, \quad t \geq 0, \quad (4.2)$$

where the kernel $K : [0, T] \times [-r, 0] \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function.

We consider the same assumptions on the coefficients b, σ , the initial condition ϕ and the Kernel K in (4.1) and (4.2).

Hypothesis ϕ There exists a positive constant L such that

$$|\phi(t) - \phi(s)| \leq L|t - s|^{H-\rho}, \quad 0 < \rho < H. \quad (4.3)$$

Hypothesis 1 There exist positive constants L_1, L_2 and L_3 such that the following properties hold.

$$|b(t, x, y) - b(s, x, y)| + |\sigma(t, x, y) - \sigma(s, x, y)| \leq L_1|t - s|. \quad (4.4)$$

$$|b(t, x_1, y_1) - b(t, x_2, y_2)| + |\sigma(t, x_1, y_1) - \sigma(t, x_2, y_2)| \leq L_2(|x_1 - x_2| + |y_1 - y_2|). \quad (4.5)$$

$$|b(t, x, y)| + |\sigma(t, x, y)| \leq L_3(1 + |x| + |y|). \quad (4.6)$$

Hypothesis 2 The function $\sigma(t, \xi)$ is Fréchet differentiable in the variable ξ . Also, there exist positive constants M_1, M_2 and M_3 such that for all $\xi, \eta \in C_r$ and $t \in [0, T]$,

$$\begin{aligned} |D_\xi \sigma(t, \xi)|_{\mathcal{L}(C_r, \mathbb{R})} &\leq M_1. \\ |D_\xi \sigma(t, \xi) - D_\xi \sigma(t, \eta)|_{\mathcal{L}(C_r, \mathbb{R})} &\leq M_2 \|\xi - \eta\|. \\ |D_\xi \sigma(t, \xi) - D_\xi \sigma(s, \xi)|_{\mathcal{L}(C_r, \mathbb{R})} &\leq M_3 |t - s|. \end{aligned} \quad (4.7)$$

Hypothesis 3 For each $t \in [0, T]$, the map

$$(s, x) \mapsto K(t, s, x), \quad (s, x) \in [-r, 0] \times \mathbb{R},$$

is continuously differentiable. Moreover, there exist positive constants K_1, K_2, K_3, K_4 such that

$$\begin{aligned} |K(t, s, x) - K(u, v, x)| &\leq K_1(|t - u| + |s - v|), \\ |K(t, s, x) - K(t, s, y)| &\leq K_2|x - y|, \\ |K(t, s, x)| &\leq K_3(1 + |x|), \\ |\nabla_{(s, x)} K(t, s, x)| &\leq K_4, \\ |\nabla_{(s, x)} K(t, s, x) - \nabla_{(s, x)} K(u, v, y)| &\leq K_4(|t - u| + |s - v| + |x - y|). \end{aligned} \quad (4.8)$$

The following theorem establishes the existence and uniqueness of the solution to (4.1) (the proof is analogous to those given in [3] and [20]), and is therefore left to the reader.

Theorem 4.1.1. *Under hypotheses 4.3 to 4.8. If ϕ is a stochastic process whose trajectories belong to the space $C^{H-\rho}[0, T]$ a.s. for any $0 < \rho < H$, then Equation (4.1) has a unique solution $X \in C^{H-\rho}[-r, T]$ a.s.,*

$$|X(u) - X(v)| \leq C(\omega)|u - v|^{H-\rho}. \quad (4.9)$$

4.2. Backward-Euler scheme for SFDE

In this section, we introduce the backward-Euler approximation of the SFDE and describe how the fractional noise and the memory term are discretized.

We assume, without loss of generality, that there exists $N_T \in \mathbb{N}$ such that $T = N_T r$. Let $h := T/N$ for some $N \in \mathbb{N}$ and consider the uniform partition of $[-r, T]$ given by

$$t_n = nh, \quad n = -N, \dots, -1, 0, 1, \dots, N \cdot N_T.$$

For simplicity, we take $T = r$, so that $N_T = 1$.

The backward-Euler scheme associated with (4.1) is defined recursively by

$$\tilde{X}^h(t_{n+1}) = \tilde{X}^h(t_n) + hb(t_{n+1}, \tilde{X}^h(t_{n+1}), \tilde{Y}(t_{n+1})) + \sigma(t_n, \tilde{X}^h(t_n), \tilde{Y}(t_n))\Delta B^H(t_n), \quad (4.10)$$

where

$$\Delta B^H(t_n) = B^H(t_{n+1}) - B^H(t_n).$$

In contrast to the classical Markovian setting, the scheme depends not only on the current value $\tilde{X}^h(t_{n+1})$ but also on the discrete approximation of the delay functional $\tilde{Y}(t_n)$, which encodes the past trajectory of the process.

The continuous interpolation is defined for $t \in [t_n, t_{n+1})$ by

$$\tilde{X}^h(t) = \tilde{X}^h(t_n) + b(t_{n+1}, \tilde{X}^h(t_{n+1}), \tilde{Y}(t_{n+1}))(t - t_n) + \sigma(t_n, \tilde{X}^h(t_n), \tilde{Y}(t_n))(B^H(t) - B^H(t_n)),$$

However, we note that $\tilde{X}^h(t)$, defined in that way, is not $\mathcal{F}_t$ -measurable. For that reason, our study will not be over the interval $[0, T]$ but over the points of the mesh.

The discrete delay functional is given by

$$\tilde{Y}(t) = \sum_{i=-N}^{-1} h K(t, t_i, \tilde{X}^h(t + t_i)). \quad (4.11)$$

Equivalently,

$$\begin{aligned} \tilde{Y}(t) &= \sum_{i=-N}^{-1} \int_{t_i}^{t_{i+1}} K(t, C_h(s), \tilde{X}^h(t + C_h(s))) ds \\ &= \int_{-r}^0 K(t, C_h(s), \tilde{X}^h(t + C_h(s))) ds. \end{aligned}$$

This representation makes explicit the link between the exact delay term and its discrete counterpart, and will be instrumental in the convergence analysis, where the discrepancy induced by the discretization of both the kernel and the trajectory must be carefully controlled.

4.3. Local truncation error

Let's consider an auxiliary process $\tilde{X}_{n+1}^h$, it is the value obtained through the scheme (4.10) when we use the exact value of the solution at the step $n + 1$, that is,

$$\tilde{X}_{n+1}^h := X(t_n) + hb(t_{n+1}, X(t_{n+1}), \hat{Y}(t_{n+1})) + \sigma(t_n, X(t_n), \hat{Y}(t_n))\Delta B^H(t_n). \quad (4.12)$$

Here, $\hat{Y}(t_m)$ represents the formula (4.11) when $\tilde{X}^h(t + t_i)$ is replaced by $X(t + t_i)$, that is,

$$\hat{Y}(t) = \sum_{i=-N}^{-1} h K(t, t_i, X(t + t_i)). \quad (4.13)$$

Equivalently,

$$\begin{aligned} \hat{Y}(t) &= \sum_{i=-N}^{-1} \int_{t_i}^{t_{i+1}} K(t, C_h(s), X(t + C_h(s))) ds \\ &= \int_{-r}^0 K(t, C_h(s), X(t + C_h(s))) ds. \end{aligned}$$

Definition 4.3.1. *The local error is defined by*

$$\delta_{n+1} := |X(t_{n+1}) - \tilde{X}_{n+1}^h|.$$

We define the consistency as follows.

Definition 4.3.2. *The scheme is said to be consistent of order p if*

$$\delta_{n+1} \leq Ch^{p+1}, \quad h \rightarrow 0.$$

In the next section, we introduce some technical lemmas needed to find out the consistency of the backward-Euler scheme.

4.4. Estimates for the numerical scheme

The following lemma tell us how close is the approximation of the memory term $\hat{Y}$ to the memory term Y .

Lemma 4.4.1. *Let $\hat{Y}$ be the approximation defined by (4.13). Then, for all $u \in [0, T]$:*

$$|Y(u) - \hat{Y}(C_h(u))| \leq C(\omega)h^{H-\rho}, \quad (4.14)$$

where $C(\omega)$ does not depend on h .

Note that if $C_h(u)$ is exchanged by $C_h(u + h)$, the upper bound remains.

Now, we're going to see that the difference between the approximations $\tilde{Y}$ and $\hat{Y}$ depends on the difference between the solution of the SFDE and backward Euler scheme.

Lemma 4.4.2. *Let $\tilde{Y}$ and $\hat{X}$ be the approximations defined by (4.11) and (4.13), respectively. Then, for each $n = 0, 1, \dots, N - 1$,*

$$|\hat{Y}(t_{n+1}) - \tilde{Y}(t_{n+1})| \leq K_2 h \sum_{i=0}^n |X(t_i) - \tilde{X}^h(t_i)|. \quad (4.15)$$

4.5. Consistency of the backward-Euler method

We now establish the consistency of this method.

Theorem 4.5.1. *Under Hypothesis 2 and 3, the backward-Euler method (4) is consistent with order $H - \rho - \beta + 1$; i.e., for each $n = 0, 1, \dots, N - 1$,*

$$\delta_{n+1} \leq C(\omega)h^{H-\rho-\beta+1}, \quad h \rightarrow 0, \quad (4.16)$$

where $C(\omega)$ does not depend on h .

Proof. First, let's calculate $X(t_{n+1}) - \tilde{X}_{n+1}^h$:

$$\begin{aligned} X(t_{n+1}) - \tilde{X}_{n+1}^h &= X(t_{n+1}) - \left(X(t_n) + hb(t_{n+1}, X(t_{n+1}), \hat{Y}(t_{n+1})) + \sigma(t_n, X(t_n), \hat{Y}(t_n))\Delta B^H(t_n) \right) \\ &= X(t_{n+1}) - X(t_n) - \left(\int_{t_n}^{t_{n+1}} b(C_h(s+h), X(C_h(s+h)), \hat{Y}(C_h(s+h)))ds \right. \\ &\quad \left. + \int_{t_n}^{t_{n+1}} \sigma(C_h(s), X(C_h(s)), \hat{Y}(C_h(s)))dB^H(s) \right) \\ &= \int_{t_n}^{t_{n+1}} b(s, X(s), Y(s))ds + \int_{t_n}^{t_{n+1}} \sigma(s, X(s), Y(s))dB^H(s) \\ &\quad - \left(\int_{t_n}^{t_{n+1}} b(C_h(s+h), X(C_h(s+h)), \hat{Y}(C_h(s+h)))ds \right. \\ &\quad \left. + \int_{t_n}^{t_{n+1}} \sigma(C_h(s), X(C_h(s)), \hat{Y}(C_h(s)))dB^H(s) \right) \\ &= \int_{t_n}^{t_{n+1}} (b(s, X(s), Y(s)) - b(C_h(s+h), X(C_h(s+h)), \hat{Y}(C_h(s+h))))ds \\ &\quad + \int_{t_n}^{t_{n+1}} (\sigma(s, X(s), Y(s)) - \sigma(C_h(s), X(C_h(s)), \hat{Y}(C_h(s))))dB^H(s), \end{aligned}$$

where

$$C_h(u) = \left\lfloor \frac{u}{h} \right\rfloor h. \quad (4.17)$$

Clearly, if $u \in [t_i, t_{i+1})$ then $C_h(u) = t_i$ and $C_h(u+h) = t_{i+1}$.

The operator $C_h(u)$ selects the last discretization point before u , ensuring that the coefficients are evaluated along the piecewise constant approximation of the trajectory.

Now, let be $\beta \in (1 - H, 1/2)$, then from Young's inequality

$$\begin{aligned} |X(t_{n+1}) - \tilde{X}_{n+1}^h| &\leq \left| \int_{t_n}^{t_{n+1}} (b(s, X(s), Y(s)) - b(C_h(s+h), X(C_h(s+h)), \hat{Y}(C_h(s+h))))ds \right| \\ &\quad + \left| \int_{t_n}^{t_{n+1}} (\sigma(s, X(s), Y(s)) - \sigma(C_h(s), X(C_h(s)), \hat{Y}(C_h(s))))dB^H(s) \right| \\ &\leq I_1 + I_2 + I_3, \end{aligned}$$

where

$$\begin{aligned} I_1 &= \int_{t_n}^{t_{n+1}} |b(s, X(s), Y(s)) - b(C_h(s+h), X(C_h(s+h)), \hat{Y}(C_h(s+h)))| ds \\ I_2 &= C(\omega) \int_{t_n}^{t_{n+1}} \frac{|\sigma(s, X(s), Y(s)) - \sigma(C_h(s), X(C_h(s)), \hat{Y}(C_h(s)))|}{(s - t_n)^\beta} ds \\ I_3 &= C(\omega) \int_{t_n}^{t_{n+1}} \int_{t_n}^s \frac{|\sigma(s, X(s), Y(s)) - \sigma(C_h(s), X(C_h(s)), \hat{Y}(C_h(s))) - (\sigma(u, X(u), Y(u)) - \sigma(C_h(u), X(C_h(u)), \hat{Y}(C_h(u))))|}{(s - u)^{\beta+1}} du ds \end{aligned}$$

Using lemma 4.4.1 and theorem 4.1.1:

$$\begin{aligned} I_1 &= \int_{t_n}^{t_{n+1}} |b(s, X(s), Y(s)) - b(C_h(s+h), X(C_h(s+h)), \hat{Y}(C_h(s+h)))| ds \\ &\leq \int_{t_n}^{t_{n+1}} (L_1|s - C_h(s+h)| + L_2|X(s) - X(C_h(s+h))| + L_2|Y(s) - \hat{Y}(C_h(s+h))|) ds \\ &\leq \int_{t_n}^{t_{n+1}} (L_1|s - C_h(s+h)| + L_2C(\omega)|s - C_h(s+h)|^{H-\rho} + L_2C(\omega)h^{H-\rho}) ds \\ &\leq \int_{t_n}^{t_{n+1}} (L_1h + L_2C(\omega)h^{H-\rho} + L_2C(\omega)h^{H-\rho}) ds \\ &\leq C(\omega)h^{H-\rho} \int_{t_n}^{t_{n+1}} ds \\ &= C(\omega)h^{H-\rho+1}, \end{aligned}$$

$$\begin{aligned} I_2 &= C(\omega) \int_{t_n}^{t_{n+1}} \frac{|\sigma(s, X(s), Y(s)) - \sigma(C_h(s), X(C_h(s)), \hat{Y}(C_h(s)))|}{(s - t_n)^\beta} ds \\ &\leq C(\omega) \int_{t_n}^{t_{n+1}} \frac{(L_1(s - C_h(s)) + L_2|X(s) - X(C_h(s))| + L_2|Y(s) - \hat{Y}(C_h(s))|)}{(s - t_n)^\beta} ds \\ &\leq C(\omega)h^{H-\rho} \int_{t_n}^{t_{n+1}} (s - t_n)^{-\beta} ds \\ &\leq C(\omega)h^{H-\rho}(t_{n+1} - t_n)^{1-\beta} \\ &= C(\omega)h^{H-\rho-\beta+1}. \end{aligned}$$

For I_3 , note that $t_n \leq u < s \leq t_{n+1}$, then $C_h(u) = C_h(s) = t_n$. In addition, we can choose $\rho > 0$ sufficiently small and $\beta \in (1 - H, 1/2)$ such that $H - \rho > \beta$. Therefore,

$$\begin{aligned} I_3 &\leq C(\omega) \int_{t_n}^{t_{n+1}} \int_{t_n}^s \frac{|\sigma(s, X(s), Y(s)) - \sigma(C_h(s), X(C_h(s)), \hat{Y}(C_h(s))) - (\sigma(u, X(u), Y(u)) - \sigma(C_h(u), X(C_h(u)), \hat{Y}(C_h(u))))|}{(s - u)^{\beta+1}} du ds \\ &\leq C(\omega) \int_{t_n}^{t_{n+1}} \int_{t_n}^s \frac{|\sigma(s, X(s), Y(s)) - \sigma(u, X(u), Y(u))|}{(s - u)^{\beta+1}} du ds \\ &\leq C(\omega) \int_{t_n}^{t_{n+1}} \int_{t_n}^s \frac{(L_1(s - u) + L_2|X(s) - X(u)| + L_2|Y(s) - Y(u)|)}{(s - u)^{\beta+1}} du ds. \end{aligned}$$

Using the Lipschitz condition on K :

$$\begin{aligned}
 |Y(s) - Y(u)| &= \left| \int_{-r}^0 K(s, w, X(s+w)) dw - \int_{-r}^0 K(u, w, X(u+w)) dw \right| \\
 &\leq \int_{-r}^0 |K(s, w, X(s+w)) - K(u, w, X(u+w))| dw \\
 &\leq \int_{-r}^0 (K_1(s-u) + K_2|X(s+w) - X(u+w)|) dw \\
 &\leq \int_{-r}^0 (K_1(s-u) + K_2C(\omega)(s-u)^{H-\rho}) dw \\
 &\leq C(s-u) + C(\omega)(s-u)^{H-\rho}.
 \end{aligned}$$

Consequently,

$$\begin{aligned}
 I_3 &\leq C(\omega) \int_{t_n}^{t_{n+1}} \int_{t_n}^s \frac{(L_1(s-u) + L_2C(\omega)(s-u)^{H-\rho} + L_2C(s-u) + L_2C(\omega)(s-u)^{H-\rho})}{(s-u)^{\beta+1}} du ds \\
 &\leq C(\omega) \int_{t_n}^{t_{n+1}} \int_{t_n}^s \frac{(C(s-u) + C(\omega)(s-u)^{H-\rho})}{(s-u)^{\beta+1}} du ds \\
 &= C(\omega) \int_{t_n}^{t_{n+1}} \int_{t_n}^s (s-u)^{-\beta} du ds + C(\omega) \int_{t_n}^{t_{n+1}} \int_{t_n}^s (s-u)^{H-\rho-\beta-1} du ds \\
 &\leq C(\omega) \int_{t_n}^{t_{n+1}} (s-t_n)^{1-\beta} ds + C(\omega) \int_{t_n}^{t_{n+1}} (s-t_n)^{H-\rho-\beta} ds \\
 &\leq C(\omega)(t_{n+1} - t_n)^{2-\beta} + C(\omega)(t_{n+1} - t_n)^{H-\rho-\beta+1} \\
 &= C(\omega)h^{2-\beta} + C(\omega)h^{H-\rho-\beta+1} \\
 &\leq C(\omega)h^{H-\rho-\beta+1}.
 \end{aligned}$$

From I_1, I_2 and I_3 , we obtain

$$\begin{aligned}
 \delta_{n+1} := |X(t_{n+1}) - \tilde{X}_{n+1}^h| &\leq C(\omega)h^{H-\rho+1} + C(\omega)h^{H-\rho-\beta+1} + C(\omega)h^{H-\rho-\beta+1} \\
 &\leq C(\omega)h^{H-\rho+1} + C(\omega)h^{H-\rho-\beta+1} \\
 &\leq C(\omega)h^{H-\rho-\beta+1}.
 \end{aligned}$$

CAPÍTULO 5

NUMERICAL SIMULATIONS

In this section, we present numerical experiments to illustrate the behavior of the approximations, the influence of the Hurst parameter, and the consistency predicted by the theoretical analysis of the proposed Euler- and backward-Euler-type methods when they are applied to a stochastic functional differential equation driven by fractional Brownian motion. As a benchmark example, we consider the stochastic delay equation

$$dX(t) = \left[\int_{t-1}^t X(s) ds + e^{-1} X(t) \right] dt + \varepsilon X(t) dB^H(t), \quad t \in [0, 2], \quad (5.1)$$

with initial condition

$$X(t) = e^t, \quad t \in [-1, 0], \quad (5.2)$$

where $B^H(t)$ denotes a fractional Brownian motion with Hurst parameter $H > 1/2$. This test equation was originally considered by Buckwar [6] in the case of standard Brownian motion and is commonly used as a benchmark problem for evaluating the strong convergence properties of numerical methods for stochastic functional differential equations. Here, we extend this example to the fractional Brownian motion setting, which introduces long-range dependence and provides a more general framework.

5.1. Euler method

The equation is discretized using the Euler-type scheme described in the Chapter (3) on a uniform time mesh with step size h . The memory integral appearing in (5.1) is approximated using Riemann sums. Since no explicit analytical solution is available in the stochastic case, a reference solution is computed using the same numerical method on a very fine mesh consisting of 2^{12} time steps. This reference solution is treated as the exact solution and is denoted by $X(T)$.

To quantify the accuracy of the numerical approximation, we compute the strong error, for each step size h , simulating 500 independent trajectories and define the empirical strong error as

$$\text{Err}(h) = \frac{1}{500} \sum_{j=1}^{500} \max_{t_i \in \{t_0, \dots, t_N\}} \left| \tilde{X}_N^{(j)} - X(T) \right|,$$

where $\tilde{X}_N^{(j)}$ denotes the numerical approximation at time T corresponding to the j -th trajectory. The convergence behavior is further analyzed using the ratio

$$\text{Ratio} = \frac{\text{Err}(h/2)}{\text{Err}(h)},$$

which provides a measure of the reduction in error as the step size is refined.

The simulations are performed for step sizes $h = 2^{-4}, 2^{-5}, 2^{-6}, 2^{-7}$ and for three different values of the noise intensity parameter $\varepsilon \in \{0,005, 0,1, 1\}$. The results are summarized in Tables 5.1–5.3 and Figures 5.1–5.3 for different values of the Hurst parameter.

$H = 0,51:$	$\varepsilon = 0,005$		$\varepsilon = 0,1$		$\varepsilon = 1$	
h	Error	Ratio	Error	Ratio	Error	Ratio
2^{-4}	$1,681 \times 10^{-1}$	–	$1,582 \times 10^{-1}$	–	$1,388 \times 10^{-1}$	–
2^{-5}	$8,560 \times 10^{-2}$	0,509	$8,049 \times 10^{-2}$	0,509	$3,935 \times 10^{-2}$	0,284
2^{-6}	$4,269 \times 10^{-2}$	0,499	$4,047 \times 10^{-2}$	0,503	$1,620 \times 10^{-2}$	0,412
2^{-7}	$2,080 \times 10^{-2}$	0,487	$2,011 \times 10^{-2}$	0,497	$2,090 \times 10^{-2}$	1,290

Cuadro 5.1: Strong error and ratio for $H = 0,51$ and different values of ε .

$H = 0,75:$	$\varepsilon = 0,005$		$\varepsilon = 0,1$		$\varepsilon = 1$	
h	Error	Ratio	Error	Ratio	Error	Ratio
2^{-4}	$1,669 \times 10^{-1}$	–	$1,331 \times 10^{-1}$	–	$7,626 \times 10^{-1}$	–
2^{-5}	$8,497 \times 10^{-2}$	0,509	$6,815 \times 10^{-2}$	0,512	$5,132 \times 10^{-1}$	0,673
2^{-6}	$4,238 \times 10^{-2}$	0,499	$3,434 \times 10^{-2}$	0,504	$3,251 \times 10^{-1}$	0,634
2^{-7}	$2,064 \times 10^{-2}$	0,487	$1,695 \times 10^{-2}$	0,493	$2,165 \times 10^{-1}$	0,666

Cuadro 5.2: Strong error and ratio for $H = 0,75$ and different values of ε .

$H = 0,9:$	$\varepsilon = 0,005$		$\varepsilon = 0,1$		$\varepsilon = 1$	
h	Error	Ratio	Error	Ratio	Error	Ratio
2^{-4}	$1,663 \times 10^{-1}$	–	$1,778 \times 10^{-1}$	–	$4,953 \times 10^{-1}$	–
2^{-5}	$8,464 \times 10^{-2}$	0,509	$9,097 \times 10^{-2}$	0,512	$2,938 \times 10^{-1}$	0,593
2^{-6}	$5,482 \times 10^{-2}$	0,500	$4,561 \times 10^{-2}$	0,501	$1,704 \times 10^{-1}$	0,580
2^{-7}	$2,056 \times 10^{-2}$	0,487	$2,230 \times 10^{-2}$	0,489	$9,333 \times 10^{-2}$	0,548

Cuadro 5.3: Strong error and ratio for $H = 0,9$ and different values of ε .

The following pictures show Sample paths of the numerical solution obtained with the Euler-type scheme and the reference solution for the stochastic functional differential equation (5.1)–(5.2) with different Hurst parameter and different values of ε . The reference solution is computed using a sufficiently fine time discretization.

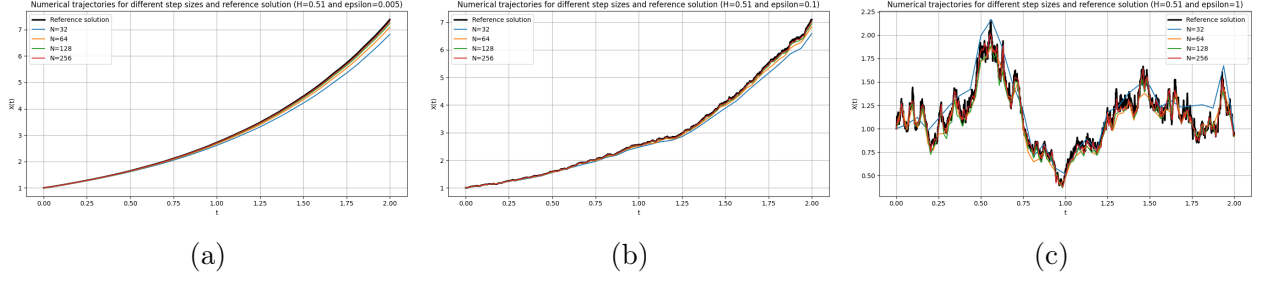


Figure 5.1: Sample path with Hurst parameter $H = 0,51$ and different values of ε .

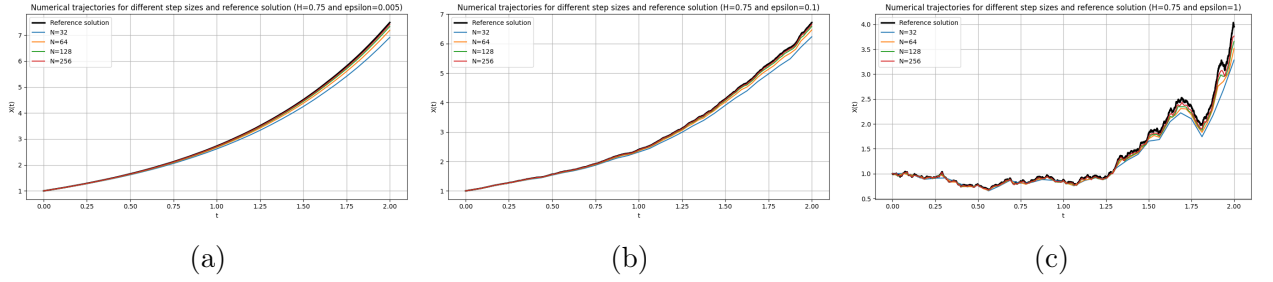


Figure 5.2: Sample path with Hurst parameter $H = 0,75$ and different values of ε .

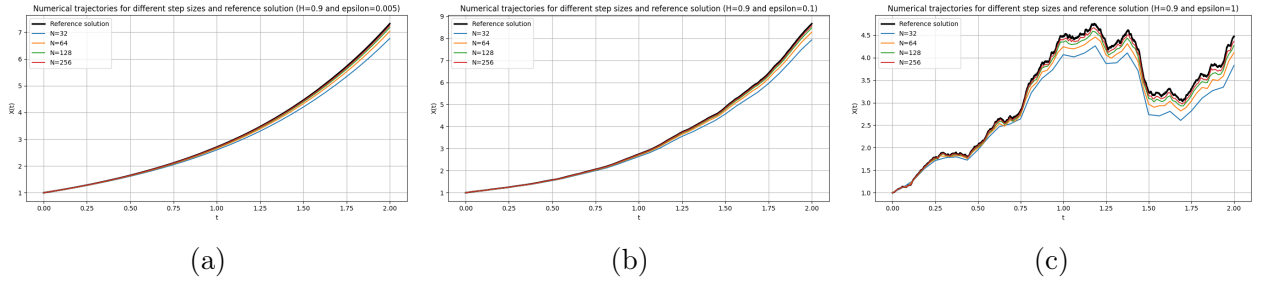


Figure 5.3: Sample path with Hurst parameter $H = 0,9$ and different values of ε .

The results clearly demonstrate the convergence of the numerical method as the step size decreases. Moreover, the observed ratios indicate a consistent reduction of the error when refining the time discretization, confirming the stability and reliability of the proposed Euler-type scheme for stochastic functional differential equations driven by fractional Brownian motion.

5.2. Back-Euler method

The tables and graphs presented here show the behavior of the approximations given by the backward-Euler scheme (4.10) to the solution of the SFDE (5.1).

The equation is discretized using the backward-Euler-type scheme described in the previous chapter on a uniform time mesh with step size h . The memory integral appearing in (5.1) is approximated using Riemann sums. As before, a reference solution is computed using the same numerical method on a very fine mesh consisting of 2^{12} time steps. This reference solution is treated as the exact solution and is denoted by $X(T)$.

To quantify the accuracy of the numerical approximation, we compute the strong error, for each step size h , simulating 500 independent trajectories and define the empirical strong error as

$$\text{Err}(h) = \frac{1}{500} \sum_{j=1}^{500} \max_{t_i \in \{t_0, \dots, t_N\}} \left| \tilde{X}_N^{(j)} - X(T) \right|,$$

where $\tilde{X}_N^{(j)}$ denotes the numerical approximation at time T corresponding to the j -th trajectory. The convergence behavior is further analyzed using the ratio

$$\text{Ratio} = \frac{\text{Err}(h/2)}{\text{Err}(h)},$$

which provides a measure of the reduction in error as the step size is refined.

The simulations are performed for step sizes $h = 2^{-4}, 2^{-5}, 2^{-6}, 2^{-7}$ and for three different values of the noise intensity parameter $\varepsilon \in \{0,005, 0,1, 1\}$. The results are summarized in Tables 5.4–5.6 and Figures 5.4–5.6 for different values of the Hurst parameter.

$H = 0,51:$	$\varepsilon = 0,005$		$\varepsilon = 0,1$		$\varepsilon = 1$	
h	Error	Ratio	Error	Ratio	Error	Ratio
2^{-4}	$1,391 \times 10^{-1}$	–	$9,891 \times 10^{-2}$	–	$1,135 \times 10^{-1}$	–
2^{-5}	$6,866 \times 10^{-2}$	0,493	$5,075 \times 10^{-2}$	0,513	$8,909 \times 10^{-2}$	0,785
2^{-6}	$3,364 \times 10^{-2}$	0,490	$2,281 \times 10^{-2}$	0,449	$1,057 \times 10^{-1}$	1,186
2^{-7}	$1,625 \times 10^{-2}$	0,483	$1,047 \times 10^{-2}$	0,459	$1,283 \times 10^{-1}$	1,214

Cuadro 5.4: Strong error and ratio for $H = 0,51$ and different values of ε .

$H = 0,75:$	$\varepsilon = 0,005$		$\varepsilon = 0,1$		$\varepsilon = 1$	
h	Error	Ratio	Error	Ratio	Error	Ratio
2^{-4}	$1,391 \times 10^{-1}$	–	$1,021 \times 10^{-1}$	–	$2,544 \times 10^{-1}$	–
2^{-5}	$6,859 \times 10^{-2}$	0,493	$4,881 \times 10^{-2}$	0,478	$2,684 \times 10^{-1}$	1,055
2^{-6}	$3,366 \times 10^{-2}$	0,491	$2,095 \times 10^{-2}$	0,429	$1,865 \times 10^{-1}$	0,695
2^{-7}	$1,626 \times 10^{-2}$	0,483	$9,029 \times 10^{-3}$	0,431	$1,421 \times 10^{-1}$	0,762

Cuadro 5.5: Strong error and ratio for $H = 0,75$ and different values of ε .

$H = 0,9:$	$\varepsilon = 0,005$		$\varepsilon = 0,1$		$\varepsilon = 1$	
h	Error	Ratio	Error	Ratio	Error	Ratio
2^{-4}	$1,392 \times 10^{-1}$	–	$1,028 \times 10^{-1}$	–	$2,464 \times 10^{-1}$	–
2^{-5}	$6,866 \times 10^{-2}$	0,493	$5,016 \times 10^{-2}$	0,488	$2,253 \times 10^{-1}$	0,914
2^{-6}	$3,369 \times 10^{-2}$	0,491	$2,400 \times 10^{-2}$	0,479	$1,668 \times 10^{-1}$	0,741
2^{-7}	$1,628 \times 10^{-2}$	0,483	$1,127 \times 10^{-2}$	0,469	$9,993 \times 10^{-2}$	0,599

Cuadro 5.6: Strong error and ratio for $H = 0,9$ and different values of ε .

The following pictures show Sample paths of the numerical solution obtained with the backward-Euler-type scheme and the reference solution for the stochastic functional differential

equation (5.1)–(5.2) with different Hurst parameter and different values of ε . The reference solution is computed using a sufficiently fine time discretization.

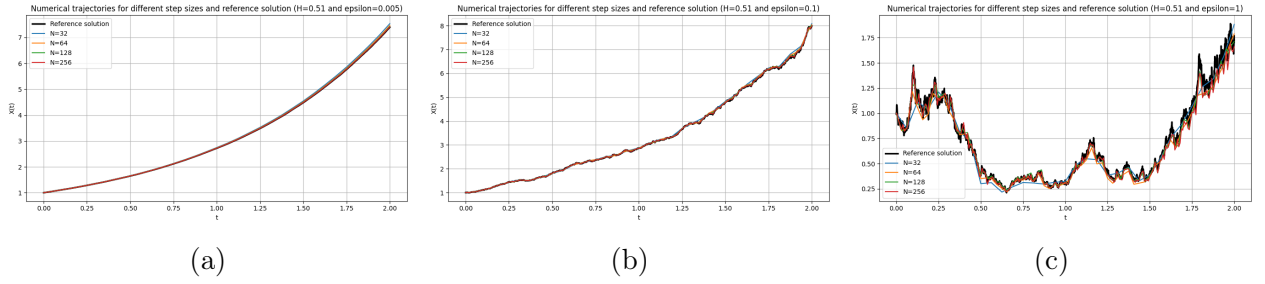


Figure 5.4: Sample path with Hurst parameter $H = 0,51$ and different values of ε .

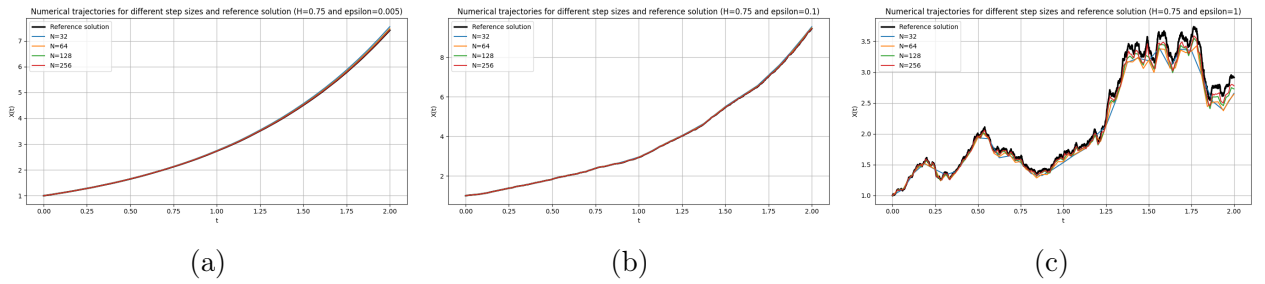


Figure 5.5: Sample path with Hurst parameter $H = 0,75$ and different values of ε .

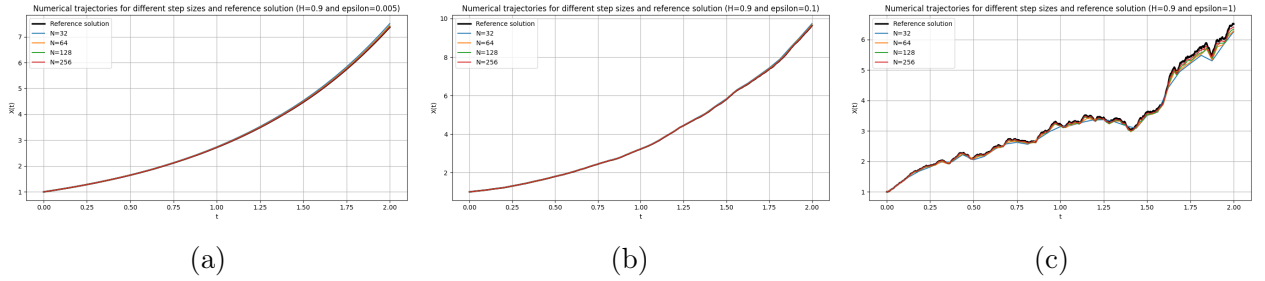


Figure 5.6: Sample path with Hurst parameter $H = 0,9$ and different values of ε .

5.3. Discussion of the numerical results

The numerical experiments presented throughout this chapter provide strong evidence of the effectiveness of the proposed numerical schemes for stochastic functional differential equations driven by a fractional Brownian motion. In all the experiments, both the explicit Euler and the backward Euler approximations reproduce satisfactorily the qualitative behavior of the reference solution, showing that the proposed discretizations are capable of capturing the combined effects of the memory term and the fractional stochastic perturbation.

The sample paths illustrate that the accuracy of the approximations improves as the mesh size decreases. This behavior is observed consistently for all the considered values of the Hurst parameter. As expected, the numerical solutions become progressively closer to the reference solution when finer discretizations are employed, indicating that the discretization error decreases with the time step.

The error tables further support these observations. For each value of the Hurst parameter, the computed strong errors exhibit an overall decreasing tendency as the step size is refined. Although minor fluctuations may appear for coarse discretizations due to the stochastic nature of the problem and the approximation of the fractional Brownian motion, the general trend clearly indicates convergence of the numerical approximations toward the reference solution.

The experimentally observed convergence rates are also in good agreement with the theoretical analysis developed in the previous chapters. In particular, the explicit Euler scheme exhibits the convergence behavior predicted by the theoretical estimates obtained in Chapter 3. Likewise, the local errors computed for the backward Euler method are consistent with the consistency analysis established in Chapter 4, providing additional numerical evidence supporting the theoretical results.

An interesting aspect revealed by the simulations is the influence of the Hurst parameter on the regularity of the sample paths. As H increases, the trajectories become noticeably smoother, reflecting the increased temporal regularity of the fractional Brownian motion. Consequently, the numerical approximations also display smoother behavior for larger values of H , which agrees with the well-known Hölder continuity properties of fractional Brownian motion.

Overall, the numerical experiments validate the proposed discretization procedures and confirm that both schemes constitute reliable approximation techniques for stochastic functional differential equations with distributed delay driven by fractional Brownian motion. The explicit Euler method benefits from its simplicity and computational efficiency, whereas the backward Euler scheme offers an implicit formulation that may become advantageous for more challenging problems, particularly those involving stronger nonlinearities or stiffness. Although only the consistency of the backward Euler method has been established in the present work, the numerical results suggest that it provides stable approximations and motivate further theoretical investigations concerning its convergence and stability properties.

Finally, it is worth emphasizing that the numerical experiments not only corroborate the theoretical findings obtained in this thesis but also indicate several promising research directions. In particular, establishing a complete convergence analysis for the backward Euler scheme, deriving stability results under weaker assumptions, and extending the proposed methods to multidimensional equations or equations with more general memory functionals constitute natural continuations of the present work.

CAPÍTULO 6

CONCLUSIONS AND FUTURE WORK

In this thesis, we investigated the numerical approximation of a class of stochastic functional differential equations driven by a fractional Brownian motion with Hurst parameter $H > 1/2$, where the hereditary effects are described through an integral kernel. The analysis was carried out within the framework of generalized Riemann-Stieltjes integration, which allows the stochastic integral with respect to the fractional Brownian motion to be interpreted pathwise.

The first main contribution of this work was the construction and analysis of an explicit Euler scheme for the considered equation. Under suitable assumptions on the drift, diffusion coefficient, and memory kernel, consistency and convergence results were established. In particular, it was shown that the numerical approximations converge pathwise to the exact solution, and an estimate for the convergence order was obtained. These results constitute the basis of a research article that has been submitted for publication.

The second contribution concerned the development of an implicit backward Euler scheme adapted to the same class of stochastic functional differential equations. By exploiting the regularity properties of the exact solution and the structure of the delay term, consistency and convergence of the implicit approximations were proved. The analysis showed that the backward Euler scheme preserves the same convergence order as the explicit Euler method. This work extends the available theory on implicit numerical methods for stochastic equations with memory driven by fractional Brownian motion and constitutes the subject of a work currently in preparation.

Although both numerical schemes were shown to converge to the exact solution, they differ substantially in their construction. The Euler method provides an explicit approximation whose implementation is straightforward and computationally inexpensive. On the other hand, the backward Euler method is implicit and therefore requires solving an algebraic equation at each time step. Nevertheless, implicit methods are generally expected to exhibit better stability properties, especially when dealing with stiff systems. From the theoretical point of view, both schemes lead to the same convergence rate, although their analytical treatments differ significantly due to the implicit character of the backward Euler approximation.

The numerical simulations presented in this thesis illustrate the theoretical findings and show that both schemes provide satisfactory approximations of the exact solution. Moreover, the experiments highlight the influence of the memory term and the fractional nature of the driving noise on the behavior of the numerical trajectories.

Several interesting questions remain open and deserve further investigation. First, it would be natural to extend the convergence analysis to more general stochastic functional differential

equations, including multidimensional systems and equations with more general delay structures. Another challenging problem is the study of numerical methods for equations driven by fractional Brownian motion with Hurst parameter $H < 1/2$, where rough path techniques or alternative stochastic integration theories are required. It would also be of interest to investigate stability properties of the backward Euler scheme and to compare them with those of the explicit Euler method. Finally, the construction and analysis of higher-order numerical methods, such as Milstein-type schemes or predictor-corrector methods, constitute promising directions for future research.

CAPÍTULO 7

APPENDIX

7.1. Auxiliary Technical Lemmas

Proof of Lemma 3.1.1

Proof. From (3.2), and by adding and subtracting $\int_{t-r}^t K(s, u-t, X(u)) du$,

$$\begin{aligned} Y(t) - Y(s) &= \int_{t-r}^t K(t, u-t, X(u)) du - \int_{s-r}^s K(s, u-s, X(u)) du \\ &= \int_{t-r}^t \left(K(t, u-t, X(u)) - K(s, u-t, X(u)) \right) du \\ &\quad + \int_{t-r}^t \left(K(s, u-t, X(u)) - K(s, u-s, X(u)) \right) du + \int_{s-r}^{t-r} K(s, u-s, X(u)) du \\ &=: I_1 + I_2 + I_3. \end{aligned}$$

Using Hypothesis 3,

$$\begin{aligned} |I_1| &\leq \int_{t-r}^t K_1 |t-s| du = K_1 r |t-s|, \\ |I_2| &\leq \int_{t-r}^t K_2 |(u-t) - (u-s)| du = K_2 r |t-s| \end{aligned}$$

and

$$|I_3| \leq \int_{s-r}^{t-r} K_3 (1 + |X(u)|) du \leq K_3 |t-s| + K_3 \int_{s-r}^{t-r} |X(u)| du.$$

Combining the three estimates yields

$$|Y(t) - Y(s)| \leq (r(K_1 + K_2) + K_3) |t-s| + K_3 \int_{s-r}^{t-r} |X(u)| du.$$

Hence, we obtain the result.

Proof of Lemma 3.3.1

Proof. From the linear growth property of K (3.8), for each $0 \leq u \leq T$,

$$\begin{aligned}
|\tilde{Y}(C_h(u))| &\leq \sum_{i=-N}^{-1} h \cdot |K(C_h(u), t_i, \tilde{X}^h(C_h(u) + t_i))| \\
&\leq \sum_{i=-N}^{-1} h \cdot K_3(1 + |\tilde{X}^h(C_h(u) + t_i)|) \\
&\leq \sum_{i=-N}^{-1} \frac{r}{N} \cdot K_3(1 + \sup_{s \in [-r, -t_1]} |\tilde{X}^h(C_h(u) + s)|) \\
&\leq \sum_{i=-N}^{-1} \frac{r}{N} \cdot K_3(1 + \sup_{x \in [-r, u-t_1]} |\tilde{X}^h(x)|) \\
&\leq rK_3(1 + \sup_{x \in [-r, u-h]} |\tilde{X}^h(x)|) \leq rK_3(1 + \tilde{F}(u-h)).
\end{aligned}$$

Proof of Lemma 3.3.2

Proof. From Hypothesis 3,

$$\begin{aligned}
|\tilde{Y}(u) - \tilde{Y}(v)| &\leq \sum_{i=-N}^{-1} h |K(u, t_i, \tilde{X}^h(u + t_i)) - K(v, t_i, \tilde{X}^h(v + t_i))| \\
&\leq \sum_{i=-N}^{-1} h(K_1|u - v| + K_2|\tilde{X}^h(u + t_i) - \tilde{X}^h(v + t_i)|) \\
&= hK_1 \sum_{i=-N}^{-1} |u - v| + K_2 \sum_{i=-N}^{-1} \int_{t_i}^{t_{i+1}} |\tilde{X}^h(u + C_h(s)) - \tilde{X}^h(v + C_h(s))| ds \\
&= rK_1|u - v| + K_2 \int_{-r}^0 |\tilde{X}^h(u + C_h(s)) - \tilde{X}^h(v + C_h(s))| ds.
\end{aligned}$$

Proof of Lemma 3.3.3

Proof. From (3.10) and (3.6), we get

$$\begin{aligned}
&|\tilde{X}^h(v) - \tilde{X}^h(C_h(v))| \\
&\leq \left| \int_{C_h(v)}^v b(C_h(u), \tilde{X}^h(C_h(u)), \tilde{Y}(C_h(u))) du \right| + \left| \int_{C_h(v)}^v \sigma(C_h(u), \tilde{X}^h(C_h(u)), \tilde{Y}(C_h(u))) dB^H(u) \right| \\
&= |b(C_h(v), \tilde{X}^h(C_h(v)), \tilde{Y}(C_h(v)))(v - C_h(v))| \\
&\quad + |\sigma(C_h(v), \tilde{X}^h(C_h(v)), \tilde{Y}(C_h(v)))| |B^H(v) - B^H(C_h(v))| \\
&\leq L_3(1 + |\tilde{X}^h(C_h(v))| + |\tilde{Y}(C_h(v))|) \times \left((v - C_h(v)) + |B^H(v) - B^H(C_h(v))| \right).
\end{aligned}$$

Using (1.5),

$$|B^H(v) - B^H(C_h(v))| \leq C(\omega)(v - C_h(v))^{H-\rho},$$

since $v - C_h(v) \leq T^{1-H+\rho}(v - C_h(v))^{H-\rho}$ we obtain

$$|\tilde{X}^h(v) - \tilde{X}^h(C_h(v))| \leq L_3(T^{1-H+\rho} + C(\omega))(1 + |\tilde{X}^h(C_h(v))| + |\tilde{Y}(C_h(v))|)(v - C_h(v))^{H-\rho}.$$

Using Lemma 3.3.1,

$$|\tilde{Y}(C_h(v))| \leq rK_3(1 + \tilde{F}(v-h)),$$

and defining $C(\omega) = L_3(T^{1-H+\rho} + C(\omega))$, we obtain the result.

Proof of Lemma 3.3.4

Proof. From (3.10), the bounds stated in Hypothesis 1, (1.4) and the fact that $C_h(w) = C_h(z)$ for $C_h(w) \leq z < w$, together with the observation that $C_h(w) - C_h(z) \leq w - z + h$, we have

$$\begin{aligned} & |\tilde{X}^h(C_h(u)) - \tilde{X}^h(v)| \\ & \leq C(\omega) \left(\int_v^{C_h(u)} (1 + |\tilde{X}^h(C_h(w))| + |\tilde{Y}(C_h(w))|) dw \right. \\ & \quad + \int_v^{C_h(u)} \frac{(1 + |\tilde{X}^h(C_h(w))| + |\tilde{Y}(C_h(w))|)}{(w - v)^\beta} dw \\ & \quad \left. + \int_v^{C_h(u)} \int_v^w \frac{|C_h(w) - C_h(z)| + |\tilde{X}^h(C_h(w)) - \tilde{X}^h(C_h(z))| + |\tilde{Y}(C_h(w)) - \tilde{Y}(C_h(z))|}{(w - z)^{\beta+1}} dz dw \right). \end{aligned}$$

Using Remark 3.3.1, see (3.14), we derive the inequality

$$\begin{aligned} & |\tilde{X}^h(C_h(u)) - \tilde{X}^h(v)| \\ & \leq C(\omega) \left((C_h(u) - v)^{1-\beta} + \int_v^{C_h(u)} \frac{|\tilde{X}^h(C_h(w))|}{(w - v)^\beta} dw + \int_v^{C_h(u)} \frac{|\tilde{Y}(C_h(w))|}{(w - v)^\beta} dw \right. \\ & \quad + h \int_v^{C_h(u)} (w - C_h(w))^{-\beta} dw + \int_v^{C_h(u)} \int_v^{C_h(w)} \frac{|\tilde{X}^h(C_h(w)) - \tilde{X}^h(z)|}{(w - z)^{\beta+1}} dz dw \\ & \quad \left. + \int_v^{C_h(u)} \int_v^{C_h(w)} \frac{|\tilde{X}^h(z) - \tilde{X}^h(C_h(z))|}{(w - z)^{\beta+1}} dz dw + \int_v^{C_h(u)} \int_v^{C_h(w)} \frac{|\tilde{Y}(C_h(w)) - \tilde{Y}(C_h(z))|}{(w - z)^{\beta+1}} dz dw \right). \end{aligned}$$

Multiplying by $(u - v)^{-\beta-1}$ and integrating over $[0, C_h(u)]$, we obtain that

$$\begin{aligned} & \int_0^{C_h(u)} |\tilde{X}^h(C_h(u)) - \tilde{X}^h(v)| (u - v)^{-\beta-1} dv \\ & \leq C(\omega) (Q^1(u) + Q^2(u) + Q^3(u) + Q^4(u) + Q^5(u) + Q^6(u) + Q^7(u)), \end{aligned} \quad (7.1)$$

where

$$\begin{aligned} Q^1(u) &= \int_0^{C_h(u)} (C_h(u) - v)^{1-\beta} (u - v)^{-\beta-1} dv, \\ Q^2(u) &= \int_0^{C_h(u)} (u - v)^{-\beta-1} \int_v^{C_h(u)} \frac{|\tilde{X}^h(C_h(w))|}{(w - v)^\beta} dw dv, \\ Q^3(u) &= \int_0^{C_h(u)} (u - v)^{-\beta-1} \int_v^{C_h(u)} \frac{|\tilde{Y}(C_h(w))|}{(w - v)^\beta} dw dv \\ Q^4(u) &= h \int_0^{C_h(u)} (u - v)^{-\beta-1} \int_v^{C_h(u)} (w - C_h(w))^{-\beta} dw dv, \\ Q^5(u) &= \int_0^{C_h(u)} (u - v)^{-\beta-1} \int_v^{C_h(u)} \int_v^{C_h(w)} \frac{|\tilde{X}^h(C_h(w)) - \tilde{X}^h(z)|}{(w - z)^{\beta+1}} dz dw dv, \\ Q^6(u) &= \int_0^{C_h(u)} (u - v)^{-\beta-1} \int_v^{C_h(u)} \int_v^{C_h(w)} \frac{|\tilde{X}^h(z) - \tilde{X}^h(C_h(z))|}{(w - z)^{\beta+1}} dz dw dv \\ Q^7(u) &= \int_0^{C_h(u)} (u - v)^{-\beta-1} \int_v^{C_h(u)} \int_v^{C_h(w)} \frac{|\tilde{Y}(C_h(w)) - \tilde{Y}(C_h(z))|}{(w - z)^{\beta+1}} dz dw dv. \end{aligned}$$

Noting that $0 < \beta < 1/2$, $C_h(u) - v \leq u - v$, it follows that

$$Q^1(u) \leq \int_0^{C_h(u)} (u - v)^{-2\beta} dv \leq \frac{u^{1-2\beta}}{1-2\beta}. \quad (7.2)$$

Exchanging the integration limits in $Q^2(u)$ and $Q^3(u)$, making a change of variable as in [16], and taking $C = \int_0^\infty (1+y)^{-\beta-1} y^{-\beta} dy$, we obtain

$$Q^2(u) = \int_0^{C_h(u)} |\tilde{X}^h(C_h(w))| \int_0^w (u - v)^{-\beta-1} (w - v)^{-\beta} dv dw \leq C \int_0^{C_h(u)} \frac{|\tilde{X}^h(C_h(w))|}{(u - w)^{2\beta}} dw. \quad (7.3)$$

Since $\tilde{F}$ is nondecreasing, we have $\tilde{F}(w - h) \leq \tilde{F}(w)$, and

$$\begin{aligned} Q^3(u) &= \int_0^{C_h(u)} |\tilde{Y}(C_h(w))| \int_0^w (u - v)^{-\beta-1} (w - v)^{-\beta} dv dw \\ &\leq C \int_0^{C_h(u)} r K_3 (1 + \tilde{F}(w - h)) (u - w)^{-2\beta} dw \\ &\leq Cr K_3 \int_0^{C_h(u)} (u - w)^{-2\beta} dw + Cr K_3 \int_0^{C_h(u)} \tilde{F}(w - h) (u - w)^{-2\beta} dw \\ &\leq C \left(u^{1-2\beta} + \int_0^{C_h(u)} \frac{\tilde{F}(w)}{(u - w)^{2\beta}} dw \right). \end{aligned} \quad (7.4)$$

From Lemma 1 in [16], it follows directly that $Q^4(u) \leq Ch^{1-\beta}$.

Exchanging the integration limits in $Q^5(u)$, we obtain

$$\begin{aligned} Q^5(u) &\leq \int_0^{C_h(u)} \int_0^{C_h(w)} \int_0^w (u - v)^{-\beta-1} \frac{|\tilde{X}^h(C_h(w)) - \tilde{X}^h(z)|}{(w - z)^{\beta+1}} dv dz dw \\ &\leq \frac{1}{\beta} \int_0^{C_h(u)} (u - w)^{-\beta} \int_0^{C_h(w)} \frac{|\tilde{X}^h(C_h(w)) - \tilde{X}^h(z)|}{(w - z)^{\beta+1}} dz dw. \end{aligned} \quad (7.5)$$

For $Q^6(u)$, we use Lemma 3.3.3, (3.13) and (3.14) to the interval $[v, C_h(u)]$ to deduce

$$Q^6(u) \leq C(\omega) h^{H-\rho-\beta} \left(1 + \tilde{F}(C_h(u)) \right). \quad (7.6)$$

Finally, we estimate $Q^7(u)$. Using the discrete structural estimate of the delay functional (Lemma 3.3.2), which mirrors Lemma 3.1.1,

$$\begin{aligned} Q^7(u) &\leq \int_0^{C_h(u)} (u - v)^{-\beta-1} \int_v^{C_h(u)} \int_v^{C_h(w)} \frac{r K_1 (C_h(w) - C_h(z))}{(w - z)^{\beta+1}} dz dw dv \\ &\quad + \int_0^{C_h(u)} (u - v)^{-\beta-1} \int_v^{C_h(u)} \int_v^{C_h(w)} \int_{-r}^0 \frac{K_2 |\tilde{X}^h(C_h(w) + C_h(s)) - \tilde{X}^h(C_h(z) + C_h(s))|}{(w - z)^{\beta+1}} ds dz dw dv. \end{aligned} \quad (7.7)$$

Using $C_h(w) - C_h(z) \leq (w - z) + h$ and arguing as in the estimate of $Q^4(u)$, we obtain the

bound

$$\begin{aligned}
Q^7(u) &\leq C + rK_1h \int_0^{C_h(u)} (u-v)^{-\beta-1} \int_v^{C_h(u)} (w-C_h(w))^{-\beta} dw dv \\
&\quad + \frac{K_2}{\beta} \int_0^{C_h(u)} (u-w)^{-\beta} \int_0^{C_h(w)} \int_{-r}^0 \frac{|\tilde{X}^h(C_h(w)+C_h(s)) - \tilde{X}^h(C_h(z)+C_h(s))|}{(w-z)^{\beta+1}} ds dz dw \\
&\leq C + \frac{K_2}{\beta} \int_0^{C_h(u)} (u-w)^{-\beta} \int_0^{C_h(w)} \int_{-r}^0 \frac{|\tilde{X}^h(C_h(w)+C_h(s)) - \tilde{X}^h(C_h(z)+C_h(s))|}{(w-z)^{\beta+1}} ds dz dw.
\end{aligned} \tag{7.8}$$

By (7.1) and the estimates given in (7.2) - (7.8),

$$\begin{aligned}
&\int_0^{C_h(u)} |\tilde{X}^h(C_h(u)) - \tilde{X}^h(v)|(u-v)^{-\beta-1} dv \\
&\leq C(\omega) \left(1 + \int_0^{C_h(u)} |\tilde{X}^h(w)|(u-w)^{-2\beta} dw + \int_0^{C_h(u)} \tilde{F}(w)(u-w)^{-2\beta} dw \right. \\
&\quad + \int_0^{C_h(u)} (u-w)^{-\beta} \int_0^{C_h(w)} \frac{|\tilde{X}^h(C_h(w)) - \tilde{X}^h(z)|}{(w-z)^{\beta+1}} dz dw + \tilde{F}(C_h(u))h^{H-\rho-\beta} \\
&\quad \left. + \int_0^{C_h(u)} (u-w)^{-\beta} \int_0^{C_h(w)} \int_{-r}^0 \frac{|\tilde{X}^h(C_h(w)+C_h(s)) - \tilde{X}^h(C_h(z)+C_h(s))|}{(w-z)^{\beta+1}} ds dz dw \right).
\end{aligned}$$

Proof of Lemma 3.3.5

Proof. For $-r < a < b < 0$, let $J(a, b) := \int_a^b |\tilde{X}^h(C_h(w)+C_h(s)) - \tilde{X}^h(C_h(z)+C_h(s))| ds$.

We partition the integration interval with respect to $s \in [-r, 0]$ into four subintervals as

$$[-r, -C_h(w)], \quad [-C_h(w), -z], \quad [-z, -C_h(z)], \quad [-C_h(z), 0].$$

Case 1 (First subinterval $[-r, -C_h(w)]$). Within this subinterval, the arguments of $\tilde{X}^h$ inside the integral $J(-r, -C_h(w))$ are negative or zero (lying within the region where the initial condition ϕ is applied). By invoking the Hölder continuity and local regularity assumptions (3.3), we obtain the following point-wise control:

$$|\tilde{X}^h(C_h(w)+C_h(s)) - \tilde{X}^h(C_h(z)+C_h(s))| \leq C_1(\omega) |C_h(w) - C_h(z)|^{H-\rho}.$$

Then

$$J(-r, -C_h(w)) \leq r C_1(\omega) |C_h(w) - C_h(z)|^{H-\rho}$$

Using $C_h(w) - C_h(z) \leq (w - z) + h$ and arguing as in Remark 3.3.1, we obtain

$$\begin{aligned}
&\int_0^{C_h(w)} (w-z)^{-\beta-1} J(-r, -C_h(w)) dz \leq r C_1(\omega) \int_0^{C_h(w)} (w-z)^{-\beta-1} (C_h(w) - C_h(z))^{H-\rho} dz \\
&\leq r C_1(\omega) (w^{-\beta+H-\rho} + h^{H-\rho} (w - C_h(w))^{-\beta}).
\end{aligned} \tag{7.9}$$

Case 2 (Second subinterval $[-C_h(w), -z]$). In this case, $-C_h(w) \leq s \leq -z \leq -C_h(z)$, which implies that $C_h(w) + C_h(s) > 0$ and $C_h(s) + C_h(z) < 0$. By adding and subtracting $\tilde{X}^h(0)$

and applying the triangle inequality, we obtain:

$$\begin{aligned}
& |\tilde{X}^h(C_h(w) + C_h(s)) - \tilde{X}^h(C_h(z) + C_h(s))| \\
& \leq C_1(\omega) \left(|\tilde{X}^h(C_h(w) + C_h(s)) - \tilde{X}^h(0)| + |C_h(z) + C_h(s)|^{H-\rho} \right) \\
& \leq C_1(\omega) \left(|\tilde{X}^h(C_h(w) + C_h(s)) - \tilde{X}^h(0)| + |C_h(w) - C_h(z)|^{H-\rho} \right) \\
& \leq C_1(\omega)(1 + \tilde{F}(C_h(w)) + |C_h(w) - C_h(z)|^{H-\rho}).
\end{aligned}$$

Recall that the length of this interval is $C_h(w) - z$. Using $C_h(w) - C_h(z) \leq (w - z) + h$ and the monotonicity of $\tilde{F}$, we obtain

$$\begin{aligned}
& \int_0^{C_h(w)} (w - z)^{-\beta-1} \int_{-C_h(w)}^{-z} |\tilde{X}^h(C_h(w) + C_h(s)) - \tilde{X}^h(C_h(z) + C_h(s))| ds dz \\
& \leq C_1(\omega) \int_0^{C_h(w)} (C_h(w) - z)(w - z)^{-\beta-1} (C_h(w) - C_h(z))^{H-\rho} dz \\
& \leq C_1(\omega)(1 + \tilde{F}(C_h(w))).
\end{aligned} \tag{7.10}$$

Case 3 (Third subinterval $[-z, -C_h(z)]$). In this case, $-C_h(w) \leq -z \leq s \leq -C_h(z)$. Hence, we have $C_h(w) + C_h(s) > 0$ and $C_h(s) + C_h(z) < 0$. Proceeding as in Case 2 and using the triangle inequality, we obtain

$$\begin{aligned}
& |\tilde{X}^h(C_h(w) + C_h(s)) - \tilde{X}^h(C_h(z) + C_h(s))| \\
& \leq C_1(\omega) \left(|\tilde{X}^h(C_h(w) + C_h(s)) - \tilde{X}^h(0)| + |C_h(w) - C_h(z)|^{H-\rho} \right) \\
& \leq C_1(\omega)(2\tilde{F}(C_h(w)) + |C_h(w) - C_h(z)|^{H-\rho}).
\end{aligned}$$

Since we work in the interval $[-z, -C_h(z)]$, whose length is smaller than h , we get

$$\begin{aligned}
& \int_0^{C_h(w)} (w - z)^{-\beta-1} \int_{-z}^{-C_h(z)} |\tilde{X}^h(C_h(w) + C_h(s)) - \tilde{X}^h(C_h(z) + C_h(s))| ds dz \\
& \leq h C_1(\omega) \int_0^{C_h(w)} (w - z)^{-\beta-1} (C_h(w) - C_h(z))^{H-\rho} dz \\
& \leq h C_1(\omega) \left(w^{H-\rho-\beta} + (w - C_h(w))^{-\beta} (1 + \tilde{F}(C_h(w))) \right).
\end{aligned} \tag{7.11}$$

Case 4 (Final subinterval $[-C_h(z), 0]$). On this subinterval it holds that $-C_h(z) \leq s \leq 0$, $-C_h(z) \leq C_h(s)$ and $-C_h(w) \leq C_h(s)$. Then $C_h(w) + C_h(s) > 0$ and $C_h(s) + C_h(z) > 0$. We leave the integral in its original form,

$$\int_{-C_h(z)}^0 |\tilde{X}^h(C_h(w) + C_h(s)) - \tilde{X}^h(C_h(z) + C_h(s))| ds. \tag{7.12}$$

In summary, from (7.9), (7.10), (7.11), and (7.12), we obtain

$$\begin{aligned}
I(w) & \leq C_1(\omega) \left(1 + \tilde{F}(C_h(w)) + h^{H-\rho} (w - C_h(w))^{-\beta} + h (w - C_h(w))^{-\beta} \tilde{F}(C_h(w)) \right. \\
& \quad \left. + \int_0^{C_h(w)} (w - z)^{-\beta-1} \int_{-C_h(z)}^0 |\tilde{X}^h(C_h(w) + C_h(s)) - \tilde{X}^h(C_h(z) + C_h(s))| ds dz \right).
\end{aligned}$$

Proof of Lemma 3.3.6

Proof. We observe that

$$|\tilde{X}^h(C_h(w) + C_h(s)) - \tilde{X}^h(C_h(z) + C_h(s))| \leq |\tilde{X}^h(C_h(w) + C_h(s)) - \tilde{X}^h(z + C_h(s))| \\ + |\tilde{X}^h(z + C_h(s)) - \tilde{X}^h(C_h(z) + C_h(s))|.$$

Noting that $C_h(z) + C_h(s) = C_h(z + C_h(s))$, then from Lemma 3.3.3,

$$\begin{aligned} & |\tilde{X}^h(z + C_h(s)) - \tilde{X}^h(C_h(z) + C_h(s))| \\ & \leq C(\omega)(1 + |\tilde{X}^h(C_h(z) + C_h(s))| + rK_3(1 + \tilde{F}(z + C_h(s) - h)))(z - C_h(z))^{H-\rho} \\ & \leq C(\omega)(1 + \tilde{F}(C_h(z)) + rK_3(1 + \tilde{F}(C_h(z))))h^{H-\rho} \\ & \leq C(\omega)(1 + \tilde{F}(C_h(z)))h^{H-\rho}. \end{aligned}$$

Integrating with respect to s over $[-C_h(z), 0]$, multiplying by $(w - z)^{-\beta-1}$ and integrating with respect to z over $[0, C_h(w)]$ we obtain

$$\begin{aligned} & \int_0^{C_h(w)} (w - z)^{-\beta-1} \int_{-C_h(z)}^0 |\tilde{X}^h(C_h(w) + C_h(s)) - \tilde{X}^h(z + C_h(s))| ds dz \\ & \leq C(\omega)h^{H-\rho} \int_0^{C_h(w)} (w - z)^{-\beta-1} (1 + \tilde{F}(C_h(z))) \int_{-C_h(z)}^0 ds dz \\ & \leq C(\omega)h^{H-\rho} \int_0^{C_h(w)} (w - z)^{-\beta-1} (1 + \tilde{F}(C_h(z))) dz \\ & \leq C(\omega)(1 + \tilde{F}(C_h(w)))h^{H-\rho} \int_0^{C_h(w)} (w - z)^{-\beta-1} dz \\ & \leq C(\omega)(1 + \tilde{F}(C_h(w)))h^{H-\rho}(w - C_h(w))^{-\beta}. \end{aligned}$$

Let us analyze now

$$|\tilde{X}^h(C_h(w) + C_h(s)) - \tilde{X}^h(z + C_h(s))|.$$

Applying inequality 1.4 on the interval $[z + C_h(s), C_h(w) + C_h(s)]$, we obtain

$$\begin{aligned} & |\tilde{X}^h(C_h(w) + C_h(s)) - \tilde{X}^h(z + C_h(s))| \\ & \leq C(\omega) \left[\int_{z+C_h(s)}^{C_h(w)+C_h(s)} \frac{1 + |\tilde{X}^h(C_h(v))| + |\tilde{Y}(C_h(v))|}{(v - (z + C_h(s)))^\beta} dv + \int_{z+C_h(s)}^{C_h(w)+C_h(s)} \int_{z+C_h(s)}^{C_h(u)} \right. \\ & \quad \left. \frac{|C_h(u) - C_h(v)| + |\tilde{X}^h(C_h(u)) - \tilde{X}^h(C_h(v))| + |\tilde{Y}(C_h(u)) - \tilde{Y}(C_h(v))|}{(u - v)^{\beta+1}} dv du \right]. \end{aligned} \quad (7.13)$$

Let us denote by I_c the double integral in (7.13). We can decompose I_c as follows:

$$\begin{aligned} I_c & \leq \int_{z+C_h(s)}^{C_h(w)+C_h(s)} \int_{z+C_h(s)}^{C_h(u)} \frac{|C_h(u) - C_h(v)|}{(u - v)^{\beta+1}} dv du \\ & \quad + \int_{z+C_h(s)}^{C_h(w)+C_h(s)} \int_{z+C_h(s)}^{C_h(u)} \frac{|\tilde{X}^h(C_h(u)) - \tilde{X}^h(C_h(v))|}{(u - v)^{\beta+1}} dv du \\ & \quad + \int_{z+C_h(s)}^{C_h(w)+C_h(s)} \int_{z+C_h(s)}^{C_h(u)} \frac{|\tilde{Y}(C_h(u)) - \tilde{Y}(C_h(v))|}{(u - v)^{\beta+1}} dv du \\ & =: I_{c1} + I_{c2} + I_{c3}. \end{aligned}$$

Next, for I_{c_1} , and using the inequality $C_h(u) - C_h(v) \leq u - v + h$, we deduce

$$\begin{aligned}
I_{c_1} &= \int_{z+C_h(s)}^{C_h(w)+C_h(s)} \int_{z+C_h(s)}^{C_h(u)} \frac{|C_h(u) - C_h(v)|}{(u-v)^{\beta+1}} dv du \\
&\leq \int_{z+C_h(s)}^{C_h(w)+C_h(s)} \int_{z+C_h(s)}^{C_h(u)} (u-v)^{-\beta} dv du + \int_{z+C_h(s)}^{C_h(w)+C_h(s)} \int_{z+C_h(s)}^{C_h(u)} \frac{h}{(u-v)^{\beta+1}} dv du \\
&\leq \frac{1}{1-\beta} \int_{z+C_h(s)}^{C_h(w)+C_h(s)} (u-z-C_h(s))^{1-\beta} du + \frac{h}{\beta} \int_{z+C_h(s)}^{C_h(w)+C_h(s)} (u-C_h(u))^{-\beta} du \\
&\leq C(C_h(w)-z)^{2-\beta} + \frac{h}{\beta} \int_{z+C_h(s)}^0 (u-C_h(u))^{-\beta} du + \frac{h}{\beta} \int_0^{C_h(w)+C_h(s)} (u-C_h(u))^{-\beta} du. \\
&\leq C(C_h(w)-z)^{2-\beta} + \frac{(C_h(w)-z)h^{1-\beta}}{\beta} \\
&\leq C((C_h(w)-z)^{2-\beta} + (C_h(w)-z)h^{1-\beta}), \tag{7.14}
\end{aligned}$$

where the last inequality follows from the expansion in (3.14) and from the fact that this inequality remains valid for negative values, since the interval $[-r, 0]$ is partitioned into the subintervals t_i together with (3.14).

For the second integral I_{c_2} , we further decompose it into two integrals:

$$\begin{aligned}
I_{c_2} &= \int_{z+C_h(s)}^{C_h(w)+C_h(s)} \int_{z+C_h(s)}^{C_h(u)} \frac{|\tilde{X}^h(C_h(u)) - \tilde{X}^h(C_h(v))|}{(u-v)^{\beta+1}} dv du \\
&\leq \int_{z+C_h(s)}^{C_h(w)+C_h(s)} \int_{z+C_h(s)}^{C_h(u)} \frac{|\tilde{X}^h(C_h(u)) - \tilde{X}^h(v)|}{(u-v)^{\beta+1}} dv du \\
&\quad + \int_{z+C_h(s)}^{C_h(w)+C_h(s)} \int_{z+C_h(s)}^{C_h(u)} \frac{|\tilde{X}^h(v) - \tilde{X}^h(C_h(v))|}{(u-v)^{\beta+1}} dv du. \tag{7.15}
\end{aligned}$$

From Lemma (3.3.3)

$$|\tilde{X}^h(v) - \tilde{X}^h(C_h(v))| \leq C(\omega) \left(1 + |\tilde{X}^h(C_h(v))| + rK_3(1 + \tilde{F}(v-h))\right) (v - C_h(v))^{H-\rho},$$

then

$$\begin{aligned}
&\int_{z+C_h(s)}^{C_h(w)+C_h(s)} \int_{z+C_h(s)}^{C_h(u)} \frac{|\tilde{X}^h(v) - \tilde{X}^h(C_h(v))|}{(u-v)^{\beta+1}} dv du \\
&\leq \int_{z+C_h(s)}^{C_h(w)+C_h(s)} \int_{z+C_h(s)}^{C_h(u)} \frac{C(\omega) \left(1 + |\tilde{X}^h(C_h(v))| + rK_3(1 + \tilde{F}(v-h))\right) h^{H-\rho}}{(u-v)^{\beta+1}} dv du \\
&\leq \int_{z+C_h(s)}^{C_h(w)+C_h(s)} \int_{z+C_h(s)}^{C_h(u)} \frac{C(\omega) \left(1 + \tilde{F}^h(C_h(u)) + rK_3(1 + \tilde{F}(C_h(u)))\right) h^{H-\rho}}{(u-v)^{\beta+1}} dv du \\
&\leq C(\omega)(1 + rK_3)(1 + \tilde{F}(C_h(w)))h^{H-\rho} \int_{z+C_h(s)}^{C_h(w)+C_h(s)} \int_{z+C_h(s)}^{C_h(u)} (u-v)^{-\beta-1} dv du \\
&\leq C(\omega)h^{H-\rho-\beta}(1 + \tilde{F}(C_h(w)))(C_h(w) - z). \tag{7.16}
\end{aligned}$$

Therefore, combining (7.14), (7.15) and (7.16), equation (7.13) takes the form

$$\begin{aligned} & |\tilde{X}^h(C_h(w) + C_h(s)) - \tilde{X}^h(z + C_h(s))| \leq C(\omega) \left((C_h(w) - z)^{1-\beta} + \int_z^{C_h(w)} \frac{\tilde{F}(C_h(v))}{(v-z)^\beta} dv \right. \\ & \quad + (C_h(w) - z)^{2-\beta} + (C_h(w) - z)h^{1-\beta} + \int_{z+C_h(s)}^{C_h(w)+C_h(s)} \int_{z+C_h(s)}^{C_h(u)} \frac{|\tilde{X}^h(C_h(u)) - \tilde{X}^h(v)|}{(u-v)^{\beta+1}} dv du \\ & \quad \left. + h^{H-\rho-\beta}(1 + \tilde{F}(C_h(w)))(C_h(w) - z) + \int_{z+C_h(s)}^{C_h(w)+C_h(s)} \int_{z+C_h(s)}^{C_h(u)} \frac{|\tilde{Y}(C_h(u)) - \tilde{Y}(C_h(v))|}{(u-v)^{\beta+1}} dv du \right). \end{aligned}$$

From Lemma (3.3.5), integrating with respect to s over $[-C_h(z), 0]$, and then multiplying by $(w-z)^{-\beta-1}$ and integrating with respect to z over $[0, C_h(w)]$, we obtain:

$$\begin{aligned} & \int_0^{C_h(w)} \int_{-C_h(z)}^0 \frac{|\tilde{X}^h(C_h(w) + C_h(s)) - \tilde{X}^h(C_h(z) + C_h(s))|}{(w-z)^{\beta+1}} ds dz \\ & \leq C(\omega)(R^1 + R^2 + R^3 + R^4 + R^5 + R^6)(w), \end{aligned}$$

where

$$\begin{aligned} R^1(w) &= \int_0^{C_h(w)} (w-z)^{-\beta-1} \int_{-C_h(z)}^0 (C_h(w) - z)^{1-\beta} ds dz, \\ R^2(w) &= \int_0^{C_h(w)} (w-z)^{-\beta-1} \int_{-C_h(z)}^0 \int_z^{C_h(w)} \frac{\tilde{F}(C_h(v))}{(v-z)^\beta} dv ds dz, \\ R^3(w) &= h^{1-\beta} \int_0^{C_h(w)} (w-z)^{-\beta-1} (C_h(w) - z) C_h(z) dz, \\ R^4(w) &= \int_0^{C_h(w)} (w-z)^{-\beta-1} \int_{-C_h(z)}^0 \int_{z+C_h(s)}^{C_h(w)+C_h(s)} \int_{z+C_h(s)}^{C_h(u)} \frac{|\tilde{X}^h(C_h(u)) - \tilde{X}^h(v)|}{(u-v)^{\beta+1}} dv du ds dz, \\ R^5(w) &= \int_0^{C_h(w)} (w-z)^{-\beta-1} \int_{-C_h(z)}^0 C(1 + rK_3)(1 + \tilde{F}(C_h(w)))h^{H-\rho-\beta}(C_h(w) - z) ds dz \\ R^6(w) &= \int_0^{C_h(w)} (w-z)^{-\beta-1} \int_{-C_h(z)}^0 \int_{z+C_h(s)}^{C_h(w)+C_h(s)} \int_{z+C_h(s)}^{C_h(u)} \frac{|\tilde{Y}(C_h(u)) - \tilde{Y}(C_h(v))|}{(u-v)^{\beta+1}} dv du ds dz. \end{aligned}$$

For the term R^1 we obtain the following.

$$R^1(w) = \int_0^{C_h(w)} (w-z)^{-\beta-1} \int_{-C_h(z)}^0 (C_h(w) - z)^{1-\beta} ds dz \leq \int_0^{C_h(w)} z(w-z)^{1-\beta} dz \leq Cw^{3-\beta}. \quad (7.17)$$

A direct computation of R^2 yields (see equation (17) in [16])

$$\begin{aligned} R^2(w) &= \int_0^{C_h(w)} C_h(z) |\tilde{F}^h(C_h(z))| \int_z^{C_h(w)} (w-z)^{-\beta-1} (v-z)^{-\beta} dv dz \\ &\leq C \int_0^{C_h(w)} \frac{|\tilde{F}^h(C_h(z))|}{(w-z)^{2\beta}} dz. \end{aligned} \quad (7.18)$$

For R^3 ,

$$R^3(w) = h^{1-\beta} \int_0^{C_h(w)} (w-z)^{-\beta-1} (C_h(w)-z) C_h(z) dz \leq C h^{1-\beta} \quad (7.19)$$

For the term R^4 , instead of computing explicitly the stepwise decomposition in s and the discrete summation, we observe that after rearranging the order of integration its structure is

$$R^4(w) = \int_0^{C_h(w)} (w-u)^{-\beta} \int_0^{C_h(u)} \frac{|\tilde{X}^h(C_h(u)) - \tilde{X}^h(C_h(v))|}{(u-v)^{\beta+1}} dv du. \quad (7.20)$$

This coincides with the fractional convolution structure appearing in inequality 1.4, and therefore we keep this operator form without expanding the discrete summation further.

For R^5 we have

$$\begin{aligned} R^5(w) &= \int_0^{C_h(w)} (w-z)^{-\beta-1} \int_{-C_h(z)}^0 C(1+rK_3)(1+\tilde{F}(C_h(w))) h^{H-\rho-\beta} (C_h(w)-z) ds dz \\ &\leq C(1+rK_3) h^{H-\rho-\beta} w(1+\tilde{F}(w)). \end{aligned} \quad (7.21)$$

Finally, for R^6 we use Lemma 3.3.2, which yields

$$|\tilde{Y}(C_h(u)) - \tilde{Y}(C_h(v))| \leq rK_1|u-v| + K_2 \int_{-r}^0 |\tilde{X}^h(C_h(u)+C_h(s)) - \tilde{X}^h(C_h(v)+C_h(s))| ds.$$

Substituting into $R^6(w)$ we obtain

$$\begin{aligned} R^6(w) &\leq C \int_0^{C_h(w)} (w-u)^{-\beta} \int_0^{C_h(u)} \frac{|u-v|}{(u-v)^{\beta+1}} dv du \\ &\quad + C \int_0^{C_h(w)} (w-u)^{-\beta} \int_0^{C_h(u)} \int_{-r}^0 \frac{|\tilde{X}^h(C_h(u)+C_h(s)) - \tilde{X}^h(C_h(v)+C_h(s))|}{(u-v)^{\beta+1}} ds dv du. \end{aligned}$$

Using Remark 3.3.1, the first term reduces to

$$C \int_0^{C_h(w)} (w-u)^{-\beta} \int_0^{C_h(u)} (u-v)^{-\beta} dv du \leq C(1+h^{1-\beta}).$$

For $u \in [0, T]$ define

$$I(u) = \int_0^{C_h(u)} (u-v)^{-\beta-1} \int_{-r}^0 |\tilde{X}^h(C_h(u)+C_h(s)) - \tilde{X}^h(C_h(v)+C_h(s))| ds dv.$$

Then observe that this coincides with the operator studied in Lemma 3.3.5 (with w replaced by u). Therefore

$$R^6(w) \leq C(\omega) \left(1 + h^{1-\beta} + \int_0^{C_h(w)} (w-u)^{-\beta} I(u) du \right).$$

Applying the bound of Lemma 3.3.5 and estimating each convolution term separately yields

$$\begin{aligned} R^6(w) &\leq C(\omega) \left[1 + h^{1-\beta} + h^{H-\rho-\beta} + h^{1-\beta} \tilde{F}(C_h(w)) + \tilde{F}(C_h(w)) w^{1-\beta} \right. \\ &\quad \left. + \int_0^{C_h(w)} (w-u)^{-\beta} \int_0^{C_h(u)} (u-v)^{-\beta-1} \int_{-C_h(v)}^0 |\tilde{X}^h(C_h(u)+C_h(s)) - \tilde{X}^h(C_h(v)+C_h(s))| ds dv du \right]. \end{aligned} \quad (7.22)$$

Finally, from (7.17), (7.18), (7.19), (7.20), (7.21), and (7.22) we obtain the result.

Proof of Corollary 3.4.1

Proof. Using the mean value theorem in several variables,

$$\sigma(t, x_1, y_1) - \sigma(t, x_2, y_2) = \int_0^1 (x_1 - x_2, y_1 - y_2) \cdot \nabla \sigma(t, \xi(\lambda), \eta(\lambda)) d\lambda,$$

where $\xi(\lambda) = (1 - \lambda)x_2 + \lambda x_1$ and $\eta(\lambda) = (1 - \lambda)y_2 + \lambda y_1$. Applying this identity at times t_1 and t_2 and subtracting, we obtain

$$\begin{aligned} & \sigma(t_1, x_1, y_1) - \sigma(t_1, x_2, y_2) - \sigma(t_2, x_3, y_3) + \sigma(t_2, x_4, y_4) \\ &= \int_0^1 (x_1 - x_2, y_1 - y_2) \cdot \nabla \sigma(t_1, \xi_1, \eta_1) d\lambda - \int_0^1 (x_3 - x_4, y_3 - y_4) \cdot \nabla \sigma(t_2, \xi_2, \eta_2) d\lambda. \end{aligned}$$

Adding and subtracting suitable terms and using the Lipschitz assumptions on $\nabla \sigma$, we obtain

$$\begin{aligned} & |\sigma(t_1, x_1, y_1) - \sigma(t_1, x_2, y_2) - \sigma(t_2, x_3, y_3) + \sigma(t_2, x_4, y_4)| \\ & \leq C(|x_1 - x_2 - x_3 + x_4| + |y_1 - y_2 - y_3 + y_4|) + C|t_1 - t_2|(|x_1 - x_2| + |y_1 - y_2|) \\ & \quad + C(|x_1 - x_2| + |y_1 - y_2|)(|x_2 - x_4| + |x_1 - x_3| + |y_2 - y_4| + |y_1 - y_3|), \end{aligned}$$

which proves the result.

Proof of Lemma 4.14

Proof.

$$\begin{aligned} |Y(u) - \hat{Y}(C_h(u))| &= \left| \int_{-r}^0 K(u, s, X(u + s)) ds - \sum_{i=-N}^{-1} K(C_h(u), t_i, X(C_h(u) + t_i)) \times h \right| \\ &= \left| \int_{-r}^0 K(u, s, X(u + s)) ds - \int_{-r}^0 K(C_h(u), C_h(s), X(C_h(u) + C_h(s))) ds \right| \\ &= \left| \int_{-r}^0 (K(u, s, X(u + s)) - K(C_h(u), C_h(s), X(C_h(u) + C_h(s)))) ds \right| \\ &\leq \int_{-r}^0 |K(u, s, X(u + s)) - K(C_h(u), C_h(s), X(C_h(u) + C_h(s)))| ds \\ &\leq \int_{-r}^0 (K_1(u - C_h(u)) + K_1(s - C_h(s)) + K_2|X(u + s) - X(C_h(u) + C_h(s))|) ds \\ &\leq \int_{-r}^0 (K_1 h + K_2 h + K_2 C(\omega) |(u - C_h(u)) + (s - C_h(s))|^{H-\rho}) ds \\ &\leq \int_{-r}^0 (Ch + C(\omega) h^{H-\rho}) ds \\ &\leq C(\omega) h^{H-\rho} \int_{-r}^0 ds \\ &= rC(\omega) h^{H-\rho} = C(\omega) h^{H-\rho}. \end{aligned}$$

Proof of Lemma 4.4.2

Proof. Under hypothesis $\mathcal{B}$,

$$\begin{aligned}
|\hat{Y}(t_{n+1}) - \tilde{Y}(t_{n+1})| &= \left| \sum_{i=-N}^{-1} K(t_{n+1}, t_i, X(t_{n+1} + t_i)) \times h - \sum_{i=-N}^{-1} K(t_{n+1}, t_i, \tilde{X}^h(t_{n+1} + t_i)) \times h \right| \\
&\leq h \sum_{i=-N}^{-1} |K(t_{n+1}, t_i, X(t_{n+1} + t_i)) - K(t_{n+1}, t_i, \tilde{X}^h(t_{n+1} + t_i))| \\
&\leq h \sum_{i=-N}^{-1} K_2 |X(t_{n+1} + t_i) - \tilde{X}^h(t_{n+1} + t_i)| \\
&= K_2 h \sum_{i=-N}^{-1} |X(t_{n+i+1}) - \tilde{X}^h(t_{n+i+1})|.
\end{aligned}$$

Given that for $i = -N, -(N+1), \dots, -(n+2)$, we have $t_{n+i+1} < 0$, then $X(t_{n+i+1}) = \tilde{X}^h(t_{n+i+1}) = \phi(t_{n+i+1})$. Therefore,

$$\begin{aligned}
|\hat{Y}(t_{n+1}) - \tilde{Y}(t_{n+1})| &\leq K_2 h \left[\sum_{i=-N}^{-(n+2)} |X(t_{n+i+1}) - \tilde{X}^h(t_{n+i+1})| + \sum_{i=-(n+1)}^{-1} |X(t_{n+i+1}) - \tilde{X}^h(t_{n+i+1})| \right] \\
&= K_2 h \sum_{i=-(n+1)}^{-1} |X(t_{n+i+1}) - \tilde{X}^h(t_{n+i+1})| \\
&= K_2 h \sum_{i=0}^n |X(t_i) - \tilde{X}^h(t_i)| \\
&= K_2 h \sum_{i=0}^n \varepsilon_i.
\end{aligned}$$

7.2. Technical Estimates for the Solution

In this section, we prove the Proposition 3.1.1. First, we establish some technical lemmas, which provide a priori estimates for the process X . Their proofs are deferred to the end of this section.

Let us introduce the following notation. For $0 < u < v \leq T$ and $s \in [-r, 0]$,

$$\begin{aligned}
|X^*(u)| &:= \sup_{-r \leq z \leq u} |X(z)|. \\
\varphi_{u,v}^X &:= \frac{|X(u) - X(v)|}{(u-v)^{\beta+1}}. \\
\Gamma_{u,v}(s) &:= \frac{|X(u+s) - X(v+s)|}{(u-v)^{\beta+1}}
\end{aligned} \tag{7.23}$$

Lemma 7.2.1. Let $0 < u < v \leq T$ and $s \in [-r, 0]$. Then,

$$\begin{aligned}
&\int_0^u \int_{-r}^0 \frac{|X(u+s) - X(v+s)|}{(u-v)^{\beta+1}} ds dv \\
&\leq C \left(1 + \int_0^u \frac{|X^*(v)|}{v^{-\beta}} dv + \int_0^u \int_{-v}^0 \frac{|X(u+s) - X(v+s)|}{(u-v)^{\beta+1}} ds dv \right).
\end{aligned}$$

Lemma 7.2.2. *Let $0 \leq v < u \leq T$ and $\varphi_{u,v}^X$ be defined by (7.23), then*

$$\begin{aligned} \int_0^u \varphi_{u,v}^X dv \leq & C \left(1 + \int_0^u |X^*(a)|(u-a)^{-2\beta} da + \int_0^u (u-a)^{-\beta} \int_0^a \varphi_{a,z}^X dz da \right. \\ & \left. + \int_0^u (u-a)^{-\beta} \int_0^a \int_{-a}^0 \Gamma_{a,z}(s) ds dz da \right). \end{aligned}$$

Lemma 7.2.3. *Let $\Gamma_{u,v}(s)$ defined by (7.23), then*

$$\begin{aligned} \int_0^u \int_{-r}^0 \Gamma_{u,v}(s) ds dv \leq & C \left(1 + \int_0^u |X^*(v)|v^{-\beta} dv + \int_0^u |X^*(a)|(u-a)^{-2\beta} da \right. \\ & \left. + \int_0^u (u-a)^{-\beta} \int_0^a \varphi_{a,z}^X dz da + \int_0^u (u-a)^{-\beta} \int_0^a \int_{-a}^0 \Gamma_{a,z}(s) ds dz da \right). \end{aligned}$$

Proof of Proposition 3.1.1

Proof. For $0 < t \leq T$, we are interested in estimating $|X(t)|$. From (3.1),

$$\begin{aligned} |X(t)| &= \left| X(0) + \int_0^t b(u, X(u), Y(u)) du + \int_0^t \sigma(u, X(u), Y(u)) dB^H(u) \right| \\ &\leq |X(0)| + \int_0^t |b(u, X(u), Y(u))| du + \\ &\quad C(\omega) \left(\int_0^t \frac{|\sigma(u, X(u), Y(u))|}{u^\beta} du + \int_0^t \int_0^u \frac{|\sigma(u, X(u), Y(u)) - \sigma(v, X(v), Y(v))|}{(u-v)^{\beta+1}} dv du \right). \end{aligned}$$

Let us denote

$$\begin{aligned} I_1 &= \int_0^t |b(u, X(u), Y(u))| du, \\ I_2 &= \int_0^t \frac{|\sigma(u, X(u), Y(u))|}{u^\beta} du, \\ I_3 &= \int_0^t \int_0^u \frac{|\sigma(u, X(u), Y(u)) - \sigma(v, X(v), Y(v))|}{(u-v)^{\beta+1}} dv du. \end{aligned} \tag{7.24}$$

For I_1 and I_2 we use the linear growth condition on the drift b and the diffusion σ (3.6) to obtain

$$\begin{aligned} I_1 &\leq L_3 \left(t + \int_0^t |X(u)| du + \int_0^t |Y(u)| du \right) \\ &\leq L_3 \left(t + T^\beta \int_0^t \frac{|X(u)|}{u^\beta} du + \int_0^t |Y(u)| du \right) \end{aligned} \tag{7.25}$$

and

$$I_2 \leq \int_0^t \frac{L_3(1 + |X(u)| + |Y(u)|)}{u^\beta} du. \tag{7.26}$$

On the other hand, using the linear growth condition on K (3.8) we obtain

$$\begin{aligned} |Y(u)| &\leq \int_{-r}^0 |K(u, s, X(u+s))| ds \leq \int_{-r}^0 K_3(1 + |X(u+s)|) ds \\ &\leq \int_{-r}^{-u} K_3(1 + |X(u+s)|) ds + \int_{-u}^0 K_3(1 + |X(u+s)|) ds. \end{aligned}$$

In the interval $[-r, -u]$; $-r + u < s + u < 0$. Similarly, in the interval $[-u, 0]$; $0 < u + s < u$. Then

$$|Y(u)| \leq K_3 r + \int_{-r}^{-u} K_3 |\phi(u+s)| ds + K_3 \int_{-u}^0 |X(u+s)| ds.$$

This leads to

$$|Y(u)| \leq K_3 r + K_3(u-r) \sup_{s \in [-r, -u]} |\phi(u+s)| + K_3 \int_{-u}^0 |X(u+s)| ds. \quad (7.27)$$

Thus, from (7.25), (7.26), and (7.27), and considering $C(\omega)$ a generic random variable and may change from line to line, we obtain that

$$\begin{aligned} I_1 + I_2 &\leq C(\omega) \left(t + T^\beta \int_0^t \frac{|X(u)|}{u^\beta} du + \int_0^t (u-r) \sup_{s \in [-r, -u]} |\phi(u+s)| + K_3 \int_0^t \int_{-u}^0 \frac{|X(u+s)|}{u^\beta} ds du \right) \\ &\leq C(\omega) \left((t-r)(1 + \sup_{s \in [-r, 0]} |\phi(u+s)|) + \int_0^t \frac{|X(u)|}{u^\beta} du + \int_0^t \int_{-u}^0 \frac{|X(u+s)|}{u^\beta} ds du \right) \\ &\leq C(\omega) \left((t-r)(1 + \sup_{s \in [-r, 0]} |\phi(u+s)|) + \int_0^t \frac{|X(u)|}{u^\beta} du + \int_0^t |X^*(u)| u^{1-\beta} du \right) \\ &\leq C(\omega) \left((t-r)(1 + \sup_{s \in [-r, -u]} |\phi(u+s)|) + \int_0^t \frac{|X(u)|}{u^\beta} du + \int_0^t \frac{|X^*(u)|}{u^\beta} du \right) \\ &\leq C(\omega) \left(1 + \int_0^t \frac{|X^*(u)|}{u^\beta} du \right). \end{aligned} \quad (7.28)$$

where $|X^*(u)|$ is given by (7.23).

Let us now analyze I_3 in (7.24)

$$\begin{aligned} I_3 &= \int_0^t \int_0^u \frac{|\sigma(u, X(u), Y(u)) - \sigma(v, X(v), Y(v))|}{(u-v)^{\beta+1}} dv du \\ &\leq \int_0^t \int_0^u \frac{L_1|u-v| + L_2|X(u) - X(v)| + L_2|Y(u) - Y(v)|}{(u-v)^{\beta+1}} dv du. \end{aligned} \quad (7.29)$$

Now, using the Lipschitz condition on K (3.8),

$$\begin{aligned} |Y(u) - Y(v)| &\leq \int_{-r}^0 |K(u, s, X(u+s)) - K(v, s, X(v+s))| ds \\ &\leq \int_{-r}^0 K_1|u-v| + K_2|X(u+s) - X(v+s)| ds \\ &\leq C|u-v| + C \int_{-r}^0 |X(u+s) - X(v+s)| ds. \end{aligned} \quad (7.30)$$

Therefore (7.29) becomes

$$I_3 \leq CT^{1-\beta} + C \int_0^t \int_0^u \frac{|X(u) - X(v)|}{(u-v)^{\beta+1}} dv du + C \int_0^t \int_0^u \int_{-r}^0 \frac{|X(u+s) - X(v+s)|}{(u-v)^{\beta+1}} ds dv du. \quad (7.31)$$

Combining (7.28) and (7.31) together with the notation in (7.23), we obtain

$$|X(t)| \leq C(\omega) \left(1 + \int_0^t \frac{|X^*(u)|}{u^\beta} du + \int_0^t \int_0^u \varphi_{u,v}^X dv du + \int_0^t \int_0^u \int_{-r}^0 \Gamma_{u,v}(s) ds dv du \right).$$

The final part of the proof of Proposition 3.1.1 follows the same argument used in the proof of Theorem 3.3.1 using Lemmas 7.2.1, 7.2.2 and 7.2.3 given above. In particular, once the previous estimates are obtained, the conclusion is derived by applying the Gronwall inequality in the same way as before. Since the same bounds and conditions hold, the argument can be repeated without modification, and therefore we omit the details.

Proof of Lemma 7.2.1

Proof. We assume that $-r < -u < -v < 0$. To facilitate the analysis, we divide the integral into the interval $[-r, 0]$ according to the relative positions of the points $-r$, $-u$, and $-v$. This decomposition allows us to treat each contribution separately.

Accordingly, we partition the interval $[-r, 0]$ into the three subintervals

$$[-r, -u], \quad [-u, -v], \quad [-v, 0].$$

We analyze each case separately.

Case I: $s \in [-r, -u]$. In this case, $-r \leq s \leq -u$. Adding u yields

$$-r + u \leq s + u \leq 0.$$

Similarly, adding v gives

$$-r + v \leq s + v \leq v - u < 0.$$

Therefore, we obtain $X(u+s) = \phi(u+s)$, $X(v+s) = \phi(v+s)$ and

$$|X(u+s) - X(v+s)| = |\phi(u+s) - \phi(v+s)| \leq |u-v|^{H-\rho}.$$

In this way

$$\int_{-r}^{-u} \frac{|X(u+s) - X(v+s)|}{(u-v)^{\beta+1}} ds \leq \int_{-r}^{-u} \frac{|u-v|^{H-\rho}}{(u-v)^{\beta+1}} ds \leq \int_{-r}^{-u} |u-v|^{H-\rho-\beta-1} ds \leq |u-v|^{H-\rho-\beta-1} (r-u).$$

Then

$$\int_0^u \int_{-r}^{-u} \frac{|X(u+s) - X(v+s)|}{(u-v)^{\beta+1}} ds dv \leq (r-u) \int_0^u |u-v|^{H-\rho-\beta-1} dv \leq (r-u) u^{H-\rho-\beta}.$$

Case II: $s \in [-u, -v]$

In this case, $-u < s < -v$. Thus, by adding u , we obtain $0 < s+u < u-v$. Similarly, by adding

v , we have $-u + v < s + v < 0$. That is, one quantity is positive and the other is negative. Therefore, $X(v + s) = \phi(v + s)$, and

$$\begin{aligned} |X(u + s) - X(0) + X(0) - X(v + s)| &= |X(u + s) - X(0)| + |X(0) - \phi(v + s)| \\ &\leq C(\omega) (|X(u + s) - X(0)| + |v + s|^{H-\rho}). \end{aligned} \quad (7.32)$$

In this way, since $-u < s$, it follows that $-s < u$. Therefore, $0 < -v - s < u - v$. Thus,

$$\begin{aligned} \int_{-u}^{-v} \frac{|X(u + s) - X(v + s)|}{(u - v)^{\beta+1}} ds &\leq \int_{-u}^{-v} \frac{|X(u + s) - X(0)|}{(u - v)^{\beta+1}} ds + \int_{-u}^{-v} \frac{|v + s|^{H-\rho}}{(u - v)^{\beta+1}} ds \\ &\leq \int_{-u}^{-v} \frac{|X(u + s) - X(0)|}{(u - v)^{\beta+1}} ds + \int_{-u}^{-v} (u - v)^{H-\rho-\beta-1} ds \\ &\leq \int_{-u}^{-v} 2 \frac{|X^*(u - v)|}{(u - v)^{\beta+1}} ds + (u - v)^{H-\rho-\beta} \\ &\leq 2 \frac{|X^*(u - v)|}{(u - v)^{\beta}} + (u - v)^{H-\rho-\beta}. \end{aligned}$$

Then

$$\begin{aligned} \int_0^u \int_{-u}^{-v} \frac{|X(u + s) - X(v + s)|}{(u - v)^{\beta+1}} ds dv &\leq C \left(\int_0^u \frac{|X^*(u - v)|}{(u - v)^{\beta}} dv \right) + \int_0^u (u - v)^{H-\rho-\beta} dv \\ &\leq C \left(\int_0^u |X^*(v)| v^{-\beta} dv + u^{H-\rho-\beta+1} \right). \end{aligned}$$

Caso III: $s \in [-v, 0]$

In this case, $-v < s < 0$. Adding u to the inequality, we obtain $u - v < s + u < u$. Similarly, by adding v , we get $0 < s + v < v$. Hence, both quantities are positive, and the integrals remain unchanged, i.e., no modification of the integration domains is required,

$$\int_0^u \int_{-v}^0 \frac{|X(u + s) - X(v + s)|}{(u - v)^{\beta+1}} ds dv. \quad (7.33)$$

Combining the three cases, we obtain the result.

Proof of Lemma 7.2.2

Proof. We will first study $|X(u) - X(v)|$, for $v < u$

$$\begin{aligned} |X(u) - X(v)| &\leq C \int_v^u (1 + |X(a)| + |Y(a)|) da + C \int_v^u \frac{1 + |X(a)| + |Y(a)|}{(a - v)^{\beta}} da \\ &\quad + \int_v^u \int_v^a \frac{|\sigma(a, X(a), Y(a)) - \sigma(z, X(z), Y(z))|}{(a - z)^{\beta+1}} dz da. \end{aligned}$$

From (7.27) we have that

$$|Y(a)| \leq K_3 r + K_3(a - r) \sup_{s \in [-r, -a]} |\phi(a + s)| + K_3 \int_{-a}^0 |X(a + s)| ds.$$

Thus

$$\begin{aligned}
& |X(u) - X(v)| \\
& \leq C \int_v^u \left(1 + |X(a)| + K_3 r + K_3(a-r) \sup_{s \in [-r, -a]} |\phi(a+s)| + K_3 \int_{-a}^0 |X(a+s)| ds \right) da \\
& \quad + C \int_v^u \frac{1 + |X(a)| + K_3 r + K_3(a-r) \sup_{s \in [-r, -a]} |\phi(a+s)| + K_3 \int_{-a}^0 |X(a+s)| ds}{(a-v)^\beta} da \\
& \quad + \int_v^u \int_v^a \frac{|\sigma(a, X(a), Y(a)) - \sigma(z, X(z), Y(z))|}{(a-z)^{\beta+1}} dz da \\
& \leq C \left(1 + \int_v^u |X^*(a)| da + \int_v^u \frac{|X^*(a)|}{(a-v)^\beta} da + \int_v^u \int_v^a \frac{|\sigma(a, X(a), Y(a)) - \sigma(z, X(z), Y(z))|}{(a-z)^{\beta+1}} dz da \right) \\
& \leq C \left(1 + \int_v^u \frac{|X^*(a)|}{(a-v)^\beta} da + \int_v^u \int_v^a \frac{|\sigma(a, X(a), Y(a)) - \sigma(z, X(z), Y(z))|}{(a-z)^{\beta+1}} dz da \right). \tag{7.34}
\end{aligned}$$

Hence,

$$\begin{aligned}
\int_0^u \frac{|X(u) - X(v)|}{(u-v)^{\beta+1}} dv & \leq C + C \left(\int_0^u \int_v^u \frac{|X^*(a)|}{(a-v)^\beta (u-v)^{\beta+1}} da dv \right. \\
& \quad \left. + \int_0^u \int_v^u \int_v^a \frac{|\sigma(a, X(a), Y(a)) - \sigma(z, X(z), Y(z))|}{(a-z)^{\beta+1} (u-v)^{\beta+1}} dz da dv \right).
\end{aligned}$$

Using inequality (16) in [16] and recalling that $\varphi_{u,v}$ is defined in (7.23), we obtain

$$\begin{aligned}
\int_0^u \varphi_{u,v}^X dv & \leq C + C \left(\int_0^u |X^*(a)| (u-a)^{-2\beta} da \right. \\
& \quad \left. + \int_0^u \int_v^u \int_v^a \frac{|\sigma(a, X(a), Y(a)) - \sigma(z, X(z), Y(z))|}{(a-z)^{\beta+1} (u-v)^{\beta+1}} dz da dv \right) \\
& \leq C + C \left(\int_0^u |X^*(a)| (u-a)^{-2\beta} da + \int_0^u \int_v^u \int_v^a \frac{|X(a) - X(z)|}{(a-z)^{\beta+1} (u-v)^{\beta+1}} dz da dv \right. \\
& \quad \left. + \int_0^u \int_v^u \int_v^a \frac{|Y(a) - Y(z)|}{(a-z)^{\beta+1} (u-v)^{\beta+1}} dz da dv \right).
\end{aligned}$$

By making a change of integration variables as in [16] equation (18), we obtain

$$\begin{aligned}
\int_0^u \varphi_{u,v}^X dv & \leq C + C \left(\int_0^u |X^*(a)| (u-a)^{-2\beta} da + \int_0^u (u-a)^{-\beta} \int_0^a \frac{|X(a) - X(z)|}{(a-z)^{\beta+1}} dz da \right. \\
& \quad \left. + \int_0^u (u-a)^{-\beta} \int_0^a \frac{|Y(a) - Y(z)|}{(a-z)^{\beta+1}} dz da \right) \\
& = C + C \left(\int_0^u |X^*(a)| (u-a)^{-2\beta} da + \int_0^u (u-a)^{-\beta} \int_0^a \varphi_{a,z}^X dz da \right. \\
& \quad \left. + \int_0^u (u-a)^{-\beta} \int_0^a \frac{|Y(a) - Y(z)|}{(a-z)^{\beta+1}} dz da \right) \\
& \leq C \left(1 + \int_0^u |X^*(a)| (u-a)^{-2\beta} da + \int_0^u (u-a)^{-\beta} \int_0^a \varphi_{a,z}^X dz da \right. \\
& \quad \left. + \int_0^u (u-a)^{-\beta} \int_0^a \int_{-a}^0 \Gamma_{a,z}(s) ds dz da \right). \tag{7.35}
\end{aligned}$$

The last inequality follows from (7.30) and Lemma 7.2.1.

Proof of Lemma 7.2.3

Proof. From Lemma (7.2.1), we obtain the following

$$\int_0^u \int_{-r}^0 \Gamma_{u,v}(s) ds dv \leq C \left(1 + \int_0^u |X^*(v)| v^{-\beta} dv + \int_0^u \int_{-v}^0 \frac{|X(u+s) - X(v+s)|}{(u-v)^{\beta+1}} ds dv \right). \quad (7.36)$$

Following the same analysis as in (7.34), we obtain

$$\begin{aligned} & |X(u+s) - X(v+s)| \\ & \leq C \left(1 + \int_{v+s}^{u+s} \frac{|X^*(a)|}{(a-(v+s))^\beta} da + \int_{v+s}^{u+s} \int_{v+s}^a \frac{|\sigma(a, X(a), Y(a)) - \sigma(z, X(z), Y(z))|}{(a-z)^{\beta+1}} dz da \right. \\ & \leq C \left(1 + \int_{v+s}^{u+s} \frac{|X^*(a)|}{(a-(v+s))^\beta} da + \int_{v+s}^{u+s} \int_{v+s}^a \frac{|X(a) - X(z)|}{(a-z)^{\beta+1}} dz da + \int_{v+s}^{u+s} \int_{v+s}^a \frac{|Y(a) - Y(z)|}{(a-z)^{\beta+1}} dz da \right). \end{aligned}$$

Writing this in (7.36), we obtain the following

$$\begin{aligned} & \int_0^u \int_{-r}^0 \Gamma_{u,v}(s) ds dv \\ & \leq C \left(1 + \int_0^u |X^*(v)| v^{-\beta} dv + \int_0^u \int_{-v}^0 \frac{|X(u+s) - X(v+s)|}{(u-v)^{\beta+1}} ds dv \right) \\ & \leq C \left(1 + \int_0^u |X^*(v)| v^{-\beta} dv + \int_0^u \int_{-v}^0 \int_{v+s}^{u+s} \frac{|X^*(a)|}{(a-(v+s))^{-\beta} (u-v)^{\beta+1}} dadsv \right. \\ & \quad \left. + \int_0^u \int_{-v}^0 \int_{v+s}^{u+s} \int_{v+s}^a \frac{|X(a) - X(z)| + |Y(a) - Y(z)|}{(a-z)^{\beta+1} (u-v)^{\beta+1}} dz dadsv \right). \quad (7.37) \end{aligned}$$

For the last integral in (7.37), a change of variables followed by direct computation yields

$$\begin{aligned} & \int_0^u \int_{-v}^0 \int_{v+s}^{u+s} \int_{v+s}^a \frac{|X(a) - X(z)| + |Y(a) - Y(z)|}{(a-z)^{\beta+1} (u-v)^{\beta+1}} dz dadsv \\ & = \int_0^u \int_0^a \int_{a-u}^0 \int_{-s}^{z-s} \frac{|X(a) - X(z)| + |Y(a) - Y(z)|}{(a-z)^{\beta+1} (u-v)^{\beta+1}} dv ds dz da \\ & \leq C \int_0^u \int_0^a \int_{a-u}^0 \frac{|X(a) - X(z)| + |Y(a) - Y(z)|}{(a-z)^{\beta+1} (u-(z-s))^\beta} ds dz da \\ & \leq C \int_0^u \int_0^a (u-z)^{1-\beta} \frac{|X(a) - X(z)| + |Y(a) - Y(z)|}{(a-z)^{\beta+1}} dz da \\ & \leq C \int_0^u \int_0^a (u-a)^{-\beta} \frac{|X(a) - X(z)| + |Y(a) - Y(z)|}{(a-z)^{\beta+1}} dz da, \quad (7.38) \end{aligned}$$

the last inequality is due to $z < a$ implies $u-a < u-z$, and then $\frac{1}{u-z} < \frac{1}{u-a}$. We study the second integral of (7.37), making a substitution, then changing the limits of the integral and

taking into account that s is non-positive, we have

$$\begin{aligned}
 & \int_0^u \int_{-v}^0 \int_{v+s}^{u+s} \frac{|X^*(a)|}{(a-(v+s))^{-\beta}(u-v)^{\beta+1}} dadsv \\
 &= \int_0^u \int_{-v}^0 \int_v^u \frac{|X^*(a+s)|}{(a-v)^{-\beta}(u-v)^{\beta+1}} dadsv = \int_0^u \int_{-a}^0 \int_{-s}^a \frac{|X^*(a+s)|}{(a-v)^{\beta}(u-v)^{\beta+1}} dv ds da \\
 &\leq \int_0^u \int_{-a}^0 \int_0^a \frac{|X^*(a)|}{(a-v)^{\beta}(u-v)^{\beta+1}} dv ds da \leq C \int_0^u \int_{-a}^0 |X^*(a)|(u-a)^{-2\beta} ds da \\
 &\leq C \int_0^u |X^*(a)|(u-a)^{-2\beta} da.
 \end{aligned} \tag{7.39}$$

Combining (7.37), (7.38) and (7.39) yields

$$\begin{aligned}
 \int_0^u \int_{-r}^0 \Gamma_{u,v}(s) ds dv &\leq C \left(1 + \int_0^u |X^*(v)|v^{-\beta} dv + \int_0^u |X^*(a)|(u-a)^{-2\beta} da \right. \\
 &\quad \left. + \int_0^u (u-a)^{-\beta} \int_0^a \frac{|X(a)-X(z)|}{(a-z)^{\beta+1}} dz da + \int_0^u (u-a)^{-\beta} \int_0^a \frac{|Y(a)-Y(z)|}{(a-z)^{\beta+1}} dz da \right).
 \end{aligned}$$

Finally, from (7.30) and Lemma 7.2.1,

$$\begin{aligned}
 \int_0^u \int_{-r}^0 \Gamma_{u,v}(s) ds dv &\leq C \left(1 + \int_0^u |X^*(v)|v^{-\beta} dv + \int_0^u |X^*(a)|(u-a)^{-2\beta} da \right. \\
 &\quad \left. + \int_0^u (u-a)^{-\beta} \int_0^a \frac{|X(a)-X(z)|}{(a-z)^{\beta+1}} dz da + \int_0^u (u-a)^{-\beta} \int_0^a \int_{-a}^0 \Gamma_{a,z}(s) ds dz da \right).
 \end{aligned}$$

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